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Remarks on the strong maximum principle involving p-Laplacian

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Abstract. Let $N \geq 1$, $1 < p < \infty$ and $p^* = \max(1, p-1)$. Let Ω be a bounded domain of $\mathbf{R}^N$. We establish the strong maximum principle for the p-Laplace operator with a nonlinear potential term. More precisely, we show that every super-solution $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ vanishes identically in Ω, if u is admissible and $u = 0$ a.e on a set of positive p-capacity relative to Ω.

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$. By $\Delta_p u$, we denote a p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \tag{1.1}$$

where $1 < p < \infty$ and $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \dots, \partial u/\partial x_N)$.

In this article, we shall study the strong maximum principle on the following quasilinear operator:

$$-\Delta_p + a(x)Q(\cdot). \tag{1.2}$$

Here $a \in L^1_{\mathrm{loc}}(\Omega)$ and $Q(\cdot)$ is a nonlinear term satisfying the following properties $[\mathbf{Q_0}]$ and $[\mathbf{Q_1}]$.

$[\mathbf{Q_0}]$: $Q(t)$ is a continuous increasing function on $[0, \infty)$ with $Q(0) = 0$.

$[\mathbf{Q_1}]$:

$$\limsup_{t \to +0} \frac{Q(t)}{t^{p-1}} < \infty. \tag{1.3}$$

Remark 1.1. *In view of* $[\mathbf{Q_0}]$, $[\mathbf{Q_1}]$ *can be replaced by the following: For any* $T > 0$, *there exists a positive number* C_T *such that*

$$Q(t) \leq C_T \cdot t^{p-1} \qquad t \in [0, T].$$

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Now let us recall some related known results on the strong maximum principle assuming that $Q(t) = t^{p-1}$ for simplicity. The classical strong maximum principle for a Laplacian asserts that if u is smooth, $u \geq 0$ and $-\Delta u \geq 0$ in a domain (a connected open set) $\Omega \subset \mathbf{R}^N$, then either $u \equiv 0$ or $u > 0$ in Ω. The same conclusion holds when $-\Delta u$ is replaced by $-\Delta u + a(x)u$ with $a \in L^s(\Omega)$, $s > N/2$. Later these results were extended to quasilinear operators $-\Delta_p u + a(x)u^{p-1}$ with $1 < p < \infty$, $a \in L^s(\Omega)$, $s > N/p$. These are consequences of weak Harnack's inequalities; see e.g. Stampacchia [21], Trudinger [23]; Theorem 5.2 and its Corollaries, Moser [19, 20] for $p = 2$, and see e.g. Stredulinsky [22], Chapter 3 for $p > 1$ (see also Vázquez [24], Theorem 5).

Another formulation of the same fact says that if $u(x) = 0$ for some point $x \in \Omega$, then $u \equiv 0$ in Ω. However the next example shows that a similar conclusion does not hold when $a \notin L^s$ for any $s > N/p$.

Example 1. *Let B_1 be a unit ball in $\mathbf{R}^N$ with a center being* 0 *and*

$$\begin{cases} u = |x|^{\alpha}, \quad \alpha > (p-N)/(p-1), \\ a(x) = c(p,\alpha)|x|^{-p}, \\ c(p,\alpha) = \alpha|\alpha|^{p-2}(\alpha p - \alpha - p + N). \end{cases} \tag{1.4}$$

Then we see $a \notin L^{N/p}(B_1)$ and $-\Delta_p u + a(x)u^{p-1} = 0$ in B_1. Clearly $u(0) = 0$ for $\alpha > 0$, but $u \not\equiv 0$ in B_1.

On the other hand, if u vanishes on a larger set, then one may conclude that $u \equiv 0$ under some weaker condition on a. When $p = 2$, such a result was obtained by Bénilan-Brezis [3] in the case where $a \in L^1(\Omega)$, $a \geq 0$ a.e. in Ω and $\operatorname{supp} u$ is a compact subset of Ω (see Theorem C1 in [3]). This maximum principle has been further extended by Ancona [1], and later a more direct proof was given by Brezis-Ponce [6] in the split of PDE's. In the present article we further study the strong maximum principle in the case where $p \in (1, \infty)$ adopting a genaral nonlinearlity $Q(t)$ in stead of t^{p-1}. To this end we prepare more notations:

We recall that a real valued function u on Ω is quasicontinuous if there exists a sequence of open subsets $\{\omega_n\}$ of Ω such that $u|_{\Omega\setminus\omega_n}$ is continuous for any $n \geq 1$ and $C_p(\omega_n, \Omega) \to 0$ as $n \to \infty$, where $C_p(\omega_n, \Omega)$ denotes a p-capacity of ω_n relative to Ω (see Definition 2.2), and we say that μ is a Radon measure on Ω if for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that $|\int_\Omega \varphi\, d\mu| \leq C_\omega \|\varphi\|_{L^\infty}$ for any $\varphi \in C_0^\infty(\omega)$. Note that if μ is a Radon measure, then the total measure of μ on ω denoted by $|\mu|(\omega)$ is finite. In order to establish the strong maximum principle (SMP) involving a Radon measure $\Delta_p u$ with $p \in (1, \infty)$, we introduce an admissible class of functions:

Definition 1.1 (Admissible class in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$). *Let $1 < p < \infty$ and $p^* = \max(1, p-1)$. A function $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ is said to be admissible if $\Delta_p u$*

is a Radon measure on Ω *and there exists a sequence* $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}(\Omega)$ *satisfying the following conditions.*

(1) $u_n \to u$ *a.e. in* Ω, $u_n \to u$ *in* $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ *as* $n \to \infty$.

(2) $\Delta_p u_n \in L^1_{\mathrm{loc}}(\Omega)$ $(n = 1, 2, \ldots)$ *and*

$$\sup_n |\Delta_p u_n|(\omega) < \infty \qquad \text{for every } \omega \subset\subset \Omega. \tag{1.5}$$

Remark 1.2. (1) *If* $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$, *then* $\Delta_p u$, $\Delta_p(u^+)$ *and* $\Delta_p(u^-)$ *are well-defined in* $D'(\Omega)$. *It follows from the condition* 1 *that* $\Delta_p u_n = \Delta_p(u_n^+) - \Delta_p(u_n^-)$ *and* $\Delta_p u_n \to \Delta_p u$ *(i.e.* $\Delta_p(u_n^{\pm}) \to \Delta_p(u^{\pm})$*) in* $D'(\Omega)$ *as* $n \to \infty$. *Moreover, it follows from the condition* 2 *and the weak compactness of measures that we have* $\Delta_p u_n \to \Delta_p u$ *(i.e.* $\Delta_p(u_n^{\pm}) \to \Delta_p(u^{\pm})$*) in the sense of measures as* $n \to \infty$. *In particular if u is admissible, then* u^+ *and* u^- *are admissible as well.*

(2) *In our main result (see Theorem* 1*, below), the admissibility is assumed only when* $p \neq 2$. *We note that when* $p = 2$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *is always admissible if* Δu *is a Radon measure on* Ω*:*
To see this, let $\rho \in C_0^{\infty}(B_1)$ *be a radial, nonegative, decreasing, mollifier. By extending* $u \in L^1(\Omega)$ *to the whole space* $\mathbf{R}^N$ *so that* $u \equiv 0$ *outside* Ω, *we define a mollification of u by*

$$u_\rho^n(x) := \rho_{1/n} * u(x) = \int_{\Omega} \rho_{1/n}(x - y)u(y)dy \qquad \forall x \in \Omega. \tag{1.6}$$

