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On Convolutions

By

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Let R^n stand for the n-dimensional Euclidean space to which our consideration of distributions will be confined. For our temporary purpose we shall denoted by $S \bar{*} T$ the convolution of distributions S and T in the generalized sense of C. Chevalley [2], and reserve the symbol $S * T$ for the one defined by L. Schwartz [7]. For any subset $A \subset (\mathscr{D}')$ let us denote by A^*, called c-dual of A in this paper, the set of distributions composable with each distribution of A in the sense of C. Chevalley. The present paper is concerned with the theory of convolution between the spaces of distributions.

Section 1 contains several auxiliary results which may serve as tools in the subsequent sections. A locally convex space E is called admissible provided that $(\mathscr{D}) \subset E \subset (\mathscr{D}')$, $(\mathscr{D})$ is dense in E and the injections $(\mathscr{D}) \to E$ and $E \to (\mathscr{D}')$ are continuous. In section 2 we give our principal theorem (Theorem 3 below) which asserts that if a given distribution T belengs to the c-dual E^* of an admissible barrelled space E, then $\check{T} * \varphi \in E'$ for any $\varphi \in (\mathscr{D})$. The converse of this statement is also true for the usual spaces of distributions as will be shown in section 3. If $T \in E^*$ and $T \bar{*} E \subset G$ where G is a permitted space or its dual, then the application $S \to S \bar{*} T$ is continuous from E into G. This makes possible to insure that if the admissible barrelled spaces E and F are composable with each other and $E \bar{*} F \subset G$, then the convolution product $S \bar{*} T$ is hypocontinuous as a bilinear application of $E \times F$ into G. By a composition operator $\mathscr{L}$ on E we mean a continuous linear application of E into $(\mathscr{D}')$ which is commutative with any translation. We show (Theorem 4 below) that then $\mathscr{L}(S) = S \bar{*} T$ where T is a certain distribution determined by $\mathscr{L}$. Finally we conclude section 3 with the discussion of determining the c-duals of the concrete spaces of distributions appeared in L. Schwartz [7].

Throughout this paper we adopt the notation of L. Schwartz [7] unless otherwise specifically mentioned. We also assume the knowledge of elements of topological linear spaces.

1. In this section we shall establish some preliminary results which will be needed in the later sections. Grothendieck has proved the

THEOREM A. ([5], Chap. I, p. 16) *Let E be a locally convex space and let F and $\{F_i\}$ $i=1,2,\cdots$, be (F) spaces. Let u be a continuous linear* *applicaiion of F into E and let, for each index i, u_i be a continuous linear application of F_i into E. Suppose $u(F)\subset\bigcup_i u_i(F_i)$. Then there exists an index i such that*

$$u(F)\subset u_i(F_i).$$

If in this case u_i is univalent, then there exists a continuous linear application v of F into F_i such that

$$u=u_i\circ v.$$

First we give here a slight generalization of this theorem. The proof will be carried out along the same line with necessary modifications.

THEOREM 1. *Let E be a locally convex space and let F, G and $\{F_i\}$, $i=1,2,\cdots$, be (F) spaces. Let u be a continuous bilinear application of $F\times G$ into E and let, for each index i, u_i be a continuous linear application of F_i into E. Suppose $u(F\times G)\subset\bigcup_i u_i(F_i)$. Then there exists an index i such that*

$$u(F\times G)\subset u_i(F_i).$$

If, in this case, u_i is univalent, then there exists a continuous bilinear application v of $F\times G$ into F_i snch that

$$u=u_i\circ v.$$

PROOF. Assume the contrary. Then there exists a sequence $\{(x_i, y_i)\}$ of elements of $F\times G$ with $u(x_i, y_i)\notin u_i(F_i)$, $i=1,2,\cdots$. Set, for any index i and j, $H_{ij}=\{x;\ u(x, y_j)\in u_i(F_i),\ x\in F\}$ and $H_i=\bigcap_j H_{ij}$. For any fixed $x\in F$, $u_x: y\to u(x, y)$ is a continuous linear application of G into E such that $u_x(G)\subset\bigcup_i u_i(F_i)$. Hence by Theorem A there exists an index i such that $u(x, G)\subset u_i(F_i)$. It follows from this, that $F=\bigcup_i H_i$. Thus for some index i, H_i and hence H_{ij}, $j=1,2,\cdots$, are all sets of the second category. For any index j, $X_j=\{(x, z);\ u(x, y_j)=u_i(z),\ x\in F,\ z\in F_i\}$ is an (F) space since it is a closed linear subset of the (F) space $F\times F_i$. The projection of X_j into $F:(x, z)\to x$ has the image H_{ij} of the second category. Hence $H_{ij}=F$ for any $j=1,2,\cdots$ ([1], p. 38). We have thus concluded that $H_i=F$ and $u(x_i, y_i)\in u_i(F_i)$, a contradiction. This proves the first part. The second statement is proved as follows. Let v be the bilinear application $F\times G$ into F_i defined by $(x, y)\to u_i^{-1}\circ u(x, y)$, then we have $u=u_i\circ v$. Owing to Theorem A we can conclude that v is separately continuous, accordingly, jointly continuous because F and G are (F) spaces. This completes the proof.

