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Chow groups of Châtelet surfaces over dyadic fields

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Abstract. A cubic Châtelet surface X over a p-adic field K is a typical surface whose Chow group $A_0(X)$ of degree-zero zero-cycles varies depending on fine conditions of the defining equation. Many researchers have computed $A_0(X)$ by a number-theoric method in many cases. We extend their computation and determine the structure of $A_0(X)$ in some new cases. It turns out that $A_0(X)$ behaves rather unexpectedly when X is defined by $y^2 - dz^2 = x(x^2 - e)$ for some $d, e \in K^* \backslash K^{*2}$ and its splitting field is wildly ramified.

1. Introduction

Let K be a perfect field. Let X be a smooth projective model of the surface

$$y^2 - dz^2 = f(x) \tag{1.1}$$

in $\mathbf{A}_K^3$, where $d \in K^*$ and $f(x) \in K[x]$ is a monic cubic separable polynomial. This is called a cubic *Châtelet surface*. Denote by $A_0(X)$ the degree-zero part of the Chow group of zero-cycles on X modulo rational equivalence.

Denote by K^{*2} the group of squares in K^*. If $d \in K^{*2}$, then X is birational to $\mathbf{P}_K^2$, and therefore $A_0(X) \cong A_0(\mathbf{P}_K^2) \cong \{0\}$ since $A_0(X)$ is a birational invariant of a smooth projective and geometrically integral surface over a perfect field ([2, Proposition 6.3]). In particular, if K is algebraically closed, then $A_0(X) \cong \{0\}$. In general, $A_0(X)$ is a 2-torsion group ([2, Proposition 6.6]).

Suppose that K is a local field of characteristic 0. Then $A_0(X)$ is finite ([1, Corollary 3.5]). It depends on arithmetical and geometrical properties of X whether $A_0(X)$ vanishes or not. It is known that $A_0(X)$ is equinumerous to the set of R-equivalence classes of K-rational points on X ([2, Remarques 6.7 (iv)]).

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The subject of this article is determination of the group $\mathrm{A}_0(X)$ under the assumptions that $d \notin K^{*2}$ and K is a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, where p is a prime number. The computation of $\mathrm{A}_0(X)$ is reduced to a number-theoric problem by Colliot-Thélène and Sansuc in [3], [10]. All of the following results rely on their method. We shall recall it in Section 2.

When $f(x)$ is irreducible, it is shown by Pisolkar in [9, Theorem 1.4] that $\mathrm{A}_0(X)$ is trivial. When $f(x)$ splits into three linear factors, the group $\mathrm{A}_0(X)$ is completely determined by Colliot-Thélène and Dalawat in [5, Proposition 4.7], [6, Section 4 and Proposition 2], [7, Proposition 3].

Henceforce, we consider the remaining case:

$$f(x) = x(x^2 - e) \qquad \text{with } e \in K^* \backslash K^{*2}.$$

Put $L = K(\sqrt{d})$ and $E = K(\sqrt{e})$. We call L the *splitting field* of X, since L is a unique minimal extension of K such that $X \times_K L$ is birational to $\mathbf{P}^2_L$. Let $v_K : K^* \to \mathbf{Z}$ be the normalized valuation of K. The following theorem is proven by Pisolkar.

Theorem 1 ([9, Theorems 1.1–1.3]). (1) *If* $L \cong E$, *then* $\mathrm{A}_0(X) \cong \{0\}$. *Henceforward, suppose* $L \not\cong E$.

(2) *If* $p \neq 2$, *then* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

(3) *If* $p = 2$ *and* L/K *is unramified, then*

$$\mathrm{A}_0(X) \cong \begin{cases} \{0\} & \text{if } v_K(e) \equiv 0 \pmod 4, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } v_K(e) \equiv 1, 3 \pmod 4. \end{cases}$$

(4) *Suppose* $K = \mathbf{Q}_2$. *If* L/K *is unramified and* $v_K(e) \equiv 2 \pmod 4$, *or if* L/K *is ramified, then* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

Theorem 1 (3) is stated for $K = \mathbf{Q}_2$ in [9], but her proof works under the assumption $p = 2$. Our first result extends Theorem 1 (4) to $K \neq \mathbf{Q}_2$.

