J. Sci. Hiroshima Univ. Ser. A
Vol. 24, No. 3, December, 1960

On $\mathfrak{m}$-adic Differentials

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(Received September 20, 1960)

Introduction

In our previous paper one of the authors developed the theory of differentials in commutative rings. In that paper he started with the definitions given in [1][1)] and applied the theory of differentials to the characterization of geometric regular local rings, and other problems. But when he was going to apply the theory to a larger class of local rings he encountered serious embarrassment. The situation may be well described if we take up the following example. Let A be the ring of the analytic functions in n complex variations $z_1, \cdots z_n$, which are regular in some neighborhood of the origin. According to the definition given in [1], the module of differentials in A (over the constant field) contains infinitely many linearly independent differentials. On the other hand if f is an element of A, the differential df can be written uniquely in the form $df = \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n$, i.e. there exist only n-analytically independent differentials $dz_1, \cdots, dz_m$ in A. This example means that the former theory is not adequate as a theory of differentials since it cannot cover the analytic case. Now it may be natural to ask whether there exists a good algebraic theory available for such a case. This is the motivation of the present work and the affirmative answer will be presented here. To develop the theory we must begin with $\mathfrak{m}$-adic rings and then the notion of $\mathfrak{m}$-adic differentials will be introduced. Naturally the newly introduced notion contains the old one as a special case. Moreover it will be seen that when we deal with the local rings of points on an algebraic variety, the new theory coincides with the former one. This is the reason why the former theory is useful to the problems in algebraic geometry. In the last paragraph we shall give the characterizations of regular local rings as an application. We hope that this theory will find good applications in the theory of algebroid varieties.

§ 1. Preliminaries.

We shall retain all the notations and terminologies used in [4]. All rings considered in this paper will be commutative containing the unity 1. Let S be an R-algebra with a ring homomorphism $f: R \to S$ such that $f(1)=1$. We shall denote by $D_R(S)$, *the module of R-differentials* in S, and R-differential

1) The number in the bracket refer to the bibliography at the end of the paper.

of an element x of S will be denoted by $d_R^S x$. $D_R(S)$ is characterized by the universal mapping property with respect to the R-derivations of S into an arbitrary S-module[2], i.e. if D is an R-derivations of S into an S-module E, then there exists an S-homomorphism h of $D_R(S)$ into E such that $Dx=h(d_R^S x)$. Let S be a ring and let $\mathfrak{m}$ be an ideal of S. We shall say that S is an $\mathfrak{m}$-*adic ring* if S is topologized by taking $\mathfrak{m}^r (r=1, 2, \dots)$ as a fundamental system of neighborhoods of zero. Let S be an $\mathfrak{m}$-adic ring and let E be an S-module. We shall say that E is an $\mathfrak{m}$-*adic S-module* if E is endowed with the topology in which $\mathfrak{m}^n E (n=1, 2, \dots)$ form a fundamental system of neighborhoods. An $\mathfrak{m}$-adic S-module is not necessarily a Hausdorff space. An $\mathfrak{m}$-adic S-module E is a Hausdorff space if, and only if, $\bigcap_{n=0}^{\infty} \mathfrak{m}^n E=0$. In this case we shall say that E is a *separated* ($\mathfrak{m}$-*adic*) *S-module*, or simply a *S_H-module*. Let S be a noetherian ring and let $\mathfrak{m}$ be an ideal of S. We shall say that the pair $(S, \mathfrak{m})$ is a *Zariski ring* if $\mathfrak{m}$ is contained in the J-radical[3] of S. Then S is a Hausdorff space as an $\mathfrak{m}$-adic ring. A Zariski ring $(S, \mathfrak{m})$ is characterized as an $\mathfrak{m}$-adic Hausdorff ring in which all the ideals are closed ([1]). The following lemmas are well known.

LEMMA 1. *Let $(S, \mathfrak{m})$ be a Zariski ring and let E be a finite S-module. Then E is an S_H-module and any submodule F of E is a closed set. Moreover the $\mathfrak{m}$-adic topology of F coincides with the induced $\mathfrak{m}$-adic topology of E* (cf. [1], Th. 1 of Exposé 18).

LEMMA 2. *Let S be an $\mathfrak{m}$-adic ring and let E be an S_H-module. Then any derivation D of S into E is a continuous map* (cf. [1], Th. 2 in Exposé 18).

In the present monograph we shall treat exclusively the $\mathfrak{m}$-adic ring which is at the same time a Haudorff space. Hence all $\mathfrak{m}$-adic rings S in this paper are assumed to satisfy the conditions $\bigcap_{r=1}^{\infty} \mathfrak{m}^r=0$ unless otherwise stated.

§ 2. $\mathfrak{m}$-adic differentials.

Let S be an R-algebra and let $\mathfrak{m}$ be an ideal of S. We shall assume that S is an $\mathfrak{m}$-adic ring. We define the module of $\mathfrak{m}$-adic R-differentials in S, denoted by $\hat{D}_R(S)$, as the S-module satisfying the following conditions.

(1) There exists an R-derivation $\hat{d}_R^S$ from S into $\hat{D}_R(S)$, ($\hat{d}_R^S$ will be called an $\mathfrak{m}$-*adic differential operator* over R)

(2) $\hat{D}_R(S)$ is generated over S by $\hat{d}_R^S x$, $x \in S$

(3) $\hat{D}_R(S)$ is a separated $\mathfrak{m}$-adic S-module

(4) Let E be an arbitrary separated $\mathfrak{m}$-adic S-module and let D be an R-derivation of S into E. Then there exists an S-linear map h from $\hat{D}_R(S)$ into E

(2) All modules in this paper are assumed to be unitary.

(3) We mean by the J-radical the intersection of all maximal ideals of S.

such that

$$Dx = h(\hat{d}_R^S x) \text{ for all } x \in S.$$

In particular (O)-adic R-differentials are no other than the differentials introduced in the previous papers. ([1]. [4]).

PROPOSITION 1. *Let S be an R-algebra and let $\mathfrak{m}$ be an ideal of S and assume that S is an $\mathfrak{m}$-adic ring. Then the module of $\mathfrak{m}$-adic R-differentias $\hat{D}_R(S)$ exists and is determined uniquely up to S-isomorphism.*[4] *Moreover $\hat{D}_R(S)$ is given by*

$$\hat{D}_R(S) = D_R(S) / \bigcap_{r=0}^{\infty} \mathfrak{m}^r D_R(S).$$

PROOF. The uniqueness is seen by the standard arguement and Lemma 2. Hence to complete the proof it will be sufficient to show the quotient module $D_R(S)/\bigcap_{r=0}^{\infty} \mathfrak{m}^r D_R(S)$ satisfies the four requirements. Let ρ be a natural homomorphism

$$\rho : D_R(S) \to D_R(S) / \bigcap_{r=0}^{\infty} \mathfrak{m}^r D_R(S)$$

and let us put $\hat{d}_R^S x = \rho(d_R^S x)$. Then we can easily verify (1), (2) and (3). The property (4) follows from the assumption that E is a separated S-module.

COROLLARY 1. *If $D_R(S)$ is a separated $\mathfrak{m}$-adic S-module, we have*

$$\hat{D}_R(S) = D_R(S).$$

COROLLARY 2. *If S is a field, $D_R(S) = \hat{D}_R(S)$.*

PPOPOSITION 2. *Assume that the ring S is noetherian and $D_R(S)$ is of finite rank over S. We have $\hat{D}_R(S) = D_R(S)$ if one of the following conditions holds:*

(a) *$(S, \mathfrak{m})$ is a Zariski ring.*

(b) *Any element of S is not a zero divisor of $D_R(S)$.*

PROOF. (a) follows from Lemma 1 and Cor. 1 of Prop. 1.

