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Classification of bi-polarized 3-folds (X, L_1, L_2) with $h^0(K_X + L_1 + L_2) = 1$

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Abstract. Let X be a complex smooth projective variety of dimension 3, and let L_1 and L_2 be ample line bundles on X. In this paper we classify (X, L_1, L_2) with $h^0(K_X + L_1 + L_2) = 1$.

1. Introduction

Let X be a complex smooth projective variety of dimension n, and let L be an ample line bundle on X. Then (X, L) is called a *polarized manifold*. There are several problems about the positivity of $h^0(K_X + tL)$, the dimension of $H^0(K_X + tL)$, for some positive integer t. In [25], P. Ionescu proposed the following conjecture.

Conjecture 1 ([25, Open problems, P. 321]). *Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$.*

It is known that Conjecture 1 is true for $\dim X \leq 3$ (see [13], [21]).

On the other hand, there is the following conjecture due to Beltrametti and Sommese, which is weaker than Conjecture 1.

Conjecture 2 ([2, Conjecture 7.2.7]). *Let (X, L) be a polarized manifold of dimension n. Assume that $K_X + (n-1)L$ is nef. Then $h^0(K_X + (n-1)L) > 0$.*

If $n \leq 4$, then Conjecture 2 is true (see [12], [19]). In general, Conjecture 2 is also true if $h^0(L) > 0$ (see [21]). We see from the adjunction theory [2] that if Conjecture 2 is true, then we can characterize (X, L) with $h^0(K_X + (n-1)L) = 0$. Namely if $h^0(K_X + (n-1)L) = 0$, then $K_X + (n-1)L$ is not nef by Conjecture 2, and by [4] and [22] we obtain that (X, L) is one of some special types. Therefore we can characterize (X, L) with $n \leq 4$ and $h^0(K_X + (n-1)L) = 0$.

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Moreover the classification of (X, L) with the following cases has been obtained.

(i) The case where $n = 3$ and $h^0(K_X + 2L) \leq 2$ (see [12] and [18]).

(ii) The case where $n = 4$ and $h^0(K_X + 3L) \leq 1$ (see [19] and [20]).

Furthermore we consider a generalization of Conjecture 2. Assume that X is smooth with $n = \dim X$ and let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Then $(X, L_1, \dots, L_{n-1})$ is called a *multi-polarized manifold of type* $n-1$. In particular, if $n = 3$, then (X, L_1, L_2) is also called a *bi-polarized manifold*.

Conjecture 3 ([15, Conjecture 5.1]). *Let* $(X, L_1, \dots, L_{n-1})$ *be a multi-polarized manifold of type* $n-1$ *with* $\dim X = n \geq 3$. *Assume that* $K_X + L_1 + \cdots + L_{n-1}$ *is nef. Then* $h^0(K_X + L_1 + \cdots + L_{n-1}) > 0$.

In [15, Theorem 5.2], we proved that Conjecture 3 is true for $n = 3$. Moreover, for $n = 3$, this implies the classification of (X, L_1, L_2) with $h^0(K_X + L_1 + L_2) = 0$ (see [15, Corollary 5.1]). As the next step, in this paper we study (X, L_1, L_2) with $h^0(K_X + L_1 + L_2) = 1$.

2. Preliminaries

Definition 1 ([14, Definition 2.1 (2), Remark 2.2 (2)], [16, Proposition 6.1.1]). Let X be a smooth projective variety of dimension 3 and let L_1 and L_2 be ample line bundles on X. Then the *first sectional geometric genus* $g_1(X, L_1, L_2)$ is defined by the following.

$$g_1(X, L_1, L_2) = 1 + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2.$$

In particular if $L_1 = L_2 = L$, then $g_1(X, L, L)$ is the *sectional genus* of (X, L), which is denoted by $g(X, L)$.

Definition 2. Let (X, L) be a polarized manifold of dimension n.

(i) We say that (X, L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n - m + 1)L = f^*A$ (resp. $K_X + (n - m)L = f^*A$) for some ample line bundle A on Y.

(ii) (X, L) is called a *classical scroll over a normal variety* Y if there exists a vector bundle $\mathscr{E}$ on Y such that $X \cong \mathbb{P}_Y(\mathscr{E})$ and $L = H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle.

(iii) We say that (X, L) is a *pure quadric fibration over a smooth projective curve* C if (X, L) is a quadric fibration over C such that the morphism $f : X \to C$ is the contraction morphism of an extremal ray.

Remark 1.

