JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 21, No. 2, JULY, 1957

A Set-Theoretical Characterization of the Torus

By

Tadashi TANAKA

(Received April 25, 1957)

Introduction

We have known the set-theoretical properties which characterize the closed 2-cells and the 2-spheres among locally connected, compact, metric continua. In this paper we shall consider the set-theoretical properties which characterize the torus.

Most of the terminologies and notations are due to G. T. Whyburn's book [1].[1]

§1. Statement of Theorem

For the convenience of the statement of Theorem, we shall give the following definitions.

DEFINITION. A set M will be said to be *locally connected* at a point p ($p \in M$ or $p \notin M$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every two points x and y of M whose distances from p are less than δ lie together in a connected subset of M of diameter less than ε.

DEFINITION. Let M be a locally connected set and N a subset of M which separates a sufficiently small region containing a point of N. Let $X = ab$ be a spanning arc of N in M, U a sufficiently small region in M containing a and V the component of $U - N$ which contains a neighborhood of a in $\langle X \rangle$. We shall say that a *spanning arc* Y *of* N *covers the end point* a *of* X if Y is an arc such that $\langle Y \rangle$ lies in V and its end points are the end points of an arc in N which has a as an interior point.

In the following sections, we shall prove

THEOREM. *In order that a locally connected, compact, metric continuum* T *be a torus it is necessary and sufficient that* $T = J_1 + J_2 + R$ *where*

(a) J_i*'s* ($i = 1, 2$) *are simple closed curves such that* $J_1 \cdot J_2 = p$ *is a single point, and* R ($= T - (J_1 + J_2)$) *is connected;*

(b) J_i *separates irreducibly a sufficiently small connected neighborhood* R_i *of* J_i *into exactly two regions* R_i^1 *and* R_i^2, *being locally connected at any point of* J_i;

(c) *if* $X = ab$ *is any spanning arc of* $J_1 + J_2$ *in* T, *then* $\langle X \rangle$ *separates* R *irreducibly, and moreover if* Y *is a spanning arc of* $J_1 + J_2$ *in* T *which*

1) Number in brackets refer to the references at the end of the paper.

covers one end point of X, then $\langle Y\rangle$ separates R between two points of $\langle X\rangle$ which are sufficiently near to a and b, respectively.

§2. Preliminary Lemmas

In the following let T, J_i, R, R_i, R_i^j be as in Theorem.

LEMMA 1. *J_2 crosses J_1 at p, that is, there exists an arc apb of J_2 such that $\langle ap\rangle$ and $\langle bp\rangle$ lie in R_1^1 and R_1^2, respectively.*

PROOF. Suppose, on the contrary, that J_2 does not cross J_1 at p. Then we may assume that $J_2\cdot R_1^1=0$. Let ab be a spanning arc of J_1+J_2 in T such that $a\in J_1$, $b\in J_2$ and a sufficiently small neighborhood of a in $\langle ab\rangle$ lie in R_1^1. Let cc' and dd' be arcs such that $(cc'+dd')\cdot(J_1+J_2+ab)=c+c'+d+d'$, where $c, d\in J_1$ and $c', d'\in ab$, and such that $cc'+(\text{arc } c'd' \text{ of } ab)+d'd$ is a spanning arc which covers the end point a of ab. Then by (c) it is easily shown that $\langle cc'\rangle$ and $\langle dd'\rangle$ lie in different components of $R-\langle ab\rangle$. On the other hand, since R_1^1 is locally connected at any point of J_1, we can join $\langle cc'\rangle$ to $\langle dd'\rangle$ by an arc in $R_1^1-\langle ab\rangle$ and hence in $R-\langle ab\rangle$. This contradiction proves Lemma 1. Q.E.D.