(b) is a consequence of the Theorem of Artin-Rees (cf. Th. 2, Exposé 2 in [1]).

PROPOSITION 3. *Let A be an affine ring over a noetherian ring R. Let $\mathfrak{a}$ be an ideal of A, and let U be a multiplicatively closed set consisting of elements $1+a$, $a \in \mathfrak{a}$. Let us put $S = A_U$ and $\mathfrak{m} = \mathfrak{a}A_U$. Then we have*

$$\hat{D}_R(S) = D_R(S).$$

PROOF. Let $A = R[u_1, \cdots u_n,]$ and let $\tilde{A}$ be a polynomial ring in n-variables

(4) Not only "algebraic"but also" topological".

$X_1, \cdots X_n$, over R. Then there exists a natural homomorphism $\varphi_{R;\tilde{A},A}$ from $D_R(\tilde{A})$ onto $D_R(A)$ (Prop. 9 of [4]). Since $D_R(\tilde{A})$ is a free module of rank n over A (Prop. 15 of [4]) we see that $D_R(A)$ is a finite A module. Since $D_R(A_U)=A_U \otimes D_R(A)$ by Prop. 10 in [4], $D_R(S)$ is also a finite module. Hence the assertion follows from (a) of Prop. 2, since $(S, \mathfrak{m})$ is a Zariski-ring.

Let S be a local ring with the maximal ideal $\mathfrak{m}$. Then the module of $\mathfrak{m}$-adic R-differentials will be called simply the module of *analytic differentials* in S. By the similar reasoning as above, we have the

PROPOSITION 4. *Let A and R be as in* Prop. 3. *Let $\mathfrak{p}$ be a prime ideal of A. Then the module of analytic differentials in the local ring $S=A_{\mathfrak{p}}$ is isomorphic to the module of R-differentials $D_R(S)$.*

From this proposition we see that when we are treating local rings which are quotient rings of points of algebraic variety, the theory of analytic differentials offers no new results.

§ 3. The properties of $\mathfrak{m}$-adic differentials.

In this paragraph we shall give generalizations of the results of §2 in [4] to $\mathfrak{m}$-adic differentials. Let S be an $\mathfrak{m}$-adic R-algebra with a ring homomorphism $f: R \to S$ and let T be an $\mathfrak{n}$-adic S-algebra with a ring homomorphism $g: S \to T$. In this case T is naturally an R-algebra with the ring homomorphism $h = g \circ f: R \to T$. Under these circumstances assume moreover that

$$g(\mathfrak{m}) \subset \mathfrak{n}. \tag{1}$$

$\hat{D}_R(T)$ is, by definition, a separated $\mathfrak{n}$-adic T-module. At the same time $\hat{D}_R(T)$ is an S-module and we can endow to $\hat{D}_R(T)$ with $\mathfrak{m}$-adic topology. Since $\mathfrak{m}^r \hat{D}_R(T) = g(\mathfrak{m})^r \hat{D}_R(T)$, $\hat{D}_R(T)$ is also an $\mathfrak{m}$-adic separated S-module under the condition (1). Let us now define a map D from S into $\hat{D}_R(T)$ by

$$Dx = \hat{d}_R^T(x).$$

This is clearly an R-derivation of S into $\hat{D}_R(T)$, hence there exists an S-homomorphism α: $\hat{D}_R(S) \to \hat{D}_R(T)$ such that

$$Dx = \hat{d}_R^T(x) = \alpha(\hat{d}_R^S(x)) \qquad \text{for } x \in S.$$

From this we can define a T-homomorphism

$$\varphi'_{R;S,T}: T \otimes_S \hat{D}_R(S) \to \hat{D}_R(T) \tag{2}$$

where $\varphi'_{R;S,T}$ is defined by the rule

$$\varphi'_{R;S,T}(\sum_i t_i \otimes \hat{d}_R^S x_i) = \sum_i t_i \hat{d}_R^T x_i \qquad \text{for } t_i \in T,\ x_i \in S.$$

In general, the T-module $T \otimes_S \hat{D}_R(S)$ is not a T_H-module. But since $\hat{D}_R(T)$ is a

T_H-module we can define the homomorphism $\hat{\varphi}_{R;S,T}$.

$$\hat{\varphi}_{R;S,T}\colon T\otimes_S\hat{D}_R(S)/\bigcap_{r=0}^{\infty}\mathfrak{n}^r(T\otimes\hat{D}_R(S))\rightarrow\hat{D}_R(T)$$

Let us us denote by $\hat{N}_{R;S,T}$ and $\hat{D}_{S,T}$ the kernel and cokernel of $\hat{\varphi}_{R;S,T}$. Then we have the following exact sequence

$$O\rightarrow\hat{N}_{R;S,T}\rightarrow T\otimes\hat{D}_R(S)/\bigcap_{r=0}^{\infty}\mathfrak{n}^r(T\otimes\hat{D}_R(S))\xrightarrow{\hat{\varphi}_{R;S,T}}\hat{D}_R(T)\rightarrow\hat{D}_{S,T}\rightarrow O \tag{3}$$

which is the natural generalization of the exact sequence (A) in [4].

PROPOSITION 5. *Retaining the notations as above, assume that* $g(\mathfrak{m})\subset\mathfrak{n}$. *Then* $\hat{D}_S(T)$ *is isomorphic to* $\hat{D}_{S,T}/(\bigcap_{r=0}^{\infty}\mathfrak{n}^r\hat{D}_{S,T})$. *Let* $T\hat{D}(S)$ *be the submodule of* $\hat{D}_R(T)$ *generated over* T *by the elements* $\hat{d}_R^Tx$, $x\in S$. *Then* $\hat{D}_S(T)$ *is isomorphic to the residue module of* $\hat{D}_R(T)$ *modulo the closure of* $T\hat{D}(S)$ *in* $\hat{D}_R(T)$.

PROOF. The first assertion can be proved in a similar way as in the proof of Prop. 1 in [4]. It will be sufficient to remark that $\hat{D}_{S,T}$ itself is easily seen to satisfy the conditions (1), (2) and (4) in §2. On the contrary $\hat{D}_{S,T}$ is not necessarily a Hausforff-space as $\mathfrak{n}$-adic T-module. Hence it is necessary to form the residue module modulo $\bigcap_{r=0}^{\infty}\mathfrak{n}^rD_{S,T}$ in order to get the module of $\mathfrak{n}$-adic differentials. The last assertion follows immediately from the fact that $\hat{D}_{S,T}\cong\hat{D}_R(T)/T\hat{D}(S)$ and the first one.

COROLLARY 1. *Assume that* $(T,\mathfrak{n})$ *is a Zariski ring and* $\hat{D}_R(T)$ *is a finite* T*-module. Then we have*

$$\hat{D}_S(T)=\hat{D}_R(T)/T\hat{D}(S)$$

PROOF. Under these assumptions $T\hat{D}(S)$ is a closed set by Lemma 1.

COROLLARY 2. *Retaining the notations and assumptions as in* Prop. 5, *if* $\hat{D}_R(S)=0$, *then* $\hat{D}_R(T)=\hat{D}_S(T)$.

