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On the Branches of Logarithmic Function of a Matrix Variable

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§ 1. Introduction

Let $\mathfrak{A}$ be the space of all the complex matrices of degree n with the usual topology, $\mathfrak{M}$ the subspace of $\mathfrak{A}$ which is composed of all the regular matrices of degree n, and then we shall consider the exponential mapping: $W \rightarrow Z = \exp W$ from $\mathfrak{A}$ onto $\mathfrak{M}$, by $\mathfrak{L}(M)$ we shall denote the complete inverse image of M of this mapping, i.e., $\mathfrak{L}(M) = \{A; \exp A = M\}$, and by $\mathfrak{A}^\circ$ we shall denote the maximal subspace of $\mathfrak{A}$ for which the mapping: $W \rightarrow Z = \exp W$ from $\mathfrak{A}$ onto $\mathfrak{M}$ is locally homeomorphic. And moreover we shall set $\mathfrak{A}^s = \mathfrak{A} - \mathfrak{A}^\circ$ (the complement of $\mathfrak{A}^\circ$ with respect to $\mathfrak{A}$), $\mathfrak{M}^s = \exp \mathfrak{A}^s$ and $\mathfrak{M}^\circ = \mathfrak{M} - \mathfrak{M}^s$.

In the previous paper ([3])[1] we have written the following facts:

(1) $\mathfrak{A}^\circ$ is the subspace of $\mathfrak{A}$ composed of all the matrices whose characteristic roots λ_i do not satisfy the condition: $\lambda_i - \lambda_j = 2l\pi\sqrt{-1}$ (l: non-zero integer) ([3], Theorem 3).

(2) $\mathfrak{M}^\circ$ is the subspace of $\mathfrak{M}$ composed of all the matrices whose minimal polynomials are of degree n, this condition is equivalent to $\dim \mathfrak{C}(M) = n$ where $\mathfrak{C}(M)$ means the set of all the matrices commutative with M.

(3) $A \in \mathfrak{C}(M) \frown \mathfrak{A}^\circ$, if and only if $\mathfrak{C}(A) = \mathfrak{C}(M)$ ([3], Theorem 5).

(4) $\mathfrak{A}^\circ$ is open and dense in $\mathfrak{A}$, and is arc-wise connected, but is not simply connected.

For a fixed matrix M of $\mathfrak{M}$, the set $\mathfrak{L}(M)$ has been already considered in the previous papers ([1], [2] and [3]). In this paper we shall consider the set $\mathfrak{L}(Z)$ for a variable matrix Z of $\mathfrak{M}$, and by making use of the correspondence between the arcs in $\mathfrak{A}$ and the arcs in $\mathfrak{M}$ under the exponential mapping we shall investigate the behavior of the points of $\mathfrak{L}(Z)$ for Z moving along arcs.

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1) Numbers in brackets refer to the references at the end of the paper.

§ 2. General form of the elements of $\mathfrak{L}(M)$

The general form of elements of $\mathfrak{L}(M)$ has been completely determined for a fixed matrix M of $\mathfrak{M}$ ([1]); however, for the present purpose, it seems to be better that the general form is slightly modified. Let M_0 be a Jordan's canonical form of M, and let $\mathfrak{R}(M, M_0)$ be the set of all the regular matrices P which transform M into its canonical form M_0, i.e., $\mathfrak{R}(M; M_0)\equiv\{P; P^{-1}MP=M_0\}$, then it is clear that $\mathfrak{R}(M; M_0)=\mathfrak{C}(M)P$ for any $P\in\mathfrak{R}(M; M_0)$.

We transform M into M_0 by a fixed matrix P_0 of $\mathfrak{R}(M; M_0)$

$$
(2.1)\qquad
\begin{aligned}
&P_0^{-1}MP=M_0=\textstyle\sum_{i=1}^{p}\dot{+}M_i,\\
&\quad M_i=\textstyle\sum_{\alpha=1}^{p_i}\dot{+}M_{i\alpha}=\lambda_iE_i+N_i, && (i=1,2,\cdots,p),\\
&\quad M_{i\alpha}=\lambda_iE_{i\alpha}+N_{i\alpha}=\begin{pmatrix}\lambda_i & 1 & & 0\\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \ddots\ 1\\ 0 & & & \lambda_i\end{pmatrix}, && (\alpha=1,2,\cdots,p_i),
\end{aligned}
$$