We define $u_n(x) = u_\rho^n(x) = \rho_{1/n} * u(x)$ $(n = 1, 2, \ldots)$. *Then the condition* 1 *is clearly satisfied. Moreover if* $p = 2$ *and* Δu *is a Radon measure, then* (1.5) *is also satisfied. For* $\omega \subset\subset \Omega$, *we see that* $\Delta u_n = (\Delta u)_n$ *and* $|\Delta u_n| = (\Delta u_n)^+ + (\Delta u_n)^- = ((\Delta u)_n)^+ + ((\Delta u)_n)^- = |(\Delta u)_n|$ *in* ω *for n sufficiently large. Hence* (1.5) *follows from the definition of the Radon measure.*

Here we give an important class of admissible functions:

Example 2. *A function* $u \in W_0^{1,p}(\Omega)$ *is admissible if* $\Delta_p u$ *is a Radon measure on* Ω.

In fact we can construct an approximating sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ *in the following way. Let* $\mu = \Delta_p u$, $F = |\nabla u|^{p-2}\nabla u \in (L^1(\Omega))^N$ *and* $F_\rho^n = (F)_\rho^n \in (C^{\infty}(\mathbf{R}^N))^N$, $n = 1, 2, \ldots$. *Let* $\omega \subset\subset \Omega$. *Then, we have* $\mu_\rho^n = (\Delta_p u)_\rho^n = \operatorname{div} F_\rho^n$ *in* ω *for a sufficiently large n. Let* $w_n \in W_0^{1,p}(\Omega)$ *be the unique weak solution of the boundary value problem for the monotone operator* Δ_p *(c.f.* [16]*):*

$$\begin{cases} \Delta_p w_n = \operatorname{div} F_\rho^n & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Then, it follows from the standard argument (in Appendix) that we have

$$w_n \to u \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty, \tag{1.8}$$

$$|\Delta w_n|(\omega) = |\mathrm{div}\, F_p^n|(\omega) = |\mu_p^n|(\omega) \to |\mu|(\omega) \qquad \text{as } n \to \infty. \tag{1.9}$$

Then, taking a subsequence if necessary, $\{w_n\}$ satisfies the conditions 1 *and* 2. *For the detail of proof, see Appendix.*

Now we describe our main result:

Theorem 1. *Let $N \geq 1$, $1 < p < \infty$ and $p^* = \max(1, p-1)$. Let Ω be a bounded domain of $\mathbf{R}^N$. Assume that Q satisfies the conditions $[\mathbf{Q_0}]$ and $[\mathbf{Q_1}]$. When $p = 2$, assume that $u \in L^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$ a.e. in Ω, $Q(u) \in L^1_{\mathrm{loc}}(\Omega)$ and Δu is a Radon measure on Ω.*

When $p \neq 2$, assume that $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$, $u \geq 0$ a.e. in Ω, $Q(u) \in L^1_{\mathrm{loc}}(\Omega)$ and u is admissible in the sense of Definition 1.1.

Then we have the following:

(1) *There exists a quasicontinuous function $\tilde{u} : \Omega \mapsto \mathbf{R}$ such that $u = \tilde{u}$ a.e. in Ω.*

(2) *Let $a \in L^1_{\mathrm{loc}}(\Omega)$, $a \geq 0$ a.e. in Ω. Assume that*

$$-\Delta_p u + a(x)Q(u) \geq 0 \qquad \text{in } \Omega \text{ in the sense of measures}, \tag{1.10}$$

i.e.,

$$\int_E \Delta_p u \leq \int_E aQ(u) \qquad \text{for every Borel set } E \subset\subset \Omega. \tag{1.11}$$

If $\tilde{u} = 0$ on a set of positive p-capacity in Ω, then $u = 0$ a.e. in Ω.

Remark 1.3. (1) *The definition p-capacity denoted by $C_p(E, \Omega)$ is given in* §2 *in connection with quasi continuity of function.*

(2) *In Example* 1, *we see that $u = |x|^\alpha$ satisfies $-\Delta_p u + a(x)u^{p-1} = 0$ in B_1. If $p > N$, then $C_p(\{0\}, B_1) > 0$ and $u(0) = 0$ hold. But we note that $a \notin L^1_{\mathrm{loc}}(B_1)$ in this case.*

(3) *When $p \leq 2 - 1/N$, as in Example* 3 *below, we cannot expect the solution of an equation of the form $\Delta_p u = f$ (a Radon measure or L^1) to be in $W^{1,1}_{loc}(\Omega)$ in general. Therefore we cannot take the gradient of u appearing in the p-Laplacian Δ_p in the distribution sense. See* [2, 4, 5].

(4) *It follows from* (1.11) *that the positive part $(\Delta_p u)^+$ should be absolutely continuous with respect to the Lebesgue measure.*

In connection with Remark 1.3 (2), we give an example, which also shows the necessity of (1.11) for the validity of Theorem 1 when $p > N$.

Example 3. *Let* $u = |x|^{\alpha}$ *for* $\alpha = (p - N)/(p - 1)$.

(1) *u satisfies*

$$\Delta_p u = \alpha|\alpha|^{p-2} c_N \delta,$$

where δ *denotes a Dirac mass and* c_N *denotes the surface area of* B_1. *It is easy to see that* $|\nabla u| \in L^1_{loc}(\Omega)$ *if and only if* $p > 2 - 1/N$.

(2) *When* $p > 2 - 1/N$, *u is admissible in* $W^{1,p^*}(B_1)$. *In fact,* $u = |x|^{\alpha}$ *is approximated by a sequence of admissible functions* $v_{\alpha(n)} = |x|^{\alpha(n)} \in L^1(B_1)$ *where* $\alpha(n) = \alpha + 1/(n(p-1))$. *Then, in the sense of measures we have*

$$\Delta_p v_{\alpha(n)} = \frac{1}{n}|\alpha(n)|^{p-2}\alpha(n)|x|^{1/n - N} \to \Delta_p u \qquad \text{as } n \to \infty.$$

Therefore there exits a sequence $\{n_{\alpha(n)}\}$ *such that* $\{n_{\alpha(n)}\} \to \infty$ *as* $n \to \infty$ *and a sequence of mollification* $\{(v_{\alpha(n)})_p^{n_{\alpha(n)}}\}$ *satisfies the conditions in Definition* 1.1.

(3) *If* $p > N$, *then* $u(0) = 0$ *and* $C_p(\{0\}, B_1) > 0$. *But* (1.11) *is not satisfied since* $\alpha > 0$.

(4) *When* $1 < p \le 2 - 1/N$ *holds, one can consider u as a renormalized solution. For the detail, see e.g.* [2, 4, 5, 17, 18].

When $Q(t) = t^{q-1}$ for $q > 1$, $Q(t)$ clearly satisfies $[\mathbf{Q_0}]$. Then the condition $[\mathbf{Q_1}]$ is satisfied if and only if $q \ge p$. In this case we can show the necessity of the condition $[\mathbf{Q_1}]$ for the validity of Theorem 1, namely we have the following.