PROOF. From the exact sequence (3), we have $\hat{D}_R(T)=\hat{D}_{S,T}$, hence $\hat{D}_{S,T}$ is a separated $\mathfrak{n}$-adic T-module. Then $\hat{D}_S(T)=\hat{D}_{S,T}$ follows from Prop. 5.

The following theorem can be proved in a similar way as Th. 1 in [4]. Hence the proof will be omitted.

THEOREM 1. *Retaining the notations and assumptions in* Prop. 5 *we have the following: Any* R*-derivation of* S *into a separated* $\mathfrak{n}$*-adic* T*-module* V *can be extended to an* R*-deriviation of* T *into* V *if, and only if,* (i) $\hat{\varphi}_{R;S,T}$ *is injective, and* (ii) $(T\otimes_S\hat{D}_R(S))/(\bigcap_{r=0}^{\infty}\mathfrak{n}^r(T\otimes\hat{D}_R(S)))$ *is the direct summand of* $\hat{D}_R(T)$. *Moreover* *if $\hat{D}_R(S)$ is known to be a finite S-module we shall have* (i) *and* (ii) *if every R-derivation of S into a finite T-module can be extended to a derivation of T.*

COROLLARY. *Let $(T, \mathfrak{n})$ be a Zariski ring and assume that $\hat{D}_R(T)$ is a finite T-module. Then $\hat{\varphi}_{R;S,T}$ is an isomorphism if and only if any derivation of S into a T_H-module V can be extended uniquely to a derivation of T into V.*

PROOF. By our assumption we see that $\hat{D}_{S,T}$ is a fiinite module, hence $\hat{D}_S(T)=\hat{D}_{S,T}$ by Prop. 5. Since $\hat{D}_S(T)$ is $\mathfrak{n}$-adic separated, the uniquenees of the extension implies $\hat{D}_S(T)=0$. Then the first assertion is an immediate consequence of the Theorem. The converse is easy.

PROPOSITION 6. *Let S be an R-algebra, let $\mathfrak{m}$ be an ideal of S and let U be a multiplicatively closed set in S. If S_U is an $(\mathfrak{m}S_U)$-adic ring,* i.e. $\bigcap_{r=0}^{\infty}(\mathfrak{m}S_U)^r=0$, *then we have*

$$\hat{D}_R(S_U)=S_U\otimes\hat{D}_R(S)/\bigcap_{r=0}^{\infty}\mathfrak{m}^r(S_U\otimes\hat{D}_R(S)).$$

PROOF. Let $T=S_U$, then T is, in a natural way, an S-algebra with a homomorphism g: $S\to T$. Let us put $\mathfrak{n}=\mathfrak{m}S_U$. By our assumption T is an $\mathfrak{n}$-adic ring and g, $\mathfrak{m}$, and $\mathfrak{n}$ satisfy the condition (1). Hence in order to prove the Theorem it is sufficient to show the following: Let D be a derivation of S into a T_H-module E. Then D can be extended in a unique way to a derivation of T into E. This is proved in [4].[5]

COROLLARY. *Let S be a noetherian $\mathfrak{m}$-adic R-algebra and assume that $\hat{D}_R(S)$ is a finite module. Let U be a multiplicatively closed set in S such that $U+\mathfrak{m}\subset U$. Then we have*

$$\hat{D}_R(S_U)=S_U\otimes_S\hat{D}_R(S).$$

PROOF. Under these assumptions the ideal $\mathfrak{n}=\mathfrak{m}S_U$ is contained in the J-radical of S_U, since any element of $1+\mathfrak{n}$ is invertible in S_U. Hence $(S_U, \mathfrak{n})$ is a Zariski ring. $S_U\otimes\hat{D}_R(S)$ is a finite module over a Zariski ring $(S_U, \mathfrak{n})$, hence separated $\mathfrak{n}$-adic S_U-module i.e. $\bigcap_{r=0}^{\infty}\mathfrak{n}^r(S_U\otimes\hat{D}_R(S)=0$. Thus the proof is complete. q.e.d.

THEOREM 2. *Let S be an R-algebra and assnme that $(S,\mathfrak{m})$* is *a Zariski ring. Let $\mathfrak{a}$ be an arbitrary ideal of S. Then if $\hat{D}_R(S)$ is a finite module the following sequence is exact,*

(5) The possibility of the extension of a derivation D of S to a derivation of T can be proved directly in the following way. Let $\alpha=v/u=v'/u'$ be an element of T where u, u' are elements of$U$. Then there exists an element u'' in U such that $u''(vu'-v'u)=0$. Applying D to this relation and multiplying u'', we get the relation $u''^2(vDu'+u'Dv-v'D'u-uDv')=0$. From this we see that $D^*(\alpha)=(uDv-vDu)/u^2$ is a well defined unique extension of D to T.

$$(4) \qquad \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{i} (S/\mathfrak{a})\otimes_S \hat{D}_R(S) \xrightarrow{\varphi'_{R;S,S/\mathfrak{a}}} \hat{D}_R(S/\mathfrak{a}) \to 0$$

where i is defined by $i\,(a)=1\otimes \hat{d}^S_R a$.

Proof. The "onto-ness" of $\varphi'_{R;R,(S/\mathfrak{a})}$ is trivial. As is easily seen we have

$$(S/\mathfrak{a})\otimes_S \hat{D}_R(S)=\hat{D}_R(S)/\mathfrak{a}\hat{D}_R(S)$$

and

$$D_R(S/\mathfrak{a})=D_R(S)/(SD(\mathfrak{a})+\mathfrak{a}D_R(S))$$

where $SD(\mathfrak{a})$ is the submodule of $D_R(S)$ generated by elements $d^S a$, $a \in \mathfrak{a}$ (cf. Prop. 9 in [4]). Then we have

$$\begin{aligned}\hat{D}_R(S/\mathfrak{a}) &= D_R(S/\mathfrak{a})/\bigcap_{r=0}^{\infty} \mathfrak{m}^r D_R(S/\mathfrak{a}) \\ &= D_R(S)/\bigcap_{r=0}^{\infty}(\mathfrak{m}^r D_R(S)+SD(\mathfrak{a})+\mathfrak{a}D_R(S)) \\ &= \hat{D}_R(S)/\bigcap_{r=0}^{\infty}(\mathfrak{m}^r \hat{D}_R(S)+S\hat{D}(\mathfrak{a})+\mathfrak{a}\hat{D}_R(S))\end{aligned}$$

where $S\hat{D}(\mathfrak{a})$ is the submodule of $\hat{D}_R(S)$ generated by the elements $\hat{d}^S a$, $a \in \mathfrak{a}$. By our assumption $\hat{D}_R(S)$ is a finite module over a Zariski ring $(S, \mathfrak{m})$, hence by Lemma 1 all the submodules of $\hat{D}_R(S)$ are closed. Hence we get the isomorphism

$$\hat{D}_R(S/\mathfrak{a})=\hat{D}_R(S)/(S\hat{D}(\mathfrak{a})+\mathfrak{a}\hat{D}_R(S)).$$

From this we get the desired exact sequence.

In the course of the proof, the assumption that $\hat{D}_R(S)$ is a finite module is used only to prove that $\bigcap_{r=0}^{\infty} (\mathfrak{m}^r\hat{D}_R(S)+S\hat{D}(\mathfrak{a})+\mathfrak{a}\hat{D}_R(S))=S\hat{D}(\mathfrak{a})+\mathfrak{a}\hat{D}_R(S)$. Hence if $\mathfrak{a}\supset\mathfrak{m}$, then we get the exact sequence (4), without using any other assumptions. Thus we get

Supplement. *Let S be an R-algebra and assume that S is an $\mathfrak{m}$-adic ring. Let $\mathfrak{a}$ be an ideal of S containing $\mathfrak{m}$. Then the sequence (4) is always exact.*

§ 4. Complete $\mathfrak{m}$-adic rings.

