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On Tori

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Introduction

A *3-manifold* M is a connected, separable metric space each of whose points has a closed neighbourhood homeomorphic to a 3-cell. It is well known that each 3-manifold is triangulable. M is said to be *closed* if it is compact and[1] Bd M is empty. In the following every thing will be considered from the semi-linear point of view.

A *solid torus* V *of genus* h ($\geqq 0$) means a 3-cell with h solid handles (i. e. Henkelkoerper vom Geschlechts h [10] p. 219), that is a compact orientable 3-manifold, whose boundary is a torus of genus h or equally an orientable closed surface of genus h, constructed by a 3-cell C, $2h$ mutually exclusive discs D_1, D'_1; $\cdots$; D_h, D'_h on Bd C and h homeomorphisms, f_i: $D_i \to D'_i$, of identification. The fundamental group $\pi_1(V)$ of V is a free group with h free generators. m_i $=$Bd D_i and D_i are called a *meridian* and a *meridian disc* of V, respectively. Furthermore the set $\{m_1, \cdots, m_h\}$ is called a *system of meridians of* V. We note that m_i's are mutually exclusive and are homologously independent on Bd V. A *system of longitudes*, conjugate to a system of meridians $\{m_1, \cdots, m_h\}$ of V, consists of a point p on Bd V and h simple closed curves l_i on Bd V such that $l_i \cdot l_j = p$, $l_i \cdot m_j = \emptyset$ for $i \neq j$ and l_i intersects m_i at only one point.

In this paper we are concerned with several problems of situation of tori of genus one in 3-sphere S^3 (§§ 1-3). Theorem 1 gives a topological characterization of S^3 by the situation of tori of genus one and Theorem 2 shows a n. a. s. condition that a polyhedral torus of genus $h \geqq 2$ bounds a solid torus in S^3. There is a question as follows: [7] (16.4) p. 330: Let T and T' be two tori of the same genus h in S^3, such that the closure of each one of the components of $S^3 - T$ and $S^3 - T'$ is a solid torus. Does there exist an isotopy of S^3 onto itself carrying T onto T'? If $h=0$ the answer is affirmative [5]. In §3 we show that it is also affirmative for $h=1$. In the last section a characterization of systems of longitudes of solid tori is shown.

1. A topological characterization of S^3

In 1924, J. W. Alexander proved that each polyhedral torus of genus one in S^3 bounds a solid torus [1] and afterward H. Schubert gave a detailed proof of the same proposition [9], §4, pp. 151–155. Conversely we show that

(1) Bd M means the boundary of M.

each closed 3-manifold with the property is topologically S^3.

THEOREM 1. *A closed 3-manifold M is topologically S^3 if and only if each polyhedral torus of genus one in M bounds a solid torus in M.*

PROOF. We may assume that M has a fixed triangulation. It is sufficient to show that there exists a polyhedral 2-sphere in M which is the common boundary of two 3-cells. Let s be a 3-simplex in M with a face t. Let a, b be two points in(2) Int t and let K be a polygonal arc joining a to b in s such that $K-(a+b)\subset$ Int s and $K+ab$ is a clover leaf (Kleeblattschlinge, [10] p. 2, Fg. 2), where ab is the segment in t from a to b. Pierce a small polyhedral tubular hole along K in s. Then we have a 3-cell $C\subset s$ with a knotted hole in it (cf. [2] p. 33, Fig, 6). Bd C is a polyhedral torus of genus one and by our assumption it bounds a solid torus V in M. Since $\pi_1(C)$ is not free, C is not a solid torus and hence $V=\overline{M-C}$. So $\overline{M-s}$ is a 3-cell and Bd s is the common boundary of s and $\overline{M-s}$.

Q. E. D.

2. Solid tori in S^3

The analogous proposition to Theorem 1, for the case where genera of tori $\geqq 2$, is untrue as shown in the following

EXAMPLE. Let V be a closed polyhedral tubular neighbourhood of a clover leaf in S^3 and s a 3-simplex in V such that $s\cdot$ Bd V is a 2-simplex t. From s we construct a 3-cell C with a knotted hole in it in the same way as in Proof of Theorem 1. Though Bd $((\overline{V-s})+C)$ is a polyhedral torus of genus 2, neither $(\overline{V-s})+C$ nor $\overline{S^3-((\overline{V-s})+C)}$ is a solid torus.

In this section we are concerned with a condition that a polyhedral torus in S^3 bounds a solid torus.