Theorem 2. *Suppose* $L \not\cong E$.

(1) *If* L/K *is unramified and* $v_K(e) \equiv 2 \pmod 4$, *then* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

(2) *Assume* $p = 2$. *If* $L/\mathbf{Q}_2$ *is totally ramified and the conductor of* L/K *(Definition* 1*) has the different parity from* $v_K(e)$, *then* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

Our second result touches on the case when the conductor of L/K has the same parity as $v_K(e)$.

Theorem 3. *Suppose that* $K = \mathbf{Q}_2(\sqrt{2})$, $L \not\cong E$, L/K *is ramified and* $v_K(d)$ *is even. Put* $m = v_K(e)$ *and take* $\varepsilon_1 \in 1 + 2\mathbf{Z}_2$ *and* $\varepsilon_2 \in \mathbf{Z}_2$ *such that* $e = \sqrt{2}^m(\varepsilon_1 + \varepsilon_2\sqrt{2})$. *Then* $\mathrm{A}_0(X) \cong \{0\}$ *only in the cases in the following table, and otherwise* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

	d mod K^{*2}	m mod 4	ε_1 mod 8	ε_2 mod 4
(i)	-1 *or* 3	0	1	2
(ii)	-1 *or* 3	0	5	2
(iii)	3	2	1	2
(iv)	-1	2	3	0
(v)	-1 *or* 3	2	5	2

Each case in this table depends on the residues of d, m, ε_1 and ε_2 modulo K^{*2}, $4\mathbf{Z}$, $8\mathbf{Z}_2$ and $4\mathbf{Z}_2$ respectively. We can find $d, e \in K^*\backslash K^{*2}$ such that $L \not\cong E$ under each condition of (i)–(v). *For instance, $d = 3$, $e = 1 + 2\sqrt{2} \in K^*$ satisfy $d, e \notin K^{*2}$, $L \not\cong E$ and* (i). *For such d and e, although $L \not\cong E$ and L/K is ramified, $\mathrm{A}_0(X)$ is trivial.*

Theorems 2 and 3 are proven in Sections 3 through 6. Theorems 1 (4) and 3 show that the structure of $\mathrm{A}_0(X)$ depends on the base field K of X when L/K is wildly ramified, in contrast to the case where $f(x)$ is irreducible or splits into three linear factors.

From now on, we use the following notation: for a local field k, denote by v_k its normalized valuation, by $\mathfrak{p}_k$ its maximal ideal, by U_k its unit group, by $U_k^{(i)}$ its i-th unit group for each integer $i > 0$, by κ_k its residue field, and define the *quadratic Hilbert symbol* $(a,b)_k \in \{\pm 1\}$ by $(a,b)_k = 1 \Leftrightarrow a \in N_{k(\sqrt{b})/k}k(\sqrt{b})^*$ for each $a, b \in k^*$.

2. Computational method

Let X be a Châtelet surface defined as above. Theorems 1 through 3 are proven based on the following method.

Theorem 4 ([3], [10]). *The group $\mathrm{A}_0(X)$ is isomorphic to the image of the map*

$$M := \{x \in K \mid x(x^2 - e) \in N_{L/K}L\} \to (K^*/N_{L/K}L^*)^2$$

$$x \mapsto \begin{cases} ([x], [x^2 - e]) & \text{if } x \neq 0, \\ ([-e], [-e]) & \text{if } x = 0, \end{cases}$$

where $[a]$ is the class of a in $K^*/N_{L/K}L^*$ for any $a \in K^*$.

This is proven by Colliot-Thélène and Sansuc in [3], [10]. Because of its importance, we shall briefly recall the outline of the proof.

Proof (outline). Let $X(K)$ be the set of K-rational points on X. Define the map $\chi : X(K) \to (K^*/N_{L/K}L^*)^2$ by compositing the following maps.

- The natural surjective map ([2, Théorème C, Remarques 6.7 (iv)])

$$X(K) \to \mathrm{A}_0(X); \qquad P \mapsto P - O,$$

where O is a singular point on the fiber above ∞ with respect to the morphism $\varphi : X \to \mathbf{P}^1_K$ associated with the function $(x, y, z) \mapsto x$ on the surface (1.1).