In this paragraph we shall deal exclusively with the noetherian rings.

Let S be an $\mathfrak{n}$-adic ring and let S^* be its completion with respect to the $\mathfrak{n}$-adic topology. We shall denote by $\mathfrak{n}^*$ the extended ideal $\mathfrak{n}S^*$. Then as is well known the topology of S^* as the limit space of S coincides with the $\mathfrak{n}^*$-adic topology and S^* is a Hausdorff $\mathfrak{n}^*$-adic ring (Th. 2, Exposé 18 in [1]).

PROPOSITION 7. $\hat{D}_S(S^*)=0$.

PROOF. Let D be a derivation of S into a S^*_H-module E. Then by Lemma 2, D is continuos and D can be extended to a derivation of S^* into E. Moreover this extension is done in a unique way. From this we have $\hat{D}_S(S^*)=0$, otherwise the differential operator $\hat{d}_S^{S^*}$ will give a non-trivial S-derivation of S^* into a S^*_H-module $\hat{D}_S(S^*)$.

Let R be an $\mathfrak{m}$-adic ring and Iet S be an R-algebra with a ring homomorphism $f: R\to S$, such that $f(1)=1$. We shall assume that f satisfies the condition

$$f(\mathfrak{m})\subset\mathfrak{n}. \tag{5}$$

Then the homomorphism f can be extended in a unique way to a ring homomorphism f^* of R^* into S^*. When we treat R-algera S in these situations we always consider S^* as an R^*-algebra with this extended homomorphism.

THEOREM 3. *Let S be an R-algebra satisfying the condition* (5). *Assume that* $(S, \mathfrak{n})$ *is a Zariski ring and $\hat{D}_R(S)$ is a finite S-module. Then we have*

$$\hat{D}_{R^*}(S^*)=\hat{D}_R(S^*)=S^*\otimes_S\hat{D}_R(S)$$

PROOF. Let E be an arbitrary finite S^*-module. Then E is a complete Hausdorff space with respect to $\mathfrak{n}^*$-adic topology. Hence any R-derviation D of S into E can be extended in a unique way to an R-derviation of S^* into E. Moreover by our assumption $S^*\otimes_S\hat{D}_R(S)$ is a finite S^*-module, hence it is an S^*_H-module, and we have an injective map

$$\hat{\varphi}_{R;S,S^*}: S^*\otimes_S\hat{D}_R(S)\to\hat{D}_R(S^*)$$

by virtue of Theorem 1. Hence to prove the theorem it is sufficient to prove the "ontoness" of $\hat{\varphi}=\hat{\varphi}_{R;S,S^*}$.

Let a^* be an arbitrary element of S^*, and $\{a_i\}$ be a Cauchy sequence in S such that $a^*=\lim_{i\to\infty} a_i$. By lemma 2 we have $\hat{d}^*a^*=\lim_{i\to\infty}\hat{d}^*a_i$, where $\hat{d}^*$ stands for $\hat{d}_R^{S^*}$ and lim is taken with respect to the $\mathfrak{n}^*$-adic topology in $\hat{D}_R(S^*)$. On the other hand $\{\hat{d}a_i\}$ is a Cauchy sequence in $\hat{D}_R(S)$, where $\hat{d}$ means $\hat{d}_R^S$, since $\{a_i\}$ is a Cauchy sequence and $\hat{d}$ is a continuous map in S_H-module (Lemma 2). Hence $1\otimes\hat{d}a_i$ is also a Cauchy sequence in $S^*\otimes_S\hat{D}_R(S)$. Since $\hat{D}_R(S)$ is a finite module, $S^*\otimes_S D_R(S)$ is a complete $\mathfrak{n}^*$-module, and there exists an element α^* in $S^*\otimes_S\hat{D}_R(S)$ such that $\alpha^*=\lim_{i\to\infty}(1\otimes\hat{d}a_i)$. Since $\hat{\varphi}$ is a continuous map we have

$$\hat{\varphi}(\alpha^*)=\lim_{i\to\infty}\hat{\varphi}(1\otimes\hat{d}a_i)=\lim_{i\to\infty}\hat{d}^*a_i=\hat{d}^*a^*.$$

Since $D_R(S^*)$ is generated by $\hat{d}^*a^*$, $a^*\in S^*$, the above consideration shows the ontoness of $\hat{\varphi}$.

The assertion $\hat{D}_R(S^*)=\hat{D}_{R^*}(S^*)$ follows from Prop. 7 and Cor. of Prop. 5.

q.e.d.

COROLLARY. *Let S and R be local rings such that R is contained in S. Then if $D_R(S)$ is a finite module we have $\hat{D}_{R^*}(S^*)=S^*\otimes_S D_R(S)$.*

THEOREM 4. *Let S be a complete local ring with the the maximal ideal $\mathfrak{n}$, and let R be a local ring with the maximal ideal $\mathfrak{m}$. Assume that S dominates R,* i.e. *$\mathfrak{n}\cap R=\mathfrak{m}$ and that the residue field $S/\mathfrak{n}$ is a finitely generated extension of $R/\mathfrak{m}$. Then $\hat{D}_R(S)$ is a finite module.*

PROOF. By the supplement to the Theorem 2, we have an exact sequence

$$\mathfrak{n}/\mathfrak{n}^2\rightarrow(S/\mathfrak{n})\otimes_S\hat{D}_R(S)\rightarrow\hat{D}_R(S/\mathfrak{n})\rightarrow 0.$$

Since $D_R(S/\mathfrak{n})=D_{(R/\mathfrak{m})}(S/\mathfrak{n})$ and $S/\mathfrak{n}$ is a finitely generated field extension of $R/\mathfrak{m}$, we see that $D_R(S/\mathfrak{n})=D_{(R/\mathfrak{m})}(S/\mathfrak{n})$ is a finite module. From this we see that

$$\hat{D}_R(S/\mathfrak{n})=D_R(S/\mathfrak{n})=D_{(R/\mathfrak{m})}(S/\mathfrak{n})=\hat{D}_{(R/\mathfrak{m})}(S/\mathfrak{n}).$$

The above exact sequence implies that $(S/\mathfrak{n})\otimes_S\hat{D}_R(S)$ is a finite dimensional vector space over the field $S/\mathfrak{n}$. The Theorem will follow from the following lemma which is proved implicitly in [2].