A locally convex space F is said to be of *type* (LF) if it is an inductive limit of an increasing sequence of (F) spaces $F_i\subset F$. If each F_i is a Banach space, we then call F of *type* (LB).

COROLLARY. *Let F be any space of type (LB) contained in $(\mathcal{D}')$ such* *that the injection $F \to (\mathcal{D}')$ is continuous. If u is a continuous linear application of $(\mathcal{D})$ into $(\mathcal{D}')$, and $u(\varphi * \psi) \in F$ for any $\varphi, \psi \in (\mathcal{D})$, then $u(\varphi) \in F$ for any $\varphi \in (\mathcal{D})$ and the linear application of $(\mathcal{D})$ into $F: \varphi \to u(\varphi)$ is continuous.*

Proof. Let $\{F_i\}$ be the sequence of definition of the space F. For any relatively compact open subset Ω in R^n, the bilinear application $u(\varphi * \psi)$ of $(\mathcal{D}_{\bar{\Omega}}) \times (\mathcal{D}_{\bar{\Omega}})$ into $(\mathcal{D}')$ is continuous. It follows from Theorem 1 that we can find an index i such that $u(\varphi * \psi) \in F_i$ for any $\varphi, \psi \in (\mathcal{D}_{\bar{\Omega}})$ and $u(\varphi * \psi)$ is a continuous application of $(\mathcal{D}_{\bar{\Omega}}) \times (\mathcal{D}_{\bar{\Omega}})$ into F_i. By a suitable choice of a parametrix h such that $\delta = \Delta^m h + \xi$ where $\xi \in (\mathcal{D}_{\bar{\Omega}})$, we have $\varphi = \Delta^m \varphi * h + \varphi * \xi$. And therefore we can apply the argument frequently used in ([7], Chap. VI) to conclude that u is a continuous application of $(\mathcal{D})$ into F. This completes the proof.

For our later use we state the following

Lemma 1. *For any given distribution T, we have:*

(1) *If $T * \varphi * \psi \in (\mathcal{O}_M)$ for any $\varphi, \psi \in (\mathcal{D})$, then $T \in (\mathcal{S}')$;*

(2) *If $T * \varphi * \psi \in (\mathcal{S})$ for any $\varphi, \psi \in (\mathcal{D})$, then $T \in (\mathcal{O}'_C)$;*

(3) *If $T * \varphi * \psi \in L^p$ for any $\varphi, \psi \in (\mathcal{D})$, where $1 \le p \le +\infty$, then $T \in (\mathcal{D}'_{L^p})$.*

Proof. As each of these statements may be proved essentially in a similar manner, we shall give here only the proof of (1). It is enough to show that $T * \varphi$ is slowly increasing for any $\varphi \in (\mathcal{D})$. To this end we consider the Banach spaces $\{F_i\}$ of the spaces of continuous functions $f(x)$ defined on R^n with the finite $\sup_{x \in R^n} \frac{|f(x)|}{1+|x|^i}$ as norm. The preceding Corollary shows us that $T * \varphi$ belongs to some F_i, as desired.

A locally convex space is said to be of *type* (β) if it is an inductive limit of a family of Banach spaces ([5], Chap. I, p. 17). Then the space is barrelled and bornological. Any semi-complete bornological space is of type (β), as shown by Grothendieck ([5], Chap. I, p. 17); so that the usual spaces of distributions are of this type.

By an *admissible space* we mean any locally convex space E of distributions with the properties:

(i) $(\mathcal{D}) \subset E \subset (\mathcal{D}')$ where the injections $(\mathcal{D}) \to E$ and $E \to (\mathcal{D}')$ are continuous;

(ii) $(\mathcal{D})$ is dense in E.

If E is admissible, then its dual E' may be considered as a linear subset of $(\mathcal{D}')$. An admissible (F) space E with the admissible dual E' is distinguished, because a sequence is dense in E' [4].