- The canonical injective homomorphism ([1], [4, Theorem 2])

$$\mathrm{A}_0(X) \to \mathrm{H}^1(K, \mathrm{Hom}(\mathrm{Pic}\,\overline{X}, \overline{K}^*)),$$

where $\overline{K}$ is an algebraic closure of K and $\mathrm{Pic}\,\overline{X}$ is the Picard group of $X \times_K \overline{K}$.

- The canonical isomorphism ([3, Théorème 5])

$$\mathrm{H}^1(K, \mathrm{Hom}(\mathrm{Pic}\,\overline{X}, \overline{K}^*)) \to (K^*/N_{L/K}L^*) \times (E^*/N_{LE/E}LE^*).$$

- The isomorphism $E^*/N_{LE/E}LE^* \to K^*/N_{L/K}L^*$ induced by $N_{E/K}$.

The map χ factors through M, since $N_{L/K}(y + \sqrt{d}z) = y^2 - dz^2$ holds for any $y, z \in K$ and all the points on each fiber of φ are mutually rationally equivalent. Computing the explicit description of the induced map $M \to (K^*/N_{L/K}L^*)^2$, we obtain the desired result. □

The following conditions (A) and (B) are used repeatedly later as criterions for determining the structure of $\mathrm{A}_0(X)$.

Corollary 1 ([9, Lemma 2.1, Corollary 2.2]). *We have*

$$\mathrm{A}_0(X) \cong \begin{cases} \{0\}, & \text{if both (A) and (B) hold,} \\ \mathbf{Z}/2\mathbf{Z}, & \text{otherwise,} \end{cases}$$

where the conditions (A) *and* (B) *are given as follows.*

(A) $x \in N_{L/K}L^*$ *or* $x^2 - e \in N_{L/K}L^*$ *for any* $x \in K^*$.

(B) $-e \in N_{L/K}L^*$.

Proof ([9]). Supposing $x \in K^*$ and $x(x^2 - e) \in N_{L/K}L^*$, then $x \in N_{L/K}L^*$ if and only if $x^2 - e \in N_{L/K}L^*$. This implies the desired result. □

Remark 1. *Let π be a uniformizer of K. The group $\mathrm{A}_0(X)$ is invariant up to isomophism under replacing d and e with $d' = d\lambda^2$ and $e' = \pi^{4m}e$ for any $\lambda \in K^*$ and $m \in \mathbf{Z}$, since the affine surface $y^2 - dz^2 = x(x^2 - e)$ is isomorphic to $y'^2 - d'z'^2 = x'(x'^2 - e')$ by the change of variables $x' = \pi^{2m}x$, $y' = \pi^{3m}y$ and $z' = \lambda^{-1}\pi^{3m}z$.*

3. Unramified case

In this section, we prove Theorem 2 (1). Assume $v_K(e) \equiv 2 \pmod 4$ and $A_0(X) \cong \{0\}$. We will show $L \cong E$. We can write $e = \pi^{4m+2}\varepsilon$ for some uniformizer π of K, $m \in \mathbf{Z}$ and $\varepsilon \in U_K$. It suffices to show $d\varepsilon \in K^{*2}$. By the perfectness of the Hilbert symbol, it is reduced to showing $(d\varepsilon, \pi)_K = (d\varepsilon, \xi)_K = 1$ for any $\xi \in U_K$.

By the assumption that L/K is unramified and the local class field theory, we have $N_{L/K}U_L = U_K$. This means

$$(d, \xi)_K = 1 \tag{2.1}$$

for any $\xi \in U_K$. By the assumption $d \notin K^{*2}$, this implies

$$(d, \pi)_K = -1. \tag{2.2}$$

By the assumption $A_0(X) \cong \{0\}$, the condition (A) in Corollary 1 holds. This means that

$$v_K(x) \in 2\mathbf{Z} \quad \text{or} \quad v_K(x^2 - e) \in 2\mathbf{Z} \qquad \text{for any } x \in K^*, \tag{2.3}$$

since $v_K : K^* \to \mathbf{Z}$ induces the isomorphism $K^*/N_{L/K}L^* \to \mathbf{Z}/2\mathbf{Z}$ by the assumption. We find

$$v_K(u^2 - \varepsilon) \in 2\mathbf{Z} \qquad \text{for any } u \in U_K \tag{2.4}$$

by applying (2.3) to $x = \pi^{2m+1}u$.