The definition of normality of a system of closed curves on a surface is due to [6] § 2, Nr. 1, p. 3. On the other hand we refer to [6] pp. 3–6, [11] pp. 206–208 and [12] § 1 for the definitions of double curves, triple points, branch points on singular discs, mutiplicity of a branch point, the diagram of a singular disc and cuts along double curves. Let A, A' be normal systems of closed curves on a surface. We shall say that A is *simpler* than A' if A has fewer crossing points than A'.

THEOREM 2. *In order that a polyhedral torus $T\subset S^3$ of genus h may bound a solid torus in S^3 it is necessary and sufficient that there exist a component K of S^3-T and h closed curves C_i on T with Property (P):*(3) *$C_i\simeq 0$ in $\bar{K}$ and $C_1,\cdots, C_h$ are homologously independent on T.*

(2) Int t means the interior of t.
(3) $\simeq$ means homotopic to.

PROOF. Necessity. A system of meridians of the solid torus bounded by T has Property (P).

Sufficiency. We divide the proof into three steps.

(i) In this step we show that there exist h simple closed curves C_i'' on T with Property (P). The details are as follows. We may suppose the system $\{C_1,\cdots,C_h\}$ is normal. If it is not already so, we obtain such a system by small deformations of C_i on T. Since $C_1\simeq 0$ in $\bar{K}$, there exists a singular disc D_1 in $\bar{K}$ such that Bd $D_1=C_1=D_1\cdot T$ and its singularities are at most crossing points in C_1, double curves, triple points and branch points (cf. [6] p. 3 and [11] pp. 206–207). We first reduce the multiplicity of each branch point to 1 (cf. [6] p. 5 or [11] (iii) p. 208). The resulted singular disc is denoted by the same notation D_1.

Let b be a branch point of D_1 and γ the double curve issuing from b. The cut of D_1 along γ ends at either (a) a branch point $b'\neq b$ or (b) a crossing point c of C_1. For Case (a) we cut off the component of $D_1-\gamma$ not containing C_1 from D_1 and have a simpler singular disc bounded by C_1. For Case (b), the cut of D_1 along γ divides it into two singular discs D_0', D_0'' such that BdD_0' + Bd D_0'' $=C_1$. For let $(\tilde{D}_1, f_1)$ be the diagram of D_1. Then $f_1^{-1}(c)$ consists of two points on Bd $\tilde{D}_1$. $f_1^{-1}(\gamma)$ is a cross cut of the disc $\tilde{D}_1$, containing the point $f_1^{-1}(b)$ and joining two points of $f^{-1}(c)$, and divides $\tilde{D}_1$ into discs $\tilde{D}_0', \tilde{D}_0''$. $D_0'=f_1(\tilde{D}_0')$, $D_0''=f_1(\tilde{D}_0'')$ are singular discs in $\bar{K}$. Let us denote Bd D_0', Bd D_0'' by A, B respectively. Since $A=f_1(\text{Bd }\tilde{D}_0'-\text{Int }f_1^{-1}(\gamma))$ and $B=f_1(\text{Bd }\tilde{D}_0''-\text{Int }f_1^{-1}(\gamma))$, we have $A+B=C_1$. We note here that $A(B)\simeq 0$ in $\bar{K}$ and $A(B)$ is simpler than C_1. Either $\{A, C_2,\cdots,C_h\}$ or $\{B, C_2,\cdots, C_h\}$ has Property (P) (say the former). For if not,[4] $A\sim\sum_{i=2}^{h}\lambda_i C_i$ on T and $B\sim\sum_{i=2}^{h}\mu_i C_i$ on T, where λ_i, μ_i are integers. Thus $C_1\sim\sum_{i=2}^{h}(\lambda_i+\mu_i)\ C_i$ on T, contrary to the fact that $C_1,\cdots, C_h$ are homologously independent on T.

Applying this process to $\{A, C_2,\cdots, C_h\}$ and the singular disc D_0', we have a system $\{A', C_2,\cdots, C_h\}$ with Property (P) such that A' is simpler than A or bounds a singular disc in $\bar{K}$ simpler than D_0'. Repeating this manner, we get a system $\{C_1', C_2,\cdots, C_h\}$ of closed curves on T with Property (P) such that C_1' is simple or bounds a singular disc D_1' without a branch point.