(i) *If* (X, L) *is a scroll over a smooth projective curve* C, *then* (X, L) *is a classical scroll over* C *(see* [2, Proposition 3.2.1]*).*

(ii) *If* (X, L) *is a scroll over a normal projective surface* S, *then* S *is smooth and* (X, L) *is also a classical scroll over* S *(see* [1, (3.2.1) Theorem] *and* [7, (11.8.6)]*).*

(iii) *Assume that* (X, L) *is a quadric fibration over a smooth curve* C *with* $\dim X = n \geq 3$. *Let* $f : X \to C$ *be its morphism. By* [1, (3.2.6) Theorem] *and the proof of* [22, Lemma (c) in Section 1], *we see that* (X, L) *is one of the following:*

(a) *A pure quadric fibration over* C.

(b) *A classical scroll over a smooth surface with* $\dim X = 3$. *(We note that this is a very restricted case. See* [23] *for detail.)*

The following notation is used in Theorem 1.

Notation 1. Let (X, L) be a pure quadric fibration over a smooth curve C with $\dim X = n$, and let $f : X \to C$ be its morphism. We put $\mathscr{E} := f_*(L)$. Then $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathscr{E}) \to C$ be the projection. Then there exists an embedding $i : X \hookrightarrow \mathbb{P}_C(\mathscr{E})$ such that $f = \pi \circ i$, $X \in |2H(\mathscr{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathscr{E})|_X$. Let $e := \deg \mathscr{E}$, $b := \deg B$ and $d := L^n$.

Here we recall the definition of k-bigness.

Definition 3 (See [2, p. 5]). Let X be a projective variety and let A be a line bundle on X. Then A is said to be *k-big* if $\kappa(A) \geq \dim X - k$.

Remark 2. *Let* (X, L) *be a polarized manifold of dimension* 3. *If* (X, L) *is a classical scroll over a smooth surface* S *with* $g(X, L) = 2$ *and* $h^1(\mathscr{O}_X) > 0$, *then* (X, L) *is* 1), 2-i), 2-ii) *or* 3) *in* [6, (2.25) Theorem]. *If* (X, L) *is the type* 1), 2-i), *or* 2-ii), *then* (X, L) *is a scroll over* S. *But if* (X, L) *is the type* 3), *then* (X, L) *is not a scroll over* S, *but a quadric fibration over a smooth elliptic curve because* $K_X + 2L$ *is not* 1*-big, but* 2*-big.*

Lemma 1. *Let* (X, L_1, L_2) *be a bi-polarized manifolds of dimension* 3. *Assume that* $\kappa(K_X + L_1 + L_2) \geq 0$ *and* $h^1(\mathscr{O}_X) > 0$. *Then* $g_1(X, L_1, L_2) \geq 2$ *holds.*

Proof. Assume that $g_1(X, L_1, L_2) \leq 1$. Then $g_1(X, L_1, L_2) = 1$ because we assume that $h^1(\mathscr{O}_X) > 0$ (see [16, Theorem 6.1.1]). Since $\kappa(K_X + L_1 + L_2) \geq 0$, we have $K_X + L_1 + L_2 = \mathscr{O}_X$ by [16, Theorem 6.1.2]. But this is impossible because $h^1(\mathscr{O}_X) > 0$. $\square$

Remark 3. *Let* (X, L_1, L_2) *be a bi-polarized manifold of dimension* 3. *Then we see from* [15, Theorem 5.1] *that*

$$h^0(K_X + L_1 + L_2) = h^0(K_X + L_1) + g_2(X, L_2) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X), \quad (1)$$

$$h^0(K_X + L_1 + L_2) = h^0(K_X + L_2) + g_2(X, L_1) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X). \quad (2)$$

Here $g_2(X, L)$ *denotes the second sectional geometric genus of* (X, L) *(see* [9, Definition 2.1]*). Moreover assume that there exist a smooth projective curve* C *and a fiber space* $f : X \to C$. *Then we have*

$$g_1(X, L_1, L_2) = g(C) + \frac{1}{2}(K_{X/C} + L_1 + L_2)L_1L_2 + (g(C) - 1)(L_1L_2F - 1). \quad (3)$$

Here F *is a general fiber of* f.

If $\dim X = 3$, then we have a lower bound for the second sectional geometric genus of (X, L).