We claim that

$$a^2 - \varepsilon b^2 \neq c \qquad \text{for any } a, b, c \in K^* \text{ such that } v_K(c) \text{ is odd.} \tag{2.5}$$

To prove (2.5), assume the existence of $a, b, c \in K^*$ such that $v_K(c)$ is odd and $a^2 - \varepsilon b^2 = c$. Then $u := ab^{-1}$ satisfies

$$v_K(u^2 - \varepsilon) = v_K\left(\frac{c}{b^2}\right) = v_K(c) - 2v_K(b) \notin 2\mathbf{Z},$$

$$2v_K(u) = v_K(u^2) = v_K\left(\frac{c}{b^2} + \varepsilon\right) = \min\left\{v_K\left(\frac{c}{b^2}\right), 0\right\} = 0,$$

and therefore $u \in U_K$. This contradicts to (2.4).

For any $\xi \in U_K$, we have

$$(\varepsilon, \pi)_K = (\varepsilon, \pi^{-1}\xi)_K = -1, \tag{2.6}$$

by applying (2.5) to $c = \pi$, $\pi^{-1}\xi$, and therefore

$$(d\varepsilon, \pi)_K = (d, \pi)_K (\varepsilon, \pi)_K = (-1)^2 = 1,$$

$$(d\varepsilon, \xi)_K = (d, \xi)_K (\varepsilon, \pi)_K (\varepsilon, \pi^{-1}\xi)_K = 1 \cdot (-1)^2 = 1$$

by (2.1), (2.2) and (2.6). This concludes the proof of Theorem 2 (1).

4. Lemmas

4.1. In this subsection, let L/K be a cyclic totally ramified extension of local fields such that the characteristic of the residue field $\kappa := \kappa_L = \kappa_K$ is $p := [L:K]$. Let σ be a generator of the Galois group of L/K. Let π_L be a uniformizer of L.

Definition 1. The *conductor* of L/K is the integer $v_L(\pi_L^{\sigma-1} - 1) + 1$.

Remark 2. *It does not depend on the choices of σ and π_L. Furthermore, it coincides with the minimal integer $i > 0$ such that $U_K^{(i)} \subset N_{L/K}L^*$.*

We quote the following proposition from the discrete valuation field theory.

Proposition 1 ([8, (1.5), Chapter 3]). *Under the assumption as above, let t be the conductor of L/K. Take $\eta \in U_L$ such that $\pi_L^{\sigma-1} = 1 + \pi_L^{t-1}\eta$. Put $\pi_K = N_{L/K}(\pi_L)$ and $\bar{\eta} = \eta \bmod \mathfrak{p}_L$. For any integer $i \geq 1$, consider the map $N_{L/K} : U_L^{(i)}/U_L^{(i+1)} \to U_K^{(i)}/U_K^{(i+1)}$ induced by $N_{L/K} : U_L^{(i)} \to U_K^{(i)}$ and isomorphisms*

$$U_L^{(i)}/U_L^{(i+1)} \to \kappa; 1 + \alpha\pi_L^i \mapsto \alpha \bmod \mathfrak{p}_L, \qquad \mathrm{Frob}_p : \kappa \to \kappa; \theta \mapsto \theta^p,$$

$$U_K^{(i)}/U_K^{(i+1)} \to \kappa; 1 + a\pi_K^i \mapsto a \bmod \mathfrak{p}_K, \qquad \psi : \kappa \to \kappa; \theta \mapsto \theta^p - \bar{\eta}^{p-1}\theta.$$

Suppose $1 \leq i < t-1$. Then the following diagrams commute.

$$\begin{array}{ccc} U_L^{(i)}/U_L^{(i+1)} & \xrightarrow{\cong} & \kappa \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}_p} \\ U_K^{(i)}/U_K^{(i+1)} & \xrightarrow[\cong]{} & \kappa \end{array} \qquad \begin{array}{ccc} U_L^{(t-1)}/U_L^{(t)} & \xrightarrow{\cong} & \kappa \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \psi} \\ U_K^{(t-1)}/U_K^{(t)} & \xrightarrow[\cong]{} & \kappa \end{array} \qquad (3.1)$$

The following lemma plays an important role in Sections 5 and 6.

