HIROSHIMA MATH. J.
44 (2014), 341–350

Addendum to "On integral quadratic forms having commensurable groups of automorphisms"

José María Montesinos-Amilibia

(Received February 12, 2014)
(Revised June 6, 2014)

Abstract. Cassels proved that projectively equivalent integral quadratic forms are commensurable. In this note, an elementary proof of the converse of this theorem, for indefinite forms, is given. This was proved in [3] for forms of Sylvester signature $+ + + \dots + -$ or $- - - \dots - +$ (*hyperbolic* forms) and it was left there, as an open problem, for non-hyperbolic indefinite forms of any Sylvester signature.

1. Introduction

We follow the notation of [3]. In particular, if F is an integral, symmetric $n \times n$ matrix with non-zero determinant, both F and the expression

$$f(x) = x^t F x$$

are refered to as an *n-ary integral quadratic form*, whose associated *bilinear form* is

$$f(x, y) = x^t F y.$$

Two n-ary integral quadratic forms F and G are *rationally equivalent* if there is an $n \times n$, rational matrix M such that $M^t F M = G$. If, moreover, M is integral, and $\det M = \pm 1$, then F and G are *integrally equivalent.*

An $n \times n$ matrix U with integral entries is an *automorph* of the integral quadratic form F if $U^t F U = F$. Then $\det U = \pm 1$. The set of automorphs of F, which will be denoted by $\mathrm{Aut}(F)$, is called the *group of automorphisms* of F.

Let F and G be two n-ary integral quadratic forms. Say that they are *commensurable*, denoted by $F \overset{c}{\sim} G$, if there is $M \in GL(n, \mathbf{R})$ such that $M^t F M = \pm G$ and $M^{-1} S_F M = S_G$, for some finite index subgroups S_F and S_G of the groups of automorphisms, $\mathrm{Aut}(F)$ and $\mathrm{Aut}(G)$, of F and G, respectively.

Supported by grant MTM 2009-07030.
2010 *Mathematics Subject Classification.* (2010) 11E04, 11E20, 57M25, 57M50, 57M60.
Key words and phrases. Integral quadratic form, automorph, commensurability class.

This definition is relevant only if the groups of automorphisms are infinite: two n-ary definite integral quadratic forms are always commensurable.

Say that F and G are *projectively equivalent*, denoted by $F \overset{P}{\sim} G$, if there is a rational matrix R and a rational number λ such that $R^t F R = \lambda G$. Equivalently, there is an integral matrix T and a nonzero integer a such that $T^t F T = aG$.

Cassels [1] proved that two rationally equivalent n-ary integral quadratic forms are commensurable.

The converse of this result, namely that commensurability implies projective equivalence, is false for definite integral quadratic forms, since they have finite groups of automorphisms. It was proved to be true for indefinite integral quadratic forms of Sylvester signature $+++\ldots+-$ or $---\ldots-+$ (*hyperbolic* forms), in [3, Theorem 3], and it was left there, as an open question, for indefinite forms of any Sylvester signature. Professor Souto gave a non-elementary proof of the affirmative answer to this question in [5]. The purpose of this *Addendum* is to offer an elementary proof of the following theorem.

Theorem 1. *Let F and G be two n-ary indefinite integral quadratic forms. Then, the following three statements are equivalent.*

(1) $F \overset{c}{\sim} G$.

(2) *There are finite index subgroups of* $\mathrm{Aut}(F)$ *and* $\mathrm{Aut}(G)$ *which are conjugate in* $GL(n, \mathbf{R})$.

(3) $F \overset{P}{\sim} G$.

2. Proof of Theorem 1

In this section we prove Theorem 1. By definition, (1) implies (2). The proof that (3) implies (1) is due to Cassels [1]. He proved that two rationally equivalent n-ary integral quadratic forms are commensurable. This is a consequence of the following more detailed statement (see [3, Proposition 2]) that we will use later in this paper.