Lemma 2. *Let* (X, L) *be a polarized manifold of dimension* 3. *Then*

$$g_2(X, L) \geq \begin{cases} h^1(\mathcal{O}_X), & \text{if } \kappa(K_X + L) \geq 0, \\ h^2(\mathcal{O}_X), & \text{if } \kappa(X) = -\infty. \end{cases}$$

Proof. See [10, Corollary 2.4] and [11, Theorem 3.2.1 and Theorem 3.3.1 (2)]. □

Lemma 3. *Let* (X, L_1, L_2) *be a bi-polarized manifold of dimension* 3. *Assume that* $h^1(\mathcal{O}_X) > 0$, $h^0(K_X + L_1 + L_2) = 1$ *and there exist a smooth projective curve* C *and a fiber space* $f : X \to C$ *such that* $L_1L_2F \geq 2$ *and* $h^1(\mathcal{O}_X) = g(C)$, *where* F *is a fiber of* f. *Then* $g(C) = 1$, $g_1(X, L_1, L_2) = 2$, $h^0(K_X + L_1) = 0$, $h^0(K_X + L_2) = 0$, $g_2(X, L_1) = 0$ *and* $g_2(X, L_2) = 0$.

Proof. Since $h^0(K_X + L_1 + L_2) = 1$, we have $h^0(K_F + (L_1)_F + (L_2)_F) > 0$. Hence we see from [17, Lemma 2.1] that $f_*(K_{X/C} + L_1 + L_2)$ is ample and we have $(K_{X/C} + L_1 + L_2)L_1L_2 > 0$ by the same argument as [8, Lemma 1.4.1]. By (3) in Remark 3 we have

$$g_1(X, L_1, L_2) \geq g(C) + 1 + (g(C) - 1)(L_1L_2F - 1) \quad (4)$$

because $(K_{X/C} + L_1 + L_2)L_1L_2$ is even. Hence $g_1(X, L_1, L_2) \geq g(C) + 1 = h^1(\mathcal{O}_X) + 1$, and by (1) and (2) in Remark 3 we have $h^0(K_X + L_1 + L_2) \geq h^0(K_X + L_1) + g_2(X, L_2) + 1$ and $h^0(K_X + L_1 + L_2) \geq h^0(K_X + L_2) + g_2(X, L_1) + 1$. Since $h^0(K_X + L_1 + L_2) = 1$, we see from Lemma 2 that $h^0(K_X + L_1) = 0$, $h^0(K_X + L_2) = 0$, $g_2(X, L_1) = 0$ and $g_2(X, L_2) = 0$. In particular

$$1 = h^0(K_X + L_1 + L_2) = g_1(X, L_1, L_2) - h^1(\mathcal{O}_X).$$

By (4) and the assumption that $L_1L_2F - 1 \geq 1$, we have $g(C) = 1$ and $g_1(X, L_1, L_2) = 2$. $\square$

Lemma 4. *Let (X, L_1, L_2) be a bi-polarized manifold of dimension 3. Assume that $K_X + 2L_1$ and $K_X + 2L_2$ are nef and 2-big, and $g_1(X, L_1, L_2) = 2$. Then the following hold.*

(i) *$g(X, L_1) = 2$ and $g(X, L_2) = 2$.*

(ii) *$L_1 \equiv L_2$.*[1]

Proof. (i) By assumption we get $(K_X + L_1 + L_2)L_1L_2 = 2$. We also note that

$$2(K_X + L_1 + L_2)L_1L_2 = (K_X + 2L_1)L_1L_2 + (K_X + 2L_2)L_1L_2.$$

Hence we have

$$(K_X + 2L_1)L_1L_2 + (K_X + 2L_2)L_1L_2 = 4. \tag{5}$$

By Hodge index Theorem [2, Proposition 2.5.1] we have

$$((K_X + 2L_1)L_1L_2)^2 \geq ((K_X + 2L_1)L_1^2)((K_X + 2L_1)L_2^2). \tag{6}$$

If $(K_X + 2L_1)L_1L_2 = 1$, then we have $(K_X + 2L_1)L_1^2 = 1$ since $K_X + 2L_1$ is nef and 2-big (see [2, Lemma 2.5.8]). But this is impossible because $(K_X + 2L_1)L_1^2$ is even by genus formula. Therefore we may assume that $(K_X + 2L_1)L_1L_2 \geq 2$. By the same argument as this, we may assume that $(K_X + 2L_2)L_1L_2 \geq 2$. By (5), we have $(K_X + 2L_1)L_1L_2 = 2$ and $(K_X + 2L_2)L_1L_2 = 2$.