Lemma 1. *Suppose that K is a totally ramified extension of $\mathbf{Q}_2$. If L/K is a quadratic ramified extension of the conductor t, then $N_{L/K}L^* \cap U_K^{(t-1)} \subset U_K^{(t)}$.*

Proof. Since L/K and $K/\mathbf{Q}_2$ are totally ramified, we have

$$\kappa_L = \kappa_K = \kappa_{\mathbf{Q}_2} = \mathbf{F}_2. \qquad (3.2)$$

This implies $U_L = U_L^{(1)}$, $U_K = U_K^{(1)}$ and $N_{L/K}L^* \cap U_K^{(t-1)} = N_{L/K}U_L^{(1)} \cap U_K^{(t-1)}$. Thus, it suffices to show for any integer $1 \le i \le t-1$ that

$$N_{L/K}U_L^{(i)} \cap U_K^{(t-1)} \subset N_{L/K}U_L^{(i+1)}. \tag{3.3}$$

(1) Case: $i < t-1$. The left diagram of (3.1) yields (3.3).

(2) Case: $i = t-1$. The map ψ in (3.1) is the zero map by (3.2), since $\bar{\eta} = 1$ by (3.2). Therefore, $N_{L/K} : U_L^{(t-1)}/U_L^{(t)} \to U_K^{(t-1)}/U_K^{(t)}$ is also the zero map, since the horizontal maps in the right diagram of (3.1) are isomorphisms. This implies (3.3). □

4.2. To prove Theorem 3, we also use the following lemma due to Pisolkar. Her proof works without assuming that $K = \mathbf{Q}_2$.

Lemma 2 ([9, Lemma 6.2]). *Suppose $L \not\cong E$. If $-1 \notin N_{L/K}L^*$, or if $e \notin N_{L/K}L^*$, then* $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$.

5. Certain ramified case

In this section, we give a proof of Theorem 2 (2) based on the idea due to Pisolkar [9]. Let t be the conductor of L/K and put $m = v_K(e)$. To prove $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$, it is sufficient to show the existence of $x \in K^*$ such that x, $x^2 - e \notin N_{L/K}L^*$ by Corollary 1. Take a unit $u \in U_K^{(t-1)} \setminus U_K^{(t)}$ and a uniformizer π of K such that $\pi \in N_{L/K}L^*$. By the assumption, $n := m - t + 1$ is even. Put $x = \pi^{n/2}u$. Since $\pi^{n/2} \in N_{L/K}L^*$ and $u \notin N_{L/K}L^*$ by Lemma 1, we have $x \notin N_{L/K}L^*$. Writing $u = 1 + \pi^{t-1}u'$ for some $u' \in U_K$, we have

$$\begin{aligned} v_K\left(u^2 - \frac{e}{\pi^n} - 1\right) &= v_K\left(2\pi^{t-1}u' + \pi^{2(t-1)}u'^2 - \frac{e}{\pi^n}\right) \\ &= \min\{[K : \mathbf{Q}_2] + t - 1, 2(t-1), t-1\} = t-1, \end{aligned}$$

and therefore $u^2 - e\pi^{-n} \in U_K^{(t-1)} \setminus U_K^{(t)}$. By using Lemma 1 again, we have $u^2 - e\pi^{-n} \notin N_{L/K}L^*$ and therefore $x^2 - e = \pi^n(u^2 - e\pi^{-n}) \notin N_{L/K}L^*$. This proves Theorem 2 (2).