$$\lambda_i\neq\lambda_j \text{ for } i\neq j \text{ and } \lambda_i\neq 0,\ (i, j=1,2,\cdots,p),$$

where M_i and $M_{i\alpha}$ are the matrices of degree n_i and $n_{i\alpha}$ respectively, E_i and $E_{i\alpha}$ are the unit matrices of degree n_i and $n_{i\alpha}$ respectively. And it is clear that $N_i^{n_i}=O_{n_i}$ and $N_{i\alpha}^{n_{i\alpha}}=O_{n_{i\alpha}}$, where O_m means the zero matrix of degree m.

Let $A\in\mathfrak{L}(M)$, then A is expressed as:

$$
(2.2)\qquad A=k(M)+PBP^{-1},\qquad P\in\mathfrak{R}(M; M_0),
$$

$$
\begin{aligned}
\text{where}\quad & k(M)=P_0\big(\textstyle\sum_{i=1}^{p}\dot{+}\sum_{\alpha=1}^{p_i}\dot{+}k(M_{i\alpha})\big)P_0^{-1},\\
& k(M_{i\alpha})=\textstyle\sum_{r=1}^{n_{i\alpha}-1}(-1)^rN_{i\alpha}^r/(r\lambda_i^r),\\
& B=\textstyle\sum_{i=1}^{p}\dot{+}\sum_{\alpha=1}^{p_i}\dot{+}\mu_{i\alpha}E_{i\alpha},\quad e^{\mu_{i\alpha}}=\lambda_i\ (\alpha=1,2,\cdots,p_i).
\end{aligned}
$$

From the definition of $k(M)$ and the result in [1] it is easily seen that $k(M)$ is independent from the choice of M_0 and P_0, and is a one-valued continuous function of M.

It is well known ([1]) that $A\in\mathfrak{L}(M)\frown\mathfrak{A}^\circ$ if and only if $\mu_{i\alpha}=\mu_{i\beta}$ ($i=1, 2,\cdots,p$; $\alpha,\beta=1,2,\cdots,p_i$). Such an element A of $\mathfrak{L}(M)$ is denoted by $\operatorname{Log} M$; in particular, if $A\in\mathfrak{L}(M)\frown\mathfrak{A}_{(-\pi,\pi]}$, such an element A is denoted by $L(M)$, where $\mathfrak{A}_{(-\pi,\pi]}$ means the set of all the matrices whose characteristic roots have their imaginary parts in the half closed interval $(-\pi,\pi]$. As regard $W=L(Z)$ as a function of a matrix variable, the function is continuous along arcs $Z=Z(t)$ (t: real) such that the characteristic roots $\zeta_i(t)$ of $Z(t)$ satisfy the condition: $-\pi<\arg\zeta_i(t)\leqq\pi$ for all t; and moreover, $W=\operatorname{Log}(Z)$ is continuous along arcs $Z=Z(t)$ (t: real), if and only if, for the characteristic roots $\zeta_i(t)$ of $Z(t)$ and the characteristic roots $\log\zeta_i(t)$ of $\operatorname{Log}(Z(t))$, $\zeta_i(t)=\zeta_j(t)$ implies $\log\zeta_i(t)=\log\zeta_j(t)$.

§ 3. Continuation along arcs

As mentioned above, $\mathfrak{A}^\circ$ is the maximal set for which the exponential mapping from $\mathfrak{A}$ onto $\mathfrak{M}$ is locally homeomorphic, and $\mathfrak{A}^\circ$ is open and dense in $\mathfrak{A}$, and arc-wise connected. By making use of these properties of $\mathfrak{A}^\circ$, we shall introduce the concept of a continuation along arc. By an arc we shall mean a set of points $W(t)$ $(0 \leqq t \leqq 1)$, where $W(t)$ is a continuous function of t in the interval $0 \leqq t \leqq 1$.

If $\mathfrak{w}: W = W(t)$ $(0 \leqq t \leqq 1)$ is an arc in $\mathfrak{A}$ such that $\exp W(t) = Z(t)$ $(0 \leqq t \leqq 1)$ for a given arc $\mathfrak{z}: Z = Z(t)$ $(0 \leqq t \leqq 1)$ in $\mathfrak{M}$, then we may say that the arc $\mathfrak{w}$ is a branch of logarithmic function $\log Z$[1] along the arc $\mathfrak{z}$.