Proposition 1. *Let F and G be two n-ary integral quadratic forms. Assume there is an integral matrix T and a nonzero integer a such that $T^t F T = aG$. Then, there are finite index subgroups S_F of* $\mathrm{Aut}(F)$ *and S_G of* $\mathrm{Aut}(G)$ *such that $M^{-1} S_F M = S_G$ and $M^t F M = \varepsilon G$, where $M = T/\sqrt{|a|}$, $\varepsilon = \frac{a}{|a|}$.*

Proof. Let $m = \det T$. If $m = \pm 1$, then F and G are integrally equivalent. Hence

$$T^{-1}\,\mathrm{Aut}(F) T = \mathrm{Aut}(aG) = \mathrm{Aut}(G).$$

If $m \neq \pm 1$, consider the subgroups

$$S_F = \{U \in \mathrm{Aut}(F) : T^{-1}UT \in \mathrm{Aut}(G)\}$$

and

$$S_G = \{V \in \mathrm{Aut}(G) : TVT^{-1} \in \mathrm{Aut}(F)\}.$$

Then, $T^{-1}S_F T = S_G$.

To show that S_F has finite index in $\mathrm{Aut}(F)$ consider the homomorphism

$$\omega : \mathrm{Aut}(F) \to GL(n, \mathbf{Z}/m\mathbf{Z}),$$

defined by $\omega(U) = U \bmod m$. Obviously, the kernel of ω has finite index in $\mathrm{Aut}(F)$. Moreover, it is contained in S_F. Indeed, if $U = I + mA$, A integral, we have $T^{-1}UT = I + mT^{-1}AT$ which, being integral, belongs to $\mathrm{Aut}(G)$. An analogous argument shows that S_G has finite index in $\mathrm{Aut}(G)$. This completes the proof.

We will need some previous results to see that (2) implies (3). The first such result will simplify condition (2).

LEMMA 1. *Let S and T be two non empty subsets of $GL(n, \mathbf{Z})$ which are conjugate in $GL(n, \mathbf{R})$. Then, they are conjugate in $GL(n, \mathbf{Q})$.*

PROOF. Let $M \in GL(n, \mathbf{R})$ such that $M^{-1}SM = T$. Then, for every $U \in S$ there exists $V_U \in T$ such that $M^{-1}UM = V_U$. Each equation $UX = XV_U$, where $X = (x_{ij})$, can be viewed as a set of n^2 linear equations with integral coefficients in the n^2 unknowns x_{ij}. The (possibly infinite) set of all these equations, for all $U \in S$, has a subset of maximal rank r. This rank is less than n^2 because there exists the solution M of the system of equations.

Then, the general solution of the system is an $n \times n$ matrix $W(t_1, \dots, t_m)$ whose entries are linear forms with rational coefficients in $m = n^2 - r$ indeterminates $t_1, \dots, t_m$, which can attain arbitrary real values.

The equation $\det W = 0$ is a polynomial equation in the variables $t_1, \dots, t_m$, which defines an algebraic subvariety $\mathscr{V}$ of $\mathbf{R}^m$. The matrix $R = W(q_1, \dots, q_m)$, where $(q_1, \dots, q_m) \in \mathbf{R}^m \backslash \mathscr{V}$ is a rational vector, belongs to $GL(n, \mathbf{Q})$ and $R^{-1}SR = T$. This concludes the proof.

Say that $U \in GL(n, \mathbf{Z})$ is *hyperbolic* if it has the eigenvalue 1 with multiplicity $n - 2$ and two real eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$. Note that U has infinite order and $\det U = 1$. The eigenvectors of eigenvalues λ and λ^{-1} span a 2-dimensional linear subspace of $\mathbf{R}^n$ that we call the *support* of U, denoted by $\mathrm{sp}(U)$. We will say that U is *supported* by $\mathrm{sp}(U)$. Each of the 1-dimensional linear subspaces of $\mathrm{sp}(U)$, spanned by the eigenvectors with eigenvalues λ and λ^{-1}, respectively, will be called a *limit vector-line* of U.

The support and the limit vector-lines of U define, respectively, a projective line and two projective points in $\mathbf{R}P^{n-1}$. They will be called, *axis* of U and *limit points* of U.