Let $\{\rho_k\} \subset (\mathcal{D})$ be a sequence of regularizations and let $\{\alpha_k\} \subset (\mathcal{D})$ be a sequence of multiplicators, where $\check{\rho}_k = \rho_k$, $\rho_k \ge 0$, $\int \rho_k(x)\, dx = 1$, support of ρ_k

tends to 0 in R^n, $0 \leq \alpha_k \leq 1$, $\alpha_k \to 1$ in $(\mathcal{E})$ and $\{\alpha_k\}$ is bounded in $(\mathcal{B})$. Now we define the applications Φ_k and ${}^t\Phi_k$ of $(\mathcal{D}')$ into $(\mathcal{D})$ by the relations

$$\Phi_k(T) = \alpha_k(T * \rho_k) \quad \text{and} \quad {}^t\Phi_k(T) = (\alpha_k T) * \rho_k.$$

An admissible space E is said to be *permitted* [8] if the following condition is satisfied:

(iii) For any $T \in E$, $\Phi_k(T) \to T$ and ${}^t\Phi_k(T) \to T$ in E.

It was shown by Y. Hirata [6] that a permitted space E is bornological if it is barrelled and $\Phi_k(T) \to T$ in Mackey sense for any $T \in E$, and that its dual E' is permitted if E is semi-reflexive.

Now we show the

THEOREM 2. *Let E and F be locally convex spaces with $F \subset (\mathcal{D}')$, and let $\mathcal{L}$ be a continuous linear application of E into $(\mathcal{D}')$ with range in F. Then $\mathcal{L}$, as an application of E into F, will be continuous under any one of the following conditions*:

(1) *E is of type (β) and we are able to introduce into F a stronger topology under which F is of type (LF) so that the injection $F \to (\mathcal{D}')$ may be continuous*:

(2) *E is a barrelled space and F is a permitted space or its dual*:

(3) *E is a semi-complete space of type (DF) and F is a permitted space.*

PROOF. The proof of Case (1) may be reduced to the case where E is a Banach space and F is a space of type (LF). Then it is a direct consequence of Theorem A. For the proofs of Case (2) and (3) we note that if u_k, $k = 1, 2, \cdots$, is a sequence of continuous linear applications of a barrelled space E into a locally convex space and $u_k(x) \to u(x)$ for each $x \in E$, then u is also continuous. This is true as well for a semi-complete space E of type (DF), since the intersection of a sequence of absolutely convex closed neighbourhoods of zero, so far as it is absorbing, is also a neighbourhood of zero [4]. If we now put $u_k(x) = \Phi_k \circ \mathcal{L}(x)$, then $u_k(x) \to \mathcal{L}(x)$ strongly or weakly according to the cases whether F is a permitted space or its dual. Therefore the proof will be completed if we show that each u_k is continuous. But this follows at once from the fact that the injection $(\mathcal{D}) \to F$ is continuous. This completes the proof.

In the next section we shall make use of a theorem of Schwartz ([8], II, p. 53) to which we shall refer as *Theorem S*. It asserts that *any continuous linear application $\mathcal{L}$ of $(\mathcal{D})$ into $\mathcal{D}')$ commutative with any translation is expressible as $\mathcal{L}(\varphi) = T * \varphi$ with a uniquely determined distribution T.* His proof is simple enough. But it will be of some interest to see that this theorem follows also from his "Kernel Theorem." Following the notation of Schwartz [8], let $K_{x,\xi} \in (\mathcal{D})_{x,\xi}$ be the kernel distribution associated with

$\mathscr{L}$. The condition that $\mathscr{L}$ is commutative with any translation is now written in the form

$$\underset{x}{\tau_h}\underset{\xi}{\tau_h}K_{x,\xi}=K_{x,\xi} \quad \text{for any} \quad h\in R^n,$$

where $\underset{x}{\tau_h}$ and $\underset{\xi}{\tau_h}$ denote the translations concerning the variables x and ξ respectively. Consider the application $\mu: y=x-\xi,\ \eta=\xi$ of (x,ξ)-space $R^n\times R^n$ onto the (y,η)-space $R^n\times R^n$. First we show that $\underset{\eta}{\tau_h}\mu K=\mu K$ for any $h\in R^n$, where μK denotes the image of K by μ. For any $\chi\in(\mathscr{D})_{y,\eta}$ we have

$$\langle\underset{\eta}{\tau_h}\mu K, \chi\rangle=\langle K, \mu*\underset{\eta}{\tau_{-h}}\chi\rangle.$$