Proposition 1.1. *Let us set* $Q(t) = t^{q-1}$ *and* $q > 1$. *Then, the condition* $[\mathbf{Q_1}]$ *is necessary for the validity of Theorem* 1.

The proof of this proposition is given in § 3 by constructing a counter-example.

We collect corollaries which follow immediately from the theorem above:

Corollary 1.1. *Let u and a be as in Theorem* 1, *and assume* (1.10) *is satisfied.*

(1) *If* $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω.

(2) *If u is continuous in* Ω *and* $u = 0$ *on a subset of* Ω *with positive p-capacity, then* $u \equiv 0$ *in* Ω.

Corollary 1.2. *Let u and a be as in Theorem* 1. *Assume that* $\Delta_p u, aQ(u) \in L^1_{\mathrm{loc}}(\Omega)$. *If*

$$-\Delta_p u + aQ(u) \ge 0 \qquad \text{a.e. in } \Omega,$$

and $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω.

Combing Theorem 1 and Remark 2.1 in §2, we have the following.

Corollary 1.3. *Let u and a be as in Theorem* 1. *Assume that* $aQ(u) \in L^1_{\mathrm{loc}}(\Omega)$. *If*

$$-\Delta_p u + aQ(u) \geq 0 \qquad \text{in the distribution sense,}$$

that is

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \leq \int_\Omega aQ(u)\varphi \qquad \text{for any } \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0, \tag{1.12}$$

and $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω.

Remark 1.4. *In view of Corollary* 1.3, *it would seem natural to replace condition* (1.11) *by* (1.12) *in Theorem* 1. *We note that condition* (1.12) *makes sense even if* $aQ(u) \notin L^1_{\mathrm{loc}}(\Omega)$ *(since* $aQ(u)\varphi \geq 0$ *a.e., the right-hand side of* (1.12) *is always well-defined, possibly taking the value* $+\infty$*). However, the strong maximum principle is not true in general (see* [6]*). See also Remark* 2.1.

This article is organized in the following way. In §2 we collect basic notations with some remarks. In §3 we prove Proposition 1.1 by constructing a counter-example, and in §4 we prove the quasicontinuity statement of Theorem 1. In §5 we prepare two lemmas including Kato's inequalities when $\Delta_p u$ is a Radon measure. Theorem 1 is finally established in §6. In Appendix we prove that $u \in W_0^{1,p}(\Omega)$ is admissible if $\Delta_p u$ is a Radon measure on Ω.

2. Preliminaries

In this subsection we collect fundamental definitions in the present article together with some remarks. Let $L^p(\Omega)$, $1 \leq p < \infty$, denote the space of Lebesugue measurable functions, defined on Ω, for which

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} < \infty.$$

By $L^p_{loc}(\Omega)$ we mean the space of functions locally integrable with power p in Ω, and by $L^\infty(\Omega)$ we mean the space of essentially bounded Lebesgue measurable functions. As a norm of f in $L^\infty(\Omega)$ we take its essential supremum. Then we define the following Sobolev spaces:

Definition 2.1 ($W^{1,p}(\Omega)$ *and* $W^{1,p}_{loc}(\Omega)$). *For each* $1 \leq p < \infty$, *we set*

$$W^{1,p}(\Omega) = \{f : \Omega \mapsto \mathbf{R} : f \in L^p(\Omega), \partial f/\partial x_i \in L^p(\Omega) \text{ for } i = 1, \dots, N\}, \tag{2.1}$$

$$W^{1,p}_{loc}(\Omega) = \{f : \Omega \mapsto \mathbf{R} : f \in L^p_{loc}(\Omega), \partial f/\partial x_i \in L^p_{loc}(\Omega) \text{ for } i = 1, \dots, N\}. \tag{2.2}$$

Here $\partial f/\partial x_i$ is taken as a distributional derivative of f for $i = 1, \dots, N$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \||\nabla u|\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}. \tag{2.3}$$

By $W_0^{1,p}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{W^{1,p}(\Omega)}$.

Definition 2.2 (*A p-capacity relative to Ω*). *Let $1 < p < \infty$. For each compact set $K \subset \Omega$ we define a p-capacity of K relative to Ω by*

$$C_p(K, \Omega) = \inf\left\{\int_\Omega |\nabla\varphi|^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K\right\}.$$

We prepare a fundamental lemma (for the proof, see e.g. [22]; Chapter 2).

Lemma 1. (1) *If $u \in W_{loc}^{1,p}(\Omega)$, then u can be redefined almost everywhere so as to be quasicontinuous.*

(2) *If $u \in W_{loc}^{1,p}(\Omega)$, then u is continuous off open sets of arbitrarily small p-capacity, and if $\varphi_n \in C^\infty(\Omega)$ and $\varphi_n \to u$ in $W_{loc}^{1,p}(\Omega)$ as $n \to \infty$, then $\varphi_{n_j} \to u$ point wise quasi-everywhere for some subsequence $\{n_j\}$.*

Remark 2.1. *Let μ be a Radon measure on Ω and f a measurable function, $f \geq 0$ a.e. in Ω. Here are two possible definitions A and B for the inequality $\mu \leq f$ in Ω:*

A: We shall write $\mu \leq_1 f$ in Ω, if $\int_E d\mu \leq \int_E f$ for every Borel set $E \subset\subset \Omega$.

B: We shall write $\mu \leq_2 f$ in Ω, if $\int \varphi\, d\mu \leq \int f\varphi$ $\forall \varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω.

(1) *If $\mu \leq_1 f$ in Ω, then $\mu \leq_2 f$ in Ω. However, the converse is not true in general. Note that $f\varphi \geq 0$ a.e., in B so that the right-hand side is always well-defined, possibly taking the value $+\infty$.*

(2) *If we assume in addition that $f \in L_{loc}^1(\Omega)$, then $\mu \leq_1 f$ in Ω, if and only if, $\mu \leq_2 f$ in Ω.*

3. Proof of Proposition 1.1

We prove Proposition 1.1 by contradiction. If $1 < q < p$ and $Q(t) = t^{q-1}$, then there exists an admissible sub-solution of (1.10) such that $\tilde{u} = 0$ on a set of positive p-capacity in Ω but not equal to 0 a.e. in B_1. In order to see this, we prepare the following.