LEMMA 3. *Let S be a complete $\mathfrak{n}$-adic ring and let E be a separated $\mathfrak{n}$-adic S-module. Assume that $(S/\mathfrak{n})\otimes_S E$ is a finite $(S/\mathfrak{n})$-module with a system of generators $1\otimes e_i(i=1,\cdots,n)$, then $e_1,\cdots,e_n$ form a system of generators of E over S.*

PROOF. Since $(S/\mathfrak{n})\otimes_S E=E/\mathfrak{n}E$, $1\otimes e_i$ can be identified with the class of e_i modulo $\mathfrak{n}E$. Then by our assumptions we have

$$E=\sum_{i=1}^{n}Se_i+\mathfrak{n}E$$

in general we have

$$\mathfrak{n}^rE=\sum_{i=1}^{n}\mathfrak{n}^re_i+\mathfrak{n}^{r+1}\qquad(r=0,1,2,\ldots\ldots).$$

Let a be an arbitrary element of E. By a standard technic we can find a sequence $\{a_r\}$ satisfying the following conditions

$$a_r\equiv a\pmod{\mathfrak{n}^{r+1}}$$

$$a_r=\sum\alpha_r^{(i)}e_i,\qquad \alpha_r^{(i)}-\alpha_{r+1}^{(i)}\equiv 0\pmod{\mathfrak{n}^{r+1}}\ (i=1,\cdots,n)$$

with $\alpha_j^{(i)}\in S$. By our assumption S is complete with respect to the $\mathfrak{n}$-adic topology in S. Let $b^i=\lim\limits_{r\to\infty}\alpha_r^{(i)}$, then it is easy to see that $a=\sum b^ie_i$, and the proof is complete.

PROPOSITION 10. *Let S be formal power series ring in n-variables $x_1,\cdots,x_n$ over a ring R, and let $\mathfrak{m}$ be the ideal of S generated by $(x_1,\cdots,x_n)$. Then the module of $\mathfrak{m}$-adic differentials $\hat{D}_R(S)$ is a free module of rank n.*

PROOF. By the supplement to Th. 2, we have the exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \rightarrow R \otimes \hat{D}_R(S) \rightarrow \hat{D}_R(S/\mathfrak{m}) \rightarrow 0.$$

Since $S/\mathfrak{m}=R$, we see that $\hat{D}_R(S/\mathfrak{m})=0$. On the other hand S is a complete $\mathfrak{m}$-adic ring, hence we see by Lemma 3 that $\hat{D}_R(S)=S\hat{d}x_1+\cdots+S\hat{d}x_n$. The proof will be complete if we show that $\hat{d}x_1,\cdots,\hat{d}x_n$ are linearly independent. For this purpose it is sufficinet to point out that the formal derivations $D_i=\frac{\partial f}{\partial x_i}$ satisfy the conditions $D_i x_j=\delta_{ij}$ and $D_i r=0$ for $r \in R$.

REMARK. Let f be a formal power series with coefficients in R. Then it is not difficult to see that we have

$$\hat{d}_R^S f=\sum_i \frac{\partial f}{\partial x_i} \hat{d}_R^S x_i$$

where $\frac{\partial f}{\partial x_i}$ stands for the formal derivative of f.

Now it is easy to give the answer to the problem raised in the introduction.

PROPOSITION 11. *Let K be the complex number field and let A be the ring composed of power series in n indeterminates over K which are convergent in some neighborhood of the origin. Then the module of analytic differentials $\hat{D}_K(A)$ is a free module of rank n.*

PROOF. Let A^* be the formal power series ring in indeterminates $X_1,\cdots,X_n$ over K. Then A^* is the completion of the local ring A. Since $\hat{d}^{A^*}X_1,\cdots,\hat{d}^{A^*}X_n$ are independent differentials in $\hat{D}_K(A^*)$, the n-differentials $\hat{d}^A X_1,\cdots,d^A X_n$ are also independent in $\hat{D}_K(A)$ (cf. the exact sequence (3)). Hence to prove the assertion it is sufficient to show that $\hat{d}^A X_1,\cdots,\hat{d}^A X_n$ generate entire $D_K(A)$. Let f be an element of A and let $\frac{\partial f}{\partial X_i}$ be formal derivative of f with respect to X_i. Then as is well known $\frac{\partial f}{\partial X_i}$ is also contained in A. Let $f^{(m)}$ be the sum of the terms of degrees $\geqq m$ in f. Then from $f=\lim_{m\to\infty} f^{(m)}$, we have

$$\hat{d}f=\lim_{m\to\infty}\hat{d}f^{(m)}=\lim_{m\to\infty}\left(\sum_i\frac{\partial f^{(m)}}{\partial X_i}\hat{d}X_i\right)=\sum_i\left(\lim_m\frac{\partial f^{(m)}}{\partial X_i}\right)\hat{d}X_i=\sum_i\frac{\partial f}{\partial X_i}\hat{d}X_i.$$

proving the assertion.

§ 5. A generalization of a lemma of Godement.

Let R be a local ring and let $\mathfrak{m}$ be the maximal ideal of R. Assume that R contains a field K such that $R/\mathfrak{m}$ is a separable extension of K. Then the sequence

$$(6) \qquad 0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m}) \otimes_R D_K(R) \to D_K(R/\mathfrak{m}) \to 0$$

is known to be exact (Th. 5. Exposé 17 in [1]). On the other hand

$$\hat{D}_K(R) \otimes_R (R/\mathfrak{m}) = D_K(R) \otimes_R (R/\mathfrak{m})$$

and $\hat{D}_K(R/\mathfrak{m}) = D_K(R/\mathfrak{m})$ since $R/\mathfrak{m}$ is a field (Cor. 2 of Prop. 1). Hence we can write the exact sequence (6) in the from

$$(7) \qquad 0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m}) \otimes_R \hat{D}_K(R) \to \hat{D}_K(R/\mathfrak{m}) \to 0.$$

We shall give a generalization of the sequence (7) in the

THEOREM 5. *Let R be a local ring and let $\mathfrak{m}$ be the maximal ideal of R. Let I be the ring contained in R which is either a field or else a discrete valuation ring such that the prime element u of I is contained in $\mathfrak{m}^2$. Assume that the residue field of R is a separable extension of the residue field K of I. Then the sequence*

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m}) \otimes_R \hat{D}_I(R) \to \hat{D}_K(R/\mathfrak{m}) \to 0$$

is exact.

PROOF. When I is a field, it is proved above, hence we assume that I is a discrete valuation ring. Let us put $S = R/\mathfrak{m}^2$. By the assumption that $u \in \mathfrak{m}^2$, K is naturally considered as a subfield of S. Let $\mathfrak{n} = \mathfrak{m}/\mathfrak{m}^2$. Applying the exact sequence (7) to the local ring S with the maximal ideal $\mathfrak{n}$, we see that the sequence

$$0 \to \mathfrak{n} \to (S/\mathfrak{n}) \otimes_S \hat{D}_K(S) \to \hat{D}_K(S/\mathfrak{n}) \to 0$$

is exact, where $\mathfrak{n}^2 = 0$. By definition we have $S/\mathfrak{n} = R/\mathfrak{m}$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{m}^2$. Hence in order to prove the assertion it is sufficient to show that

$$(S/\mathfrak{n}) \otimes_S \hat{D}_K(S) \cong (R/\mathfrak{m}) \otimes_R \hat{D}_I(R).$$

Let D be a derivation of R into $\hat{D}_K(S)$ defined by

$$Dx = \hat{d}_K^S \bar{x}$$

where $\bar{x}$ is the class of x mod $\mathfrak{m}^2$. Since D is trivial on I and $\hat{D}_K(S)$ is a S_H-module, we can find an R-homomorphism $\alpha : \hat{D}_I(R) \to \hat{D}_K(S)$ such that

$$\hat{d}_K^S \bar{x} = \alpha(\hat{d}_I^R x), \ x \in R$$

α induces in a natural way an $(R/\mathfrak{m})$-homomorphism

$$1 \otimes \alpha : (R/\mathfrak{m}) \otimes_R \hat{D}_I(R) \to (S/\mathfrak{n}) \otimes_S \hat{D}_K(S).$$