Then a simple calculation yields that

$$(\mu*\underset{\eta}{\tau_{-h}}\chi)(x,\xi)=(\underset{x}{\tau_{-h}}\underset{\xi}{\tau_{-h}}\mu*\chi)(x,\xi).$$

Therefore,

$$\begin{aligned}\langle\underset{\eta}{\tau_h}\mu K, \chi\rangle&=\langle\underset{x}{\tau_h}\underset{\xi}{\tau_h}K, \mu*\chi\rangle\\&=\langle K, \mu*\chi\rangle=\langle\mu K, \chi\rangle\end{aligned}$$

as desired. This implies that μK is independent of η and so there exists a uniquely determined distribution T such that

$$\langle\mu K, \chi\rangle=\int T_y\Big(\int\chi(y,\eta)\,d\eta\Big)\,dy.$$

Then we obtain that

$$\begin{aligned}\langle K\varphi, \psi\rangle&=\langle K, \varphi\psi\rangle=\langle\mu K, \mu*^{-1}\varphi\psi\rangle\\&=\int T_y\Big(\psi(\zeta)\varphi(\zeta-y)\,d\zeta\Big)\,dy\\&=\langle T, \check{\varphi}*\psi\rangle=\langle T*\varphi, \psi\rangle\end{aligned}$$

for any $\varphi(\xi)\in(\mathscr{D})_\xi$ and $\psi(x)\in(\mathscr{D})_x$. Therefore,

$$\mathscr{L}(\varphi)=\int K_{x,\xi}\varphi(\xi)d\xi=T*\varphi.$$

This completes the proof.

2. In this section we develop our main results. We first recall the notion of the generalized convolution of Chevalley [2]. If S and T are any given distributions satisfying the following condition:

(*) $\qquad S*\varphi(x)\cdot\check{T}*\psi(x)$ belongs to L for any $\varphi, \psi\in(\mathscr{D})$,

then we say that *S and T are composable with each other* or that *the generalized convolution $U=S\bar{*}T$ is defined*, where U is uniquely determined by the relation

$$\langle U*\varphi, \psi\rangle=\int S*\varphi(x)\cdot\check{T}*\psi(x)\,dx.$$

To see this, let us consider the bilinear form $B(\varphi,\psi)$ defined by

$$B(\varphi,\psi)=\int S*\varphi(x)\cdot\check{T}*\psi(x)\,dx.$$

Applying now Theorem 2 to $(\mathscr{D})$ and L, $B(\varphi,\psi)$ will become hypocontinuous since $(\mathscr{D})$ is barrelled. Hence it determines uniquely a continuous linear application $\mathcal{L}$ of $(\mathcal{D})$ into $(\mathcal{D}')$ such that $B(\varphi, \psi)=\langle \mathcal{L}(\varphi), \psi \rangle$. We show that $\mathcal{L}$ is commutative with any translation τ_h.

$$\begin{aligned}\langle \mathcal{L}(\tau_h\varphi), \psi\rangle &= \int S*\tau_h\varphi(x)\cdot \check{T}*\psi(x)\,dx \\ &= \int \tau_h S*\varphi(x)\cdot \check{T}*\psi(x)\,dx \\ &= \int S*\varphi(x)\cdot \check{T}*\tau_{-h}\psi(x)\,dx \\ &= \langle \mathcal{L}(\varphi), \tau_{-h}\psi\rangle = \langle \tau_h\mathcal{L}(\varphi), \psi\rangle.\end{aligned}$$

Accordingly $\mathcal{L}\tau_h=\tau_h\mathcal{L}$, as desired. Then Theorem S shows us that the distribution U in question does exist.

Chevalley has shown [2] that *if* $S*\varphi$ *and* T *are composable for any* $\varphi\in(\mathcal{D})$, *then* S *and* T *are also composable.* This is an immediate consequence of Corollary to Theorem 1. In fact it is sufficient to consider the application $u(\varphi)=S*\varphi\cdot\check{T}*\psi$ of $(\mathcal{D})$ into $(\mathcal{D}')$.