Denote by S_t^{m-1} the $(m-1)$-sphere $S_t^{m-1} = \{v \in \mathbf{R}^m : v^t v = t\}$. Any n-ary indefinite quadratic form is real equivalent to a diagonal quadratic form

$$F = \left\langle \overbrace{1, \dots, 1}^{r}, \overbrace{-1, \dots, -1}^{s} \right\rangle, \qquad n = r + s,$$

and we will say that it has *signature* (r, s). A vector $v = (x_1, \dots, x_r, y_1, \dots, y_s) \in \mathbf{R}^n$, lying in the intersection of S_1^{n-1} with the cone $v^t F v = 0$, satisfies the equations

$$x_1^2 + \cdots + x_r^2 = \frac{1}{2}$$

$$y_1^2 + \cdots + y_s^2 = \frac{1}{2}.$$

Therefore, this intersection is $S_{1/2}^{r-1} \times S_{1/2}^{s-1}$.

The hyperquadric $\mathbf{Q}_F \subset \mathbf{R}P^{n-1}$ defined by the cone $v^t F v = 0$ is homeomorphic to the quotient of $S_{1/2}^{r-1} \times S_{1/2}^{s-1}$ by the antipodal map. The subspaces $\mathbf{R}^r \times \{0\}$ and $\{0\} \times \mathbf{R}^s$ define linear projective subvarieties of $\mathbf{R}P^{n-1}$, denoted by $\mathbf{L}_+$ and $\mathbf{L}_-$. Given a particular point $\mathbf{q} \in \mathbf{Q}_F$ it determines univocally two points $\mathbf{q}_+ \in \mathbf{L}_+$ and $\mathbf{q}_- \in \mathbf{L}_-$, defined as follows. The point $\mathbf{q}$ and $\mathbf{L}_-$ (resp. $\mathbf{L}_+$) span a projective subvariety intersecting $\mathbf{L}_+$ (resp. $\mathbf{L}_-$) in one point, denoted by $\mathbf{q}_+$ (resp. $\mathbf{q}_-$).

The point $\mathbf{q}$ has a neighborhood basis $\mathscr{B}_q$ in $\mathbf{Q}_F$ whose members are constructed as follows. Take a basis of convex neighborhoods $\mathbf{Y}_+$ (resp. $\mathbf{Y}_-$) of $\mathbf{q}_+$ (resp. $\mathbf{q}_-$) in $\mathbf{L}_+$ (resp. $\mathbf{L}_-$). Join every point in $\mathbf{Y}_+$ with every point in $\mathbf{Y}_-$ with a projective line. The union of all points lying in these projective lines is the *join* of $\mathbf{Y}_+$ and $\mathbf{Y}_-$, denoted by $\mathbf{Y}_+\mathbf{Y}_-$. We assume that $\mathbf{Y}_+$ and $\mathbf{Y}_-$ are so small that the intersection of $\mathbf{Y}_+\mathbf{Y}_-$ with $\mathbf{Q}_F$ has always two connected components. Denote by $(\mathbf{Y}_+\mathbf{Y}_-)_q$ the connected component containing $\mathbf{q}$. The set of the different $(\mathbf{Y}_+\mathbf{Y}_-)_q$, thus obtained, is the neighborhood basis $\mathscr{B}_q$ of $\mathbf{q}$ in $\mathbf{Q}_F$.

All these concepts are transferred, with analogous notation, to any indefinite n-ary integral quadratic form F of signature (r, s), via the real equivalence with the standard diagonal form of the same signature, discussed above.

Denote by $e_1, \dots, e_n$ the columns of the identity matrix of order n. The following proposition is the key ingredient for the proof of Theorem 1.

Proposition 2. *Let* $F = \langle a_1, \dots, a_r, -b_1, \dots, -b_s \rangle$, *where* $a_1, \dots, a_r$, $b_1, \dots, b_s$ *are positive, square-free integers and* $0 < s \le r < n$, $n = r + s > 2$.

