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Biharmonic hypersurfaces in Riemannian symmetric spaces I

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Abstract. We classify biharmonic geodesic spheres in the Cayley projective plane. Our results completes the classification of all biharmonic homogeneous hypersurfaces in simply connected compact Riemannian symmetric spaces of rank 1. In addition we show that complex Grassmannian manifolds, and exceptional Lie groups F_4 and G_2 admit proper biharmonic real hypersurfaces.

1. Introduction

Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds. A smooth map $\phi : M \to N$ is said to be *harmonic* if it is a critical point of the *energy functional*

$$E(\phi) = \int \frac{1}{2} |\mathrm{d}\phi|^2 \mathrm{d}v_g$$

under compactly supported variations. The Euler-Lagrange equation of this variational problem is

$$\tau(\phi) = \mathrm{tr}_g(\nabla \mathrm{d}\phi) = 0,$$

where $\nabla \mathrm{d}\phi$ is the second fundamental form of ϕ. Here the vector field $\tau(\phi)$ along ϕ is called the *tension field* of ϕ (see [11]).

In case ϕ is an isometric immersion, $\tau(\phi)$ is a constant multiple of the mean curvature vector field of ϕ. Thus an isometric immersion ϕ is harmonic if and only if it is minimal.

In some mapping spaces, harmonic maps do not exist. For instance, Eells and Wood [12] showed that the space $\mathrm{Map}_1(\mathbf{T}^2, \mathbf{S}^2)$ of smooth maps from a 2-torus $\mathbf{T}^2$ into a 2-sphere $\mathbf{S}^2$ with degree 1 does not contain harmonic maps. To find a good representative of any homotopy class in such a mapping space,

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alternative geometric variational problem would be proposed. As one of the candidate, bienergy functional has been studied extensively.

For a smooth map ϕ, its *bienergy functional* is defined by

$$E_2(\phi) = \int \frac{1}{2} |\tau(\phi)|^2 \mathrm{d}v_g.$$

Critical points of the bienergy functional are called *biharmonic maps*. Clearly, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called *proper biharmonic maps*.

Many examples of biharmonic immersions into spheres have been obtained. For more informations on biharmonic maps, we refer to the reader [23].

Since the sphere is a compact Riemannian symmetric space of rank 1, biharmonic immersions into compact Riemannian symmetric spaces would be of some interest.

The first example of proper biharmonic immersions into compact Riemannian symmetric space of rank 1 other than sphere was discovered by the second named author of the present paper. In [26], he classified biharmonic Lagrangian surfaces of constant mean curvature in complex projective plane. Next, Ichiyama, Urakawa and the first named author of the present paper gave some explicit examples of proper biharmonic (real) hypersurfaces in complex projective space as well as quaternion projective space [15]–[16].

In [13], Fetcu, Loubeau, Montaldo and Oniciuc studied biharmonic submanifolds in complex projective spaces.

As far as the authors know, no examples of biharmonic immersions into compact Riemannian symmetric spaces of rank *greater than* 1 are known.

The purpose of this paper is to provide *new examples* of proper biharmonic hypersurfaces in Riemannian symmetric spaces.

Firstly, we shall show that the complex Grassmannian manifolds $\mathrm{Gr}_k(\mathbf{C}^n)$ with $2 \leq k < n$ admit proper biharmonic real hypersurfaces.

Secondly, we determine all the biharmonic homogeneous hypersurfaces in the Cayley projective plane. Our result together with [15]–[16] gives a classification of *all* biharmonic homogeneous hypersurfaces in compact Riemannian symmetric spaces of rank 1.

Furthermore, we study biharmonic hypersurfaces in the exceptional Lie groups F_4 and G_2. We shall show that these compact Lie groups contain proper biharmonic hypersurfaces.

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Throughout the paper we denote by $\Gamma(E)$ the space of all smooth sections of a vector bundle E.

2. Preliminaries

2.1. Let (M^m, g) and $(N^n, \tilde{g})$ be Riemannian manifolds and $\phi : M \to N$ a smooth map. Then the map ϕ induces a vector bundle $\phi^* TN$ over M by

$$\phi^* TN = \bigcup_{p \in M} T_{\phi(p)} N,$$

where TN is the tangent bundle of N. A section of $\phi^* TN$ is called a *vector field along* ϕ.

The Levi-Civita connection $\tilde{\nabla}$ of N induces a unique connection ∇^ϕ on $\phi^* TN$ which satisfies the condition

$$\nabla^\phi_X (V \circ \phi) = (\tilde{\nabla}_{\mathrm{d}\phi(X)} V) \circ \phi,$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(TN)$, see [11, p. 4].

The *second fundamental form* $\nabla \mathrm{d}\phi$ of ϕ is defined by

$$(\nabla \mathrm{d}\phi)(Y, X) = \nabla^\phi_X \, \mathrm{d}\phi(Y) - \mathrm{d}\phi(\nabla_X Y), \qquad X, Y \in \Gamma(TM). \tag{1}$$

Here ∇ denotes the Levi-Civita connection of M. One can see that the second fundamental form is symmetric.

Definition 1. Let $\phi : (M, g) \to (N, \tilde{g})$ be a smooth map. The *tension field* $\tau(\phi)$ of ϕ is a section of $\phi^* TN$ defined by

$$\tau(\phi) = \mathrm{tr}_g (\nabla \mathrm{d}\phi).$$

2.2. A smooth map $\phi : (M, g) \to (N, \tilde{g})$ is said to be *harmonic* if it is a critical point of the *energy functional*:

$$E(\phi) = \int \frac{1}{2} |\mathrm{d}\phi|^2 \mathrm{d}v$$

under compactly supported variations. The Euler-Lagrange equation of this variational problem is

$$\tau(\phi) = 0.$$

More generally, a smooth map ϕ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\phi) = \int \frac{1}{2} |\tau(\phi)|^2 \mathrm{d}v.$$

under compact supported variations. The Euler-Lagrange equation of this variational problem is

$$\Delta_\phi \tau(\phi) + \mathrm{tr}_g \, \tilde{R}(\mathrm{d}\phi, \tau(\phi)) \mathrm{d}\phi = 0. \tag{2}$$

Here $\tilde{R}$ is the Riemannian curvature of N. The operator Δ_ϕ is the *rough Laplacian* acting on the space $\Gamma(\phi^*TN)$ defined by

$$\Delta_\phi := -\sum_{i=1}^{m}\{\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}\},$$

where $\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame field on M.

2.3. Now let us consider an isometric immersion $\phi : (M^m, g) \to (N^{m+1}, \tilde{g})$ of codimension 1, *i.e.*, hypersurface immersion. We choose a local unit normal vector field ξ. Then the second fundamental form $\nabla d\phi$ can be written as $(\nabla d\phi)(X, Y) = h(X, Y)\xi$, where $h \in \Gamma(T^*M \odot T^*M)$ is the function-valued second fundamental form. The *Gauss formula* becomes

$$\nabla^\phi_X \phi_* Y = \phi_*(\nabla_X Y) + h(X, Y)\xi, \qquad X, Y \in \Gamma(TM) \tag{3}$$

and the *Weingarten formula* reads:

$$\tilde{\nabla}_{\phi_* X}\xi = -\phi_*(AX), \qquad X \in \Gamma(TM). \tag{4}$$

The endomorphism field A is called the *shape operator* derived from ξ. The *mean curvature function* H of the hypersurface $\phi : M \to N$ is defined by $H = \frac{1}{m}\operatorname{tr} h$. A hypersurface M is said to be *minimal* if $H = 0$. The mean curvature function and the tension filed $\tau(\phi)$ are related by

$$\tau(\phi) = mH\xi.$$

This formula implies that a hypersurface immersion $\phi : M \to N$ is minimal if and only if it is a harmonic map.