On the other hand the map $\varphi: R\to(R/\mathfrak{m})\otimes\hat{D}_I(R)$ defined by $\varphi(x)=1\otimes\hat{d}_I^R x$, $x\in R$ is a derivation which is trivial on I. Moreover if $x\in\mathfrak{m}^2$, then $\hat{d}_I^R x$ is contained in $\mathfrak{m}\hat{D}_I(R)$, hence $\varphi(x)=0$. This means that φ is a derivation of $S=R/\mathfrak{m}^2$, which is trivial on $K=I/(u)$. Since $R/\mathfrak{m}$ is an S-algebra with the natural homomorphism $S\to R/\mathfrak{m}$, and $(R/\mathfrak{m})\otimes_R\hat{D}_I(R)$ is an S_H-module, we can find an S-homomorphism $\beta:\hat{D}_K(S)\to(R/\mathfrak{m})\otimes_R\hat{D}_I(R)$ such that

$$\varphi(x)=1\otimes\hat{d}_I^R x=\beta(d_K^S\bar{x})$$

where $x\in R$ and $\bar{x}$ denote the class of x modulo $\mathfrak{m}^2$. β induces the natural homorphism

$$1\otimes\beta:(S/\mathfrak{n})\otimes_S\hat{D}_K(S)\to(R/\mathfrak{m})\otimes_R\hat{D}_I(R)$$

and we have

$$(1\otimes\beta)(1\otimes\alpha)(1\otimes\hat{d}_I^R x)=(1\otimes\beta)(1\otimes\hat{d}_K^S\bar{x})=(1\otimes\beta)(\hat{d}_K^S\bar{x})=1\otimes\hat{d}_I^R x,\quad x\in R$$

and by the similar calculus we get

$$(1\otimes\alpha)(1\otimes\beta)(1\otimes\hat{d}_K^S\bar{x})=(1\otimes\alpha)(1\otimes\hat{d}_I^R x)=1\otimes\hat{d}_I^S\bar{x}\quad\text{for } x\in R$$

which proves that $1\otimes\alpha$ gives the isomorphism of $(R/\mathfrak{m})\otimes_R\hat{D}_I(R)$ and $(S/\mathfrak{n})\otimes_S\hat{D}_K(S)$.

REMARK. In this theorem, the assumption $u\in\mathfrak{m}^2$ is essential. In fact let I be a discrete valuation ring with the prime element u, and let R be the formal power series ring $I[[x_1,\cdots,x_n]]$ over I. Then $\mathfrak{m}=(u,x_1,\cdots,x_n)$ and $\mathfrak{m}/\mathfrak{m}^2$ is an $(n+1)$-dimensional vector space over $R/\mathfrak{m}=I/(u)=K$, $\hat{D}_I(R)$ is a free module of rank n, and hence $(R/\mathfrak{m})\otimes\hat{D}_I(R)$ is an n-dimensional vector space over K.

§ 6. Characterizations of regular local rings. (equal characteristic case)

Let R be a local ring and let $\mathfrak{m}$ be the maximal ideal of R. We shall say that a field K is a Cohen field of R if K is contained in R and $R/\mathfrak{m}\cong K$.

THEOREM 6. *Let R be a complete regular local ring of rank n and let $\mathfrak{m}$ be its maximal ideal. Let k be a field contained in R such that $R/\mathfrak{m}$ is a finitely generated separable extension of dimension r over k. Then $\hat{D}_k(R)$ is a free module of rank $n+r$. Moreover let $(x_1,\cdots,x_n)$ be a regular system of parameters of R and let $\alpha_1,\cdots,\alpha_r$ be elements of R such that the residue classes of α's modulo $\mathfrak{m}$ form a separating transcendent base of $R/\mathfrak{m}$ over k. Then $\hat{d}x_1,\cdots,\hat{d}x_n,\hat{d}\alpha_1,\cdots,\hat{d}\alpha_r$ form a free base of $\hat{D}_K(R)$, where $\hat{d}$ stand for $\hat{d}_k^R$.*

PROOF. Let $K'=k(\alpha_1,\cdots,\alpha_r)$. Then K' is a subfield of R and $R/\mathfrak{m}$ is a separable extension of K'. Hence there exists a Cohen field K containing K'

([1], [3] Exposé 18). Then $R=K[[x_1,\cdots,x_n]]$ and $\hat{D}_K(R)$ is a free module of rank n with the base $\hat{d}_K x_1,\cdots,\hat{d}_K x_n$. Let E be a finite R-module. Then E is a complete and any derivation of K into E can be extended to a derivation of R into E. Since $\hat{D}_k(R)$ is a finite wodule (Th. 4) We can apply Theorem 1 to our case and we get the exact sequence

$$0\rightarrow R\otimes_K D_k(K)\rightarrow \hat{D}_k(R)\rightarrow \hat{D}_K(R)\rightarrow 0.$$

Since $D_k(K)$ is a free module with the base $\hat{d}_k^K\alpha_1,\cdots,\hat{d}_k^K\alpha_r$, the assertion follows immediately from this.

COROLLARY. *In Theorem 6 let us take off the assumption that R is complete and instead assume that $\hat{D}_k(R)$ is a finite module. Then we have the same conclusion as in Theorem 6.*

PROOF. Let R^* be the completion of R. From the exact sequence

$$0\rightarrow \mathfrak{m}/\mathfrak{m}^2\rightarrow (R/\mathfrak{m})\otimes_R \hat{D}_k(R)\rightarrow \hat{D}_k(R/\mathfrak{m})\rightarrow 0.$$

We see that $(R/\mathfrak{m})\otimes\hat{D}_k(R)$ is generated by $1\otimes\hat{d}x_1,\cdots,1\otimes\hat{d}x_n, 1\otimes\hat{d}\alpha_1,\cdots,1\otimes\hat{d}\alpha_r$. Since $\hat{D}_k(R)$ is a finite module, $\hat{D}_k(R)$ is generated by $\hat{d}x_1,\cdots,\hat{d}x_n, \hat{d}\alpha_1,\cdots,\hat{d}\alpha_r$. The same assumption also implies that $\hat{D}_k(R^*)=R^*\otimes_R\hat{D}_k(R)$ by Theorem 3. By Theorem 6 $\hat{D}_k(R^*)$ is a free module with the base $\hat{d}^*x_i, \hat{d}^*\alpha_j$ $(i=1,\cdots n, j=1,\cdots,r)$ and the above isomorphism is given by corresponding $\hat{d}^*x_i, \hat{d}^*\alpha_j$ to $1\otimes\hat{d}x_i, 1\otimes\hat{d}\alpha_j$. The assertion follows from this.

THEORREM 7. *Let R be a complete local ring and let $\mathfrak{m}$ be its maximal ideal. Let k be a field contained in R such that $R/\mathfrak{m}$ is a finitely generated extension of k. Assume that one of the following conditions is satisfied:*

(1) Characteristic of R is >0, R is an integral domain, k is perfect and $R/\mathfrak{m}$ is algebraic over k.

(2) Characteristic of R is zero.

Under these conditions, if $\hat{D}_k(R)$ is a free module of finite rank, then R is a regular local ring.