LEMMA 2. *Let $X=ab$ be a spanning arc of J_1+J_2 in T. Then $R-\langle X\rangle$ consists of exactly two components.*

PROOF. Suppose, on the contrary, that $R-\langle X\rangle$ has at least three components D_0, D_1, D_2, then we may take the following three arcs $Y_0=a_0b_0$, $Y_1=a_1b_1$, $Y_2=a_2b_2$: $\langle Y_0\rangle$, $\langle Y_1\rangle$, $\langle Y_2\rangle$ lie in D_0, D_1, D_2, respectively, and their end points are on X in the order $a, b_1, a_0, a_1, a_2, b_0, b_2, b$. Then the arc $Z=(\text{arc } ab_1 \text{ of } X)+Y_1+(\text{arc } a_1a_2 \text{ of } X)+Y_2+(\text{arc } b_2b \text{ of } X)$ is a spanning arc of J_1+J_2 in T. Every component of $R-\langle Z\rangle$ intersects X but all points of $X-Z$ lie in one component of $R-\langle Z\rangle$, so that $R-\langle Z\rangle$ is connected, contradicting (c). Q.E.D.

LEMMA 3. *Let $X=ab$ be a spanning arc of J_1+J_2 in T, and let $Y=cd$ be an arc such that $Y\cdot(J_1+J_2+X)=c+d$, where $c\in\langle X\rangle$ and $d\in(J_1+J_2)-X$. Then $\langle Y\rangle$ separates irreducibly the component of $R-\langle X\rangle$ containing $\langle Y\rangle$ into exactly two regions.*

PROOF. Let X_1 and X_2 be the two arcs into which c separates X. Then X_1+Y and X_2+Y are spanning arcs of J_1+J_2 in T. Hence we have the following three separations:

$$R-\langle X\rangle=R_X^1+R_X^2, \qquad \text{where } R_X^1\supset\langle Y\rangle,$$
$$R-\langle X_1+Y\rangle=R_{X_1Y}^1+R_{X_1Y}^2, \text{ where } R_{X_1Y}^2\supset R_X^2,$$
$$R-\langle X_2+Y\rangle=R_{X_2Y}^1+R_{X_2Y}^2, \text{ where } R_{X_2Y}^2\supset R_X^2.$$

To prove this lemma, it suffices to show that $R_X^1-\langle Y\rangle=R_{X_1Y}^1+R_{X_2Y}^1$ and this is a separation. It is obvious by (c) that $R_{X_1Y}^1\cdot R_{X_2Y}^1=0$, and by the definitions of $R_{X_1Y}^1$ and $R_{X_2Y}^1$ that $R_X^1-\langle Y\rangle\supseteq R_{X_1Y}^1+R_{X_2Y}^1$. To see $R_X^1-\langle Y\rangle\subseteq R_{X_1Y}^1+R_{X_2Y}^1$, suppose, on the contrary, that there exists a point x such that $x\in R_X^1-\langle Y\rangle$ but $x\notin R_{X_1Y}^1+R_{X_2Y}^1$. Since $R_{X_1Y}^2$ contains both x and R_X^2, there exists an arc xx_2 in $R^2_{X_1Y}$, where $x_2 \in \langle X_2 \rangle$. Similarly there exists an arc xx_1 in $R^2_{X_2Y}$, where $x_1 \in \langle X_1 \rangle$. We may assume that $xx_1 \cdot X_1 = x_1$, $xx_2 \cdot X_2 = x_2$ and $xx_1 \cdot xx_2 = x$. Moreover, we construct an open arc $\langle y_1y_2 \rangle$ in R^2_X whose end points are on X in the order a, x_1, x_2, y_1, y_2, b. Then $Z = (\text{arc } ax, \text{ of } X) + x_1x + xx_2 + (\text{arc } x_2y_1 \text{ of } X) + y_1y_2 + (\text{arc } y_2b \text{ of } X)$ is a spanning arc of $J_1 + J_2$ in T. But it is easily shown that Y does not separate R, contradicting (c). Q.E.D.

LEMMA 4. *Let X be a simple closed curve in T such that $X \cdot (J_1 + J_2) = a$ is a single point and X crosses either J_1 or J_2 at a. Then $\langle X \rangle$ $(= X - a)$ separates R irreducibly into exactly two regions.*