In the latter case, let c be a crossing point on C_1' and γ the double curve issuing from c. Then γ ends at a crossing point $c'\neq c$ on C_1'. Let $(\tilde{D}_1', f_1')$ be the diagram of D_1' and let $f_1'^{-1}(c)=\tilde{c}+\tilde{c}_1$; $f_1'^{-1}(c')=\tilde{c}'+\tilde{c}_1'$ and $f_1'^{-1}(\gamma)=\tilde{\gamma}+\tilde{\gamma}_1$ where $\tilde{\gamma}(\tilde{\gamma}_1)$ is a cross cut of $\tilde{D}_1'$ from $\tilde{c}(\tilde{c}_1)$ to $\tilde{c}'(\tilde{c}_1')$ (see Fig. 1). $\tilde{\gamma}$ and $\tilde{\gamma}_1$ are mutually exclusive. Bd $\tilde{D}_1'$ is divided into four arcs $\tilde{c}\ \tilde{c}_1$, $\tilde{c}_1\ \tilde{c}_1'$, $\tilde{c}_1'\ \tilde{c}'$ and $\tilde{c}'\ \tilde{c}$. Let $\omega_1, \omega_2, \omega_3$ be discs in $\tilde{D}_1'$ bounded by $\tilde{c}'\ \tilde{c}+\tilde{\gamma}$, $\tilde{c}_1'\ \tilde{c}'+\tilde{\gamma}+\tilde{c}\ \tilde{c}_1+\tilde{\gamma}_1$, $\tilde{c}_1\ \tilde{c}_1'+\tilde{\gamma}_1$, respectively. We denot the closed curves $f_1'(\tilde{c}_1\ \tilde{c}_1')f_1'(\tilde{c}'\ \tilde{c})$, $(f_1'(\tilde{c}'\ \tilde{c}))^{-1}\ f_1'(\tilde{c}_1'\ \tilde{c}')\ f_1'(\tilde{c}'\ \tilde{c})f_1'(\tilde{c}\ \tilde{c}_1)$ by α_1, α_2 respectively. Then $C_1'\simeq\alpha_1\alpha_2$ on T and $C_1'\sim\alpha_1+\alpha_2$ on T. We construct a singular disc D_1'' in $\bar{K}$ as follows: By cutting $\tilde{D}_1'$ along $\tilde{\gamma}$ and $\tilde{\gamma}_1$, it is divided

(4) $\sim$ means homologous to.

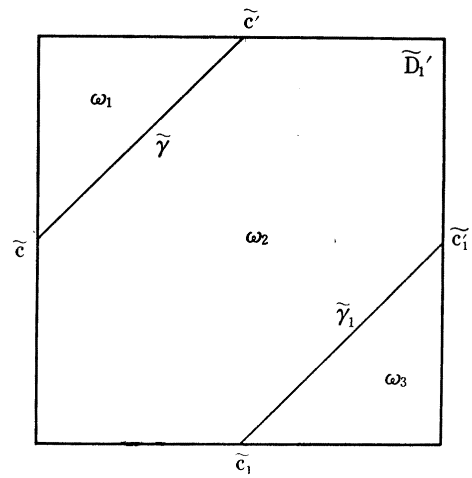

Fig. 1

into three pieces $\omega_1, \omega_2, \omega_3$. Next we glue again ω_1, ω_3 to ω_2 along $\tilde{\gamma}_1, \tilde{\gamma}$ respectively. Thus we have a disc $\omega_3+\omega_2+\omega_1$ and define a mapping of the disc into $\bar{K}$ by f_1' in a natural way. Since α_2 is homotopic to $(f_1'(\tilde{c}'\,\tilde{c}))^{-1} f_1'(\tilde{c}_1'\tilde{c}')(f_1'(\tilde{c}_1\,\tilde{c}_1'))^{-1} f_1'(\tilde{c}\,\tilde{c}_1)$ in $\bar{K}$ and the latter bounds the singular disc $f_1'(\omega_3+\omega_2+\omega_1)$ in $\bar{K}$, it results $\alpha_2 \simeq 0$ in $\bar{K}$. Moreover $\alpha_1 \simeq 0$ in $\bar{K}$, because it bounds the singular disc $f_1'(\omega_1+\omega_3)$ in $\bar{K}$. Since $C_1', C_2, \cdots, C_h$ are homologously independent on T, either $\{\alpha_1, C_2, \cdots, C_h\}$ or $\{\alpha_2, C_2, \cdots, C_h\}$ has Property (P). α_1 is simpler than C_1' and α_2 is slightly adjusted to get a closed curve on T simpler than C_1'.