First we consider $(K_X + 2L_1)L_1L_2 = 2$. Here we note that $K_XL_2^2$ is even by [2, Lemma 1.1.11]. Therefore $(K_X + 2L_1)L_2^2$ is even. Since $(K_X + 2L_1)L_2^2 > 0$ (see [2, Lemma 2.5.8]), we have $(K_X + 2L_1)L_2^2 \geq 2$. Hence by (6) we have $(K_X + 2L_1)L_1^2 \leq 2$, that is, $g(X, L_1) \leq 2$. But since $K_X + 2L_1$ is nef and 2-big, we have $(K_X + 2L_1)L_1^2 > 0$. Since $(K_X + 2L_1)L_1^2$ is even, we have $g(X, L_1) \geq 2$. Hence we get $g(X, L_1) = 2$.

Next we consider $(K_X + 2L_2)L_1L_2 = 2$. By the same argument as above we get $g(X, L_2) = 2$. Therefore we get the assertion of (i).

(ii) First we note that $(K_X + L_1 + L_2)L_1^2 \geq 2$ and $(K_X + L_1 + L_2)L_2^2 \geq 2$ hold. Actually since $K_X + 2L_1$ and $K_X + 2L_2$ are nef and 2-big, we have $(K_X + 2L_1)L_i^2 > 0$ and $(K_X + 2L_2)L_i^2 > 0$ for $i = 1, 2$. Since $K_XL_i^2$ is even, we see that $(K_X + 2L_1)L_i^2$ and $(K_X + 2L_2)L_i^2$ are even for $i = 1, 2$. Hence $(K_X + 2L_1)L_i^2 \geq 2$ and $(K_X + 2L_2)L_i^2 \geq 2$ for $i = 1, 2$. Therefore $2(K_X + L_1 + L_2)L_i^2 \geq 4$, that is, $(K_X + L_1 + L_2)L_i^2 \geq 2$ for $i = 1, 2$.

[1] The symbol $\equiv$ denotes the numerical equivalence.

Since $g_1(X, L_1, L_2) = 2$, we have $(K_X + L_1 + L_2)L_1L_2 = 2$. Hence we see from [2, Proposition 2.5.1] that $(K_X + L_1 + L_2)L_1^2 = 2$ and $(K_X + L_1 + L_2)L_2^2 = 2$ hold. So we get $L_1^2L_2 = L_1^3$ and $L_1L_2^2 = L_2^3$ because $(K_X + 2L_1)L_1^2 = 2$ and $(K_X + 2L_2)L_2^2 = 2$ hold by $g(X, L_1) = 2$ and $g(X, L_2) = 2$ (see (i) above). If $L_1^2L_2 > L_1L_2^2$ holds, then we have $(L_1^2L_2)(L_2^3) > (L_1L_2^2)^2$. But this is impossible by [2, Proposition 2.5.1]. If $L_1^2L_2 < L_1L_2^2$ holds, then we have $(L_1L_2^2)(L_1^3) > (L_1^2L_2)^2$. But this is also impossible by [2, Proposition 2.5.1]. Hence $L_1^2L_2 = L_1L_2^2$ holds. Since $L_1L_2^2 = L_2^3$, we get $(L_1^2L_2)(L_2^3) = (L_1L_2^2)^2$ and by [2, Corollary 2.5.4] we get $L_1 \equiv L_2$.

Therefore we get the assertion of (ii). $\square$

3. Main result

THEOREM 1. *Let (X, L_1, L_2) be a bi-polarized manifold of dimension* 3. *Assume that $h^0(K_X + L_1 + L_2) = 1$. Then, exchanging L_1 and L_2 if necessary, (X, L_1, L_2) is one of the following types.*

(i) *(X, L) is a Del Pezzo manifold for some ample line bundle L on X and $L_j = L$ for $j = 1, 2$.*

(ii) *$(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3), \mathcal{O}_{\mathbb{P}^3}(1))$.*

(iii) *$(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2), \mathcal{O}_{\mathbb{Q}^3}(1))$.*

(iv) *$X \cong \mathbb{P}^2 \times \mathbb{P}^1$, $L_1 = p_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) + p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $L_2 = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) + p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where p_i is the ith projection.*

(v) *(X, L_1) is a scroll over a smooth elliptic curve, and $(L_2)_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of the projection.*

(vi) *(X, L_1) is a quadric fibration over a smooth elliptic curve C, $L_1 \equiv L_2$ and (X, L_1) is one of the following types. (Here we use Notation* 1.*)*