6. Another example

In this section, we prove Theorem 3. The base field $K = \mathbf{Q}_2(\sqrt{2})$ of X is a quadratic ramified extension of $\mathbf{Q}_2$ with a uniformizer $\sqrt{2}$ and unit group

$$U_K = U_K^{(1)} = \{a + b\sqrt{2} \mid a \in 1 + 2\mathbf{Z}_2, b \in \mathbf{Z}_2\}. \tag{6.1}$$

Nontrivial elements of K^*/K^{*2} are in one-to-one correspondence with quadratic extensions of K by the map $xK^{*2} \mapsto K(\sqrt{x})$. Elementary arguments yield that K/K^{*2} is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^4$, and generated by residue classes of $\sqrt{2}$, -1, 3 and $1-\sqrt{2}$. By the assumption that $v_K(d)$ is even and L/K is ramified, we have

$$d \equiv -1,\ 3,\ \pm(1-\sqrt{2}),\ \pm 3(1-\sqrt{2}) \pmod{K^{*2}},$$

since $K(\sqrt{-3})/K$ is unramified. By Remark 1, it suffices to consider these six values for d.

Case 1: $d = \pm(1-\sqrt{2})$ **or** $\pm 3(1-\sqrt{2})$**.** In this case, by the projection formula and the explicit formula over $\mathbf{Q}_2$ for the Hilbert symbols, we have

$$(-1, d)_K = (-1, N_{K/\mathbf{Q}_2}(d))_{\mathbf{Q}_2} = (-1,-1)_{\mathbf{Q}_2} = -1. \tag{6.2}$$

This implies $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$ by Lemma 2.

Case 2: $d = -1$ **or** 3**.** In this case, since $(1+\sqrt{2}+\sqrt{d})\sqrt{2}^{-1}$ is a uniformizer of L, we can directly show that the conductor of L/K is 2. Therefore we have

$$N_{L/K}L^* \cap U_K = U_K^{(2)} = \{a + b\sqrt{2} \mid a \in 1 + 2\mathbf{Z}_2, b \in 2\mathbf{Z}_2\} \tag{6.3}$$

by (6.1) and Lemma 1. By a similar calculation as (6.2), we have

$$\sqrt{2} \notin N_{L/K}L^* \quad \text{if } d = -1 \qquad \text{and} \qquad \sqrt{2} \in N_{L/K}L^* \quad \text{if } d = 3. \tag{6.4}$$

Note that the following cases are excluded.

(vi) $\varepsilon_1 \equiv 1 \pmod 8$, $\varepsilon_2 \equiv 0 \pmod 4$.

(vii) $\varepsilon_1 \equiv 3 \pmod 8$, $\varepsilon_2 \equiv 0 \pmod 4$ and $d = 3$.

(viii) $\varepsilon_1 \equiv 7 \pmod 8$, $\varepsilon_2 \equiv 0 \pmod 4$ and $d = -1$.

Indeed, by the assumptions that $e \notin K^{*2}$, $L \ncong E$, m is even and $d \in \{-1, 3\}$, the units ε, $-\varepsilon$ and 3ε do not belong to K^{*2} and therefore to

$$U_K^{(5)} = \{a + b\sqrt{2} \mid a \in 1 + 8\mathbf{Z}_2, b \in 4\mathbf{Z}_2\} = \exp(\mathfrak{p}_K^5) = \exp(2\mathfrak{p}_K^3) = U_K^{(3)} \cap K^{*2}.$$

Case 2.1: m **is odd.** In this case, we have $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$ by Theorem 2 (2).

Case 2.2: m **is even.** Put $r = \sqrt{2}^{m/2}$ and $\varepsilon = \varepsilon_1 + \varepsilon_2\sqrt{2}$. Then we have $e = r^2\varepsilon$, $\varepsilon_1 \in 1 + 2\mathbf{Z}_2$ and $\varepsilon_2 \in \mathbf{Z}_2$.

Case 2.2.1: $\varepsilon_2 \equiv 1 \pmod 2$**.** In this case, we have $\varepsilon \in U_K \backslash U_K^{(2)}$. This implies $\varepsilon \notin N_{L/K}L^*$ by (6.3) and therefore $e = r^2\varepsilon \notin N_{L/K}L^*$. Thus, $\mathrm{A}_0(X) \cong \mathbf{Z}/2\mathbf{Z}$ by Lemma 2.