Repeating the same manner, we get the desired system of simple closed curves on T. We denote it by $\{C_1, C_2, \cdots, C_h\}$.

(ii) In this step we shall show that there exists a system of mutually exclusive simple closed curves on T with Property (P). Since C_i's are simple and $C_i \simeq 0$ in $\bar{K}$, we can find discs D_i such that $D_i \cdot T = C_i = \mathrm{Bd}\, D_i$ and such that D_i, D_j are in general position in the sense that $D_i \cdot D_j$ consists of a finite number of double curves along which D_i and D_j cross $(1 \leqq i \neq j \leqq h)$. Therefore each double curve in $\sum D_i \cdot D_j$ is a simple closed curve or an arc. By using induction on the number of the simple closed curves in $\sum D_i \cdot D_j$, from $\{D_1, \cdots, D_h\}$ we have h discs, denoted by the same notations D_i, such that $\sum D_i \cdot D_j$ consists of mutually exclusive arcs. The details are as follows: Let t be a simple closed curve in $D_i \cdot D_j$ such that the interior of the disc D_0 in D_i bounded by t contains no curve in $D_i \cdot D_j$. We note here $t \subset \mathrm{Int}\, D_i \cdot \mathrm{Int}\, D_j$. Let D_0' be the disc in D_j bounded by t. If the disc $D_j - D_0' + D_0$ is pushed away from D_i in a neighbourhood of D_0, there results a disc D_j' bounded by C_j such that $D_j' \cdot D_i$ are in general position and $D_i \cdot D_j'$ has fewer simple closed curves than $D_i \cdot D_j$. We denote also the set of the discs thus obtained by $D_1, D_2, \cdots, D_h$.

By using induction on the number of arcs in $\sum D_i \cdot D_j$, we finish the work of the step. If $C_1 \cdot C_j$ contains a crossing point c, the double curve γ issuing from c ends at the other crossing point c' on $C_1 \cdot C_j$. Let $\alpha_1, \alpha_2 (\alpha_1', \alpha_2')$ be arcs into which C_1 (C_j) is divided, and let us denote the closed curves $\alpha_2\,\alpha_1'$, $\alpha_2\,\alpha_1'^{-1}$ by C_1', C_1'' respectively. Then either $C_1', C_2, \cdots, C_h$ or $C_1'', C_2, \cdots, C_h$ are homologously independent on T (say the former). For if not, $C_1' \sim \sum_{i=2}^{h}\lambda_i\, C_i$ and $C_1'' \sim \sum_{i=2}^{h}\mu_i\, C_i$ on T. Thus we have $C_1 = C_1' + C_1'' = \sum_{i=2}^{h}(\lambda_i+\mu_i)C_i$ on T, contrary to the fact that C_1, $C_2, \cdots, C_h$ are homologously independent on T. Obviously $C_1' \simeq 0$ in $\bar{K}$. Though C_1' may not be simple, by a slight adjustment of C_1' we have a system $\{C_1', C_2, \cdots, C_h\}$ of closed curves on T with Property (P) which is simpler than $\{C_1, C_2, \cdots, C_h\}$. By repeating the process (i), (ii) there results the desired system.

(iii) In this step we finish the proof of the theorem. Let V_i $(i=1,\cdots,h)$ be mutually exclusive 3-cells in $\bar{K}$ such that for each i, $V_i \cdot T$ is an annulus, one component of whose boundary is C_i and Bd $V_i \supset D_i$. Since $C_1, \cdots, C_h$ are homologously independent on T, Bd $(K - \sum_{i=1}^{h} V_i)$ is a 2-sphere in S^3. Thus $\overline{K - \sum_{i=1}^{h} V_i}$ is a 3-cell and $\bar{K} = \overline{K - \sum_{i=1}^{h} V_i} + \sum_{i=1}^{h} V_i$ is a solid torus of genus h.

3. Tori of genus one in S^3

LEMMA. *Let M be a closed 3-manifold which is the sum of two compact 3-manifolds V and V', such that $V \cdot V' = Bd\ V = Bd\ V'$ is a polyhedral closed surface and $\pi_1(V, Bd\ V) = 1 = \pi_1(V', Bd\ V')$. Then in order that M may be simply connected, it is necessary and sufficient that for each closed curve C in V there exists a closed curve C_0 on $Bd\ V$ such that*[5] $C_0 \overset{*}{\simeq} C$ *in V and $C_0 \simeq 0$ in V'.*

PROOF. Sufficiency is obvious.
To prove necessity it is enough to show that there exists a singular disc D, bounded by C, such that $f^{-1}(V \cdot D)$ is an annulus, one of whose boundary curves is Bd $\tilde{D}$, where $(\tilde{D}, f)$ is the diagram of D. We divide the proof into two steps.