PROOF. Let K be a Cohen field of R containing k and let $\alpha_1,\cdots,\alpha_r$ be a transcendent base of K over k (in the case (1), this is an empty set). Let $u_1,\cdots,u_n$ be a minimal set of generators of $\mathfrak{m}$. By Theorem 5 we have the following exact sequence

$$0\rightarrow \mathfrak{m}/\mathfrak{m}^2\rightarrow (R/\mathfrak{m})\otimes_R \hat{D}_k(R)\rightarrow \hat{D}_k(R/\mathfrak{m})\rightarrow 0.$$

From this we see that $\hat{d}\alpha_1\cdots,\hat{d}\alpha_r, \hat{d}u_1,\cdots,\hat{d}u_n$ are a minimal base, hence also a free base of $\hat{D}_k(R)$ over R where $\hat{d}$ stands for $\hat{d}_k^R$. Let A be a formal power series ring in n-variables $X_1, X_2,\cdots, X_n$ over K and let φ be a K-homomorphism of A onto R such that

$$\varphi(X_i)=u_i\ (i=1, 2,\cdots, n).$$

Let $\mathfrak{P}$ be the kernel of φ. Then $\mathfrak{P}$ has the following property.

(*) "*Let f be an element of $\mathfrak{P}$, then the formal derivatives $\dfrac{\partial f}{\partial x_i}$'s are still contained in $\mathfrak{P}$.*"

In fact let f be a formal power series and let f_m be the sum of the terms of degree $\leqq m$ in f. Then "f is contained in $\mathfrak{P}$" is equivalent to saying that $\{f_m(n)\}$ is a zero sequence in R. Since $f_m(u)\equiv 0$ (mod $\mathfrak{m}^{m+1}$), we have

$$\sum_i(\partial f_m/\partial u_i)\hat{d}u_i+\sum_j(\partial f_m/\partial \alpha_j)\hat{d}\alpha_j\equiv 0 \qquad (\text{mod } \mathfrak{m}^m\hat{D}_k(R))$$

Since $\hat{D}_k(R)$ is a free module with the base $\hat{d}u_i$, and $\hat{d}\alpha_j$, we see that $\partial f_m/\partial u_i\equiv 0$ (mod. $\mathfrak{m}^m$) for all i's. This proves that $\partial f/\partial X_i$ are contained in $\mathfrak{P}$.

From this we see immediately $\mathfrak{P}=0$ in the case (2). The proof of the theorem will be complete, if we show the following

LEMMA 5. *Let K be a perfect field of characteristic $p(>0)$ and $\mathfrak{P}$ be a prime ideal of the formal power series ring over K in n-variables $X_1,\cdots, X_n$. Assume that $\mathfrak{P}$ has the property (*). Then $\mathfrak{P}$ must be a zero ideal.*

PROOF. Let f be an element of $\mathfrak{P}$ different from zero. Then we can express f in the form

$$f=\sum_{(i_1,\cdots,i_n)} f_{i_1\cdots i_n}X_1^{i_1}\cdots X_n^{i_n}$$

where $f_{i_1\cdots i_n}$'s are elements in $k[[X_1^p,\cdots, X_n^p]]$ and $0\leqq i_\alpha\angle p$. Arranging n-tuples $(i_1,\cdots, i_n)$ in lexicographical order and applying the operations

$$\partial^{i_1+\cdots+i_n}/\partial X_1^{i_1}\cdots\partial X_n^{i_n}$$

step by step from the highest order of n-tuples, we see easily that all the coefficients $f_{i_1\ldots i_n}$'s are contained in $\mathfrak{P}$. The above considerations show that $\mathfrak{P}$ is generated by the elements in $\mathfrak{P}\cap K[[X_1^p,\cdots\cdots, X_n^p]]=\mathfrak{P}_0$. Moreover the coefficient field K is perfect and $\mathfrak{P}$ is a prime ideal, hence if f is in $\mathfrak{P}_0$, $f^{1/p}$ is also in $\mathfrak{P}$. Let us now assume that $\mathfrak{P}$ is not the zero ideal and let f be an element of $\mathfrak{P}_0$ which has the lowest initial term. Then $f^{1/p}$ is in $\mathfrak{P}$. But this is impossible since $\mathfrak{P}$ is generated by the elements in $\mathfrak{P}_0$ and the initial term of $f^{1/p}$ has the degree less than that of f. Thus the proof is complete.

COROLLARY. *Let R be a local ring and let $\mathfrak{m}$ be its maximal ideal and let R^* be its completion. Let k be a field contained in R such that $R/\mathfrak{m}$ is a finitely generated extension of k. Assume that one of the following conditions is satisfied:*

(1) *Characteristic of k is >0, R is analytically irreducible, k is perfect and $R/\mathfrak{m}$ is algebraic over k,*

(2) *Characteristic of k is zero.*

Then if $\hat{D}_k(R)$ is a free module of finite rank, R is a regular local ring.

Proof. It holds that $\hat{D}_k(R^*)=R^*\otimes_R\hat{D}_k(R)$ by Theorem 3 since $\hat{D}_k(R)$ is of finite rank by the assumption. Hence $\hat{D}_k(R^*)$ is a free module. Then by the above Theorem R^* is a regular local ring. Hence R is also a regular local ring. q. e. d.

In Theorem 7, (1), we have assumed that $R/\mathfrak{m}$ is algebraic over k. We don't know whether this assumption can be replaced or not by "*$R/\mathfrak{m}$ is a finitely generated separable extension* of k". On the other hand the remaining two assumptions are inevitable as is shown in the following example.

Example. Let $\tilde{R}=k[[X, Y]]$ be a formal power series ring in two variables over a field k. First assume that k is not perfect. Then there exists an element a in k such that $a^{1/p}$ is not in k. The principal ideal $\mathfrak{P}=(X^p+aY^p)$ is a prime ideal of R. Hence the quotient ring $R=\tilde{R}/\mathfrak{P}$ is a complete local domain which is not regular. Let x and y be the classes of X and Y in R. Then as is easily seen, $\hat{D}_k(R)$ is a free module with the base $\hat{d}x$ and $\hat{d}y$.

If k is not perfect, the above defined principal ideal is not prime, hence the local ring R contains a zero divisor. But the similar considerations show that $\hat{D}_k(R)$ is a free module of rank 2 with the base $\hat{d}x$ and $\hat{d}y$.

§ 7. Characterizations of regular local rings. (unequal characteristic case)

In this paragraph we shall treat exclusively the local rings of characteristic 0 with the residue field of prime characteristic p.

Lemma. 6. *Let $(R, \mathfrak{m})$ be a complete local ring and let I be a discrete valuation ring with the prime element u. Assume that R dominates I and the residue field of R is separably algebraic over I/uI. Then we have the following*:

(1) *There exists a complete discrete valvation ring I' containing I such that I' has the same prime element u as I and $I'/uI'=R/\mathfrak{m}$.*

(2) *For the valuation ring I' satisfying* (1) *we have*

$$\hat{D}_I(R)=\hat{D}_{I'}(R).$$

Proof. Let $\mathfrak{S}$ be a set of discrete valuation rings in R containing I such that they have the common prime element u as I. As we can see easily, $\mathfrak{S}$ is the inductive set and there exists a maximal element I' in $\mathfrak{S}$. I' is subspace of R (Th. 6 in [3]), hence I' must be complete, because otserwise the completion of I' will be contained in $\mathfrak{S}$ and it will contradict the fact that I' is a maximal element in $\mathfrak{S}$.

We shall show that I' has the same residue field as R. Assume the contrary. Then there exists an element α in $R/\mathfrak{m}$ which is separably algebraic over I'/uI'. Let $f(X)$ be a monic polynomial in $I'[X]$ such that the polynomial

$\bar{f}(X)$ obtained from $f(X)$ by reducing the coefficients modulo uI' is an irreducible equation over I'/uI' such that $\bar{f}(\alpha)=0$. Since $\bar{f}(X)$ is separable, we see, by Hensel's lemma, that $f(X)$ has a linear factor X-a such that a is an element of the class α. Then, by Lemma 4 in [3], $I'[a]$ is also a discrete valuation ring with the same prime elemnt u as I' and this is a contradiction.