2.4. From Gauss formula and Weingarten formula, we have the *Gauss equation* which describes the relation between the Riemannian curvatures R of M and $\tilde{R}$ of N:

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + h(X, W)h(Y, Z) - h(X, Z)h(Y, W). \tag{5}$$

From (5) we can have the relationship between the Ricci tensor fields Ric of the hypersurface M and the $\widetilde{\mathrm{Ric}}$ of the ambient space N:

$$\widetilde{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) + g(AX, AY) - mHh(X, Y) + \tilde{R}(X, \xi, Y, \xi), \tag{6}$$

and the relationship between the scalar curvatures ρ and $\tilde{\rho}$ is

$$\tilde{\rho} = \rho + |A|^2 - m^2H^2 + 2\,\widetilde{\mathrm{Ric}}(\xi, \xi). \tag{7}$$

In particular if the ambient space N is of constant curvature c, we have

$$\rho = cm(m-1) + m^2H^2 - |A|^2. \tag{8}$$

3. Biharmonic hypersurfaces in Einstein manifolds

3.1. As we have seen before, harmonicity of isometric immersions is equivalent to minimality of isometric immersions. Thus biharmonic isometric immersions are generalizations of minimal immersions.

In [25], Ou obtained the following criterion for biharmonicity of hypersurfaces in Einstein manifolds.

Theorem 1 ([25]). *A hypersurface (M, g) in an Einstein manifold $(N, \tilde{g})$ is biharmonic if and only if its mean curvature function H is a solution to the following PDEs*

$$\Delta H - H|A|^2 + \frac{H}{m+1}\tilde{\rho} = 0, \qquad 2A(\operatorname{grad} H) + \frac{m}{2}\operatorname{grad} H^2 = 0. \tag{9}$$

From this criterion we have the following useful result:

Theorem 2 ([25]). *Let $\phi : (M^m, g) \to (N^{m+1}, \tilde{g})$ $(m \geq 2)$ be a hypersurface with shape operator A in an Einstein manifold N with $\widetilde{\mathrm{Ric}} = \lambda\tilde{g}$. Assume that the mean curvature H of the hypersurface is constant. Then ϕ is biharmonic if and only if either ϕ is minimal or non-minimal with*

$$|A|^2 = \lambda. \tag{10}$$

Furthermore, in the latter case, both the ambient space and the hypersurface must have positive scalar curvatures:

$$\tilde{\rho} = (m+1)\lambda > 0, \qquad \rho = (m-2)\lambda + m^2H^2 > 0.$$

4. Riemannian symmetric spaces

4.1. Hereafter we assume that the ambient space N is an *irreducible* compact Riemannian symmetric space G/K with compact semi-simple G. Let us denote by B the Killing form of G. Then since G is semi-simple, B is negative definite on the Lie algebra $\mathfrak{g}$ of G. Thus $-B$ is a $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Moreover the tangent space T_oN of N at the origin $o = K$ is identified with the orthogonal complement $\mathfrak{p}$ of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$. The orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is a reductive decomposition of $\mathfrak{g}$, that is, $\mathfrak{p}$ satisfies $\mathrm{ad}(\mathfrak{k})\mathfrak{p} \subset \mathfrak{p}$. Moreover, since N is a symmetric space, we have

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The restriction $-B|_{\mathfrak{p}}$ of $-B$ to $\mathfrak{p}$ induces a G-invariant Riemannian metric $\tilde{g}$ on N. This Riemannian metric is called the *Killing metric* of N. The *rank* of a Riemannian symmetric space $N = G/K$ is the maximum dimension of flat totally geodesic submanifold of N.

4.2. The Ricci tensor $\widetilde{\mathrm{Ric}}$ of N with respect to the Killing metric $\tilde{g}$ computed at the origin as follows (see Besse [4, Theorem 7.73], Kobayashi-Nomizu [21]):

$$\widetilde{\mathrm{Ric}}_o = -\frac{1}{2}B|_{\mathfrak{p}}.$$

This formula shows that N is an Einstein manifold. This together with the formula (10) implies the following criterion.

THEOREM 3. *Let $N = G/K$ be an irreducible compact semi-simple Riemannian symmetric space equipped with the Killing metric. Then a hypersurface $\phi : M \to G/K$ with constant mean curvature is proper biharmonic if and only if its shape operator A has constant square norm*

$$|A|^2 = \frac{1}{2}.$$

In [15]–[16], biharmonic homogeneous hypersurfaces in the sphere $\mathbf{S}^n$, the complex projective space $\mathbf{C}P^n$ and the quaternion projective space $\mathbf{H}P^n$ have been classified. From the next section we start our study on biharmonic homogeneous hypersurfaces in other compact Riemannian symmetric spaces.

5. Complex Grassmannian manifolds

5.1. Let us denote by $\mathrm{Gr}_k(\mathbf{C}^m)$ the Grassmannian manifold of all complex linear k-subspaces in complex Euclidean m-space $\mathbf{C}^m$. The Grassmannian manifold $\mathrm{Gr}_k(\mathbf{C}^m)$ is represented by $\mathrm{Gr}_k(\mathbf{C}^m) = \mathrm{SU}(m)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(m-k))$ as a homogeneous space. We equip the Grassmannian manifold with the Killing metric $\tilde{g}$. Then the resulting homogeneous Riemannian space is a real $2k(m-k)$-dimensional compact Riemannian symmetric space of rank $\min(k, m-k)$. Moreover $\mathrm{Gr}_k(\mathbf{C}^m)$ admits a $\mathrm{SU}(m)$-invariant complex structure J which is compatible to the Killing metric. Hence $(\mathrm{Gr}_k(\mathbf{C}^m), \tilde{g}, J)$ is a Hermitian symmetric space of type AIII.

In this section we consider real hypersurfaces in $\mathrm{Gr}_k(\mathbf{C}^m)$ with $m > k > 2$. The case $k = 2$ will be studied in the next section.

5.2. The inclusion $\mathbf{C}^{m-1} \subset \mathbf{C}^m$ induces a totally geodesic imbedding of $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ into $\mathrm{Gr}_k(\mathbf{C}^m)$. Tubes of radius $r < \sqrt{m}\pi$ around $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ are real hypersurfaces. In particular these hypersurfaces are homogeneous and of the form:

$$\mathrm{SU}(m-1)/\mathrm{S}(\mathrm{SO}(1)\times \mathrm{U}(k-1)\times \mathrm{U}(m-k-1)).$$

The principal curvatures $\{\lambda_i\}$ and their multiplicities $\{m_i\}$ of the tube are given by García, Hullet and Sánchez [14] as follows:

$$\lambda_1 = -\frac{1}{\sqrt{m}}\cot\left(\frac{r}{\sqrt{m}}\right), \qquad m_1 = 1,$$

$$\lambda_2 = -\frac{1}{2\sqrt{m}}\cot\left(\frac{r}{2\sqrt{m}}\right), \qquad m_2 = 2(k-1),$$

$$\lambda_3 = \frac{1}{2\sqrt{m}}\tan\left(\frac{r}{2\sqrt{m}}\right), \qquad m_3 = 2(m-k-1),$$

$$\lambda_4 = 0, \qquad m_4 = 2(k-1)(m-k-1)$$

for $r \in (0, \sqrt{m}\pi)$.