(i) In this step it is shown that there exists a singular disc D_0, bounded by C, such that $f_0^{-1}(V \cdot D_0)$ is connected, where $(\tilde{D}_0, f_0)$ is the diagram of D_0. Let D be a singular disc in M bounded by C and $(\tilde{D}, f)$ the diagram of D. If $D \subset V$, it is trivial to find such a C_0 on Bd V. Since f is semi-linear (see Introduction), $f^{-1}(V \cdot D)$ consists of a finite number of simplexes of $\tilde{D}$. A slight adjustment of D in a small neighbourhood of Bd V is enough to cause the adjusted D to be such a polyhedron as follows: Each component of $f^{-1}(V \cdot D)$ is a perforated disc (i. e. a set homeomorphic to a 2-sphere minus the sum of a finite number of mutually exclusive open discs). Let K be the component of $f^{-1}(V \cdot D)$ containing Bd $\tilde{D}$ and let $K_1, \cdots, K_h$ be the components of Bd K except for Bd $\tilde{D}$. Let $\tilde{D}_i$ be the component of $\tilde{D} - f^{-1}(V \cdot D)$ with $K_i, K_{i1}, \cdots, K_{im(i)}$ as boundary curves. Then there exist mutually exclusive[6] cross cuts $\tilde{C}_{ij}$ of $\tilde{D}_i$

(5) $\overset{*}{\simeq}$ means free homotopy.

(6) A cross cut C of a perforated disc D is an arc joining two points on Bd D such that Int $C \subset$ Int D.

joining a point $\tilde{b}_i$ on K_i to $\tilde{b}_{ij}$ on $K_{ij}(j=1,\cdots, m(i))$. Now $f(\tilde{C}_{ij})$ is a (not necessarily simple) curve in $\overline{M-V}$ joining $f(\tilde{b}_i)$ to $f(\tilde{b}_{ij})$ on Bd V. Since $\pi_1(\overline{M-V}$, Bd $V)=1$, $f(\tilde{C}_{ij})$ is homotopic in V' to a curve C'_{ij} on Bd V joining $f(\tilde{b}_i)$ to $f(\tilde{b}_{ij})$. Let $(\tilde{D}'_j, g_j)$ be the diagram of the homotopy, that is the diagram of the singular disc bounded by $f(\tilde{C}_{ij})$ and C'_{ij}. Bd $\tilde{D}'_j$ is divided into arcs α, α' with common end points such that $g_j(\alpha)=f(\tilde{C}_{ij})$ and $g_j(\alpha')=C'_{ij}$. It may be supposed without loss of generality $\alpha=\tilde{C}_{ij}$ and $g_j(p)=f(p)$ for every point p on α. We have a diagram of a singular disc from f, g_j and $\tilde{D}+\tilde{D}'_j$ as follows: We first consider the duplication of $\tilde{D}'_j$ and pull $\tilde{D}+\tilde{D}'_j$ along $\tilde{D}'_j-\alpha'$. Thus we have a diagram $(\tilde{D}', f')$ of a singular disc bounded by C such that $f'^{-1}(V\cdot f'(\tilde{D}'))=f^{-1}(V\cdot f(\tilde{D}))+\alpha'$. $f'^{-1}(V\cdot f'(\tilde{D}'))$ has fewer components than $f^{-1}(V\cdot f(\tilde{D}))$.

By using induction we obtain a singular disc $f_0(\tilde{D}_0)$ in M, with boundary C, such that $f_0^{-1}(V\cdot f_0(\tilde{D}_0))$ is connected, where $(\tilde{D}_0, f_0)$ is the diagram of $f_0(\tilde{D}_0)$.

(ii) Our method in (i) may be applied to $(\tilde{D}_0, f_0)$ and the components of $\tilde{D}_0-f_0^{-1}(V\cdot f_0(\tilde{D}_0))$, because $\pi_1(V, \text{Bd } V)=1$. Therefore, we have a singular disc $f(\tilde{D})$ in M, with boundary C, such that $f^{-1}(V\cdot f(\tilde{D}))$ is an annulus $\tilde{A}$ in $\tilde{D}$ with boundary Bd $\tilde{D}+\tilde{C}_0$. Let $C_0=f(\tilde{C}_0)$. Then since C, C_0 are the boundary curves of the singular annulus $f(\tilde{A})$, we have $C\overset{*}{\simeq}C_0$ in V.