Example 4. *Let $\Omega = B_1$. Let m be a nonnegative integer such as $m \leq N - 1$. Let $\mathscr{M}_0 = \{0\}$ and let $\mathscr{M}_m \subset \mathbf{R}^N$ for $m > 0$ be an m dimensional linear subspace defined by*

$$\mathscr{M}_m = \{y = (y_1, y_2, \dots, y_N) \in \mathbf{R}^N : y_{m+1} = y_{m+2} = \cdots y_N = 0\}, \tag{3.1}$$

and we put $K_m = \mathscr{M}_m \cap \overline{B_{1/2}}$. *Let us define*

$$d_m(x) = dist(x, \mathscr{M}_m) \equiv \left(\sum_{k=m+1}^{N} x_k^2 \right)^{1/2}. \tag{3.2}$$

Then clearly $d_m \in C^\infty(\mathbf{R}^N \backslash \mathscr{M}_m)$, *Lipschitz continuous in* B_1 *and* $|\nabla d_m(x)| = 1$ *in* $\mathbf{R}^N \backslash \mathscr{M}_m$. *Now we construct a null solution* U_m *for* (1.2) *in* B_1 *of the form*

$$U_m(x) = d_m(x)^\alpha \tag{3.3}$$

as before. By a direct calculation we see that U_m *is admissible for a large* $\alpha > 0$ *and* $\Delta_p U_m$ *satisfies*

$$-\Delta_p U_m + a(x) Q(U_m) = 0 \qquad in \ D'(B_1), \tag{3.4}$$

where

$$a(x) = \frac{\Delta_p U_m}{Q(U_m)} = \frac{U_m^{q-1}}{Q(U_m)} \alpha^{p-1} (d_m \Delta d_m + (\alpha - 1)(p-1)) d_m^{\alpha(p-q)-p}.$$

Here we note that

$$d_m(x) \Delta d_m(x) = N - m - 1 \qquad and \qquad Q(t) = t^{q-1}. \tag{3.5}$$

Then we have for a sufficiently large $\alpha > 0$

$$0 \le a(x) \le C d_m(x)^{\alpha(p-q)-p} \in L^1(B_1), \qquad for\ some\ positive\ constant\ C.$$

Clearly U_m *is an admissible sub-solution of* (1.10) *such that* $U_m = 0$ *on* K_m *but not equal to* 0 *a.e. in* B_1.

PROOF OF PROPOSITION 1.1. In Example 4, let us choose a nonnegative integer m such that $N - p < m \le N - 1$. Then, it follows from a fundamental property of relative p-capacity that $C_p(K_m, B_1) > 0$. Clearly $U_m = 0$ on $K_m \subset \mathscr{M}_m$ and $U_m \not\equiv 0$. Therefore U_m becomes a counter-example. For precise properties of relative p-capacity, see e.g. Lemma 1 in Dupaigne-Ponce [8], Proposition 3.1 in Horiuchi [10]. □

4. Proof of the quasicontinuity statement of Theorem 1

In this section we prove the quasicontinuity statement of Theorem 1. Given $k > 0$, we denote by $T_k : \mathbf{R} \to \mathbf{R}$ a truncation function

$$T_k(s) := \begin{cases} k & \text{if } s \ge k, \\ s & \text{if } -k < s < k, \\ -k & \text{if } s \le -k. \end{cases} \tag{4.1}$$

Recall the following satndard inequality (see, e.g., Lemma 1 in [6]):

Lemma 2. *Assume that* $u \in L^1_{\mathrm{loc}}(\Omega)$ *and* Δu *is a Radon measure. Then*

$$T_k(u) \in W^{1,2}_{loc}(\Omega), \qquad \forall k > 0. \tag{4.2}$$

Moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, *there exists positive constant* C *such that*

$$\int_\omega |\nabla T_k(u)|^2 \le k\left(\int_{\omega'} |\Delta u| + C\int_{\omega'} |u|\right), \tag{4.3}$$

where positive constant C *are independent on each* u. *Moreover, there exists a quasicontinuous function* $\tilde{u} : \Omega \mapsto \mathbf{R}$ *such that* $u = \tilde{u}$ *a.e. in* Ω.

When $p = 2$, the existence of a quasicontinuous function $\tilde{u}$ (the statement 1 of Theorem 1) follows from Lemma 2. When $p \neq 2$, this fact is a consequence of Lemma 3 below:

Lemma 3. *Let* $\Omega \subset \mathbf{R}^{\mathbf{N}}$ *be an open set. Assume that* $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ *is admissible. Then*

$$T_k(u) \in W^{1,p}_{loc}(\Omega), \qquad \forall k > 0. \tag{4.4}$$

Moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, *there exists positive constant* C *such that*

$$\int_\omega |\nabla T_k(u)|^p \le Ck\left(\int_{\omega'} |\Delta_p u| + \int_{\omega'} |\nabla u|^{p-1}\right), \tag{4.5}$$

where positive constant C *are independent on each* u. *Moreover, there exists a quasicontinuous function* $\tilde{u} : \Omega \to \mathbf{R}$ *such that* $u = \tilde{u}$ *a.e. in* Ω.

Proof of Lemma 3. We shall split the proof into two steps.

Step 1. Proof of (4.4) and (4.5).

By the hypotheses on u and ∇u, we can take the gradient of u appearing in the p-Laplacian Δ_p in the distribution sense. Then, it follows from a standard argument that $\Delta_p u = \Delta_p(u^+) - \Delta_p(u^-)$ in $\mathscr{D}'(\Omega)$. In fact, we see that for any $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}
\langle \Delta_p u, \varphi\rangle &= \langle \mathrm{div}\,|\,(|\nabla u|^{p-2}\nabla(u^+ - u^-)), \varphi\rangle \\
&= -\int_\Omega (|\nabla u|^{p-2}(\nabla u^+ - \nabla u^-)\cdot \nabla\varphi \\
&= -\int_\Omega |\nabla u^+|^{p-2}\nabla u^+ \cdot \nabla\varphi + \int_\Omega |\nabla u^-|^{p-2}\nabla u^- \cdot \nabla\varphi \\
&= \langle \Delta_p(u^+) - \Delta_p(u^-), \varphi\rangle.
\end{aligned}$$

Hence we may assume that $u \ge 0$ a.e. in Ω from now on.

Since u is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^{\infty}(\Omega)$ satisfying the conditions 1 and 2 in Definition 1.1. Since u^+ is also admissible, we may assume $u_n \geq 0$ by taking u_n^+ for u_n, namely:

(1) $0 \leq u_n \to u$ a.e. in Ω, $u_n \to u$ in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$ as $n \to \infty$.

(2) $\Delta_p u_n \in L^1_{\mathrm{loc}}(\Omega)$ $(n = 1, 2, \dots)$ and

$$\sup_n |\Delta_p u_n|(\omega) < \infty \qquad \text{for every } \omega \subset\subset \Omega. \tag{4.6}$$

For $k > 0$ fixed, we have $T_k(u_n) \in W^{1,p}_{loc}(\Omega)$ and

$$\nabla T_k(u_n) = \chi_{[|u_n|<k]} \nabla u_n, \tag{4.7}$$

where $\chi_{[|u_n|<k]}$ denotes the characteristic function of the set $[|u_n| < k]$.