For any subset $A\subset(\mathcal{D}')$, we denote by A^*, called the *c-dual* of A, the set of distributions composable with any distribution of A. It is obvious that (1) $A\subset B$ implies $A^*\supset B^*$, (2) $A\subset A^{**}$, (3) $A^*=A^{***}$ (4) $A^{\vee *}=A^{*\vee}$. Thus the mapping $A\to A^{**}$ is a *closure operation* in the sense of Birkhoff. We say that A *is c-closed* or *c-reflexive* if $A=A^{**}$. Moreover we have $A^*\supset(\mathcal{E}')$ and $A^*\bar{*}(\mathcal{E}')\subset A^*$. In particular $\tau_h A^*=A^*$ and $DA^*\subset A^*$ for any translation τ_h and for any derivation D. By the result of Chevalley cited above $T*(\mathcal{D})\subset A^*$ is equivalent to $T\in A^*$.

THEOREM 3. *Let E be any admissible barrelled space. If $T\in E^*$ then,*

(1) *$\check{T}*\varphi\in E'$ for any $\varphi\in(\mathcal{D})$ and the application $\varphi\to\check{T}*\varphi$ of $(\mathcal{D})$ into E' is continuous;*

(2) *the application $S\to S\bar{*}T$ of E into $(\mathcal{D}')$ is continuous.*

PROOF. First we prove (2). Let $\{\rho_k\}$ be a sequence of regularizations, and put $\mathcal{L}_k(S)=(S\bar{*}T)*\rho_k$. Then $\mathcal{L}_k(S)\to S\bar{*}T$ for any $S\in E$. Hence as noted in the proof of Theorem 2, it is enough to observe that each $\mathcal{L}_k$ is continuous. But in view of Theorem 2 the linear application of E into L defined by $S\to S*\rho_k(x)\cdot\check{T}*\varphi(x)$ is continuous for any $\varphi\in(\mathcal{D})$. Thus

$$\langle \mathcal{L}_k(S), \varphi\rangle = \int S*\rho_k(x)\cdot\check{T}*\varphi(x)\,dx$$

is a continuous linear form of S, that is $\mathcal{L}_k(S)$ is weakly and hence strongly continuous, as desired. Then the conjugate ${}^t\mathcal{L}(\varphi)=\check{T}*\varphi$ of $(\mathcal{D})$ into E' is also continuous. This completes the proof.

We note that in the statement of the theorem the continuity of the application $\varphi\to\check{T}*\varphi\in E'$ is an immediate consequence of Theorem 2 if E is permitted. On the other hand, as clear from the proof, the continuity of the application $S\to S\bar{*}T$ follows from the mere assumption that E is a barrelled space or a semi-complete space of type (DF) because the assumption that E is admissible is not used there.

COROLLARY. *Let E, F and G be locally convex spaces contained in* $(\mathcal{D}')$ with continuous injections. *If $E\subset F^*$ and $E\bar{*}F\subset G$, then the bilinear application $(S, T)\to S\bar{*}T$ of $E\times F$ into G is hypocontinuous under any one of the following conditions*:

(1) *E and F are spaces of type (β) and we are able to introduce into G a stronger topology under which G is of type (LF) so that the injection $G\to(\mathcal{D}')$ may be continuous*:

(2) *E and F are barrelled spaces, and G is a permitted space or its dual.*

PROOF. It follows from the combination of the above remark and Theorem 2 that the bilinear application is separately continuous under any one of the conditions stated in the theorem, and therefore hypocontinuous since E and F are barrelled. This completes the proof.

As to the usual spaces of distributions, the condition that $\check{T}*\varphi\in E'$ for any $\varphi\in(\mathcal{D})$ is also sufficient for $T\in E^*$ as will be shown in section 3, so that we shall say that an admissible space E is *c-regular* if $T\in E^*$ is equivalent to the condition: $\check{T}*\varphi\in E'$ for any $\varphi\in(\mathcal{D})$.

Let E be an admissible barrelled space, and let T be any element of E^*. Then the linear application $\mathcal{L}(S)=S\bar{*}T$ of E in to $(\mathcal{D}')$ is continuous and obviously $\mathcal{L}(\tau_h\varphi)=\tau_h\mathcal{L}(\varphi)$ holds true for any $\varphi\in(\mathcal{D})$. We shall say that any linear continuous application $\mathcal{L}$ of a locally convex space (with the condition (i) of the preceding section) into $(\mathcal{D}')$ is a *composition operator* if $\mathcal{L}(\tau_h\varphi)=\tau_h\mathcal{L}(\varphi)$ for any $\varphi\in(\mathcal{D})$.