Biharmonic tubes around $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ are classified as follows:

Theorem 4. *Let M_r be a tube around $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ of radius r in $\mathrm{Gr}_k(\mathbf{C}^m)$, $2 < k < m$. Then M_r is minimal if and only if the radius is*

$$r = 2\sqrt{m}\tan^{-1}\sqrt{\frac{2k-1}{2m-2k-1}}.$$

The only proper biharmonic tubes M_r around $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ in $\mathrm{Gr}_k(\mathbf{C}^m)$ are the tube of radius

$$r = 2\sqrt{m}\tan^{-1}\sqrt{\frac{m+1\pm\sqrt{(m-2k)^2+4m}}{2m-2k-1}} < \sqrt{m}\pi. \qquad \square$$

Proof. First we look for minimal tubes. The mean curvature H of the tube M_r of radius r around $\mathrm{Gr}_k(\mathbf{C}^{m-1})$ is computed as

$$\begin{aligned}\{2k(m-k)-1\}H &= \lambda_1 + 2(k-1)\lambda_2 + 2(m-k-1)\lambda_3 \\ &= -\frac{1}{\sqrt{m}}\frac{1-t^2}{2t} - \frac{k-1}{\sqrt{m}}\frac{1}{t} + \frac{m-k-1}{\sqrt{m}}t \\ &= -\frac{1}{2\sqrt{m}t}\{-(2m-2k-1)t^2 + 2k-1\},\end{aligned}$$

where we put $t = \tan\{r/(2\sqrt{m})\}$. Thus M_r is minimal if and only if

$$r = 2\sqrt{m}\tan^{-1}\sqrt{\frac{2k-1}{2m-2k-1}} < \sqrt{m}\pi.$$

Next we look for proper biharmonic tubes. The square norm $|A|^2$ is computed as

$$|A|^2 = \lambda_1^2 + 2(k-1)\lambda_2^2 + 2(m-k-1)\lambda_3^2$$
$$= \frac{1}{m}\left(\frac{1-t^2}{2t}\right)^2 + \frac{k-1}{2m}\cdot\frac{1}{t^2} + \frac{m-k-1}{2m}\cdot t^2 = \frac{1}{2}$$

From this we have

$$t^2 = \frac{m+1 \pm \sqrt{D}}{2m-2k-1},$$

where

$$D = (m-2k)^2 + 4m > 0.$$

Since $(m+1)^2 - D = (2k-1)\{2(m-k)-1\} > 0$, we have $m+1-\sqrt{D} > 0$. Thus we obtain

$$r = 2\sqrt{m}\tan^{-1}\sqrt{\frac{m+1 \pm \sqrt{(m-2k)^2+4m}}{2m-2k-1}} < \sqrt{m}\pi. \qquad \square$$

6. Grassmannian manifolds of two-planes

6.1. In this section we study biharmonic real hypersurfaces in the Grassmannian manifold $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ of all 2-planes in the complex Euclidean $(m+2)$-space. The Grassmannian $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ is a real $4m$-dimensional Hermitian symmetric space of rank 2. Note that $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ is a *quaternionic symmetric space*.

Berndt [1] initiated the study of real hypersurfaces in $\mathrm{Gr}_2(\mathbf{C}^{m+2})$. To adapt our computation to [1], in this section, we normalize the metric so that the maximal sectional curvature of $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ is 8. After this normalization the resulting Riemannian symmetric space $(\mathrm{Gr}_2(\mathbf{C}^{m+2}), \tilde{g})$ is a Kähler-Einsten manifold with Ricci tensor field $\widetilde{\mathrm{Ric}} = 4(m+2)\tilde{g}$.

It should be remarked that $\mathrm{Gr}_2(\mathbf{C}^3)$ is the projective space $\mathbf{C}P^2$ equipped with the Fubini-Study metric of constant holomorphic sectional curvature 8. Next, in case $m=2$, $\mathrm{Gr}_2(\mathbf{C}^4)$ is identified with the real Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^6)$ of all *oriented* 2-planes in Euclidean 6-space $\mathbf{R}^6$ because of the isomorphism $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$. Moreover $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^6)$ is identified with the complex quadric $\mathcal{Q}_4$ in the complex projective 5-space. The case $\mathrm{Gr}_2(\mathbf{C}^4) = \widetilde{\mathrm{Gr}}_2(\mathbf{R}^6)$ will be investigated separately in the next section, thus hereafter we assume that $m > 2$.

The proper biharmonicity of constant mean curvature real hypersurfaces can be rephrased as follows:

Proposition 1. *Let M be a real hypersurface in* $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ *with constant mean curvature. Then M is proper biharmonic if and only if the length $|A|$ of the shape operator satisfies $|A|^2 = 4(m+2)$.*

In [1], Berndt studied two remarkable classes of real hypersurfaces in the Grassmannian manifold $\mathrm{Gr}_2(\mathbf{C}^{m+2})$ with $m \geq 3$.

6.2. Let us consider a totally geodesic (Kähler) imbedding of $\mathbf{C}^{m+1}$ into $\mathbf{C}^{m+2}$. Then this imbedding induces a totally geodesic Kähler imbedding $\mathrm{Gr}_2(\mathbf{C}^{m+1}) \subset \mathrm{Gr}_2(\mathbf{C}^{m+2})$. Take a tube $M = M_r$ around $\mathrm{Gr}_2(\mathbf{C}^{m+1})$ of radius $r \in (0, \pi/(2\sqrt{2}))$. Then M is a real hypersurface of $\mathrm{Gr}_2(\mathbf{C}^{m+2})$. The tube has at most four constant principal curvatures

$$\alpha = 2\sqrt{2}\cot(2\sqrt{2}r), \qquad \beta = \sqrt{2}\cot(\sqrt{2}r), \qquad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \qquad \mu = 0.$$

The multiplicities of these principal curvatures are

$$m_\alpha = 1, \qquad m_\beta = 2, \qquad m_\lambda = m_\mu = 2m-2.$$

Note that in case $r = \pi/(4\sqrt{2})$, $\alpha = \gamma = 0$, the number of distinct principal curvatures is 3.

Theorem 5. *The tube M_r around* $\mathrm{Gr}_2(\mathbf{C}^{m+1})$ *of radius*

$$r = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{3}}{\sqrt{2m-1}}$$

is minimal for any $m \geq 3$.

The tubes M_r around $\mathrm{Gr}_2(\mathbf{C}^{m+1})$ *of radius*

$$r = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{m+3 \pm \sqrt{m^2+12}}{2m-1}}$$

are proper biharmonic for any $m \geq 3$.