Given $\omega \subset\subset \omega' \subset\subset \Omega$, let $\varphi \in C_0^{\infty}(\omega')$ be such that $0 \leq \varphi \leq 1$ in ω' and $\varphi \equiv 1$ on ω. First, using (4.7) and integrating by parts, we have

$$\begin{aligned} I &= \int |\nabla T_k(u_n)|^p \varphi^p = \int |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T_k(u_n) \varphi^p \\ &= -\int T_k(u_n) \operatorname{div}(\varphi^p |\nabla u_n|^{p-2} \nabla u_n) \qquad (\Delta_p u_n \in L^1_{\mathrm{loc}}(\Omega), |\nabla u_n| \in L^{p^*}_{\mathrm{loc}}(\Omega)) \\ &= -\int T_k(u_n) \Delta_p u_n \varphi^p - \int T_k(u_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi^p \\ &= J_1 + J_2. \end{aligned}$$

Since $T_k(u_n) \leq k$, we have

$$|J_1| = \left| \int T_k(u_n) \Delta_p u_n \varphi^p \right| \leq k \int |\Delta_p u_n| \varphi^p, \tag{4.8}$$

and

$$|J_2| \leq p \int |\nabla T_k(u_n)|^{p-1} T_k(u_n) |\nabla \varphi| \varphi^{p-1} \leq pk \int |\nabla u_n|^{p-1} |\nabla \varphi|. \tag{4.9}$$

From (4.8) and (4.9) we have

$$I \leq k\left(\int |\Delta_p u_n| \varphi^p + p \int |\nabla u_n|^{p-1} |\nabla \varphi| \right).$$

In particular,

$$\int_{\omega} |\nabla T_k(u_n)|^p \leq Ck\left(\int_{\omega'} |\Delta_p u_n| + \|\nabla \varphi\|_{L^{\infty}} \int_{\omega'} |\nabla u_n|^{p-1} \right). \tag{4.10}$$

It follows from the condition 1 on u_n and the statement 2 of Lemma 8 that we have

$$\int_{\omega'} | |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u| \to 0 \qquad \text{as } n \to \infty. \tag{4.11}$$

Therefore we see as $n \to \infty$

$$\int \Delta_p u_n \varphi = -\int |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \to -\int |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int \Delta_p u \varphi, \tag{4.12}$$

that is,

$$\Delta_p u_n \to \Delta_p u \qquad \text{in } \mathscr{D}'(\Omega) \text{ as } n \to \infty. \tag{4.13}$$

Together with $\sup_n |\Delta_p u_n|(\omega') < \infty$ and $|\Delta_p u|(\omega') < \infty$ for any $\omega' \subset\subset \Omega$, we see

$$\Delta_p u_n \to \Delta_p u \qquad \text{in the sense of measures.} \tag{4.14}$$

By the weak compactness of Radon measures and the uniqueness of weak limit, we also have

$$\lim_{n\to\infty} |\Delta_p u_n| = |\Delta_p u| \qquad \text{in the sense of measures.}$$

Letting $n \to \infty$, we conclude that $T_k(u) \in W^{1,p}_{loc}(\omega)$ and the inequality (4.5) holds.

Step 2. Under the assumptions of the Lemma 1, there exists a function $\tilde{u} : \Omega \mapsto \mathbf{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.

Proof of Step 2. By (4.4), for each $k > 0$ there exists $\widetilde{T_k(u)} : \Omega \mapsto \mathbf{R}$ quasicontinuous such that $T_k(u) = \widetilde{T_k(u)}$ a.e. in Ω. Let $v_k := \frac{1}{k} T_k(u)$, so that $v_k \to 0$ as $k \to \infty$ in $L^q(\Omega)$, $\forall q \in [1, \infty)$. In fact, we see that

$$\begin{cases} \text{when} \quad q = 1, \int_\Omega |v_k| = \frac{1}{k}\int_\Omega |T_k(u)| \le \frac{1}{k}\int_\Omega |u| \to 0 \quad (k \to \infty), \\ \text{when} \quad q > 1, \int_\Omega |v_k|^q = \frac{1}{k^q}\int_\Omega |T_k(u)|^{q-1}|T_k(u)| \le \frac{1}{k}\int_\Omega |u| \to 0 \quad (k \to \infty). \end{cases}$$

By (4.5), we see that

$$\int_\omega |\nabla v_k|^p \to 0 \qquad (k \to \infty)\ \forall \omega \subset\subset \Omega.$$

In particular, $v_k \to 0$ in $W^{1,p}_{loc}(\Omega)$, which implies that there exists a subset $P \subset \Omega$ with 0 p-capacity such that

$$\tilde{v}_k(x) = \frac{1}{k}\widetilde{T_k(u)}(x) \to 0, \qquad \forall x \in \Omega \backslash P.$$

We conclude that

$$C_p\left(\left\{|\widetilde{T_k(u)}| > \frac{k}{2}\right\}, \Omega\right) = C_p\left(\left\{\widetilde{|v_k|} > \frac{1}{2}\right\}, \Omega\right) \to 0 \qquad (k \to \infty). \tag{4.15}$$

Set

$$w(x) := \begin{cases} \sup_{k \in N}\{\widetilde{T_k(u)}(x)\} & \text{if } \sup_{k \in N}|\widetilde{T_k(u)}(x)| < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{4.16}$$

so that $w = u$ a.e. in Ω. By (4.15) and the quasicontinuity of the functions $\widetilde{T_k(u)}$, it is easy to see that w is quasicontinuous in Ω. This concludes the proof of the Lemma 3. □

5. Kato's inequalities when $\Delta_p u$ is a Radon measure

We retain the same notations as in the previous section. Since $T_k|_{\mathbf{R}_+}$ is concave, by the standard L^1-version of Kato's inequality (see [6, 11, 13]) we have the following lemma.

Lemma 4. *Assume that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\Delta_p v \in L^1_{\mathrm{loc}}(\Omega)$ and $v \geq 0$ a.e. in Ω. Then, we have*

$$\Delta_p(T_k(v)) \leq t_k(v)\Delta_p v \qquad \text{in } D'(\Omega), \tag{5.1}$$

where the function $t_k : \mathbf{R}_+ \mapsto \mathbf{R}$ is given by

$$t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases} \tag{5.2}$$

Proof. Let $\{\Phi_n\}$ be a sequence of smooth concave functions in $\mathbf{R}$ such that $\Phi_n(t) = t$ if $t \leq k$ and $|\Phi_n(t) - k| \leq 1/n$ if $t > k$. In particular, $0 \leq \Phi_n' \leq 1$ in $\mathbf{R}$. Then we define

$$\Phi_{n,\eta}(t) = \Phi_n(t) + \eta t \qquad \text{for } \eta > 0. \tag{5.3}$$

We may assume that v is smooth by the approximation argument. By a direct calculation

$$\begin{aligned}\Delta_p(\Phi_{n,\eta}(v)) &= \Phi_{n,\eta}'(v)^{p-1}\Delta_p v + (p-1)\Phi_{n,\eta}'(v)^{p-2}\Phi_{n,\eta}''(v)|\nabla v|^p \\ &\leq \Phi_{n,\eta}'(v)^{p-1}\Delta_p v \qquad (\Phi_{n,\eta}'' = \Phi_n'' \leq 0 \text{ by concavity of } \Phi_n)\end{aligned}$$

Letting $\eta \to 0$, we clearly have

$$\Delta_p(\Phi_n(v)) \leq \Phi_n'(v)^{p-1}\Delta_p v \qquad \text{in } \Omega. \tag{5.4}$$

As $n \to \infty$, we finnally get the inequality (5.1). □

Then we prove the following lemma which is due to Ancona [1] and Brezis-Ponce [6], if $p = 2$.