Q. E. D.

COROLLARY. *Let M be a simply connected, closed 3-manifold which is the sum of two tori of genus h, V, V' such that $V\cdot V'=$Bd $V=$Bd V' is a polyhedral torus. Then to each closed curve C in V there exists a closed curve C' on Bd V such that $C\overset{*}{\simeq}C'$ in V and $C'\simeq 0$ in V'.*

Let X be a topological space and let T, T' be two subsets of X. Suppose that T and T' are homeomorphic and that there exists a mapping h of the topological product of X and the unit interval I onto X, such that $h(p\times 0)=p$ for every p in X, $h(T\times 1)=T'$ and such that for each t in I the mapping $p\times t\rightarrow h(p\times t)$ is a homeomorphism of $X\times t$ onto X. Then we call h an *isotopy* of X onto itself taking T onto T'.

THEOREM 3. *Let T, T' be polyhedral tori of genus one in S^3 and V, V' solid tori in S^3 with boundary T, T' respectively. If $\pi_1(\overline{S^3-V}, T)=1=\pi_1(\overline{S^3-V'}, T')$, there exists an isotopy of S^3 onto itself taking T onto T'.*

PROOF. we divide the proof into two parts.

(i) Let S, S' be polyhedral 2-spheres in S^3, Q, Q' 3-cells in S^3 bounded by S, S' respectively. Let Q_0 (Q'_0) be a 3-cell in Q (Q') such that $Q_0\cdot$ Bd $Q(Q'_0\cdot$ Bd $Q')$ consists of mutually exclusive 2-cells D_1, $D_2(D'_1, D'_2)$ and $Q_0(Q'_0)$ is a straight hole of $Q(Q')$, that is, $\overline{Q-Q_0}$ $(\overline{Q'-Q'_0})$ is a solid torus of genus one. Then there exists an isotopy h of S^3 taking Q, D_1, D_2, Q_0 onto Q', D'_1, D'_2, Q'_0 respectively.

This is proved as follows: There is an isotopy h_1 of S^3 which deforms Q onto Q', and takes D_1, D_2 onto D'_1, D'_2 respectively (by [8], Theorem 5). Since both $h_1(Q_0\times 1)$ and Q_0' are straight holes in Q_0', we can find an isotopy of Q' which deforms $h_1(Q_0\times 1)$ onto Q_0' and leaves $S'=$ Bd Q' fixed.

(ii) Let L be a longitude of V (see Introduction). By the above lemma there exists a closed curve L' on T such that $L \stackrel{*}{\simeq} L'$ in V and $L'\simeq 0$ in $\overline{S^3-V}$. Since T is of genus one, we can find a simple closed curve L'' on T, which is homotopic to L' on T, and a disc D in $\overline{S^3-V}$ such as $D\cdot T=L''$. Now let Q_0 be a 3-cell in $\overline{S^3-V}$ such that Q_0 contains D, $Q_0\cdot V\supset L''$ and $T\cdot Q_0$ is an annulus A. Let D_1, D_2 be the components of Bd Q_0- Int A. Q_0+V is a 3-cell Q. Q', D_1', D_2', Q_0' and A' are also defined for V' in the same way. By (i) there exists an isotopy h of S^3 onto itself taking Q, D_1, D_2, Q_0 onto Q', D_1', D_2', Q_0', respectively. Hence T is deformed onto T' under h.

Q. E. D.

4. A system of longitudes

The purpose of this section is to prove the following

THEOREM 4. *Let V be a solid torus of genus h and $W_1,\cdots,W_h$ simple closed curves on Bd V such that $W_i\cdot W_j=x_0$ $(i\neq j)$. In order that $\{W_1,\cdots,W_h\}$ be a system of longitudes of V, it is necessary and sufficient that it represents a system of generators of $\pi_1(V)$.*

By the theorem we see that if a 3-manifold M is the sum of two solid tori V, V' such that $V\cdot V'=$ Bd $V=$ Bd V' and(7) a system of meridians of V' represents a system of generators of $\pi_1(V)$, M is topologically S^3.

PROOF. Necessity is obvious.
The proof of sufficiency will be given in the form of a series of lemmas. Let G be a free group. The cardinal number of free generators of G is called the *rank* of G [4], I p. 127.