Proof. The mean curvature H of M_r is computed as

$$\begin{aligned}(4m-1)H &= \alpha + 2\beta + (2m-2)\lambda \\ &= \sqrt{2}\{3 - (2m-1)\tan^2(\sqrt{2}r)\}/\tan(\sqrt{2}r).\end{aligned}$$

Hence M_r is minimal if and only if the radius of M is

$$r = \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{\frac{3}{2m-1}} < \frac{\pi}{2\sqrt{2}}.$$

Next the square norm $|A|^2$ of the shape operator is

$$\begin{aligned}|A|^2 &= \alpha^2 + 2\beta^2 + (2m-2)\lambda^2 \\ &= \frac{2}{\tan^2(\sqrt{2}r)}\{(2m-1)\tan^4(\sqrt{2}r) - 2\tan^2(\sqrt{2}r) + 3\}.\end{aligned}$$

Thus M_r is proper biharmonic if and only if $|A|^2 = 4(m+2)$, that is,

$$(2m-1)\tan^4(\sqrt{2}r) - 2(m+3)\tan^2(\sqrt{2}r) + 3 = 0.$$

From this we get

$$\tan^2(\sqrt{2}r) = \frac{m+3 \pm \sqrt{m^2+12}}{2m-1} > 0.$$

Hence

$$r = \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{\frac{m+3 \pm \sqrt{m^2+12}}{2m-1}} < \frac{\pi}{2\sqrt{2}}. \qquad \square$$

6.3. The *quaternion projective space* $\mathbf{H}P^n$ is the manifold of all quaternion lines through the origin in the quaternion linear space $\mathbf{H}^{n+1}$. Since $\mathbf{H}^{n+1} \cong \mathbf{C}^{2n+2}$ as a right complex linear space, every element of $\mathbf{H}P^n = \mathrm{Gr}_1(\mathbf{H}^{n+1})$ is regarded as an element of $\mathrm{Gr}_2(\mathbf{C}^{2n+2})$. This induces a totally geodesic imbedding $\mathbf{H}P^n \subset \mathrm{Gr}_2(\mathbf{C}^{m+2})$ with $m = 2n$. The tube $M = M_r$ around $\mathbf{H}P^n$ of radius $r \in (0, \pi/4)$ is a real hypersurface that has five constant principal curvatures

$$\alpha = -2\tan(2r), \qquad \beta = 2\cot(2r), \qquad \gamma = 0, \qquad \lambda = \cot r, \qquad \mu = -\tan r$$

with multiplicities

$$m_\alpha = 1, \qquad m_\beta = 3, \qquad m_\gamma = 3, \qquad m_\lambda = m_\mu = 4n-4.$$

Theorem 6. *The tube around* $\mathbf{H}P^n$ *of radius*

$$r = \tan^{-1}\frac{2\sqrt{n}-1}{\sqrt{4n-1}} < \frac{\pi}{4}$$

is minimal in $\mathrm{Gr}_2(\mathbf{C}^{2n+2})$.

The only biharmonic tubes around $\mathbf{H}P^n$ *in* $\mathrm{Gr}_2(\mathbf{C}^{2n+2})$ *are the minimal ones.*

PROOF. The mean curvature H is

$$
\begin{aligned}
(8n-1)H &= \alpha + 3\beta + (4n-4)(\lambda + \mu) \\
&= -2\tan(2r) + 6\cot(2r) + (4n-4)(\cot r - \tan r) \\
&= \frac{1}{t(1-t^2)}\{-4t^2 + 3(1-t^2)^2 + 4(n-1)(1-t^2)^2\},
\end{aligned}
$$

where $t = \tan r$. Thus M_r is minimal if and only if the radius r of M satisfies

$$
r = \tan^{-1}\sqrt{\frac{4n+1 \pm 4\sqrt{n}}{4n-1}} = \tan^{-1}\frac{2\sqrt{n} \pm 1}{\sqrt{4n-1}}.
$$

Here we notice that

$$
0 < \tan^{-1}\frac{2\sqrt{n}-1}{\sqrt{4n-1}} < \frac{\pi}{4} < \tan^{-1}\frac{2\sqrt{n}+1}{\sqrt{4n-1}}.
$$

Next we look for proper biharmonic tubes. The square norm $|A|^2$ of the shape operator is computed as

$$
\begin{aligned}
|A|^2 &= \alpha^2 + 3\beta^2 + (4n-4)(\lambda^2 + \mu^2) \\
&= 4\tan^2(2r) + 12\cot^2(2r) + (4n-4)(\cot^2 r + \tan^2 r) \\
&= \frac{1}{t^2(t^2-1)^2}\{16t^4 + 3(1-t^2)^4 + (4n-4)(t^4+1)(t^2-1)^2\}.
\end{aligned}
$$

Thus M_r is proper biharmonic if and only if

$$
n = \frac{t^8 + 12t^6 - 42t^4 + 12t^2 + 1}{4(t^2-1)^4} \qquad (0 < t < 1),
$$

whose right hand side attains its maximum $5/4$ at $t = \sqrt{2-\sqrt{3}}$. Since $n \geq 2$, this equation can have no real solution. $\square$

REMARK 1. When $r = \pi/(4\sqrt{2})$, M_r has three distinct principal curvatures

$$
\alpha = 0, \qquad \beta = \sqrt{2}, \qquad \gamma = 0, \qquad \lambda = -\sqrt{2}.
$$

Thus this tube satisfies

$$
H = -\frac{2\sqrt{2}(m-2)}{4m-1} \neq 0, \qquad |A|^2 = 4m.
$$

Hence M_r is not biharmonic.

7. Complex quadrics

In [3], Berndt and Suh studied real hypersurfaces in the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2})$ of *oriented* 2-planes in Euclidean $(m+2)$-space. As is well known, the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2})$ is identified with the complex quadric:

$$\mathcal{Q}_m = \{[z_1 : z_2 : \cdots : z_{m+2}] \in \mathbf{C}P^{m+1} \mid z_1^2 + z_2^2 + \cdots + z_{m+2}^2 = 0\}$$

in the complex projective $(m+1)$-space.

Now we equip the ambient projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4, then $\mathcal{Q}_m = \mathrm{SO}(m+2)/\mathrm{SO}(2) \times \mathrm{SO}(m)$ is a Hermitian symmetric space of rank 2 and maximal sectional curvature 4 with respect to the induced metric $\tilde{g}$. The Ricci tensor is given by $\widetilde{\mathrm{Ric}} = 2m\tilde{g}$ (see [21, Example 10.6 in Chapter XI]).

Hereafter we assume that $m \geq 3$. For $m = 2k$, the map

$$[z_1 : z_2 : \cdots : z_{k+1}] \mapsto [z_1 : z_2 : \cdots : z_{k+1} : iz_1 : iz_2 : \cdots : iz_{k+1}]$$

defines a totally geodesic complex immersion of $\mathbf{C}P^k$ into $\mathcal{Q}_{2k} \subset \mathbf{C}P^{2k+1}$.

For $r \in (0, \pi/2)$, the tube around $\mathbf{C}P^k$ is a homogeneous real hypersurface with principal curvatures:

$$\lambda_1 = 2\cot(2r), \qquad \lambda_2 = 0, \qquad \lambda_3 = -\tan r, \qquad \lambda_4 = \cot r$$

and multiplicities

$$m_1 = 1, \qquad m_2 = 2, \qquad m_3 = m_4 = 2k - 2.$$

In case $m = 2$, *i.e.*, $k = 1$, then we have $\mathbf{C}P_1 \subset \mathcal{Q}_2 = \mathbf{S}^2 \times \mathbf{S}^2$. The principal curvatures of a tube around $\mathbf{C}P^1$ are 0 and $2\cot(2r)$.