PROOF. Let us suppose that X crosses J_1 at a. Then there exist two open arcs $\langle ab_1 \rangle$ and $\langle ab_2 \rangle$ of X which lie in R^1_1 and R^2_1, respectively. Let ee' be an arc such that $ee' \cdot (J_1 + J_2 + X) = e + e'$ and $\langle ee' \rangle \subset R^2_1$, where $e \in (J_1 - a)$ and $e' \in (X - a)$. Replacing the arc ae' of X lying in R^2_1 by ee', we obtain a spanning arc $Y = (\text{arc } ae' \text{ of } X) + e'e$ of $J_1 + J_2$ in T. By virtue of (b) we may construct a spanning arc of $J_1 + J_2$ in T $Z = cc' + (\text{arc } c'd' \text{ of } X) + d'd$, which covers the end point a of Y, as in the proof of Lemma 1. Here we may suppose that $\langle Z \rangle$ separates $\langle ab_1 \rangle$ and $\langle ee' \rangle$ and hence $\langle ab_1 \rangle$ and $\langle ab_2 \rangle$ in R. By Lemma 3 we have the following separations:

$$R - \langle Z \rangle = R^1_Z + R^2_Z, \text{ where } R^1_Z \supset \langle ab_1 \rangle \text{ and } R^2_Z \supset \langle ab_2 \rangle,$$

$$R^1_Z - \langle \text{arc } ac' \text{ of } X \text{ lying in } \overline{R^1_Z} \rangle = R^1_c + R^1_d, \text{ where } \overline{R^1_c} \supset cc' \text{ and } \overline{R^1_d} \supset dd',$$

$$R^2_Z - \langle \text{arc } ad' \text{ of } X \text{ lying in } \overline{R^2_Z} \rangle = R^2_c + R^2_d, \text{ where } \overline{R^2_c} \supset cc' \text{ and } \overline{R^2_d} \supset dd'.$$

Then it is easily shown that $R - \langle X \rangle = (R^1_c + \langle cc' \rangle + R^2_c) + (R^1_d + \langle dd' \rangle + R^2_d)$ is the separation desired to prove Lemma 4. Q.E.D.

LEMMA 5. *Let X be either a spanning arc of $J_1 + J_2$ in T or a simple closed curve as in Lemma 4. Then every component of $R - \langle X \rangle$ has at least a point of $(J_1 + J_2) - X$ as a limit point.*

The proof of this lemma is exactly same as the proof of (5.5) ([2], pp. 92–93).

§3. Proof of Theorem

The necessity of the condition of Theorem follows at once from the definition of torus. To prove the sufficiency we shall make use of the fundamental properties of relative distance spaces (Cf. [1], pp. 154–162) and the characterization of 2-cells (Cf. [1], pp. 119–120). The proof is given below in sectionalized form.

Since $(T - J_1)$ is a region in a locally connected metric continuum T, the relative distance space $(T - J_1)^*$ for $(T - J_1)$ and the relative distance transformation $f((T - J_1)) = (T - J_1)^*$ may be defined by changing all distances $\rho(x, y)$ in $(T - J_1)$ into distances $\rho^*(x, y)$, where $\rho^*(x, y)$ is the greatest lower bound of the aggregate of diameters of all connected sets in $(T - J_1)$ which contain both x and y. Then we have known that $(T-J_1)^*$ is uniformly locally connected and $f((T-J_1))=(T-J_1)^*$ is topological. Hence $f^{-1}((T-J_1)^*)=(T-J_1)$ can be extended to give the continuous transformation ω of $(\widetilde{T-J_1})^*$ into T, where $(\widetilde{T-J_1})^*$ stands for the complete enclosure of $(T-J_1)^*$. In the following we denote points of $(T-J_1)$ by $x, y, \cdots$ and corresponding points in $(T-J_1)^*$ by $x^*, y^*, \cdots$.

(i) *$(\widetilde{T-J_1})^*-(T-J_1)^*$ consists of two disjoint simple closed curves $J_1^{1(*)}$ and $J_1^{2(*)}$ such that $\omega(J_1^{1(*)})=\omega(J_1^{2(*)})=J_1$ and such that the transformations $\omega(J_1^{1(*)})=J_1$ and $\omega(J_1^{2(*)})=J_1$ are topological.*