THEOREM 4. *Let E be any c-regular barrelled space. Then with any composition operator $\mathcal{L}$ on E we may associate a unique distribution $T\in E^*$ such that $\mathcal{L}(S)=S\bar{*}T$ for any $S\in E$.*

PROOF. On account of Theorem S, there exists a unique distribution T such that $\mathcal{L}(\varphi)=T*\varphi$ for any $\varphi\in(\mathcal{D})$. The conjugate ${}^t\mathcal{L}(\varphi)=\check{T}*\varphi$ is a linear application of $(\mathcal{D})$ into E'. This implies that $T\in E^*$ since E is c-regular. Owing to the continuity of $\mathcal{L}$ we have at once $\mathcal{L}(S)=S\bar{*}T$ for any $S\in E$. This completes the proof.

We remark that in this theorem if the range of $\mathcal{L}$ is contained in a permitted space or its dual, then $\mathcal{L}$, as an application of E into this space, is continuous.

COROLLARY. *Let E be a c-regular barrelled space. If for given $T_i\in E^*$, $i=1, 2$, $T_1\bar{*}E\subset E$ and $T_2\bar{*}E\subset E$ hold true, then T_1 and T_2 are composable* *and further* $T_1 \overline{*}\, T_2 \in E^*$ *and* $(T_1 \overline{*}\, T_2) \overline{*}\, S = T_1 \overline{*}\, (T_2 \overline{*}\, S)$ *for any* $S \in E$. *Therefore if* $E \subset E^*$ *and* $E \overline{*}\, E^* \subset E$ *then* E *is a module over* E^*.

PROOF. From the hypotheses we see that T_1 and $T_2 * \varphi$ are composable for any $\varphi \in (\mathcal{D})$, and therefore T_1 and T_2 are composable. Now consider the linear application $\mathcal{L}$ of E into $(\mathcal{D}')$ defined by $S \to T_1 \overline{*}\, (T_2 \overline{*}\, S)$. We show that $\mathcal{L}$ is continuous. To this end let $\mathcal{L}_k(S) = \mathcal{L}(S) * \rho_k$ where $\{\rho_k\}$ is a sequence of regularizations. As noted in the proof of Theorem 2, it is enough to show that $\mathcal{L}_k$ is continuous since $\mathcal{L}_k(S) \to \mathcal{L}(S)$ in $(\mathcal{D}')$. But in virtue of Theorem 2 the application $S \to (T_2 \overline{*}\, S) * \rho_k(x) \cdot \check{T}_1 * \varphi(x)$ of E into L is continuous and therefore the linear form:

$$S \to \langle \mathcal{L}_k(S), \varphi \rangle = \int (T_2 * S) * \rho_k(x) \cdot \check{T}_1 * \varphi(x)\, dx$$

is continuous for any $\varphi \in (\mathcal{D})$. This means that $\mathcal{L}_k$ is continuous, as desired. Then $\mathcal{L}$ is a composition operator since $\mathcal{L}(\tau_h \varphi) = \tau_h \mathcal{L}(\varphi)$ for any $\varphi \in (\mathcal{D})$ so that $\mathcal{L}(S) = S \overline{*}\, T$ for any $S \in E$ where T is a uniquely determined distribution $\in E^*$. This in turn implies that $T \overline{*}\, S = T_1 \overline{*}\, (T_2 \overline{*}\, S)$ for any $S \in E$ and, in particular, $T * \varphi = T_1 \overline{*}\, (T_2 * \varphi) = (T_1 \overline{*}\, T_2) * \varphi$ for any $\varphi \in (\mathcal{D})$. Hence $T = T_1 \overline{*}\, T_2$ and thus $T_1 \overline{*}\, T_2 \in E^*$. The rests of the statements are clear. This completes the proof.

3. This section is devoted to the determination of c-duals of the various spaces of distributions discussed in Schwartz [7]. In the following theorem the symbol $E \Rightarrow F$ is used to denote the relation $F = E^*$.

THEOREM 5.

(1) $(\mathcal{D}) \Rightarrow (\mathcal{D}') \Rightarrow (\mathcal{E}') \Rightarrow (\mathcal{D}')$;

(2) $(\mathcal{E}) \Rightarrow (\mathcal{E}') \Rightarrow (\mathcal{D}') \Rightarrow (\mathcal{E}')$;

(3) $(\mathcal{S}) \Rightarrow (\mathcal{S}') \Rightarrow (\mathcal{O}'_C) \Rightarrow (\mathcal{S}')$;

(4) $$\begin{array}{ccccccc} (\mathcal{O}_C) & \Rightarrow & (\mathcal{O}'_C) & \Rightarrow & (\mathcal{S}') & \Rightarrow & (\mathcal{O}'_C); \\ & \Nearrow & & \Nearrow & & & \\ (\mathcal{O}_M) & & (\mathcal{O}'_M) & & & & \end{array}$$