THEOREM 7. *The only minimal tube around $\mathbf{C}P^k$ in $\mathcal{Q}_{2k}$ is the tube of radius $\pi/4$.*

The only proper biharmonic tubes around $\mathbf{C}P^k$ in $\mathcal{Q}_{2k}$ are tubes of radius

$$\tan^{-1}\frac{\sqrt{2k} \pm 1}{\sqrt{2k-1}}.$$

PROOF. For $k \geq 1$, we have

$$\begin{aligned}(4k-1)H &= \lambda_1 + (2k-2)(\lambda_3 + \lambda_4)\\ &= \frac{1-t^2}{t} + (2k-2)\left(-t + \frac{1}{t}\right),\end{aligned}$$

where $t = \tan r$. Thus $H = 0$ if and only if $t^2 = 1$, that is $r = \pi/4$.

Next we have

$$|A|^2 = \frac{(1-t^2)^2}{t^2} + (2k-2)\left(t^2 + \frac{1}{t^2}\right)$$
$$= \frac{1}{t^2}\{t^4 - 2t^2 + 1 + (2k-2)(t^4+1)\}$$
$$= \frac{1}{t^2}\{(2k-1)t^4 - 2t^2 + (2k-1)\}.$$

Hence the biharmonicity equation $|A|^2 = 4k$ is

$$(2k-1)t^4 - 2(2k+1)t^2 + (2k-1) = 0.$$

Hence

$$t^2 = \frac{2k+1 \pm \sqrt{8k}}{2k-1}.$$

Thus we obtain

$$r = \tan^{-1}\sqrt{\frac{2k+1 \pm \sqrt{8k}}{2k-1}} = \tan^{-1}\frac{\sqrt{2k} \pm 1}{\sqrt{2k-1}} \qquad \square$$

Corollary 1. *Let M_r be the tube of radius r around* $\mathbf{C}P^2 = \mathrm{Gr}_2(\mathbf{C}^3) \subset \mathrm{Gr}_2(\mathbf{C}^4)$. *Then M_r is*

- *minimal in* $\mathrm{Gr}_2(\mathbf{C}^4)$ *if and only if* $r = \pi/4$.
- *proper biharmonic in* $\mathrm{Gr}_2(\mathbf{C}^4)$ *if and only if* $r = \tan^{-1}(3)$ *or* $r = \tan^{-1}(1/3)$.

8. Cayley projective plane

8.1. Let us denote by $\mathfrak{O}$ the division algebra of octonions (also called the *Cayley algebra*). Denote by $\mathfrak{J}$ the real linear space of all 3 by 3 Hermitian matrices of octonions. On this linear space the *Jordan product* $\circ$ is defined by

$$X \circ Y := \frac{1}{2}(XY + YX), \qquad X, Y \in \mathfrak{J}.$$

The real algebra $\mathfrak{J}$ equipped with Jordan product is called the *exceptional Jordan algebra*. The automorphism group F_4 of the Jordan algebra $\mathfrak{J}$ is a simply connected compact simple Lie group of dimension 52.

The projective plane $\mathfrak{O}P^2$ over $\mathfrak{O}$ is called the *Cayley projective plane*. The projective plane $\mathfrak{O}P^2$ is realized as

$$\{X \in \mathfrak{J} \mid X^2 = X,\ \mathrm{tr}\, X = 1\}$$

and represented by $F_4/\mathrm{Spin}(9)$. The negative of the Killing form of F_4 induces an invariant Riemannian metric on $\mathfrak{O}P^2$ so that $\mathfrak{O}P^2$ is a Riemannian symmetric space of rank 1. To adapt our discussion with Dimitric's paper [10], we normalize the metric such that the maximal sectional curvature of $\mathfrak{O}P^2$ is 4 (see also [29, § 5]). More precisely, the metric is induced form $-\frac{1}{72}B$. Then the resulting Riemannian symmetric space $(\mathfrak{O}P^2, \tilde{g})$ is a homogeneous Einstein manifold with Ricci tensor $\widetilde{\mathrm{Ric}} = 36\tilde{g}$.

In $\mathfrak{O}P^2$, we have the following criterion.

Proposition 2. *Let $M^{15} \subset \mathfrak{O}P^2$ be a hypersurface of constant mean curvature. Then M is proper biharmonic if and only if $|A|^2 = 36$.*

The homogeneous hypersurfaces in $\mathfrak{O}P^2$ are (essentially) classified by Iwata [19] (see also [2, 24]). There exist only two families of homogeneous hypersurfaces in $\mathfrak{O}P^2$. Geodesic spheres in $\mathfrak{O}P^2$ and tubes around a totally geodesic quaternion projective plane $\mathbf{H}P^2 \subset \mathfrak{O}P^2$.

8.2. Geodesic spheres. Now let us investigate biharmonicity of geodesic spheres in $\mathfrak{O}P^2$.

Let M_r be a geodesic sphere in $\mathfrak{O}P^2$ of radius r. Then M_r has two constant principal curvatures. The principal curvatures and their multiplicities are given as follows (see Dimitric [10], Murphy [24] and Verhóczki [27]):

$$\lambda = \cot r, \qquad \mu = 2\cot(2r), \qquad r \in (0, \pi/2),$$

$$m_\lambda = 8, \qquad m_\mu = 7.$$

Theorem 8. *Let M_r be a geodesic sphere in $\mathfrak{O}P^2$ of radius r. Then M_r is*

- *minimal if and only if the radius is*

$$r = \tan^{-1}\sqrt{\frac{15}{7}}.$$

- *proper biharmonic if and only if the radius is*

$$r = \tan^{-1}\sqrt{\frac{25 \pm 2\sqrt{130}}{7}}.$$

Proof. The mean curvature H is computed as

$$15H = 8\lambda + 7\mu = (15 - 7\tan^2 r)/\tan r.$$

Thus M is minimal if and only if its radius is

$$0 < r = \tan^{-1}\sqrt{\frac{15}{7}} < \frac{\pi}{2}.$$

Next, we have

$$|A|^2 = 8\lambda^2 + 7\mu^2 = \frac{1}{\tan^2 r}\{8 + 7(1 - \tan^2 r)^2\}.$$

Thus M is proper biharmonic if and only if $t = \tan r$ is a positive solution to

$$7t^4 - 50t^2 + 15 = 0.$$

Hence we get

$$0 < r = \tan^{-1}\sqrt{\frac{25 \pm 2\sqrt{130}}{7}} < \frac{\pi}{2}. \qquad \square$$

Remark 2. Geodesic spheres in $\mathfrak{O}P^2$ have been used to construct examples of Riemannian manifolds with special properties. For instance, a geodesic sphere M_r is of 1-*type submanifold* in the sense of Chen [6] via the 1st standard imbedding of $\mathfrak{O}P^2$ if and only if the radius is $r = \tan^{-1}(\sqrt{17}/\sqrt{7})$ (see Dimitric [10]).

Jensen [20] constructed Einstein metrics of non-constant curvature on the 15-sphere $\mathbf{S}^{15}$ (see also Ziller [32]). The resulting Einstein manifolds are realized as geodesic spheres in $\mathfrak{O}P^2$ of radius $r = \tan^{-1}(2\sqrt{2}/\sqrt{3})$. Thus biharmonic geodesic spheres and minimal geodesic spheres are neither Einstein nor 1-type (*cf.* [5]).