Let $A \in \mathfrak{L}(M)$ and $A' \in \mathfrak{L}(M')$, if A and A' are joined by an arc $\mathfrak{w}: W = W(t)$ $(0 \leqq t \leqq 1)$ such that $W(t) \in \mathfrak{A}^\circ$ $(0 < t < 1)$, $W(0) = A$ and $W(1) = A'$, then we shall say that the point A' of $\mathfrak{L}(M')$ is obtained from the point A of $\mathfrak{L}(M)$ by a continuation along the arc $\mathfrak{z}: Z = Z(t) = \exp W(t)$ $(0 \leqq t \leqq 1)$ which joins M and M'; also then, we may say that the point A and A' of $\mathfrak{L}(M)$ and $\mathfrak{L}(M')$ respectively are jointed by a regular branch $\operatorname{Log} Z$ of $\log Z$ along the arc $\mathfrak{z}$ which joins M and M'.

PROPOSITION 1. *Any points A and A' of $\mathfrak{L}(M)$ and $\mathfrak{L}(M')$ respectively are joined by a regular branch* $\log Z$ *of* $\operatorname{Log} Z$ *along an arc which joins M and M'.*

PROOF. To prove this we have only to show that any point A of $\mathfrak{L}(M)$ is obtained from the point O of $\mathfrak{L}(E)$ by a continuation along an arc. If $A \in \mathfrak{A}^\circ$, then A and O is jointed by an arc in $\mathfrak{A}^\circ$, since $\mathfrak{A}^\circ$ is arc-wise connected. If $A \in \mathfrak{A}^s$, then there exists a point A_1 in a neighborhood of A such that A and A_1 are joined by an arc $\mathfrak{w}_1: W = W_1(s)$, $(0 \leqq s \leqq 1)$, where $W_1(0) = A$, $W_1(1) = A_1$ and $W_1(s) \in \mathfrak{A}^\circ$ $(0 < s < 1)$; and A_1 and O are joined by an arc in $\mathfrak{A}^\circ$. Therefore, our assertion is proved.

PROPOSITION 2. *Any points A_1 and A_2 of $\mathfrak{L}(M)$ are joined by a regular branch of* $\log Z$ *along a closed arc through M.*

PROOF. Since $A_1, A_2 \in \mathfrak{L}(M)$, for the arc $\mathfrak{w}: W = W(t)$ $(0 \leqq t \leqq 1)$ such that $W(0) = A_1$ and $W(1) = A_2$, the corresponding arc $\mathfrak{z}: Z = \exp W(t)$ is a closed arc through M. From this and Proposition 1, the proposition is obtained.

§ 4. Local continuation

To investigate, in more detail, the behavior of the points of $\mathfrak{L}(Z)$ for Z moving along arcs, we shall introduce the concept of a local continuation around M.

Let A_1 and A_2 be the points of $\mathfrak{L}(M)$, if A_2 is obtained from A_1 by a continuation along a closed arc through M which is in a small neighbor-

1) By $\log Z$ we shall mean the multi-valued inverse function of $\exp Z$.

hood of M, we shall say that A_2 is obtained from A_1 by a local continuation around M.

We shall consider a decomposition of $\mathfrak{L}(M)$ as follows:

$$\mathfrak{L}(M)=\bigcup_B \mathfrak{L}(M; B), \tag{4.1}$$

where $\mathfrak{L}(M; B)\equiv\{A; A=k(M)+PBP^{-1}, P\in\mathfrak{K}(M; M_0)\}$ in the notation of (2.2). As for this decomposition, it is clear that $\mathfrak{L}(M; B_1)=\mathfrak{L}(M; B_2)$, if and only if $B_2=QB_1Q^{-1}$ for some $Q\in\mathfrak{C}(M)$, which is equivalent to the fact that B_2 is obtained from B_1 by the arrangement of the blocks $\mu_{i\alpha}^{(1)}E_{i\alpha}$ of B_1 such that the degree $n_{i\alpha}$ are equal to each other; and that either $\mathfrak{L}(M; B_1)=\mathfrak{L}(M; B_2)$ or $\mathfrak{L}(M; B_1)\frown\mathfrak{L}(M; B_2)=\phi$. And moreover the number of $\mathfrak{L}(M; B)$ is countable; and either $\mathfrak{L}(M, B)\subset\mathfrak{A}^\circ$ or $\mathfrak{L}(M, B)\frown\mathfrak{A}^\circ=\phi$.