Lemma 5. *Let $\Omega \subset \mathbf{R}^N$ be an open set. Assume that Q satisfies* [$\mathbf{Q_0}$] *and* [$\mathbf{Q_1}$]. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ if $p = 2$ and let $u \in W^{1,p^*}_{\mathrm{loc}}(\Omega)$ if $p \neq 2$. Assume that $u \geq 0$ a.e. in Ω, $Q(u) \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta_p u$ is a Radon measure on Ω. Moreover if $p \neq 2$, assume that u is admissible in the sense of Definition* 1.1. *Then,*

$$\Delta_p(T_k(u)) \qquad \text{is a Radon measure } \forall k > 0.$$

Moreover, for any $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. in Ω, we have

$$\Delta_p(T_k(u)) - aQ(T_k(u)) \leq (\Delta_p u - aQ(u))^+. \tag{5.5}$$

Proof of Lemma 5. We shall establish this lemma in the case where $p \neq 2$, using the same notation as in the proof of Lemma 3. Since $u = u^+$ is admissible in $W^{1,p^*}_{\mathrm{loc}}(\Omega)$, there exists a nonnegative sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfying the conditions 1 and 2 in Definition 1.1. When $p = 2$ holds, the same argument with Lemma 2 works by replacing u_n by u^n; a sequence of mollifications of u defined by (1.6) in Remark 1.2. By Lemma 4, we have

$$\Delta_p(T_k(u_n)) \leq t_k(u_n)\Delta_p u_n \qquad \text{in } D'(\Omega), \forall n, \tag{5.6}$$

where the function $t_k : \mathbf{R}_+ \mapsto \mathbf{R}$ is given by (5.2). Since $T_k(s) \geq t_k(s)s$, $\forall s \geq 0$, $u_n \geq 0$ and $a \geq 0$ a.e. in Ω, it follows from (5.6) that

$$\begin{aligned} \Delta_p T_k(u_n) - aQ(T_k(u_n)) &\leq t_k(u_n)\Delta_p u_n - aQ(T_k(u_n)) \\ &\leq t_k(u_n)(\Delta_p u_n - aQ(u_n)) \\ &\leq (\Delta_p u_n - aQ(u_n))^+ \qquad \text{in } \mathscr{D}'(\Omega). \end{aligned} \tag{5.7}$$

In other worlds, we have for $\forall \varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω

$$\int \Delta_p(T_k(u_n))\varphi - aQ(T_k(u_n))\varphi \leq \int (\Delta_p u_n - aQ(u_n))^+\varphi, \tag{5.8}$$

and we have

$$\int \Delta_p(T_k(u_n))\varphi - aQ(T_k(u_n))\varphi = -\int |\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) \cdot \nabla\varphi + aQ(T_k(u_n))\varphi.$$

Note that $a \in L^\infty$ and $T_k(u) \in W^{1,p}_{loc}(\Omega)$ by Lemma 3. Letting $n \to 0$ we get

$$\text{the left side of (5.8)} = -\int |\nabla T_k(u)|^{p-2}\nabla T_k(u) \cdot \nabla\varphi + aQ(T_k(u))\varphi. \tag{5.9}$$

In the proof of Lemma 3 we showed that $\Delta_p u_n \to \Delta_p u$ in $\mathscr{D}'(\Omega)$ and $|\Delta_p u_n| \to |\Delta_p u|$ on any open set $\omega' \subset\subset \Omega$ in the sense of measures as $n \to \infty$, where $|\Delta_p u|$ denotes the total measure of $\Delta_p u$. Therefore, by letting $n \to \infty$ we have

$$\int \Delta_p(T_k(u))\varphi - aQ(T_k(u))\varphi \leq \int (\Delta_p u - aQ(u))^+ \varphi. \tag{5.10}$$

So that, $\Delta_p(T_k(u))$ is a Radon measure. $\square$

LEMMA 6. *Assume the same hypotheses of Lemma 5. Let $a \in L^1_{\mathrm{loc}}(\Omega)$, $a \geq 0$ a.e. in Ω. Assume that*

$$-\Delta_p u + aQ(u) \geq 0 \qquad \text{in } \Omega \text{ in the sense of measures,}$$

i.e.,

$$\int_E \Delta_p u \leq \int_E aQ(u) \qquad \text{for every Borel set } E \subset\subset \Omega.$$

Then

$$-\Delta_p(T_k(u)) + aQ(T_k(u)) \geq 0 \qquad \text{in } \mathscr{D}'(\Omega),\ \forall k > 0. \tag{5.11}$$

PROOF OF LEMMA 6. By the preceding Lemma applied with $a_i := T_i(a)$, where i is a positive integer, we know that

$$\Delta_p(T_k(u)) - a_i Q(T_k(u)) \leq (\Delta_p u - a_i Q(u))^+ \qquad \text{in } \mathscr{D}'(\Omega). \tag{5.12}$$

On the other hand, from (5.10), for every Borel set $E \subset \Omega$, we get

$$\int_E \Delta_p u - a_i Q(u) \leq \int_E (a - a_i)Q(u). \tag{5.13}$$

Since $(a - a_i)Q(u) \geq 0$ a.e. in Ω, for every Borel set $E \subset \Omega$, (5.12) implies that

$$0 \leq \int_E (\Delta_p u - a_i Q(u))^+ \leq \int_E (a - a_i)Q(u). \tag{5.14}$$

Hence, $(\Delta_p u - a_i Q(u))^+$ is a nonnegative measure, which is absolutely continuous with respect to Lebesgue measure. Therefore, we have

$$(\Delta_p u - a_i Q(u))^+ \in L^1(\Omega) \qquad \forall i = 1, 2, \ldots. \tag{5.15}$$

We now return to (5.13) to conclude that

$$0 \leq (\Delta_p u - a_i Q(u))^+ \leq (a - a_i)Q(u) \qquad \text{a.e. in } \Omega.$$

In particular,

$$(\Delta_p u - a_i Q(u))^+ \downarrow 0 \qquad a.e.\ in\ \Omega\ as\ i \to \infty. \tag{5.16}$$

It follows from (5.15) and (5.16) that

$$(\Delta_p u - a_i Q(u))^+ \to 0 \qquad in\ L^1(\Omega)\ as\ i \to \infty. \tag{5.17}$$

Finally, for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω by (5.12) and (5.17) we have

$$\int \Delta_p(T_k(u))\varphi - aQ(T_k(u))\varphi \leq \int (\Delta_p u - a_i Q(u))^+ \varphi \to 0 \qquad (i \to \infty). \tag{5.18}$$

Then we conclude

$$-\Delta_p(T_k(u)) + aQ(T_k(u)) \geq 0 \qquad in\ \mathscr{D}'(\Omega)\ \forall k > 0. \qquad \square$$

6. End of Proof of Theorem 1

It follows from Lemma 1 that, under the hypotheses of theorem, there exists $\tilde{u} : \Omega \mapsto \mathbf{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω. Let us assume that $\tilde{u} = 0$ on a set of positive capacity $E \subset \Omega$. We shall prove that $u = 0$ a.e. in Ω. We split the proof into two steps:

Step 1.