Remark 3. Take a totally geodesic $\mathfrak{O}P^1 = \mathbf{S}^8 \subset \mathfrak{O}P^2$. Then its tube of radius $\tilde{r} < \pi/4$ is a homogeneous hypersurface. The tube around $\mathfrak{O}P^1$ is nothing but a geodesic sphere of radius $r = \pi/2 - \tilde{r}$.

8.3. Tubes around $\mathbf{H}P^2$. The totally geodesic quaternion projective plane $\mathbf{H}P^2$ in $\mathfrak{O}P^2 = \mathrm{F}_4/\mathrm{Spin}(9)$ is represented by $\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)/\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)$. Here we use notation $\mathrm{Sp}(3)\cdot\mathrm{Sp}(1) := \mathrm{Sp}(3)\times\mathrm{Sp}(1)/\mathbf{Z}_2$ and $\mathrm{Spin}(4)\cdot\mathrm{Spin}(3) = \mathrm{Spin}(4)\times\mathrm{Spin}(3)/\mathbf{Z}_2$.

For a positive constant $r < \pi/4$, tubes around $\mathbf{H}P^2$ of radius r are hypersurfaces in $\mathfrak{O}P^2$. The principal curvatures and their multiplicities are given by Verhoczki [30, § 5]:

$$\lambda_1 = 2\tan(2r), \qquad m_1 = 4,$$

$$\lambda_2 = \tan r, \qquad m_2 = 4,$$

$$\lambda_3 = -2\cot(2r), \qquad m_3 = 3,$$

$$\lambda_4 = -\cot r, \qquad m_3 = 4.$$

THEOREM 9. *The only minimal tube around* $\mathbf{H}P^2$ *in* $\mathfrak{O}P^2$ *is the tube of radius*

$$r = \tan^{-1}\frac{\sqrt{11}-2}{\sqrt{7}}.$$

There are no proper biharmonic tubes around $\mathbf{H}P^2$ *in* $\mathfrak{O}P^2$.

PROOF. The mean curvature is computed as

$$\begin{aligned} 15H &= 4\lambda_1 + 4\lambda_2 + 3\lambda_3 + 4\lambda_4 \\ &= \frac{16}{1-t^2}t + 4t - \frac{3}{t}(1-t^2) - \frac{4}{t} \\ &= -\frac{1}{t(1-t^2)}(7t^4 - 30t^2 + 7), \end{aligned}$$

where we put $t = \tan r$.

Thus M_r is minimal if and only if

$$t^2 = \frac{15 \pm 4\sqrt{11}}{7}.$$

Hence

$$r = \tan^{-1}\sqrt{\frac{15 \pm 4\sqrt{11}}{7}} = \tan^{-1}\frac{\sqrt{11} \pm 2}{\sqrt{7}}.$$

However we notice that

$$0 < \tan^{-1}\frac{\sqrt{11}-2}{\sqrt{7}} < \frac{\pi}{4} < \tan^{-1}\frac{\sqrt{11}+2}{\sqrt{7}}$$

Thus the radius of the minimal tube is

$$r = \tan^{-1}\frac{\sqrt{11}-2}{\sqrt{7}}.$$

Next, the square norm $|A|^2$ is computed as:

$$\begin{aligned} |A|^2 &= 4\lambda_1^2 + 4\lambda_2^2 + 3\lambda_3^3 + 4\lambda_4^2 \\ &= \frac{4}{t^2(1-t^2)^2}\left\{16t^4 + t^4(1-t^2)^2 + \frac{4}{3}(1-t^2)^4 + (1-t^2)^2\right\}. \end{aligned}$$

Thus the biharmonic equation $|A|^2 = 1/2$ is rewritten as:

$$f(t) := 7t^8 - 56t^6 + 162t^4 - 56t^2 + 7 = 0. \tag{11}$$

This equation has no real solution. In fact $f(t) > 0$ for all $t \in \mathbf{R}$. $\square$

8.4. Compact Riemannian symmetric spaces of rank 1. In [15, 16], biharmonic homogeneous hypersurfaces in the unit sphere $\mathbf{S}^n$ and the complex projective space $\mathbf{C}P^n(4)$ of constant holomorphic sectional curvature 4 are classified. In addition biharmonic curvature-adapted real hypersurfaces of constant principal curvatures in the quaternion projective space $\mathbf{H}P^n(4)$ of maximal sectional curvature 4 are classified. Comparing these results with the classification of homogeneous real hypersurfaces in $\mathbf{H}P^n(4)$ due to D'Atri [9] and Iwata [18] and combining with Theorem 8 and Theorem 9, we obtain the following complete classification of all proper biharmonic homogeneous hypersurfaces in simply connected compact Riemannian symmetric spaces of rank 1.

THEOREM 10. *The proper biharmonic homogeneous hypersurfaces in simply connected compact Riemannian symmetric spaces of rank* 1 *are given as follows:*

- *Totally umbilical small hyperspheres of radius* $r = 1/\sqrt{2}$ *in the unit sphere* $\mathbf{S}^n$.
- *The product immersion* $\mathbf{S}^{n-p}(1/\sqrt{2}) \times \mathbf{S}^{p-1}(1/\sqrt{2}) \subset \mathbf{S}^n$ *with* $n - p \neq p - 1$.
- *Tubes* M_r *around totally geodesic subspace* $\mathbf{C}P^p \subset \mathbf{C}P^n(4)$ *of radius*

$$r = \cot^{-1}\left\{\frac{p+q+3 \pm \sqrt{(p-q)^2 + 4(p+q+2)}}{1+2q}\right\}^{1/2} < \frac{\pi}{2}.$$

  in the complex projective space $\mathbf{C}P^n(4)$ *of constant holomorphic sectional curvature* 4, *where* $0 \leq p \leq n-1$, $q := n-1-p$.
- *Tubes* M_r *around the Plücker imbedding* $\mathrm{Gr}_2(\mathbf{C}^5) \subset \mathbf{C}P^9(4)$ *of radius* $r < \pi/4$ *which is determined by the equation* $41t^2 + 43t^4 + 41t^2 - 15 = 0$ *with* $t = \cot r$.
- *Tubes* M_r *around* $\mathrm{SO}(10)/\mathrm{U}(5) \subset \mathbf{C}P^{15}(4)$ *of radius* $r < \pi/4$ *which is determined by the equation* $13t^6 - 107t^4 + 43t^2 - 9 = 0$.
- *Geodesic spheres of radius* r *in the quaternion projective space* $\mathbf{H}P^n$ *of maximal sectional curvature* 4. *Here* $t = \cot r$ *is a positive solution to*

$$(4n-1)r^4 - 2(2n+7)r^2 + 3 = 0.$$

- *Tubes* M_r *around* $\mathbf{C}P^n \subset \mathbf{H}P^n(4)$ *of radius* r *which is determined by the equation*

$$(2n-1)t^8 - 8(n+1)t^6 - (6n+11)t^4 - 2(2n-1)t^2 - 12 = 0$$

  for $t = \cot r$.