- (vi.1) *(X, L_1) is a pure quadric fibration over C with $(b, e, d) = (1, 0, 1)$.*
- (vi.2) *(X, L_1) is a pure quadric fibration over C with $(b, e, d) = (0, 1, 2)$.*
- (vi.3) *(X, L_1) is a pure quadric fibration over C with $(b, e, d) = (-1, 2, 3)$.*
- (vi.4) *(X, L_1) is a classical scroll $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ over $S \cong \mathbb{P}_B(\mathcal{F})$, where $\mathcal{E} \cong \rho^*\mathcal{G} \otimes H(\mathcal{F})$ for some semistable vector bundles $\mathcal{F}$ and $\mathcal{G}$ of rank two on an elliptic curve B. Here ρ denotes the morphism $S \to B$. Moreover $(c_1(\mathcal{F}), c_1(\mathcal{G})) = (1, 0)$ or $(0, 1)$. X is the fiber product of $\mathbb{P}_B(\mathcal{F})$ and $\mathbb{P}_B(\mathcal{G})$ over B. In this case $L_1^3 = 3$.*

(vii) *(X, L_1) is a scroll over a smooth projective surface S and $L_1 \equiv L_2$. Then there exists an ample vector bundle $\mathcal{E}$ on S such that $(X, L_1) \cong (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, and $(S, \mathcal{E})$ is one of the following types.*

(vii.1) *S is the Jacobian variety of a smooth projective curve C of genus two and $\mathscr{E} \cong \mathscr{E}_2(C,o) \otimes N$ for some numerically trivial line bundle N on S, where $\mathscr{E}_2(C,o)$ is the Jacobian bundle of rank* 2 *for some point o on C (see* [6, (2.18)]*). In this case* $L_1^3 = 1$.

(vii.2) *$S \cong \mathbb{P}_C(\mathscr{F})$ for some stable vector bundle $\mathscr{F}$ of rank two on an elliptic curve C with $c_1(\mathscr{F}) = 1$. There is an exact sequence $0 \to \mathcal{O}_S(2H(\mathscr{F}) + \rho^*G) \to \mathscr{E} \to \mathcal{O}_S(H(\mathscr{F}) + \rho^*T) \to 0$, where G and T are line bundles on C and ρ is the morphism $S \to C$. Then one of the following holds.*

(vii.2.1) $\deg T = 1$, $\deg G = -2$ *and* $L_1^3 = 1$.

(vii.2.2) $\deg T = 0$, $\deg G = -1$ *and* $L_1^3 = 2$.

Proof. (I) Assume that $\kappa(K_X + L_1) \geq 0$ or $\kappa(K_X + L_2) \geq 0$.

First we assume that $\kappa(K_X + L_1) \geq 0$. Then we see from Lemma 2 that $g_2(X, L_1) - h^1(\mathcal{O}_X) \geq 0$ holds. Hence by (2) in Remark 3 we have $g_1(X, L_1, L_2) \leq 1$. But in this case $h^0(K_X + L_1 + L_2) = 1$ is impossible by [16, Theorems 6.1.1 and 6.1.2] because $\kappa(K_X + L_1) \geq 0$. By the same argument as this, we see from (1) in Remark 3 that $\kappa(K_X + L_2) \geq 0$ is also impossible.

(II) Assume that $\kappa(K_X + L_1) = -\infty$ and $\kappa(K_X + L_2) = -\infty$.

(II.1) Suppose that $h^1(\mathcal{O}_X) = 0$. Then (1) in Remark 3 $g_1(X, L_1, L_2) \leq 1$ because $g_2(X, L_2) \geq 0$ by Lemma 2. We see from [15, Corollary 5.1] and [16, Theorems 6.1.1 and 6.1.2] that (X, L_1, L_2) satisfies $K_X + L_1 + L_2 = \mathcal{O}_X$. Then by [16, Theorem 6.1.3] (X, L_1, L_2) is one of the following four types, and these cases satisfy $h^0(K_X + L_1 + L_2) = 1$.

(A) (X, L) is a Del Pezzo manifold for some ample line bundle L on X and $L_j = L$ for $j = 1, 2$.

(B) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3), \mathcal{O}_{\mathbb{P}^3}(1))$.

(C) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2), \mathcal{O}_{\mathbb{Q}^3}(1))$.

(D) $X \cong \mathbb{P}^2 \times \mathbb{P}^1$, $L_1 = p_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) + p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $L_2 = p_1^*(\mathcal{O}_{\mathbb{P}^2}(1)) + p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$, where p_i is the ith projection.

(II.2) Assume that $h^1(\mathcal{O}_X) > 0$. Then for any $i = 1$ and 2, we see from adjunction theory (see e.g. [2, Proposition 7.2.2, Theorems 7.2.3, 7.2.4, 7.3.2 and 7.3.4]) that (X, L_i) is one of the following types.