(5) $(\mathcal{D}_{L^p}) \Rightarrow (\mathcal{D}'_{L^{p'}}) \Rightarrow (\mathcal{D}'_{L^p}) \Rightarrow (\mathcal{D}'_{L^{p'}})$;

(6) $$\begin{array}{ccccccc} (\mathcal{B}) & \Rightarrow & (\mathcal{D}'_{L^1}) & \Rightarrow & (\mathcal{B}') & \Rightarrow & (\mathcal{D}'_{L^1}); \\ & \Nearrow & & & & \Nearrow & \\ (\dot{\mathcal{B}}) & & & & (\dot{\mathcal{B}}') & & \end{array}$$

(7) $(\mathcal{D}_+) \Rightarrow (\mathcal{D}'_+) \Rightarrow (\mathcal{D}'_+)$;

(8) $(\mathcal{D}_-) \Rightarrow (\mathcal{D}'_-) \Rightarrow (\mathcal{D}'_-)$.

PROOF. The proof will be carried out in several steps. For the proof of (1) and (2) it is sufficient to show that $(\mathcal{E}) \Rightarrow (\mathcal{E}')$. $(\mathcal{E}') \subset (\mathcal{E})^*$ is well-known. Conversely suppose that $T \in (\mathcal{E})^*$. Then Theorem 3 implies that $T * \varphi \in (\mathcal{E}')$ for any $\varphi \in (\mathcal{D})$. Now we show that $T \in (\mathcal{E}')$. Let $\{\Omega_k\}$ be an increasing sequence of relatively compact open subsets of R^n such that

$R^n = \bigcup_k \Omega_k$. Since by Theorem 2 $\mathcal{L}(\varphi) = T * \varphi$ is a continuous endomorphism of $(\mathcal{D})$, it follows from Theorem A that we can find an integer k_1 such that $\mathcal{L}(\mathcal{D}_{\bar{\Omega}_1}) \subset \mathcal{D}_{\bar{\Omega}_{k_1}}$. Now take a sequence of regularizations $\{\rho_k\}$ with $\rho_k \in (\mathcal{D}_{\bar{\Omega}_1})$, $k=1,2,\cdots$. Then $T * \rho_k \to T$ in $(\mathcal{D}')$ and the supports of $T * \rho_k$ are all contained in $\bar{\Omega}_{k_1}$. Therefore $T \in (\mathcal{E}')$, as desired. This was also shown by Ehrenpreis [3], but our method may also be applied to the proof of $(\mathcal{D}'_+) \Rightarrow (\mathcal{D}'_+)$. The details are omitted. On the other hand it is well-known $(\mathcal{D}'_+) \subset (\mathcal{D}_+)^*$. For any $T \in (\mathcal{D}_+)^*$ we have $T * \varphi \in (\mathcal{D}'_+)$ and therefore $T * \varphi \in (\mathcal{D}_+)$ and in turn $T \in (\mathcal{D}'_+)$. This proves $(\mathcal{D}_+) \Rightarrow (\mathcal{D}'_+)$. The proof of (7) is thus completed and (8) may be proved in a similar manner.

It is known that any two different types of spaces of (3) are composable. Let $T \in (\mathcal{S})^*$ then $T * \varphi \in (\mathcal{S}')$ for any $\varphi \in (\mathcal{D})$ and therefore $T * \psi * \varphi \in (\mathcal{O}_M)$ for any $\varphi, \psi \in (\mathcal{D})$. Hence $T \in (\mathcal{S}')$ by Lemma 1. This implies that $(\mathcal{S}) \Rightarrow (\mathcal{S}')$. $(\mathcal{S}) \subset (\mathcal{O}'_C)$ implies that $(\mathcal{S}') = (\mathcal{S})^* \supset (\mathcal{O}'_C)^* \supset (\mathcal{S}')$, and therefore $(\mathcal{O}'_C) \Rightarrow (\mathcal{S}')$. Now if $T \in (\mathcal{S}')^*$ then $T * \varphi \in (\mathcal{S})$ for any $\varphi \in (\mathcal{D})$. Hence by definition $T \in (\mathcal{O}'_C)$. This proves $(\mathcal{S}') \Rightarrow (\mathcal{O}'_C)$. The proof of (3) is thus completed.

