HIROSHIMA MATH. J.
46 (2016), 79–85

A note on a result of Lanteri about the class of a polarized surface

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(Received April 6, 2015)
(Revised June 26, 2015)

ABSTRACT. Let S be a smooth complex projective surface, H be a very ample divisor on S, and $m(S,H)$ be its class. In this short note we prove that $m(S,H) \geq H^2 + 2g(S,H) + 2$ under the assumption that $m(S,H) > H^2$ and $g(S,H) \geq 2$, where $g(S,H)$ denotes the sectional genus of (S,H). Moreover we classify (S,H) with $m(S,H) = H^2 + 2g(S,H) + 2$. This result is an improvement of a result of Lanteri.

1. Introduction

Let S be a smooth complex projective surface, H be a very ample divisor on S, and $m(S,H)$ be its class, i.e. the degree of the dual variety of S (embedded via H). Then some relations between $m(S,H)$ and H^2 have been studied by many authors (for example, [4], [5], [6], [7] and [9]). Among other things, in [6, (2.5) Proposition], Lanteri proved $m(S,H) \geq H^2 + 2g(S,H) + 1$ under the assumption that $m(S,H) > H^2$ and $g(S,H) \geq 2$. Here $g(S,H)$ denotes the sectional genus of (S,H), which is defined by the following formula.

$$g(S,H) = 1 + \frac{1}{2}(K_S + H)H.$$

In his paper, Lanteri also said that it is not known whether this result is the best possible or not (see [6, p. 85]). In this short note, we improve this inequality and we show that $m(S,H) \geq H^2 + 2g(S,H) + 2$ holds under the assumption that $m(S,H) > H^2$ and $g(S,H) \geq 2$. Moreover we classify (S,H) with $m(S,H) = H^2 + 2g(S,H) + 2$.

The author is supported by JSPS KAKENHI Grant Number 24540043.

2010 *Mathematics Subject Classification.* Primary 14C20; Secondary 14N15, 14J26.

Key words and phrases. Polarized surface, very ample divisors, class, adjunction theory, sectional genus, Δ-genus.

2. Preliminaries

In this paper, we work over the field of complex numbers $\mathbf{C}$. We use the customary notation in algebraic geometry. The words "line bundles" and "(Cartier) divisors" are used interchangeably. If a smooth projective surface S is a $\mathbf{P}^1$-bundle over a smooth projective curve C, then there exists a vector bundle $\mathscr{E}$ on C such that $S \cong \mathbf{P}_C(\mathscr{E})$. Let $H(\mathscr{E})$ be the tautological line bundle of $\mathbf{P}_C(\mathscr{E})$. For a smooth projective surface S and a very ample divisor H on S, let $g(S,H)$ be the sectional genus of (S,H), K_S be the canonical divisor of S, $m(S,H)$ be the class of (S,H), and $\chi(S)$ be the topological Euler characteristic. Let $q(S)$ be the irregurality of S and $p_g(S)$ be the geometric genus of S.

It is known that these invariants satisfy the following (see [6, (1.3)]):

$$m(S,H) - H^2 = \chi(S) + 4(g(S,H) - 1). \tag{1}$$

By using the genus formula and Noether's formula, we also have

$$m(S,H) = 12\chi(\mathcal{O}_S) - K_S^2 + 4(g(S,H) - 1) + H^2. \tag{2}$$

3. Main result

THEOREM 1. *Let (S,H) be a polarized surface such that H is very ample. Let $m(S,H)$ be the class of (S,H). Assume that $m(S,H) > H^2$ and $g(S,H) \geq 2$. Then $m(S,H) \geq H^2 + 2g(S,H) + 2$ holds. If this equality holds, then $(S,H) = (\mathbf{P}_C(\mathscr{E}), 2C_0 + F)$, where C is a smooth elliptic curve, $\mathscr{E}$ is a normalized vector bundle of rank two on C with* $\deg \mathscr{E} = 1$, *and C_0 (resp. F) is a section of S with $\mathcal{O}_S(C_0) \cong H(\mathscr{E})$ (resp. a fiber).*

PROOF. (A) First we will prove that $m(S,H) \geq H^2 + 2g(S,H) + 2$. Here we note that

$$m(S,H) \geq H^2 + 2g(S,H) + 1 \tag{3}$$

holds by [6, (2.5) Proposition].