- *Tubes* M_r *around* $\mathbf{H}P^k \subset \mathbf{H}P^n(4)$ *of radius* r *which is determined by the equation*

$$(4n-4k-1)t^4 - 2(2n+4)t^2 + 4k + 3 = 0, \qquad 1 \le k \le n-1$$

for $t = \cot r$.
- *Geodesic spheres of radius* $r = \tan^{-1}\sqrt{\frac{25 \pm 2\sqrt{130}}{7}}$ *in Cayley projective plane* $\mathfrak{O}P^2$ *of maximal sectional curvature* 4.

As a corollary we obtain the following classification of all proper biharmonic geodesic spheres in simply connected compact Riemannian symmetric spaces of rank 1:

COROLLARY 2. *The proper biharmonic geodesic spheres in simply connected compact Riemannian symmetric spaces of rank* 1 *are given as follows:*
- *Totally umbilical small spheres of radius* $r = 1/\sqrt{2}$ *in the unit sphere* $\mathbf{S}^n$.
- *Geodesic spheres of radius*

$$r = \cot^{-1}\sqrt{\frac{n+2 \pm \sqrt{n^2+2n+5}}{2n-1}}$$

in the complex projective space $\mathbf{C}P^n$ *of constant holomorphic sectional curvature* 4.
- *Geodesic spheres of radius* r *which is a positive solution to*

$$(4n-1)r^4 - 2(2n+7)r^2 + 3 = 0$$

in the quaternion projective space $\mathbf{H}P^n$ *of constant quaternion sectional curvature* 4.
- *Geodesic spheres of radius* $r = \tan^{-1}\sqrt{\frac{25 \pm 2\sqrt{130}}{7}}$ *in Cayley projective plane* $\mathfrak{O}P^2$ *of maximal sectional curvature* 4.

Chen and Vanhecke developed differential geometric study of geodesic spheres. In particular they obtained the following characterization of real space forms in terms of geodesic spheres.

THEOREM 11 ([7]). *A Riemannian manifold* N *of* $\dim N \ge 3$ *is of constant curvature if and only if every geodesic sphere is a parallel hypersurface.*

As we have seen, every simply connected compact Riemannian symmetric space of rank 1 admits proper biharmonic geodesic spheres. Thus we are interested in the following conjecture:

CONJECTURE. A Riemannian manifold N of $\dim N \ge 3$ is locally isometric to a compact Riemannian symmetric space of rank 1 if and only if it contains biharmonic geodesic spheres.

9. Exceptional Lie group G_2

The automorphism group

$$G_2 = \{g \in GL(\mathfrak{O}) \mid g(\boldsymbol{x}\boldsymbol{y}) = g(\boldsymbol{x})g(\boldsymbol{y}),\ \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{O}\}$$

of $\mathfrak{O}$ is a compact Lie group of dimension 14. Since every $g \in G_2$ is a linear isometry, G_2 is a closed subgroup of $O(8)$. Moreover, since every element of G_2 fixes the identity element of $\mathfrak{O}$, G_2 is a closed subgroup of $O(7) = \{g \in O(8) \mid g(\mathrm{id}) = \mathrm{id}\}$. The exceptional Lie group G_2 with the Killing form is a compact Riemannian symmetric space of rank 2 of the form $G_2 \times G_2/G_2$. With respect to the Killing metric, G_2 has maximal sectional curvature $1/16$. For a positive $r < 2\sqrt{3}\pi$, tubes M_r of radius r around $SU(3)$ are hypersurfaces of G_2.

Principal curvatures of M_r are given by Verhóczki [29]:

$$\begin{aligned}
\lambda_1 &= \frac{1}{8\sqrt{3}}\left(-2\cot\frac{r}{2\sqrt{3}} + \sqrt{4\cot^2\frac{r}{2\sqrt{3}} + 3}\right),\\
\lambda_2 &= \frac{1}{8\sqrt{3}}\left(-2\cot\frac{r}{2\sqrt{3}} - \sqrt{4\cot^2\frac{r}{2\sqrt{3}} + 3}\right),\\
\lambda_3 &= \frac{1}{4\sqrt{3}}\tan\frac{r}{4\sqrt{3}},\\
\lambda_4 &= -\frac{1}{4\sqrt{3}}\cot\frac{r}{4\sqrt{3}},\\
\lambda_5 &= 0
\end{aligned}$$

with multiplicities

$$m_1 = m_2 = 4, \qquad m_3 = m_4 = 1, \qquad m_5 = 3.$$

Theorem 12. *A tube M_r around* $SU(3)$ *in* G_2 *is minimal if and only if the radius is $\sqrt{3}\pi$. The only proper biharmonic tubes around* $SU(3)$ *in* G_2 *are tubes of radius*

$$4\sqrt{3}\tan^{-1}\frac{1}{\sqrt{5}} \qquad \text{or} \qquad 4\sqrt{3}\tan^{-1}\sqrt{5}.$$

Proof. The mean curvature is computed as

$$\begin{aligned}
13H &= 4(\lambda_1 + \lambda_2) + \lambda_3 + \lambda_4\\
&= -\frac{1}{\sqrt{3}}\cot\frac{r}{2\sqrt{3}} + \frac{1}{4\sqrt{3}}\tan\frac{r}{4\sqrt{3}} - \frac{1}{4\sqrt{3}}\cot\frac{r}{4\sqrt{3}}\\
&= \frac{3}{4\sqrt{3}\tan\frac{r}{4\sqrt{3}}}\left(\tan^2\frac{r}{4\sqrt{3}} - 1\right).
\end{aligned}$$

Hence M_r is minimal if and only if

$$r = 4\sqrt{3}\tan^{-1} 1 = \sqrt{3}\pi < 2\sqrt{3}\pi.$$

Next we compute $|A|^2$.

$$|A|^2 = 4(\lambda_1^2 + \lambda_2^2) + \lambda_3^2 + \lambda_4^2 = \frac{1}{48t^2}(5t^4 - 2t^2 + 5),$$

where we put $t = \tan\frac{r}{4\sqrt{3}}$. The biharmonicity equation $|A|^2 = 1/2$ is rewritten as $5t^4 - 26t^2 + 5 = 0$. Hence

$$t^2 = \frac{13 \pm 12}{5}.$$

Namely $t^2 = 1/5$ or 5. Thus we have

$$r = 4\sqrt{3}\tan^{-1}\frac{1}{\sqrt{5}} \qquad \text{or} \qquad r = 4\sqrt{3}\tan^{-1}\sqrt{5}$$

Both the values satisfy the inequality $r < 2\sqrt{3}\pi$. □

10. Exceptional Lie group F_4

The exceptional Lie group F_4 equipped with Killing metric is a 52-dimensional compact Riemannian symmetric space of rank 4 with the form $\mathrm{F}_4 \times \mathrm{F}_4/\mathrm{F}_4$. The maximal sectional curvature of F_4 is $1/36$. The exceptional Lie group F_4 has totally geodesic submanifold Spin(9). The maximal sectional curvature of Spin(9) with respect to the induced metric is $1/36$.

For a positive $r < 3\sqrt{2}\pi$, tubes M_r of radius r around Spin(9) are hypersurfaces of F_4.