(i) A scroll over a smooth projective curve. In this case $K_X + 2L_i$ is not nef.

(ii) A quadric fibration over a smooth curve. In this case $K_X + 2L_i$ is nef and 2-big.

(iii) A scroll over a smooth projective surface. In this case $K_X + 2L_i$ is nef and 1-big.

(iv) M is a $\mathbb{P}^2$-bundle over a smooth curve C and for any fiber F' of it, $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, where (M, A) is the reduction of (X, L_i). In this case $K_X + 2L_i$ is nef and big.

(i.1) Assume that (X, L_1) is a scroll over a smooth projective curve C. Let $f : X \to C$ be its fibration. Here we note that $g(C) = h^1(\mathcal{O}_X) > 0$. Then

$$\begin{aligned} g_1(X, L_1, L_2) &= 1 + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2 \\ &= g(C) + \frac{1}{2}(K_{X/C} + L_1 + L_2)L_1L_2 + (g(C) - 1)(L_1L_2F - 1), \end{aligned}$$

where F is a fiber of f.

By assumption we have $(L_1)_F = \mathcal{O}_{\mathbb{P}^2}(1)$, and we put $(L_2)_F = \mathcal{O}_{\mathbb{P}^2}(a)$.

(i.1.1) If $a = 1$, then $g_1(X, L_1, L_2) = h^1(\mathcal{O}_X)$ (see [14, Example 2.1 (H)]), $g_2(X, L_1) = 0$ (see [9, Example 2.10 (8)]) and $h^0(K_X + L_2) = 0$. But then $h^0(K_X + L_1 + L_2) = 0$ and this case cannot occur.

(i.1.2) If $a \geq 3$, then $h^0((K_X + L_2)_F) \neq 0$. Therefore $h^0(K_X + L_2) \neq 0$ by [3, Lemma 4.1]. But this is impossible by Lemma 3.

(i.1.3) By (i.1.1) and (i.1.2) we have $a = 2$. Then by Lemma 3 we have $g(C) = 1$, $g_1(X, L_1, L_2) = 2$, $h^0(K_X + L_1) = 0$, $h^0(K_X + L_2) = 0$, $g_2(X, L_1) = 0$ and $g_2(X, L_2) = 0$.

(i.2) Assume that (X, L_2) is a scroll over a smooth curve. Then by the same argument as (i.1), we have $(L_2)_F = \mathcal{O}_{\mathbb{P}^2}(1)$ and $(L_1)_F = \mathcal{O}_{\mathbb{P}^2}(2)$, $g(C) = 1$, $g_1(X, L_1, L_2) = 2$, $h^0(K_X + L_1) = 0$, $h^0(K_X + L_2) = 0$, $g_2(X, L_1) = 0$ and $g_2(X, L_2) = 0$. This gives the type (v) in Theorem 1.

By (i.1) and (i.2) above, we may assume that (X, L_i) is either (ii), (iii) or (iv) for $i = 1$ and 2. In particular

$$K_X + 2L_1 \text{ and } K_X + 2L_2 \text{ are nef and 2-big.} \tag{7}$$

(ii) Assume that (X, L_1) is a quadric fibration over a smooth projective curve C. Let $f : X \to C$ be its fibration. Here we note that $g(C) = h^1(\mathcal{O}_X) > 0$. By assumption we have $(L_1)_F = \mathcal{O}_{\mathbb{Q}^2}(1)$, and we put $(L_2)_F = \mathcal{O}_{\mathbb{Q}^2}(b)$.

By Lemma 3, we have $g_1(X, L_1, L_2) = 2$. So by (7) and Lemma 4 we have $g(X, L_1) = 2$, $g(X, L_2) = 2$ and $L_1 \equiv L_2$. By Remark 1 (iii), (X, L_1) is one of the following types.

(a) A pure quadric fibration over C.

(b) A classical scroll over a smooth surface.

If (X, L_1) is the type (a), then we see from [5, (3.7)] that (X, L_1) is one of the types (vi.1), (vi.2) and (vi.3) in Theorem 1.

If (X, L_1) is the type (b), then we see from [6, (2.25) Theorem] that (X, L_1) is the type (vi.4) in Theorem 1.

(iii) Assume that (X, L_1) is a scroll over a smooth projective surface S.

Then by [9, Example 2.10 (8)] we have $g_2(X, L_1) = h^2(\mathcal{O}_X)$ and by (2) in Remark 3 we have

$$h^0(K_X + L_1 + L_2) = h^0(K_X + L_2) + h^2(\mathcal{O}_X) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X). \quad (8)$$

We also note that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$. Let $f : X \to S$ be the projection.