(A.i) Assume that $\kappa(S) \geq 0$. Then by [3, Theorems 2.1 and 3.1 and Corollary 4.3][1] we get

$$g(S,H) \geq \begin{cases} 3q(S), & \text{if } \kappa(S) = 0 \text{ or } 1, \\ 2q(S), & \text{if } \kappa(S) = 2. \end{cases} \tag{4}$$

By [6, (2.1) Proposition], we get

$$m(S,H) - H^2 \geq 4(g(S,H) - q(S)) + 2p_g(S) + \rho(S) - 2. \tag{5}$$

[1] We note that a line bundle L is 1-very ample if and only if L is very ample.

Here $\rho(S)$ denotes the Picard number of S. In particular $\rho(S) \geq 1$. By using (4) and (5) we have

$$m(S,H) - H^2 \geq 2g(S,H) + 2(g(S,H) - 2q(S)) + 2p_g(S) + \rho(S) - 2$$
$$\geq 2g(S,H) + 2p_g(S) + \rho(S) - 2. \tag{6}$$

Assume that $m(S,H) = H^2 + 2g(S,H) + 1$. Then by (6) we see that one of the following holds.

- $p_g(S) = 0$ and $\rho(S) \leq 3$.
- $p_g(S) = 1$ and $\rho(S) = 1$.

Claim 1. *$q(S) \leq 1$ holds.*

Proof. Assume that $p_g(S) = 1$. Then $q(S) \leq 2$ because $\chi(\mathcal{O}_S) \geq 0$. If $q(S) = 2$, then $\chi(\mathcal{O}_S) = 0$ and we get $\kappa(S) \leq 1$. By (4) we have $g(S,H) \geq 3q(S)$ and by (6) we get

$$m(S,H) - H^2 \geq 2g(S,H) + 2(g(S,H) - 2q(S)) + 2p_g(S) + \rho(S) - 2$$
$$\geq 2g(S,H) + 2q(S) + 2p_g(S) + \rho(S) - 2$$
$$\geq 2g(S,H) + 5,$$

but this is a contradiction. So we get $q(S) \leq 1$ if $p_g(S) = 1$.

Assume that $p_g(S) = 0$. Since $\kappa(S) \geq 0$, we have $\chi(\mathcal{O}_S) \geq 0$. Hence we have $q(S) \leq 1$. Therefore we get the assertion of Claim 1.

If $g(S,H) \geq 2q(S) + 2$, then by (6)

$$m(S,H) - H^2 \geq 2g(S,H) + 3,$$

but this is impossible. So we get $g(S,H) \leq 2q(S) + 1$ and by Claim 1 we have $g(S,H) \leq 3$. Since H is very ample with $g(S,H) \leq 3$ and $\kappa(S) \geq 0$, we see from [1, Theorems 8.7.1, 8.9.1 and 10.2.7] that $S \subset \mathbf{P}^3$ is a quartic surface in $\mathbf{P}^3$ and $H = \mathcal{O}_S(1)$. Then $g(S,H) = 3$, $H^2 = 4$, $q(S) = 0$ and $\mathcal{O}_S(K_S) = \mathcal{O}_S$. But then by (5)

$$m(S,H) - H^2 \geq 4g(S,H) - 4q(S) + 2p_g(S) + \rho(S) - 2$$
$$= 2g(S,H) + 2g(S,H) - 4q(S) + 2p_g(S) + \rho(S) - 2$$
$$\geq 2g(S,H) + 7,$$

and this is impossible. Therefore $m(S,H) - H^2 \geq 2g(S,H) + 2$ holds for the case where $\kappa(S) \geq 0$.

(A.ii) Assume that $\kappa(S) = -\infty$.

(A.ii.1) If $K_S + H$ is not nef, then by [10, (1.5) Proposition and (1.5.2) Corollary] or [8, 1.3 Remark] (S, H) is one of the following three types.

(a) $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$.

(b) $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$.

(c) A scroll over a smooth projective curve.

If (S, H) is either (a) or (b), then $g(S, H) = 0$ and this contradicts the assumption that $g(S, H) \geq 2$.