Claim. Under the hypotheses of the theorem, if we assume in addition that $u \in L^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Proof. Since $u \in L^\infty(\Omega)$, we have $aQ(u) \in L^1_{\mathrm{loc}}(\Omega)$. It follows from (1.10) and Remark 2.1(2) that

$$-\Delta_p u + aQ(u) \geq 0 \qquad \text{in } \mathscr{D}'(\Omega). \tag{6.1}$$

By Lemma 2 and Lemma 3, we know that $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $\forall k > 0$. Since $\sup_k\{T_k(u)\} = u$ holds, we have $u \in W^{1,p}_{loc}(\Omega)$ as well. Therefore it follows from Remark 1.2(2) that u is automatically admissible when $p = 2$. Then, for any $p \in (1, \infty)$ we can choose a nonnegative sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfying the conditions 1 and 2 in Definition 1.1. We set

$$-\Delta_p u + aQ(u) = -\Delta_p u_n + aQ(u_n) + f_n + g_n, \tag{6.2}$$

where $f_n = \Delta_p u_n - \Delta_p u$ and $g_n = a(Q(u) - Q(u_n))$. Since $aQ(u) \in L^1_{\mathrm{loc}}(\Omega)$ and $u \in L^\infty(\Omega)$, we see that $g_n \to 0$, (as $n \to \infty$) in $L^1_{\mathrm{loc}}(\Omega)$, and it follows from (4.13) that $f_n \to 0$ (as $n \to \infty$) in $\mathscr{D}'(\Omega)$. Then we have the next lemma.

Lemma 7. *Under the hypotheses of the theorem, if we assume in addition that $u \in L^\infty(\Omega)$, then for any $\delta > 0$ we have*

(1) $\lim_{n\to\infty} \int \frac{f_n}{(\delta + u_n)^{p-1}} \varphi = 0$ for any φ in $C_0^\infty(\Omega)$.

(2) $\lim_{n\to\infty} \int \frac{g_n}{(\delta + u_n)^{p-1}} \varphi = 0$ for any φ in $C_0^\infty(\Omega)$.

Proof. Since $g_n \to 0$ in $L^1(\Omega)$, the assertion 2 is clear. By a direct calculation, we have for any n

$$\begin{aligned}\int \frac{f_n}{(\delta + u_n)^{p-1}} \varphi &= \int \frac{-\Delta_p u_n + \Delta_p u}{(\delta + u_n)^{p-1}} \varphi \\ &= \int (-|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi \frac{1}{(\delta + u_n)^{p-1}} \\ &\quad + \int (-|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \nabla u_n \frac{\varphi}{(\delta + u_n)^p} (1 - p). \end{aligned} \tag{6.3}$$

Here we recall that $u = T_k(u) \in W^{1,p}_{loc}(\Omega)$, if $k \geq \|u\|_{L^\infty}$. Hence by Young's nequality we have

$$\begin{aligned}\left| \int \frac{f_n}{(\delta + u_n)^{p-1}} \varphi \right| &\leq \int (|\nabla u_n|^{p-1} + |\nabla u|^{p-1}) |\nabla \varphi| \frac{1}{(\delta + u_n)^{p-1}} \\ &\quad + \int (|\nabla u_n|^p + |\nabla u|^{p-1} |\nabla u_n|) |\varphi| \frac{p-1}{(\delta + u_n)^p} \\ &\leq C(\delta, \varphi) \int_{\operatorname{supp} \varphi} (|\nabla u|^p + |\nabla u_n|^p) < \infty, \end{aligned} \tag{6.4}$$

where $C(\delta, \varphi)$ denotes a positive number independent of n. Then the assertion 1 follows from the dominated convergence theorem (see e.g. [9]; Section 4 in Chapter II). □

Let $\omega \subset\subset \omega' \subset\subset \Omega$ and let $0 \leq \varphi \in C_0^\infty(\omega')$ with $\varphi \geq 1$ on ω. Multiplying (6.2) by $\varphi^p/(u_n + \delta)^{p-1}$ with $0 \leq \varphi \in C_0^\infty(\omega')$, we get

$$\frac{\varphi^p \Delta_p u_n}{(\delta + u_n)^{p-1}} \leq \frac{aQ(u_n)\varphi^p}{(\delta + u_n)^{p-1}} + \frac{(f_n + g_n)\varphi^p}{(\delta + u_n)^{p-1}} \qquad \text{in } \omega', \ \forall n. \tag{6.5}$$

Then we have

$$\begin{aligned}\int \frac{|\nabla u_n|^p}{(\delta+u_n)^p}\varphi^p &= \int \frac{|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n \varphi^p}{(\delta+u_n)^p}\\ &= \frac{1}{1-p}\int \varphi^p |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\left(\frac{1}{(u_n+\delta)^{p-1}}\right)\\ &= \frac{1}{p-1}\int \frac{\Delta_p u_n}{(u_n+\delta)^{p-1}}\varphi^p + \frac{p}{p-1}\int \frac{\varphi^{p-1}\nabla\varphi\cdot|\nabla u_n|^{p-2}\nabla u_n}{(u_n+\delta)^{p-1}}\\ &\le \frac{1}{p-1}\int \frac{aQ(u_n)\varphi^p}{(\delta+u_n)^{p-1}} + \frac{p}{p-1}\int \frac{|\nabla\varphi|\,|\nabla u_n|^{p-1}\varphi^{p-1}}{(u_n+\delta)^{p-1}}\\ &\quad + \frac{1}{p-1}\int \frac{(f_n+g_n)\varphi^p}{(\delta+u_n)^{p-1}}.\end{aligned}$$

By Young's inequality and Lemma 7, for any $\eta > 0$, there is some $C_\eta \ge 0$ such that

$$\int \frac{|\nabla\varphi|\,|\nabla u_n|^{p-1}\varphi^{p-1}}{(u_n+\delta)^{p-1}} \le \eta\int \frac{|\nabla u_n|^p\varphi^p}{(\delta+u_n)^p} + C_\eta\int|\nabla\varphi|^p \qquad (\forall\eta>0, \forall n).$$

Hence there exists a positive number C independent of n such that we have

$$\begin{aligned}\int \frac{|\nabla u_n|^p\varphi^p}{(\delta+u_n)^p} &\le C\left(\int \frac{aQ(u_n)\varphi^p}{(\delta+u_n)^{p-1}} + \int|\nabla\varphi|^p + \int\frac{(f_n+g_n)\varphi^p}{(\delta+u_n)^{p-1}}\right)\\ &\le C\left(\int a\varphi^p + \int|\nabla\varphi|^p + \int\frac{(f_n+g_n)\varphi^p}{(\delta+u_n)^{p-1}}\right) \qquad ([\mathbf{Q_0}],[\mathbf{Q_1}], u\in L^\infty). \quad (6.6)\end{aligned}$$

Since $\nabla u_n/(u_n+\delta) = \nabla\log(\delta+u_n) = \nabla\log(u_n/\delta+1)$, the estimate above may be rewritten as

$$\int_\Omega \left|\nabla\log\left(1+\frac{u_n}{\delta}\right)\right|^p\varphi^p \le C\int_\Omega\left(a\varphi^p + |\nabla\varphi|^p + \frac{(f_n+g_n)\varphi^p}{(\delta+u_n)^{p-1}}\right), \qquad (6.7)$$

$$\log\left(\frac{u_n}{\delta}+1\right) \in W^{1,p}_{loc}(\Omega) \qquad \forall\delta>0,\ \forall n.$$