LEMMA 1 (A corollary of Gruško's theorem [4], II p. 59). Let G be a free group of rank h. *A* system of generators of G consisting of h elements is a system of free generators.

LEMMA 2. Let G be a free group of rank h with h free generators $a_1,\cdots,a_h$, and let Γ be the group of automorphisms of G. Then Γ is generated by the automorphisms of G as the following types (using Nielsen's symbols), [5] § 1, [3] p. 89:

$$\mathrm{P}=[a_2,\ a_1,\ a_3,\cdots,\ a_h]$$
$$\mathrm{Q}=[a_2,\ a_3,\cdots,a_h,\ a_1]$$
$$\mathrm{O}=[a_1^{-1},a_2,\cdots\cdots,a_h]$$
$$\mathrm{U}=[a_1a_2,a_2,\cdots\cdots,a_h]$$

(7) Strictly speaking, each meridian of the system should be joined by a curve to the base point of π_1 (Bd V).

LEMMA 3. Let V be a solid torus of genus h and let m^* be a simple closed curve on BdV such that $m^* \simeq 0$ in V and m^* not ~ 0 on Bd V. Then m^* is a meridian of V.

PROOF. Let $\{m_1, \cdots, m_h\}$ be a system of meridians of V. By the same method as in the proof of Theorem 2, we find a system of meridians $\{m_1', \cdots, m_h'\}$, each of whose elements does not intersect m^*, from the system $\{m_1, \cdots, m_h\}$. Next it can be seen that $h-1$ elements in $\{m_1', \cdots, m_h'\}$ are homologouslly independent on Bd V together with m^*.

Q. E. D.

LEMMA 4. Let V be a solid torus of genus h, $\{l_1, \cdots, l_h\}$ a set of closed curves in V representing a system of generators of $\pi_1(V)$. Then there exists a system of longitudes, $\{l_1', \cdots, l_h'\}$, such that $l_i \simeq l_i'$ in V $(i=1, \cdots, h)$.

PROOF. We denote the element of $\pi_1(V)$, represented by an oriented closed curve l which contains the base point x_0 of $\pi_1(V)$, by $\tilde{l}$. By lemma 1 $\{\tilde{l}_1, \cdots, \tilde{l}_h\}$ is a system of free generators. Let $\{l_1'', \cdots, l_h''\}$ be a system of longitudes of V. By virtue of Lemma 2 $\{\tilde{l}_1, \cdots, \tilde{l}_h\}$ is obtained from $\{\tilde{l}_1'', \cdots, \tilde{l}_h''\}$ by a finite series of the operations of type P, Q, O, U, To each operation is associated a geometric operation as follows: To P and Q corresponds the change of order of elements of $\{l_1'', \cdots, l_h''\}$ and to O the inversion of the orientation of l_1''. The result is still a system of longitudes. For the operation of type U, we first consider a system $\{l_1''l_2'', l_2'', \cdots, l_h''\}$ and next we find a simple closed curve l_1^* on Bd V so that $l_1^* \simeq l_1''l_2''$ in V and $\{l_1^*, l_2'', \cdots, l_h''\}$ is a system of longitudes. Such an l_1^* is obtained as follows: Let $\{m_1, \cdots, m_h\}$ be a system of meridians conjugate to $\{l_1'', \cdots, l_h''\}$, D_i meridian discs bounded by m_i and p_i the points at which l_i'', m_i meet. If we cut V along D_i, there results a 3-cell on whose boundary there exist mutually exclusive $2h$ discs D_i' and D_i'', two copies of D_i, and $2h$ arcs x_0p_i' and x_0p_i'' joining x_0 to p_i' and p_i'' respectively, where p_i', p_i'' are two copies

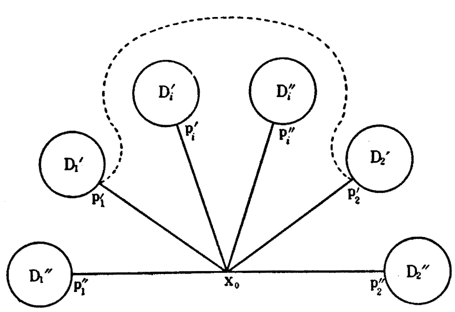

Fig. 2

of p_i and x_0p_i', x_0p_i'' compose l_i'' in V. Then l_1^* will be found on Bd V so that it is divided into x_0p_1'', x_0p_2'' and the dotted arc, as shown in Fig. 2, whose interior does not meet $\sum_{i=1}^{h}(D_i' + D_i'' + x_0p_i' + x_0p_i'')$.