For let $x^{(*)}$ be any point of $(\widetilde{T-J_1})^*-(T-J_1)^*$ and let $\{x_i^*\}$ be a sequence converging to $x^{(*)}$ in $(T-J_1)^*$. Then the corresponding sequence $\{x_i\}$ converges to a point $\omega(x^{(*)})$ of J_1, and all but a finite number of the points of $\{x_i\}$ lie in only one of R_1^i's, denoted by R_1^j. By virtue of (b), for $x^{(*)}$ R_1^j corresponds independently to the choice of $\{x_i^*\}$. Let $J_1^{j(*)}$ be the set of all points of $(\widetilde{T-J_1})^*-(T-J_1)^*$ to which R_1^j corresponds. Then it is obvious that $J_1^{1(*)}\cdot J_1^{2(*)}=0$. To see $\omega(J_1^{j(*)})=J_1$, it suffices to show that $\omega(J_1^{j(*)})\supseteq J_1$. Let y be any point of J_1 and let $\{y_i\}$ be a sequence converging to y in R_1^j. By (b), $\{y_i^*\}$ converges to a point $y^{(*)}$ of $J_1^{j(*)}$, so that $\omega(y^{(*)})=y$. Let $z^{(*)}$ be any point of $J_1^{j(*)}$ such that $\omega(z^{(*)})=y$ and let $\{z_i^*\}$ be a sequence converging to $z^{(*)}$ such that $z_i \in R_1^j$. Then both $\{y_i\}$ and $\{z_i\}$ converge to y, so that $y^{(*)}=z^{(*)}$. Therefore $\omega(J_1^{j(*)})=J_1$ is one-to-one. Next to show that $J_1^{j(*)}$ is compact, let $\{x_i^{(*)}\}$ be any infinite set of $J_1^{j(*)}$ and for each $x_i^{(*)}$ let $\{x_{ik}^*\}$ be a sequence converging to $x_i^{(*)}$ such that $x_{ik}\in R_1^j$. Since T is compact we may assume that $\{x_{ii}\}$ converges to a point of J_1. Then $\{x_{ii}^*\}$ converges to a point of $J_1^{j(*)}$, and also $\{x_i^{(*)}\}$ converges to the same point. Therefore $J_1^{j(*)}$ is compact. Thus the transformations $\omega(J_1^{1(*)})=J_1$ and $\omega(J_1^{2(*)})=J_1$ are topological.

(ii) *$(\widetilde{T-J_1})^*$ is a locally connected, compact continuum.*

For, firstly, to prove that $(\widetilde{T-J_1})^*$ is compact, it suffices to show that any infinite set $\{x_i^*\}$ of $(T-J_1)^*$ such that $\{x_i\}$ converges to a point of J_1, has a limit point in $(\widetilde{T-J_1})^*$, because the transformations $\omega(J_1^{1(*)})=J_1$, $\omega(J_1^{2(*)})=J_1$ and $\omega((T-J_1)^*)=(T-J_1)$ are topological. Moreover, we may assume that $x_i\in R_1^j$. Then by virtue of (b) $\{x_i^*\}$ is a fundamental sequence, and $\{x_i^*\}$ has a limit point in $(\widetilde{T-J_1})^*$. Secondly, since $(T-J_1)^*$ is uniformly locally connected and conditionaly compact, $(\widetilde{T-J_1})^*$ has property S and hence $(\widetilde{T-J_1})^*$ is locally connected. Thus (ii) is proved.

It follows at once from Lemma 1 that $J_2^{(*)}$ $(=\omega^{-1}(J_2))$ is an arc spanning $J_1^{1(*)}$ and $J_1^{2(*)}$ in $(\widetilde{T-J_1})^*$. Furthermore, it is easily shown that the sets $J_2^{(*)}$, $\omega^{-1}(R_2)$, $\omega^{-1}(R_2^1)$ and $\omega^{-1}(R_2^2)$ in $(\widetilde{T-J_1})^*$ satisfy the condition (b). Therefore, in exactly the same manner, we may obtain the complete enclosure

$\tilde{R}^* = \{\widetilde{(T-J_1)^* - J_{2(*)}}\}^*$ of the relative distance space for $\{\widetilde{(T-J_1)}^* - J_{2(*)}\}$ and the continuous transformation ω' of $\tilde{R}^*$ onto $\widetilde{(T-J_1)}^*$. Hence we have also the following two results:

(i′) *$\tilde{R}^* - \{\widetilde{(T-J_1)}^* - J_{2(*)}\}^*$ consists of two disjoint arcs $J^1_{2(**)}$ and $J^2_{2(**)}$ such that $\omega'(J^1_{2(**)}) = \omega'(J^2_{2(**)}) = J_{2(*)}$ and such that the transformations $\omega'(J^1_{2(**)}) = J_{2(*)}$ and $\omega'(J^2_{2(**)}) = J_{2(*)}$ are topological.*

(ii′) *$\tilde{R}^*$ is a locally connected, compact continuum.*

Obviously $J^1_{1(**)} (= \omega'^{-1}(J^1_{1(*)}))$ and $J^2_{1(**)} (= \omega'^{-1}(J^2_{1(*)}))$ are arcs.