Case 2.2.2: $\varepsilon_2 \equiv 0 \pmod 2$**.** In this case, we have $-\varepsilon \in U_K^{(2)}$. Therefore, the condition (B) in Corollary 1 holds by (6.3). We consider whether (A) holds or not. Take $x \in K^*$ and write $x = r\sqrt{2}^n u$, $u = a + b\sqrt{2}$ for some $n \in \mathbf{Z}$, $u \in U_K$, $a \in 1 + 2\mathbf{Z}_2$ and $b \in \mathbf{Z}_2$.

- Supposing $n \neq 0$, then

$$r^{-2}(x^2 - e) = (2^n a^2 + 2^{n+1} b^2 - \varepsilon_1) + (2^{n+1} ab - \varepsilon_2)\sqrt{2} \in U_K^{(2)} \quad \text{if } n > 0,$$
$$2^{|n|} r^{-2}(x^2 - e) = (a^2 + 2b^2 - 2^{|n|}\varepsilon_1) + (2ab - 2^{|n|}\varepsilon_2)\sqrt{2} \in U_K^{(2)} \quad \text{if } n < 0,$$

  and therefore $x^2 - e \in N_{L/K}L^*$ by (6.3) since $2 \in K^{*2}$.
- Supposing $n = 0$, then $x^2 - e \in N_{L/K}L^*$ if and only if $u^2 - \varepsilon \in N_{L/K}L^*$, since $x^2 - e = r^2(u^2 - \varepsilon)$.

Therefore, $\mathrm{A}_0(X) \cong \{0\}$ if and only if either ru or $u^2 - \varepsilon$ belongs to $N_{L/K}L^*$ for any $u \in U_K$. Set

$$U^* = \begin{cases} U_K \backslash U_K^{(2)} & \text{if } d = -1 \text{ and } m \equiv 0 \pmod 4, \text{ or if } d = 3, \\ U_K^{(2)} & \text{if } d = -1 \text{ and } m \equiv 2 \pmod 4. \end{cases} \tag{6.5}$$

Then $\mathrm{A}_0(X) \cong \{0\}$ if and only if

$$u^2 - \varepsilon \in N_{L/K}L^* \qquad \text{for any } u \in U^*. \tag{6.6}$$

Indeed, $ru \in N_{L/K}L^*$ for any $u \in U_K \backslash U^*$, since $r \in N_{L/K}L^*$ if and only if $d = -1$ and $m \equiv 0 \pmod 4$, or if $d = 3$ by (6.4). For each $u = a + b\sqrt{2} \in U^*$ $(a \in 1 + 2\mathbf{Z}_2, b \in \mathbf{Z}_2)$, put

$$i(u) = \mathrm{ord}_2(a^2 + 2b^2 - \varepsilon_1), \qquad j(u) = \mathrm{ord}_2(2ab - \varepsilon_2).$$

Then a unit part of $u^2 - \varepsilon$ is written by

$$\frac{u^2 - \varepsilon}{2^i} = \frac{a^2 + 2b^2 - \varepsilon_1}{2^i} + \frac{2ab - \varepsilon_2}{2^i}\sqrt{2} \qquad \text{if } i = i(u) \le j(u),$$
$$\frac{u^2 - \varepsilon}{2^j\sqrt{2}} = \frac{2ab - \varepsilon_2}{2^j} + \frac{a^2 + 2b^2 - \varepsilon_1}{2^{j+1}}\sqrt{2} \qquad \text{if } i(u) > j(u) = j.$$

Therefore, by (6.3) and (6.4), we obtain the following criterions:

$$\left.\begin{array}{ll} i(u) < j(u) & \Rightarrow u^2 - \varepsilon \in N_{L/K}L^*, \\ i(u) = j(u) & \Rightarrow u^2 - \varepsilon \notin N_{L/K}L^*, \\ i(u) = j(u) + 1 \text{ and } d = -1 & \Rightarrow u^2 - \varepsilon \in N_{L/K}L^*, \\ i(u) = j(u) + 1 \text{ and } d = 3 & \Rightarrow u^2 - \varepsilon \notin N_{L/K}L^*, \\ i(u) > j(u) + 1 \text{ and } d = -1 & \Rightarrow u^2 - \varepsilon \notin N_{L/K}L^*, \\ i(u) > j(u) + 1 \text{ and } d = 3 & \Rightarrow u^2 - \varepsilon \in N_{L/K}L^*. \end{array}\right\} \tag{6.7}$$

The relation among functions i, j and $j+1$ is compiled in the following table.