To prove the assertion (2) it is sufficient to show that $\hat{D}_I(I')=0$ by Corollary 2 of Prop. 5. Since I' is complete, I' contains the completion I^* of I and $\hat{D}_I(I')=\hat{D}_{I^*}(I')$ by the same Cor. 2. Hence it is without restriction to assume that I is complete. Let k and k' be the quotient field of I and I' respectively, and let L be the algebraic closure of k in k'. Then the integral closure B of I in I' is given by $B=I'\cap L$ hence B is also a discrete valuation ring with the prime element u. Moreover we can see that the residue fields of B and I' coincide by Hensel's lemma. Then I' must be the completion of B and $\hat{D}_B(I')=0$ by Theorem 3. Hence our problem is reduced to prove $\hat{D}_I(B)=0$. Now let L' be any subfield of L such that $[L':k]<\infty$, and let $B'=B\cap L'$. Then if $\hat{D}_I(B)\neq 0$, we have $\hat{D}_I(B')\neq 0$ for some B'. But B' is, by its construction, an unramified extension of I', hence $\hat{D}_I(B')=D_I(B')=0$ by Cor. 1 of Th. 10 in [4]. Thus the proof is complete.

THEOREM 8. *Let R be a regular local ring of rank n and let $\mathfrak{m}$ be its maximal ideal. Let I be a discrete valuation ring dominated by R and let u be a prime element of I. Assume that $R/\mathfrak{m}$ is a finitely generated separable extension of dimension r over the field I/uI and $u\notin\mathfrak{m}^2$. Then if $\hat{D}_I(R)$ is a finite module, $\hat{D}_I(R)$ is a free module of rank $n+r-1$.*

PROOF. Let $\alpha_1, \alpha_2,\cdots, \alpha_r$ be elements of R such that their residue classes $\bar{\alpha}_1, \bar{\alpha}_2,\cdots, \bar{\alpha}_r$ modulo $\mathfrak{m}$ are a separating transcendent base of $R/\mathfrak{m}$ over I/uI. Then it is easy to see that there exists a discrete valuation ring I_1, dominated by R such that u is a prime element of I_1 and $I_1/uI_1=(I/uI)(\bar{\alpha}_1,\cdots, \bar{\alpha}_r)$. Let R^* be the completion of R and let I' be a complete discrete valuation ring constructed for R^* and I_1 as in lemma 6. Since $u\notin\mathfrak{m}^2$, R^* is a formal power series ring $I'[[x_1,\cdots, x_{n-1}]]$. Hence $\hat{D}_{I'}(R^*)$ is a free R^*-module and hence $\hat{D}_{I_1}(R^*)$ is also a free R^*-module and $\hat{d}x_1,\cdots, \hat{d}x_{n-1}$ are a free base, where $\hat{d}$ stands for $\hat{d}_{I_1}^{R^*}$. On the other hand I_I is a quotient ring of $I[\alpha_1,\cdots, \alpha_r]$ and $\hat{D}_I(I[\alpha_1,\cdots, \alpha_r])$ is a free $I[\alpha_1,\cdots, \alpha_r]$-module. Hence $\hat{D}_I(I_1)$ is a free I_1-module with the base $\hat{d}'\alpha_1,\cdots,\hat{d}'\alpha_r$ where we denote $\hat{d}_I^{I_1}$ by $\hat{d}'$. The remaining reasoning is the same as that of Theorem 6 and its Corollary, as we may take x_i's in R such that $u, x_1,\cdots, x_{n-1}$ form a regular system of parameters.

REMARK. In Th. 8 the assumption $u\notin\mathfrak{m}^2$ is essential. Let I be a Cohen ring[6] of R such that $u\in\mathfrak{m}^2$. Then applying Th. 5 to this case, we see that $\hat{D}_I(R)$ is not a free module. (cf. Th. 9).

(6) We mean by a Cohen ring a coefficient ring in the sense of [3].

THEOREM 9. *Let R be a local ring with the maximal ideal $\mathfrak{m}$. Let I be a discrete valuation ring dominated by R and let u be a prime element of I. Assume that $R/\mathfrak{m}$ is a finitely generated separable extension of I/uI. Then if $\hat{D}_I(R)$ is a free module of finite rank, R is a regular local ring and $u \notin \mathfrak{m}^2$.*

PROOF. Let R^* be the completion of R. And take α_i's, I_1 and I' as in the proof of Th. 8.

First we shall assume that $u \in \mathfrak{m}^2$. Then by Th. 5 the following exact sequence holds:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m}) \otimes_R \hat{D}_I(R) \to \hat{D}_I(R/\mathfrak{m}) \to 0.$$

Hence by the same reasoning as in Th. 7, we see that R^* is a formal power series ring with coefficients in I, But in such a case u cannot be contained in $\mathfrak{m}^2$, contradicting our assumption.

Henceforth we shall assume that $u \notin \mathfrak{m}^2$. Let $u = u_1, u_2, \cdots, u_n$ be a minimal set of generators of $\mathfrak{m}$. Then from the exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to (R/\mathfrak{m}) \otimes_R \hat{D}_I(R) \to \hat{D}_I(R/\mathfrak{m}) \to 0$$

and the fact that $\hat{d}u_1 = 0$, we see that $\hat{D}_I(R)$ is generated over R by $\hat{d}u_2, \cdots, \hat{d}u_n$ and $\hat{d}\alpha_1, \cdots, \hat{d}\alpha_r$. On the other hand if we put $R' = R/(u_1)$ and $\mathfrak{m}' = \mathfrak{m}/(u_1)$, the assumption in Th. 5 are satisfied by R', m' and $I' = I/(u_1)$ and we have the exact sequence

$$0 \to \mathfrak{m}'/\mathfrak{m}'^2 \to (R'/\mathfrak{m}') \otimes_{R'} \hat{D}_{I'}(R') \to \hat{D}_{I'}(R'/\mathfrak{m}') \to 0.$$

Let us denote by u'_i the class of u_i and by α'_i the class of α_i mod. $\mathfrak{m}^2$ Then $u'_2, \cdots, u'_n$ are a minimal base of $\mathfrak{m}'$, hence $\hat{d}'u'_2, \cdots, \hat{d}'u'_n, \cdots, \hat{d}'\alpha'_r$ are a minimal set of generatores for $\hat{D}_{I'}(R')$ where $\hat{d}'$ stands for $\hat{d}^{R'}_{I'}$. On the other hand we have a natural homomorphism (cf. the proof of Th. 5)

$$\varphi : \hat{D}_I(R) \to \hat{D}_{I'}(R')$$

such that $\varphi(\hat{d}u_i) = \hat{d}'u'_i (i = 2, \cdots, n)$ and $\varphi(\hat{d}\alpha_i) = \hat{d}'\alpha'_i (i = 1, 2, \cdots r)$. Hence we see that $\hat{d}u_2, \cdots, \hat{d}u_n$ and $\hat{d}\alpha_1, \cdots, \hat{d}\alpha_r$ are a minimal set of generators of $\hat{D}_I(R)$, hence the free base of $\hat{D}_I(R)$ by the assumption in the Theorem. The remaining reasoning is the same as that of Th. 7 and its corollary.

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