(iii) *$\tilde{R}^*$ is a closed 2-cell.*

For, firstly, it follows at once from (i) and (i′) that $\tilde{R}^*$ is the sum of the simple closed curve $J = J^1_{1(**)} + J^1_{2(**)} + J^2_{1(**)} + J^2_{2(**)}$ and the region $R_{(**)}$ bounded by J, where $\omega'^{-1}(\omega^{-1}(R))$ is denoted by $R_{(**)}$. Secondly, it is known from Lemma 2 and 4 that if γ is any spanning arc of J in $\tilde{R}^*$, then $\langle\gamma\rangle$ cuts $R_{(**)}$ into exactly two regions R_a and R_b. Thirdly, R_a and R_b are bounded by $\gamma+\alpha$ and $\gamma+\beta$, respectively, where α and β the two arcs into which the end points of γ separate J. For by Lemma 5, each of R_a and R_b has at least a point of $\langle\alpha\rangle + \langle\beta\rangle$ as a limit point. If R_a has a point of $\langle\alpha\rangle$ (or $\langle\beta\rangle$) as a limit point, then R_a has every point of α (or β) as a limit point by virtue of (b), and the same as for R_a is true for R_b. On the other hand, any point of $\langle\alpha\rangle + \langle\beta\rangle$ can not be a limit point of both R_a and R_b. Therefore it follows from Lemma 4 and (c) that the boundaries of R_a and R_b are $\gamma+\alpha$ and $\gamma+\beta$, respectively. Thus $\tilde{R}^*$ is a closed 2-cell.

(iv) *T is a torus.*

To show this we shall consider the relation between T and $\tilde{R}^*$. At first we assign to both $J^1_{1(**)}$ and $J^2_{1(**)}$ the same direction from $J^1_{2(**)}$ to $J^2_{2(**)}$, and to both $J^1_{2(**)}$ and $J^2_{2(**)}$ the same direction from $J^1_{1(**)}$ to $J^2_{1(**)}$. Then the transformation $\omega'(\tilde{R}^*) = \widetilde{(T-J_1)}^*$ has the following properties:

(1′) $\omega'(\{\tilde{R}^* - (J^1_{2(**)} + J^2_{2(**)})\}) = \{\widetilde{(T-J_1)}^* - J_{2(*)}\}$ is topological;

(2′) both $\omega'(J^1_{2(**)}) = J_{2(*)}$ and $\omega'(J^2_{2(**)}) = J_{2(*)}$ are topological, and assign to $J_{2(*)}$ the same direction.

And also the transformation $\omega(\widetilde{(T-J_1)}^*) = T$ has the following properties:

(1) $\omega(\{\widetilde{(T-J_1)}^* - (J^1_{1(*)} + J^2_{1(*)})\}) = (T-J_1)$ is topological;

(2) both $\omega(J^1_{1(*)}) = J_1$ and $\omega(J^2_{1(*)}) = J_1$ are topological, and assign to J_1 the same direction.

Here that both $\omega(J^1_{1(*)}) = J_1$ and $\omega(J^2_{1(*)}) = J_1$ assign to J_1 the same direction follows at once from the fact that J_2 separates R_2.

Thus T is a torus.

In conclusion I wish to express my hearty thanks to Prof. K. Morinaga for his kind guidance.

References

[1] Whyburn, G. T.: Analytic Topology, Amer. Math. Soc. Colloquium Publications, vol. 28 (1942).

[2] Wilder, R. L.: Topology of Manifolds, Amer. Math. Soc. Colloquium Publications, vol. 32 (1948).

Mathematical Institute,
Hiroshima University