U^*	ε_1 mod 8	$\varepsilon_2 \equiv 0\ (4)$	$\varepsilon_2 \equiv 2\ (4)$
$U_K \backslash U_K^{(2)}$	1	(excluded)	$i < j$ (i), (iii)
	3	$i > j+1$	$i(u) = j(u)$ for some u
	5	$i = j$	$i < j$ (ii), (v)
	7	$i = j+1$	$i(u) = j(u)$ for some u
$U_K^{(2)}$	1	(excluded)	$i > j+1$
	3	$i < j$ (iv)	$i = j$
	5	$i(u) = j(u)$ for some u	$i = j+1$ (v)
	7	(excluded)	$i = j$

Indeed, the relation in each case can be shown by a similar way as follows.

- If $U^* = U_K \backslash U_K^{(2)}$, $\varepsilon_1 \equiv 1, 5 \pmod 8$ and $\varepsilon_2 \equiv 2 \pmod 4$, then

$$a^2 + 2b^2 - \varepsilon_1 \equiv 2 \pmod 4, \qquad 2ab - \varepsilon_2 \equiv 0 \pmod 4$$

for any $u = a + b\sqrt{2} \in U^*$ $(a, b \in 1 + 2\mathbf{Z}_2)$, and therefore $i(u) < j(u)$.

- Suppose $U^* = U_K \backslash U_K^{(2)}$, $\varepsilon_1 \equiv 3 \pmod 8$ and $\varepsilon_2 \equiv 2 \pmod 4$. Then the equations

$$a^2 + 2b^2 - \varepsilon_1 \equiv 2ab - \varepsilon_2 \equiv 8 \pmod{16}$$

have a common solution $(a, b) \in (1 + 2\mathbf{Z}_2)^2$. Indeed, one of the solutions is given by the following table which depends on the residues of ε_1 and ε_2 modulo 16.

ε_1	ε_2	a	b
3	2	3	7
3	6	3	5
3	10	3	3
3	14	3	1

ε_1	ε_2	a	b
11	2	1	5
11	6	1	7
11	10	1	1
11	14	1	3

For such a and b, the unit $u := a + b\sqrt{2} \in U^*$ satisfies $i(u) = j(u)$.

- If $U^* = U_K^{(2)}$, $\varepsilon_1 \equiv 1 \pmod 8$ and $\varepsilon_2 \equiv 2 \pmod 4$, then

$$a^2 + 2b^2 - \varepsilon_1 \equiv 0 \pmod 8, \qquad 2ab - \varepsilon_2 \equiv 2 \pmod 4,$$

for any $u = a + b\sqrt{2} \in U^*$ $(a \in 1 + 2\mathbf{Z}_2,\ b \in 2\mathbf{Z}_2)$, and therefore $i(u) > j(u) + 1$.

- Suppose $U^* = U_K^{(2)}$, $\varepsilon_1 \equiv 5 \pmod 8$ and $\varepsilon_2 \equiv 0 \pmod 4$. Then the equations

$$a^2 + 2b^2 - \varepsilon_1 \equiv 2ab - \varepsilon_2 \equiv 4 \pmod 8$$

have a common solution $(a, b) \in (1 + 2\mathbf{Z}_2) \times 2\mathbf{Z}_2$, for example,

$$(a,b) = \begin{cases} (1,2) & \text{if } \varepsilon_2 \equiv 0 \pmod 8, \\ (1,4) & \text{if } \varepsilon_2 \equiv 4 \pmod 8. \end{cases}$$

For such a and b, the unit $u := a + b\sqrt{2} \in U^*$ satisfies $i(u) = j(u)$. Thus, only in the cases (i)–(v) in the statement, the condition (6.6) holds by (6.5) and (6.7), and therefore $A_0(X) \cong \{0\}$. Recall that the cases (vi)–(viii) are excluded by the assumption $L \not\cong E$.

Under each condition of (i)–(v), we can find d, $e \in K^* \backslash K^{*2}$ such that $L \not\cong E$ by elementary arguments. This completes the proof of Theorem 3.

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