Let $v = \sum_{i=1}^{r} \lambda_i e_i$ *be a vector with integral coordinates* λ_i, *and let* $u = \sum_{i=r+1}^{n} \mu_j e_j$ *be a vector with integral coordinates* μ_i. *Assume* $-f(v)f(u)$ *is not the square of an integer. Then, there is a hyperbolic automorph* U *of* F *supported by the* 2-*dimensional linear space spanned by* v *and* u.

Proof. Working with the r-ary positive definite, integral quadratic form f restricted to the r-dimensional linear space W generated by $e_1, \dots, e_r$, we can complete $v = v_1$ to a vector basis $v_1, \dots, v_r$ of W such that the $r \times r$ matrix $T = (\lambda_{ik})$ is integral, where $v_k = \sum_{i=1}^{r} \lambda_{ik} e_i$, and $f(v_i, v_j) = 0$ if $i \neq j$ (Gram-Schmidt process [4, p. 83]). Similarly, working with the s-ary positive definite, integral quadratic form $-f$ restricted to the s-dimensional linear space W_1 generated by $e_{r+1}, \dots, e_n$, complete $u = u_1$ to a vector basis $u_1, \dots, u_s$ of W_1 such that the $s \times s$ matrix $T_1 = (\mu_{ik})$ is integral, where $u_k = \sum_{i=r+1}^{n} \mu_{ik} e_i$, and $f(u_i, u_j) = 0$ if $i \neq j$. Consider the n-ary integral quadratic form

$$H = T^t \langle a_1, \dots, a_r \rangle T \oplus T_1^t \langle -b_1, \dots, -b_s \rangle T_1 = \langle c_1, \dots, c_r, -d_1, \dots, -d_s \rangle,$$

where $c_k = f(v_k)$ is a positive integer and $d_h = -f(u_h)$ is a positive integer. Note that $H = S^t F S$, where $S = T \oplus T_1$ (the columns of S are the vectors $v_1, \dots, v_r, u_1, \dots, u_s$). Thus, F and H are rationally equivalent.

Moreover, by hypothesis, $c_1 d_1$ is not the square of an integer. The binary integral quadratic form $\langle c_1, -d_1 \rangle$ possesses the automorph

$$V = \frac{1}{2}\left(w \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & d_1 \\ c_1 & 0 \end{bmatrix} \right),$$

where w and t are positive integers satisfying the Pell equation $w^2 - c_1 d_1 t^2 = 4$ in such a way that they make V integral. Since $c_1 d_1$ is not a square, the Pell equation has infinite solutions, and the automorph V is hyperbolic ([2] or [1]).

Now we can complete V to a hyperbolic automorph V_1 of H by letting V_1 act on the vectors

$$e_2, \dots, e_r, \qquad e_{r+2}, \dots, e_s$$

by the identity map. Then the linear space spanned by e_1 and e_{r+1} is the support $\mathrm{sp}(V_1)$ of V_1.

Since $H = S^t F S$, there are finite index subgroups S_F and S_H of $\mathrm{Aut}(F)$ and $\mathrm{Aut}(H)$ such that $S^{-1} S_F S = S_H$ (Proposition 1). Some power V_1^{α} of V_1 belongs to S_H. Then, $U = S V_1^{\alpha} S^{-1}$ is an automorph of F.

Now,

$$Se_1 = v, \qquad Se_{r+1} = u,$$

and

$$Uv_i = SV_1^{\alpha} S^{-1} v_i = SV_1^{\alpha} e_i = Se_i = v_i, \qquad \text{for } i = 2, \dots, r$$

and, similarly, $Uu_i = u_i$, for $i = 2, \dots, s$. Hence U is a hyperbolic automorph of F supported by

$$\mathrm{sp}(U) = S(\mathrm{sp}(V_1)),$$

which is spanned by u and v. This completes the proof.

The proof of Proposition 2 is constructive.