To prove (4) it is sufficient to show that $(\mathcal{O}'_M) \Rightarrow (\mathcal{S}')$, $(\mathcal{O}_C) \Rightarrow (\mathcal{O}'_C)$ and $(\mathcal{O}_M) \Rightarrow (\mathcal{O}'_C)$. First we remark that $(\mathcal{O}'_M)$ (resp. $(\mathcal{O}_C)$) is contained in $(\mathcal{O}'_C)$ (resp. $(\mathcal{O}_M)$) with continuous injection. Then $(\mathcal{O}'_C) \supset (\mathcal{O}'_M) \supset (\mathcal{S})$ implies that $(\mathcal{S}') = (\mathcal{O}'_C)^* \subset (\mathcal{O}'_M)^* \subset (\mathcal{S})^* = (\mathcal{S}')$ and therefore $(\mathcal{O}'_M) \Rightarrow (\mathcal{S}')$. Now it is well-known that $(\mathcal{O}'_C) \subset (\mathcal{O}_C)^*$. If $T \in (\mathcal{O}_C)^*$ then $T * \varphi \in (\mathcal{O}'_C)$ for any $\varphi \in (\mathcal{D})$. This yields that $T * \varphi * \psi \in (\mathcal{S})$ for any $\varphi, \psi \in (\mathcal{D})$ so that by Lemma 1, $T \in (\mathcal{O}'_C)$. This proves that $(\mathcal{O}_C) \Rightarrow (\mathcal{O}'_C)$. On the other hand $(\mathcal{O}_C) \subset (\mathcal{O}_M) \subset (\mathcal{S}')$ implies that $(\mathcal{O}'_C) = (\mathcal{O}_C)^* \supset (\mathcal{O}_M)^* \supset (\mathcal{S}')^* = (\mathcal{O}'_C)$ that is $(\mathcal{O}_M) \Rightarrow (\mathcal{O}'_C)$. We have thus proved (4).

Let $T \in (\mathcal{D}_{L^p})^*$, $1 \leq p < +\infty$, then $T * \varphi \in (\mathcal{D}'_{L^{p'}})$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, and hence $T * \varphi * \psi \in L^{p'}$ for any $\varphi, \psi \in (\mathcal{D})$. Lemma 1 asserts that $T \in (\mathcal{D}'_{L^{p'}})$. Since it is known that $(\mathcal{D}_{L^p})$ and $(\mathcal{D}'_{L^{p'}})$ are composable, we may obtain that $(\mathcal{D}_{L^p}) \Rightarrow (\mathcal{D}'_{L^{p'}})$. Then the relations $(\mathcal{D}_{L^p}) \subset (\mathcal{D}'_{L^p})$ and $(\mathcal{D}'_{L^{p'}}) \subset (\mathcal{D}'_{L^p})^*$ imply that $(\mathcal{D}'_{L^p}) \Rightarrow (\mathcal{D}'_{L^{p'}})$. Thus we have proved (5).

Finally we turn to the proof of (6). Clearly $(\dot{\mathcal{B}})^* \supset (\mathcal{D}'_{L^1})$ and $(\mathcal{B}')^* \supset (\mathcal{D}'_{L^1})$. But for any $T \in (\dot{\mathcal{B}})^*$ we have $T * \varphi \in (\mathcal{D}'_{L^1})$ and hence $T * \varphi * \psi \in L^1$ for any $\varphi, \psi \in (\mathcal{D})$. Lemma 1 yields that $T \in (\mathcal{D}'_{L^1})$ and hence we have $(\dot{\mathcal{B}}) \Rightarrow (\mathcal{D}'_{L^1})$. By the relations $(\dot{\mathcal{B}}) \subset (\mathcal{B}) \subset (\mathcal{B}')$ it follows that $(\mathcal{D}'_{L^1}) = (\dot{\mathcal{B}})^* \supset (\mathcal{B})^* \supset (\mathcal{B}')^* \supset (\mathcal{D}'_{L^1})$. This proves $(\mathcal{B}) \Rightarrow (\mathcal{D}'_{L^1})$ and $(\mathcal{B}') \Rightarrow (\mathcal{D}'_{L^1})$. Owing to the relations $(\dot{\mathcal{B}}) \subset (\dot{\mathcal{B}}') \subset (\mathcal{B}')$ we have $(\mathcal{D}'_{L^1}) = (\dot{\mathcal{B}})^* \supset (\dot{\mathcal{B}}')^* \supset (\mathcal{B}')^* = (\mathcal{D}'_{L^1})$ and hence $(\dot{\mathcal{B}}') \Rightarrow (\mathcal{D}'_{L^1})$. This proves (6). The theorem is thus completely proved.

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