(iii.1) Assume that $\kappa(S) = 2$. Then $\chi(\mathcal{O}_S) \geq 1$. Hence by (8)

$$\begin{aligned} h^0(K_X + L_1 + L_2) &= h^0(K_X + L_2) + \chi(\mathcal{O}_X) + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2 \\ &\geq h^0(K_X + L_2) + 1 + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2. \end{aligned}$$

Since $h^0(K_X + L_1 + L_2) = 1$, we get $(K_X + L_1 + L_2)L_1L_2 = 0$, that is, $g_1(X, L_1, L_2) = 1$. But by Lemma 1 this is impossible.

(iii.2) Assume that $\kappa(S) = 1$. Then $\chi(\mathcal{O}_S) \geq 0$.

(iii.2.1) If $\chi(\mathcal{O}_S) \geq 1$, then $\chi(\mathcal{O}_X) \geq 1$ and by the same argument as (iii.1), this is impossible.

(iii.2.2) If $\chi(\mathcal{O}_S) = 0$. Then by (8)

$$\begin{aligned} 1 &= h^0(K_X + L_1 + L_2) \\ &= h^0(K_X + L_2) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \\ &= h^0(K_X + L_2) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S). \end{aligned} \quad (9)$$

Let $\pi : S \to T$ be its elliptic fibration, where T is a smooth projective curve. Let $h := \pi \circ f : X \to S \to T$. We note that $(K_{X/T} + L_1 + L_2)L_1L_2 \geq 2$ holds by the same argument as the first part of the proof of Lemma 3. We also note that $g(T) \leq h^1(\mathcal{O}_S) \leq g(T) + 1$. Then

$$\begin{aligned} &g_1(X, L_1, L_2) \\ &\quad = g(T) + \frac{1}{2}(K_{X/T} + L_1 + L_2)L_1L_2 + (g(T) - 1)(L_1L_2F_h - 1) \\ &\quad \geq g(T) + 1 + (g(T) - 1)(L_1L_2F_h - 1). \end{aligned} \quad (10)$$

(iii.2.2.1) If $h^1(\mathcal{O}_S) = g(T)$, then $g(T) > 0$, and by (10) we have $g_1(X, L_1, L_2) \geq h^1(\mathcal{O}_S) + 1$. By (9) we have $h^0(K_X + L_2) = 0$ and $h^2(\mathcal{O}_S) = 0$. Hence $h^1(\mathcal{O}_S) = 1$ because $\chi(\mathcal{O}_S) = 0$. Since $h^0(K_X + L_1 + L_2) = 1$, we have $g_1(X, L_1, L_2) = 2$.

(iii.2.2.2) If $h^1(\mathcal{O}_S) = g(T) + 1$ and $g(T) \geq 1$, then by (10) $g_1(X, L_1, L_2) \geq g(T) + 1 = h^1(\mathcal{O}_S)$. Since $h^1(\mathcal{O}_S) = g(T) + 1 \geq 2$, we have $h^2(\mathcal{O}_S) \geq 1$. Hence by (9) we have $h^0(K_X + L_2) = 0$, $h^2(\mathcal{O}_S) = 1$ and $g_1(X, L_1, L_2) = h^1(\mathcal{O}_X) = 2$. Moreover we have $h^1(\mathcal{O}_S) = 2$ and $g(T) = 1$.

(iii.2.2.3) If $h^1(\mathcal{O}_S) = g(T) + 1$ and $g(T) = 0$, then $h^1(\mathcal{O}_S) = 1$. Moreover we have $g_1(X, L_1, L_2) \geq 2$ by Lemma 1. Therefore by (9) we have $h^0(K_X + L_2) = 0$, $h^2(\mathcal{O}_S) = 0$ and $g_1(X, L_1, L_2) = 2$.

Hence we see from the argument above that if $\kappa(S) = 1$, then $g_1(X, L_1, L_2) = 2$ holds. Therefore by (7) and Lemma 4 we have $g(X, L_1) = 2$, $g(X, L_2) = 2$ and $L_1 \equiv L_2$.

(iii.3) Assume that $\kappa(S) = 0$. Then $\chi(\mathcal{O}_S) \geq 0$. First of all, since $h^1(\mathcal{O}_S) > 0$, S is birationally equivalent to either a bielliptic surface or an Abelian surface. Here we note that $g_1(X, L_1, L_2) \geq 2$ by Lemma 1.