The principal curvatures of M_r are given explicitly by Cskós and Verhóczki [8]:

$$\lambda_1 = \frac{1}{6\sqrt{2}}\tan\frac{r}{6\sqrt{2}},$$

$$\lambda_2 = \frac{1}{12\sqrt{2}}\tan\frac{r}{12\sqrt{2}},$$

$$\lambda_3 = 0,$$

$$\lambda_4 = -\frac{1}{6\sqrt{2}}\cot\frac{r}{6\sqrt{2}},$$

$$\lambda_5 = -\frac{1}{12\sqrt{2}}\cot\frac{r}{12\sqrt{2}}$$

with multiplicities

$$m_1 = 7, \qquad m_2 = 8, \qquad m_3 = 21, \qquad m_4 = 7, \qquad m_5 = 8.$$

THEOREM 13. *A tube M_r around $\mathfrak{O}P^2$ in* F_4 *is minimal if and only if the radius is*

$$12\sqrt{2}\tan^{-1}\frac{\sqrt{22}-\sqrt{7}}{\sqrt{15}}.$$

The only proper biharmonic tubes around $\mathfrak{O}P^2$ in F_4 *are tubes of radius*

$$12\sqrt{2}\tan^{-1}\left(\sqrt{\frac{51-2\sqrt{35}-2\sqrt{553-51\sqrt{35}}}{23}}\right),$$

or

$$12\sqrt{2}\tan^{-1}\left(\sqrt{\frac{51+2\sqrt{35}-2\sqrt{553+51\sqrt{35}}}{23}}\right).$$

PROOF. The mean curvature H is computed as

$$\begin{aligned}
51H &= 7\lambda_1 + 8\lambda_2 + 7\lambda_4 + 8\lambda_5 \\
&= \frac{7}{6\sqrt{2}}\left(\frac{2t}{1-t^2} - \frac{1-t^2}{2t}\right) + \frac{2}{3\sqrt{2}}\left(t - \frac{1}{t}\right) \\
&= \frac{7}{6\sqrt{2}}\frac{4t^2-(1-t^2)^2}{2t(1-t^2)} + \frac{2}{3\sqrt{2}}\frac{t^2-1}{t} \\
&= -\frac{7}{6\sqrt{2}}\frac{t^4-6t^2+1}{2t(1-t^2)} + \frac{2}{3\sqrt{2}}\frac{t^2-1}{t}.
\end{aligned}$$

Here we put $t = \tan\frac{r}{12\sqrt{2}}$. From this, M_r is minimal if and only if

$$15t^4 - 58t^2 + 15 = 0.$$

Hence

$$t^2 = \frac{29 \pm 2\sqrt{154}}{15}.$$

Since $0 < r < 3\sqrt{2}\pi$, we need to choose

$$t = \sqrt{\frac{29-2\sqrt{154}}{15}} = \frac{\sqrt{22}-\sqrt{7}}{\sqrt{15}}.$$

Next we look for proper biharmonic tubes. The square norm $|A|^2$ is computed as

$$|A|^2 = \frac{7}{72}\left(\left(\frac{2t}{1-t^2}\right)^2 + \left(\frac{1-t^2}{2t}\right)^2\right) + \frac{4}{72}(t^2 + t^{-2}).$$

The biharmonic equation $|A|^2 = 1/2$ can be written as

$$23t^8 - 204t^6 + 474t^4 - 204t^2 + 23 = 0.$$

Since $0 < r < 3\sqrt{2}\pi$, we obtain

$$t = \sqrt{\frac{51 - 2\sqrt{35} - 2\sqrt{553 - 51\sqrt{35}}}{23}}$$

or

$$t = \sqrt{\frac{51 + 2\sqrt{35} - 2\sqrt{553 + 51\sqrt{35}}}{23}}. \qquad \square$$

11. Exceptional symmetric space $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$

In this section we consider the real 28-dimensional Riemannian symmetric space $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$ with isotropy subgroup $\mathrm{Sp}(3)\cdot\mathrm{Sp}(1) = \mathrm{Sp}(3)\times\mathrm{Sp}(1)/\mathbf{Z}_2$ of rank 4 (type FI). Note that $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$ is a quaternionic symmetric space. The maximal sectional curvature of $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$ with respect to Killing metric is 1/9. Note that $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$ is quaternionic symmetric. This symmetric space has totally geodesic submanifolds $\widetilde{\mathrm{Gr}}_4(\mathbf{R}^9)$. The maximal sectional curvature of $\widetilde{\mathrm{Gr}}_4(\mathbf{R}^9) \subset \mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$ is 1/9. The tube around $\widetilde{\mathrm{Gr}}_4(\mathbf{R}^9)$ of radius $r < \frac{3}{\sqrt{2}}\pi$ is homogeneous and has principal curvatures ([30, §4]):

$$\lambda_1 = \frac{1}{3\sqrt{2}}\tan\frac{r}{3\sqrt{2}}, \qquad m_1 = 4,$$

$$\lambda_2 = \frac{r}{6\sqrt{2}}\tan\frac{r}{6\sqrt{2}}, \qquad m_2 = 4,$$

$$\lambda_3 = 0, \qquad m_3 = 12,$$

$$\lambda_4 = -\frac{1}{3\sqrt{2}}\cot\frac{r}{3\sqrt{2}}, \qquad m_4 = 3,$$

$$\lambda_5 = -\frac{r}{6\sqrt{2}}\cot\frac{r}{6\sqrt{2}}, \qquad m_5 = 4.$$

THEOREM 14. *There are no proper biharmonic tubes around* $\widetilde{\mathrm{Gr}}_4(\mathbf{R}^9)$ *in* $\mathrm{F}_4/\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)$. *The only minimal tube is a tube of radius*

$$r = 6\sqrt{2}\tan^{-1}\frac{\sqrt{11}-2}{\sqrt{7}}.$$

PROOF. From the table of principal curvatures, we have

$$\begin{aligned}
27H &= 4\lambda_1 + 4\lambda_2 + 3\lambda_4 + 4\lambda_5 \\
&= \frac{1}{3\sqrt{2}}\left(\frac{8t}{1-t^2} + 2t - \frac{3(1-t^2)}{2t} - \frac{2}{t}\right) \\
&= -\frac{1}{6\sqrt{2}t(1-t^2)}(7t^4 - 30t^2 + 7),
\end{aligned}$$

where $t = \tan(r/(6\sqrt{2}))$. Hence $H = 0$ if and only if

$$t = \sqrt{\frac{15 \pm 4\sqrt{11}}{7}} = \frac{\sqrt{11} \pm 2}{\sqrt{7}}.$$

Since $r < 3\pi/\sqrt{2}$, we need to choose

$$r = 6\sqrt{2}\tan^{-1}\frac{\sqrt{11}-2}{\sqrt{7}}.$$

Next we look for proper biharmonic tubes. The square norm $|A|^2$ is computed as

$$|A|^2 = \frac{1}{18}\left(\frac{16t^2}{(1-t^2)^2} + t^2 + \frac{3(1-t^2)^2}{4t^2} + \frac{1}{t^2}\right).$$

The biharmonic equation $|A|^2 = 1/2$ can be written as

$$7t^8 - 56t^6 + 162t^4 - 56t^2 + 7 = 0.$$

As we have seen before in (11), this equation has no real solutions. $\square$

In a separate publication [17], we shall study biharmonic homogeneous hypersurfaces in compact Riemannian symmetric spaces associated with the exceptional simple Lie groups E_6 and G_2 as well as real Grassmannian manifolds and quaternion Grassmannian manifolds.

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