If (S, H) is the type (c), then by (1) and (2)

$$\begin{aligned} m(S,H) - H^2 &= \chi(S) + 4(g(S,H) - 1) \\ &= 12\chi(\mathcal{O}_S) - K_S^2 + 4(g(S,H) - 1) \\ &= 12(1 - q(S)) - 8(1 - q(S)) + 4(q(S) - 1) \\ &= 0. \end{aligned}$$

But this contradicts the assumption that $m(S, H) > H^2$. So we may assume that $K_S + H$ is nef.

(A.ii.2) Assume that $K_S + H$ is nef.

(A.ii.2.1) If $S \cong \mathbf{P}^2$, then by (1)

$$\begin{aligned} m(S,H) - H^2 &= \chi(S) + 4(g(S,H) - 1) \\ &= 3 + 4(g(S,H) - 1) \\ &= 2g(S,H) + 2g(S,H) - 1 \\ &\geq 2g(S,H) + 3 \end{aligned}$$

because of the assumption that $g(S, H) \geq 2$.

(A.ii.2.2) We assume that $S \not\cong \mathbf{P}^2$. Then $\rho(S) \geq 2$ and by (5) we have

$$\begin{aligned} m(S,H) - H^2 &\geq 2g(S,H) + 2(g(S,H) - 2q(S)) + 2p_g(S) + \rho(S) - 2 \\ &\geq 2g(S,H) + 2(g(S,H) - 2q(S)). \end{aligned} \tag{7}$$

Since $K_S + H$ is nef, we have

$$\begin{aligned} 0 \leq (K_S + H)^2 &= K_S^2 + 2K_S H + H^2 \\ &= K_S^2 + 4(g(S,H) - 1) - H^2 \\ &\leq 8(1 - q(S)) + 4(g(S,H) - 1) - H^2 \\ &= 4(g(S,H) - 2q(S) + 1) - H^2. \end{aligned} \tag{8}$$

In particular we see from (8) that

$$g(S,H) \geq 2q(S). \tag{9}$$

Assume that $m(S,H) - H^2 = 2g(S,H) + 1$. Then by (7) and (9) we have $g(S,H) = 2q(S)$ and we also get $H^2 \leq 4$ by (8). But by [1, Proposition 8.10.1] and the assumption we see that $S \subset \mathbf{P}^3$ is a quartic surface in $\mathbf{P}^3$ and $H = \mathcal{O}_S(1)$. Then $g(S,H) = 3$. But this contradicts the equality $g(S,H) = 2q(S)$.

Hence by (3) we get $m(S,H) - H^2 \geq 2g(S,H) + 2$.

(B) Next we will classify (S,H) with $m(S,H) - H^2 = 2g(S,H) + 2$. Since H is very ample, we have $h^0(H) \geq 3$. If $h^0(H) = 3$, then $(S,H) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$ and $m(S,H) = 0$. But this is impossible. Hence $h^0(H) \geq 4$.

(B.i) Assume that $\kappa(S) \geq 0$. If $g(S,H) \geq 2q(S) + 2$, then by (6) we have

$$m(S,H) - H^2 \geq 2g(S,H) + 4 - 1 = 2g(S,H) + 3,$$

but this is impossible. So we get $g(S,H) = 2q(S)$ or $2q(S) + 1$ by (4).

(B.i.1) Assume that $g(S,H) = 2q(S)$. Then by (6)

$$\begin{aligned} 2g(S,H) + 2 &= m(S,H) - H^2 \\ &\geq 2g(S,H) + 2(g(S,H) - 2q(S)) + 2p_g(S) + \rho(S) - 2 \\ &= 2g(S,H) + 2p_g(S) + \rho(S) - 2. \end{aligned} \tag{10}$$

So we get $p_g(S) \leq 1$. Since $\kappa(S) \geq 0$, we have $q(S) \leq 2$ and $g(S,H) = 2q(S) \leq 4$.

(B.i.1.1) If $g(S,H) \leq 3$, then by the classification of (S,H) with $g(S,H) \leq 3$ ([1, Theorems 8.7.1, 8.9.1 and 10.2.7]) we see that S is a quartic surface in $\mathbf{P}^3$ and $H = \mathcal{O}_S(1)$. Then $g(S,H) = 3$, $H^2 = 4$, $q(S) = 0$ and $\mathcal{O}_S(K_S) = \mathcal{O}_S$. But this case is impossible because $g(S,H) = 3 \neq 2q(S)$.