(iii.3.1) If S is birationally equivalent to a bielliptic surface, then $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S) = 1$. Hence $g_1(X, L_1, L_2) - h^1(\mathcal{O}_X) \geq 1$. Since $h^0(K_X + L_1 + L_2) = 1$, we get $g_1(X, L_1, L_2) = 2$ by (1) in Remark 3 and Lemma 2.

(iii.3.2) If S is birationally equivalent to an Abelian surface, then $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S) = 2$ and $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_S) = 1$. Hence $g_1(X, L_1, L_2) - h^1(\mathcal{O}_X) \geq 0$. On the other hand, by Lemma 2, we have $g_2(X, L_1) \geq h^2(\mathcal{O}_X) = 1$. Since $h^0(K_X + L_1 + L_2) = 1$, we get $g_1(X, L_1, L_2) = 2$ by (2) in Remark 3.

(iii.3.3) We see from (iii.3.1), (iii.3.2), (7) and Lemma 4 that $g(X, L_1) = 2$, $g(X, L_2) = 2$ and $L_1 \equiv L_2$.

(iii.4) Assume that $\kappa(S) = -\infty$.

Lemma 5. *If $\kappa(S) = -\infty$, then $g_1(X, L_1, L_2) = 2$.*

Proof. Since $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X) > 0$, we can take the Albanese fibration $\alpha : S \to C$, where C is a smooth projective curve with $g(C) \geq 1$. Here we note that $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X) = g(C)$. Let $h := \alpha \circ f$. Since $h^0(K_X + L_1 + L_2) = 1$, we have $h_*(K_{X/C} + L_1 + L_2) \neq 0$. Since $(K_{X/C} + L_1 + L_2)L_1L_2$ is even, we get $(K_{X/C} + L_1 + L_2)L_1L_2 \geq 2$ by the same argument as the first part of the proof of Lemma 3, and $g_1(X, L_1, L_2) = g(C) + \frac{1}{2}(K_{X/C} + L_1 + L_2)L_1L_2 + (g(C) - 1)(L_1L_2F_h - 1) \geq g(C) + 1 = h^1(\mathcal{O}_X) + 1$, where F_h is a fiber of h. Hence $h^0(K_X + L_2) = 0$ and $h^2(\mathcal{O}_X) = 0$ by (8) because $h^0(K_X + L_1 + L_2) = 1$. By [3, Lemma 4.1], $h^0(K_X + L_2) = 0$ implies $h^0(K_{F_h} + (L_2)_{F_h}) = 0$ for any general fiber F_h of h. Hence by [13, Theorem 2.8] we see that $\kappa(K_{F_h} + (L_2)_{F_h}) = -\infty$. In particular $K_{F_h} + (L_2)_{F_h}$ is not nef.

Claim 1. *$(F_h, (L_2)_{F_h})$ is a scroll over $\mathbb{P}^1$.*

Proof. First we note that $h^1(\mathcal{O}_{F_h}) = 0$. So, by [24, 1.3 Remark], we obtain that $(F_h, (L_2)_{F_h})$ is either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or a scroll over $\mathbb{P}^1$. But we note that $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are impossible because $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$. So we get the assertion of Claim 1. □

By Claim 1, we infer that F_h is a Hirzebruch surface. Hence we have $((L_1)_{F_h})^2 \geq 2$ and $((L_2)_{F_h})^2 \geq 2$ because $(L_1)_{F_h}$ and $(L_2)_{F_h}$ are very ample.

Therefore $((L_1)_{F_h})((L_2)_{F_h}) \geq 2$ by the Hodge index theorem. By Lemma 3, we get $g(C) = 1$ and $g_1(X, L_1, L_2) = 2$. □

By (7) and Lemma 4 we have $g(X, L_1) = 2$, $g(X, L_2) = 2$ and $L_1 \equiv L_2$. By the above argument and [6, (2.25) Theorem], (X, L_1, L_2) is one of the types (vii.1), (vii.2.1) and (vii.2.2) in Theorem 1 (see Remark 2).

(iv) Assume that (X, L_1) is the case (iv), that is, M is a $\mathbb{P}^2$-bundle over a smooth curve C and $(F', A_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber F' of it, where (M, A) is the reduction of (X, L_1). Let $p : M \to C$ be the projection and $\mu : X \to M$ be the reduction map. Let $f : X \to C$ be the morphism $p \circ \mu$. By Lemma 3, we have $g_1(X, L_1, L_2) = 2$ and $g(C) = 1$. So by (7) and Lemma 4 we have $g_1(X, L_1) = 2$. But since $g(M, A) = g(X, L_1) = 2$ this is impossible by [5, (1.8)]. □

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