Concerning this decomposition we shall prove

THEOREM 1. *The decomposition $\mathfrak{L}(M)=\bigcup_B\mathfrak{L}(M; B)$ of $\mathfrak{L}(M)$ is invariant by a local continuation around M. For any points A_1 and A_2 of $\mathfrak{L}(M; B)$, A_2 is obtained from A_1 by a local continuation around M.*

PROOF. It is easily seen that this decomposition is invariant by any local continuation around M. We shall prove that for any points A_1 and A_2 of $\mathfrak{L}(M; B)$, A_2 is obtained from A_1 by a local continuation around M. For this purpose, without loss of generality we may assume $M=M_0$, consequently, $P_0=E$ and $\mathfrak{K}(M; M_0)=\mathfrak{C}(M)$. Then we have

$$A_1=k(M)+P_1BP_1^{-1} \quad \text{and} \quad A_2=k(M)+P_2BP_2^{-1} \tag{4.2}$$

where $P_1, P_2\in\mathfrak{C}(M)$ and $B=\sum_{i=1}^{p}\dot{+}\sum_{\alpha=1}^{p_i}\dot{+}\mu_{i\alpha}E_{i\alpha}$. Here if we set

$$P(t)=\exp((1-t)\operatorname{Log}P_1+t\operatorname{Log}P_2), \quad (0\leqq t\leqq 1), \tag{4.3}$$

since $\operatorname{Log}P_1$, $\operatorname{Log}P_2\in\mathfrak{C}(M)$, then we have

$$P(t)\in\mathfrak{C}(M)\frown\mathfrak{M}, \ (0\leqq t\leqq 1), \ P(0)=P_1 \text{ and } P(1)=P_2. \tag{4.4}$$

So if we consider an arc $\mathfrak{w}_1$: $W=W_1(t)=P(t)(k(M)+B)P(t)^{-1}$, then $W_1(0)=A$, $W_1(1)=A_2$ and $W_1(t)\in\mathfrak{L}(M; B)$, $(0\leqq t\leqq 1)$, and hence we have the corresponding arc $\mathfrak{z}_1: Z=Z_1(t)=\exp W_1(t)=M$ $(0\leqq t\leqq 1)$.

Next, for any neighborhood $\mathfrak{V}(M; \varepsilon)\equiv\{X; \|X-M\|<\varepsilon\}$ of M, where $\|X\|\equiv(\operatorname{Tr}({}^t\bar{X}X))^{\frac{1}{2}}$, we can take an arc $\mathfrak{w}: W=W(t)=P(t)U(t)P(t)^{-1}$ $(0\leqq t\leqq 1)$, where $U(0)=U(1)=k(M)+B$, $U(t)\in\mathfrak{A}^\circ$ $(0<t<1)$, $\|U(t)-(k(M)+B)\|<\eta/\Gamma^2$, Γ is a positive number such that $\|P(t)\|$, $\|P(t)^{-1}\|<\Gamma$ for $0\leqq t\leqq 1$, and η is a positive number such that $\|W(t)-W_1(t)\|<\eta$ implies $\|\exp W(t)-\exp W_1(t)\|<\varepsilon$. Since $\|P(t)\|$ and $\|P(t)^{-1}\|$ are continuous in a closed interval $0\leqq t\leqq 1$, it is clear that such a positive number Γ exists; and the existence of η, clearly, follows from the continuity of the exponential mapping. Then we have $W(0)=A_1$, $W(1)=A_2$ and $W(t)\in\mathfrak{A}^\circ$ $(0<t<1)$. For this arc $\mathfrak{w}$, we make correspond an arc $\mathfrak{z}: Z=\exp W(t)$, then we have

$$\|W(t)-W_1(t)\|\leqq\|P(t)\|\ \|U(t)-(k(M)+B)\|\ \|P(t)^{-1}\|<\eta, \tag{4.5}$$

and therefore we have

$$\|Z(t)-M\|=\|\exp W(t)-\exp W_1(t)\|<\varepsilon.$$

That is, $Z(t)\in\mathfrak{B}(M;\varepsilon)$ for $0\leqq t\leqq 1$. Thus, our assertion is proved.