(B.i.1.2) If $g(S,H) = 4$, then $q(S) = 2$ and $p_g(S) = 1$. In this case $\chi(\mathcal{O}_S) = 0$. Here we note that $(K_S + H)H = 6$ because $g(S,H) = 4$ in this case. Since $\kappa(S) \geq 0$, we have $H^2 \leq 6$.

(B.i.1.2.1) Assume that $H^2 \leq 5$.

(B.i.1.2.1.1) If $h^0(H) = 4$, then S is a hypersurface of degree d in $\mathbf{P}^3$ and $H = \mathcal{O}_{\mathbf{P}^3}(1)|_S$, where $d = H^2$. Moreover $K_S = (K_{\mathbf{P}^3} + \mathcal{O}_{\mathbf{P}^3}(d))|_S = \mathcal{O}_{\mathbf{P}^3}(d-4)|_S$. Since $\kappa(S) \geq 0$, we have $d = 4$ or 5.

If $d = 5$, then $\mathcal{O}(S) = \mathcal{O}_{\mathbf{P}^3}(5)$ and by the following exact sequence

$$0 \to K_{\mathbf{P}^3} \to K_{\mathbf{P}^3} + S \to K_S \to 0$$

we have

$$p_g(S) = h^0(K_S) \geq h^0(K_{\mathbf{P}^3} + S) = h^0(\mathcal{O}_{\mathbf{P}^3}(1)) = 4.$$

But this is a contradiction.

So we may assume that $d = 4$. But then $K_S = \mathcal{O}_S$ and we have $d = H^2 = 6$ because $(K_S + H)H = 6$. This is also impossible.

(B.i.1.2.1.2) If $h^0(H) \geq 5$, then

$$\Delta(S, H) = 2 + H^2 - h^0(H) \leq 2 < 4 = g(S, H). \tag{11}$$

On the other hand we have

$$H^2 \geq 2\Delta(S, H) + 1.$$

So by [2, (3.5) Theorem 3)] we have $g(S, H) = \Delta(S, H)$, but this contradicts (11).

(B.i.1.2.2) Assume that $H^2 = 6$. Then $K_S H = 0$. Hence we have $\kappa(S) = 0$ and S is minimal because H is ample. Since $q(S) = 2$ and $p_g(S) = 1$, we see that S is an Abelian surface. But then

$$h^0(H) = \frac{H^2}{2} = 3$$

and this is impossible because $h^0(H) \geq 4$.

(B.i.2) Assume that $g(S, H) = 2q(S) + 1$. Then we see from (6) that $p_g(S) = 0$. Hence $q(S) \leq 1$ and $g(S, H) = 2q(S) + 1 \leq 3$. By the classification of (S, H) with $g(S, H) \leq 3$ and $\kappa(S) \geq 0$ ([1, Theorems 8.7.1, 8.9.1 and 10.2.7]) we have $q(S) = 0$ and $g(S, H) = 3$. But this is impossible because here we assume $g(S, H) = 2q(S) + 1$.

(B.ii) Assume that $\kappa(S) = -\infty$. By the same argument as in (A.ii) above we may assume that $K_S + H$ is nef and $S \not\cong \mathbf{P}^2$. We also note that $g(S, H) - 2q(S) = 0$ or 1 by (7). Hence we get $H^2 \leq 8$ by (8). By the classification of (S, H) with $H^2 \leq 8$ (see e.g. [11, (3.1) Table]), we infer that if $m(S, H) > H^2$, $g(S, H) \geq 2$ and $m(S, H) = H^2 + 2g(S, H) + 2$, then $(S, H) = (\mathbf{P}_C(\mathcal{E}), 2C_0 + F)$, where C is a smooth elliptic curve and $\mathcal{E}$ is a normalized vector bundle of rank two on C with $\deg \mathcal{E} = 1$, and C_0 (resp. F) is a section of S with $\mathcal{O}_S(C_0) \cong H(\mathcal{E})$ (resp. a fiber).

Acknowledgement

The author would like to thank the referee for giving some comments.

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