Hence, by a finite series of these operations we get the desired system of longitudes.

Q. E. D.

LEMMA 5. Let l be a longitude of a solid torus V of genus h and C a simple closed curve on Bd V such that $l \stackrel{*}{\simeq} C$ in V. Then there exists a meridian m^* of V interjecting C at only one point.

PROOF. Let m be a meridian of V conjugate to l and D a meridian disc bounded by m. l and D meet at only one point p by definition. We may suppose that $\{C, m\}$ is normal.

To prove the lemma it is enough to show that there exists a simple closed curve m' on Bd V such that $m' \cdot C$ has fewer points than $m \cdot C$, $m' \simeq 0$ in V and[8] $s(m', C) = \pm 1$. Since $l \stackrel{*}{\simeq} C$ in V, there exist an annulus A with $\tilde{l}$, $\tilde{C}$ as boundary curves, and a mapping $f\colon A \to V$ such that f takes $\tilde{l}$, $\tilde{C}$ homeomorphically onto l, C respectively and such that $f(\text{Int } A) \subset \text{Int } V$. Let us denote the set of crossing points of $\{m, C\}$ by F. By a slight modification of $f(A)$ in the vicinity of D, it results that D and the adjusted $f(A)$ are in general position, in the sense that $f(A) \cdot D$ consists of a finite number of double curves along which $f(A)$ and D actually cross. And we may suppose that $f^{-1}(D \cdot f(A))$ consists of mutually exclusive simple curves, such as an arc from $f^{-1}(p)$ to a point q of $f^{-1}(F)$, closed curves contained in Int A and cross cuts, called to be of type T, joining two points of $f^{-1}(F)$. Let K be a cross cut of type T such that the disc on A, bounded by K and one of the arcs K' on $\tilde{C}$ with $K \cdot \tilde{C}$ as end points, does not contain the other cross cuts of type T. Now let J be the subarc of m, having $f(K \cdot \tilde{C})$ as end points and not containing $f(q)$. By a slight deformation on Bd V of the simple closed curve, $m + f(K') - \text{Int } J$ we get the desired m'.

PROOF OF THEOREM. $\{W_1, \cdots, W_h\}$ is a system of free generators of $\pi_1(V)$ (Lemma 1). Hence by Lemma 4 W_1 is a simple closed curve homotopic in V to a logitude of V. Therefore we find a meridian m_i'' meeting W_i at only one point (Lemma 5). From the system $\{m_1'', \cdots, m_h''\}$ we can easily get a system of meridians of V so that $\{W_1, \cdots, W_h\}$ is a conjugate system of longitudes.

Q. E. D.

The following example shows a transformation of a system of longitudes of a solid torus of genus 2 in S^3.

EXAMPLE. A clover leaf L_1 in S^3 is a simple closed curve on the boundary of an unknotted polyhedral torus W of genus one. Let L_2' be a meridian of W and p a point of $L_1 \cdot L_2'$. Moreover let L_2 be a simple closed curve containing p and obtained by slightly pushing $L_2' - p$ into $S^3 - W$ (Fig. 3). Let V be a small polyhedral tubular neighbourhood of $L_1 + L_2$. Then an adjusted $\{L_1, L_2\}$ is a

(8) s (m', C) means the interjection number of m and C.

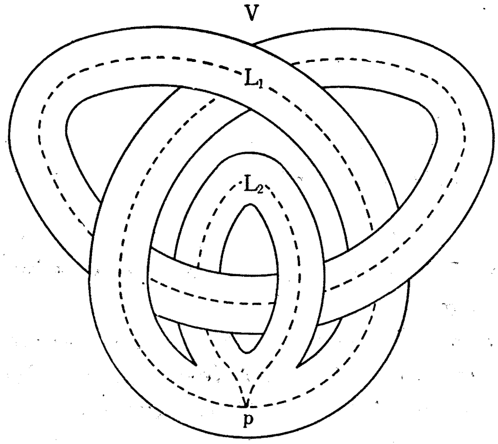

Fig. 3

system of longitudes of V. We note that even though L_1 is knotted, $\overline{S^3-V}$ is also a solid torus, and that by a transformation of type U in Lemma 2 there is obtained a system of longitudes of V whose elements are unknotted.

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