Letting $n\to\infty$ we have

$$\int\left|\nabla\log\left(\frac{u}{\delta}+1\right)\right|^p\varphi^p \le C\int(a\varphi^p + |\nabla\varphi|^p) \qquad \forall\varphi\in C_0^\infty(\Omega). \qquad (6.8)$$

Let $E \subset \Omega$ be a set of positive capacity such that $\tilde{u} = 0$ on E. Without any loss of generality, we may assume that $E \subset \omega \subset\subset \omega' \subset\subset \Omega$ and ω is an open connected. We have

$$\int_\omega \left|\nabla \log\left(\frac{u}{\delta} + 1\right)\right|^p \le C \int_\Omega (a\varphi^p + |\nabla\varphi|^p). \tag{6.9}$$

Since the quasicontinuous representative $\log(\widetilde{u/\delta + 1}) = \log(\tilde{u}/\delta + 1)$ of $\log(u/\delta + 1)$ equals 0 on $E \subset \Omega$ with $C_p(E, \Omega) > 0$, it follows from a variant of Poincare's inequality that there exists positive number C (depending only on E and Ω) such that

$$\int_\omega \left|\log\left(\frac{u}{\delta} + 1\right)\right|^p \le C \int a\varphi^p + |\nabla\varphi|^p \qquad \forall\delta > 0. \tag{6.10}$$

In particular, the integral in the left-hand side remains bounded as $\delta \downarrow 0$. On the other hand,

$$\log\left(\frac{u}{\delta} + 1\right)^p \to +\infty \qquad a.e.\ in\ \omega \backslash \{u = 0\}\ as\ \delta \downarrow 0. \tag{6.11}$$

By (6.10) and (6.11), we conclude that $u = 0$ a.e. in ω. Since ω is an arbitrary connected neighborhood of E in ω', we conclude that $u = 0$ a.e. in Ω. $\square$

STEP 2. From Lemma 5, we know that $\Delta_p(T_1(u))$ is a Radon measure and

$$-\Delta_p(T_1(u)) + aQ(T_1(u)) \ge 0 \qquad in\ \mathscr{D}'(\Omega).$$

We note that $\Delta_p(T_1(u_n) \to \Delta_p(T_1(u))$ in $D'(\Omega)$ and $aQ(T_1(u_n)) \to aQ(T_1(u))$ in $L^1_{\mathrm{loc}}(\Omega)$ as $n \to \infty$. In addition, $\widetilde{T_1(u)} = T_1(u) = 0$ on $E \subset \Omega$ with $C_p(E, \Omega) > 0$. By Setp 1, we have $T_1(u) = 0$ a.e. in Ω and so $u = 0$ a.e. in Ω. After all we have the desired result. $\square$

7. Appendix

PROOF OF THE PROPERTIES (1.8) AND (1.9). We prove that $u \in W_0^{1,p}(\Omega)$ is admissible if $\Delta_p u$ is a Radon measure on Ω. Let $w_n \in W_0^{1,p}(\Omega)$ be the unique weak solution of the boundary value problem for the monotone operator Δ_p (see e.g. [16]): For $n = 1, 2, \dots,$

$$\begin{cases} \Delta_p w_n = \operatorname{div} F_\rho^n & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

where $F = |\nabla u|^{p-2}\nabla u \in (L^1(\Omega))^N$ and $F_\rho^n = (|\nabla u|^{p-2}\nabla u)_\rho^n \in (C^\infty(\mathbf{R}^N))^N$ is a mollification of F defined by (1.6). Let us set $\omega \subset\subset \Omega$ and $\Delta_p u = \operatorname{div} F = \mu$.

Since $|\mu|(\omega) < \infty$, we see $\operatorname{div} F_p^n = (\operatorname{div} F)_p^n = (\Delta_p u)_p^n = \mu_p^n$ in ω provided that n is sufficiently large. Hence we clearly have for every $\omega \subset\subset \Omega$

$$|\Delta w_n|(\omega) = |\operatorname{div} F_p^n|(\omega) = |\mu_p^n|(\omega) \to |\mu|(\omega) \qquad \text{as } n \to \infty.$$

This proves (1.9). Next we show (1.8), that is:

$$w_n \to u \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{7.2}$$

We need the next elementary lemma, see e.g. [5, 12].

Lemma 8. *Let* $1 < p < \infty$.

(1) *There exist positive constants* $c_1(p)$ *and* $c_2(p)$ *depending on* p *such that for every* $\xi, \eta \in \mathbf{R}^N$ *we have*

$$|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} \le c_1(p)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta). \tag{7.3}$$

In particular if $p > 2$ *we have*

$$|\xi - \eta|^p \le c_1(p)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta), \tag{7.4}$$

and if $1 < p < 2$ *we have for any* $\varepsilon \in (0, 1)$

$$|\xi - \eta|^p \le c_2(p)\varepsilon^{(p-2)/p}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) + \varepsilon(|\xi| + |\eta|)^p. \tag{7.5}$$

(2) *There exist positive constants* $d_1(p)$, $d_2(p)$ *and* $d_3(p)$ *depending on* p *such that for every* $\xi, \eta \in \mathbf{R}^N$ *we have*

$$\begin{cases} \big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \le d_1(p)|\xi - \eta|(|\xi| + |\eta|)^{p-2} & \text{if } p \ge 2, \\ \big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \le d_2(p)|\xi - \eta|^{p-1} & \text{if } 1 < p < 2. \end{cases} \tag{7.6}$$

In particular if $p > 2$, *then we have for any* $\varepsilon \in (0, 1)$,

$$\big| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \big| \le d_3(p)\varepsilon^{-1/(p-2)}|\xi - \eta|^{p-1} + \varepsilon(|\xi| + |\eta|)^{p-1}. \tag{7.7}$$

First we treat the case where $p \ge 2$: By using $w_n - u \in W_0^{1,p}(\Omega)$ as a test function, we have

$$\begin{aligned} -\langle \Delta_p w_n - \Delta_p u, w_n - u \rangle &= \int (|\nabla w_n|^{p-2}\nabla w_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(w_n - u) \\ &\ge c_1(p)^{-1} \int |\nabla(w_n - u)|^p \qquad \text{(by (7.4))}. \end{aligned} \tag{7.8}$$

In the left-hand side, using Young's inequality for $\delta > 0$ we have

$$\begin{aligned} &-\langle \Delta_p w_n - \Delta_p u, w_n - u \rangle \\ &\quad = \int (F_p^n - F) \cdot \nabla(w_n - u) \\ &\quad \le C(\delta) \int |F_p^n - F|^{p'} + \delta \int |\nabla(w_n - u)|^p \qquad \text{for some } C(\delta) > 0. \end{aligned} \tag{7.9}$$

Noting $|F_p^n|^{p'}$ and $|F|^{p'}$ with $p' = p/(p-1)$ are bounded in $L^1(\Omega)$, it follows from (7.6) and the dominated convergence theorem we see that $w_n \to u$ in $W_0^{1,p}(\Omega)$. Then, taking a subsequence if necessary, $\{w_n\}$ satisfies the property $w_n \to u$, $\nabla w_n \to \nabla u$ a.e. as $n \to \infty$.

Lastly we treat the case where $1 < p < 2$. In this case the proof can be done using (7.5) instead of (7.4) in a quite similar way. Hence we omit the detail.

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