If $M\in\mathfrak{M}^\circ$, then $\mathfrak{L}(M;B)$ is always composed of only one point, and the points of $\mathfrak{L}(M)$ are always invariant by any local continuation around M. On the other hand, if $M\in\mathfrak{M}^s$, for some B, the set $\mathfrak{L}(M;B)$ contains an uncountable number of points; and by Theorem 1, these points are always obtained from any point of $\mathfrak{L}(M;B)$ by a local continuation around M.

By Proposition 2, any point of $\mathfrak{L}(M)$ are obtained from any point of $\mathfrak{L}(M)$ by a continuation along a closed arc through M. Especially, $k(M)+B$ is obtained from $L(M)$ by a continuation along a closed arc through M. And moreover, by Proposition 1 the point $L(M')$ of $\mathfrak{L}(M')$ is obtained from the point $L(M)$ of $\mathfrak{L}(M)$ by a continution along an arc from M to M'.

Summarizing these results we have

THEOREM 2. *Any points of $\mathfrak{L}(M';B')$ are obtained from any points of $\mathfrak{L}(M;B)$ by composing the following continuations along arcs:*

(*1*) *a continuation from $L(M)$ to $L(M')$ along a fixed arc from M to M',*

(*2*) *a continuation along a fixed closed arc through M,*

(*2'*) *a continuation along a fixed closed arc through M',*

(*3*) *local continuations around M,*

(*3'*) *local continuations around M'.*

Finally we shall prove

THEOREM 3. *Let $S\in\mathfrak{A}-\mathfrak{M}$, i.e., if* $\det S=0$, *then there exists a closed arc: $Z=Z(t)$ $(0\leqq t\leqq 1)$ of $\mathfrak{M}$, in any small neighborhood of S, such that $Z(0)=Z(1)=M$, a point of $\mathfrak{L}(M)$ is transferred to another point of $\mathfrak{L}(M)$ by a continuation along this closed arc.*

PROOF. To prove this, without loss of generality, we assume that

$$S=S_1\dot{+}S_2,\quad \det S_2\neq 0\quad\text{and}\quad S_1=\overbrace{\begin{pmatrix}0&\eta_1&&&0\\&0&\eta_2&&\\&&0&\ddots&\\&&&\ddots&\eta_{m-1}\\0&&&&0\end{pmatrix}}^{m},\tag{4.6}$$

where $\eta_k=0$ or 1, $(k=1,2,\cdots,m-1)$. For any neighborhood $\mathfrak{B}(S;\varepsilon)\equiv\{X;\ \|X-S\|<\varepsilon\}$ of S, if we take

$$M=M_1\dot{+}S_2,\quad M_1=\lambda E_1+S_1,\quad 0<\lambda<\frac{1}{\sqrt{m}}\varepsilon,\tag{4.7}$$

and

$$\begin{gathered}Z(t)=(\lambda e^{2t\pi\sqrt{-1}}E_1+S_1)\dot{+}S_2,\quad(0\leqq t\leqq 1),\\ W(t)=(k(Z_1(t))+(\log\lambda+2(t+l)\pi\sqrt{-1})E_1)\dot{+}L(S_2),\\ Z_1(t)=\lambda e^{2t\pi\sqrt{-1}}E_1+S_1,\quad(l:\ \text{integer}),\end{gathered}\tag{4.8}$$

then we have

$$\exp W(t)=Z(t) \quad \text{and} \quad W(t)\in\mathfrak{A}^\circ \ (0\leqq t\leqq 1).$$

Therefore, a point $W(0)$ of $\mathfrak{L}(M)$ is transferred to another point $W(1)$ of $\mathfrak{L}(M)$ by a continuation along a closed arc $Z=Z(t)$. Thus, the theorem is proved.

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References

[1] K. Morinaga and T. Nôno, *On the logarithmic functions of matrices I*, J. Sci. Hiroshima Univ., (A), Vol. 14, No. 2 (1950), pp. 107–114.

[2] K. Morinaga and T. Nôno, *On the matrix space*, J. Sci. Hiroshima Univ., (A), Vol. 19, No. 1 (1955), pp. 171–179.

[3] T. Nôno, *On the singularity of general linear groups*, J. Sci. Hiroshima Univ., Vol. 20, No. 3 (1957), pp. 115–123.

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