Example 1. *Let F be the diagonal indefinite integral quadratic form $\langle 1, 1, -1, -1\rangle$. We want to obtain a hyperbolic automorph U of F with support generated by $v_1 = 2e_1 + e_2$ and $u_1 = e_3$. It must exist because $-f(v_1)f(u_1) = 5$ is not the square of an integer. To obtain U, we first complete v_1 to a basis $v_1, v_2 = e_1 - 2e_2$ and, then, we complete u_1 to the basis $u_1, u_2 = e_4$. Then*

$$S = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $H = S^t FS = \langle 5, 5, -1, -1\rangle$. The binary integral quadratic form $\langle c_1, -d_1\rangle = \langle 5, -1\rangle$ possesses the automorph

$$V = \frac{1}{2}\left(w\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & d_1 \\ c_1 & 0 \end{bmatrix}\right),$$

where w and t are positive integers satisfying the Pell equation $w^2 - 5t^2 = 4$, so that V is integral. Take $w = 18$, $t = 8$. Then,

$$V = \begin{bmatrix} 9 & 4 \\ 20 & 9 \end{bmatrix}$$

is an automorph of $\langle 5, -1\rangle$.

Now, we can complete V to the automorph

$$V_1 = \begin{bmatrix} 9 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 20 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

of h, by letting V_1 act on the vectors e_2, e_4 by the identity map. The rational automorph SV_1S^{-1} of F is

$$SV_1S^{-1} = \begin{bmatrix} \frac{37}{5} & \frac{16}{5} & 8 & 0 \\ \frac{16}{5} & \frac{13}{5} & 4 & 0 \\ 8 & 4 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and its second power,

$$U = SV_1^2S^{-1} = \begin{bmatrix} 129 & 64 & 144 & 0 \\ 64 & 33 & 72 & 0 \\ 144 & 72 & 161 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

is already an integral automorph of $F = \langle 1, 1, -1, -1 \rangle$. *The eigenvalues of this automorph* U *are* $\lambda = 161 + 72\sqrt{5}$, $\lambda^{-1} = 161 - 72\sqrt{5}, 1$ *and* 1, *corresponding, respectively, to the eigenvectors*

$$a = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1, 0\right)$$

$$b = \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 1, 0\right)$$

$$e_4 = (0, 0, 0, 1)$$

and

$$v_2 = (1, -2, 0, 0).$$

The support of U *is generated by the first two eigenvectors. It is also generated by* $v_1 = \sqrt{5}a$ *and* $u_1 = \frac{1}{2}(a + b)$.

Proposition 3. *Let* F *be an* n*-ary indefinite integral quadratic form. Assume* $U \in \mathrm{Aut}(F)$ *is hyperbolic. Then, the limit points of* U *belong to the hyperquadric* $\mathbf{Q}_F$.

Proof. Let $\mathbf{q} = [q]$ be a limit point of U. Then $Uq = \tau q$, where $\tau \neq \pm 1$ is a real number. Then

$$f(q) = f(Uq) = \tau^2 f(q)$$

implies $f(q) = 0$. This concludes the proof.

Proposition 4. *Let* F *be a binary positive definite integral quadratic form that is not necessarily primitive. Let* u_1, u_2 *be two* $\mathbf{Q}$*-linearly independent elements of* $\mathbf{Z}^2$. *Then, there is a pair of non-negative integers* x, y, *such that*

$$f(xu_1 + yu_2)$$

is not the square of an integer.

Proof. If $f(u_1)$ or $f(u_2)$ is not the square of an integer, there is nothing to prove. Assume then, that $f(u_1) = \lambda^2$ and $f(u_2) = \mu^2$, where $\lambda > 0$, $\mu > 0$ are integers. Let $v_1 = \mu u_1$ and $v_2 = \lambda u_2$. Then $f(v_1) = f(v_2) = \alpha^2$, where

$\alpha = \lambda\mu > 0$. If $f(v_1 + v_2) = 2\alpha^2 + 2f(v_1, v_2)$ is not the square of an integer, there is nothing to prove. Assume then, that $f(v_1 + v_2)$ is the square of a, necessarily even, positive integer 2β. Then $f(v_1, v_2) = 2\beta^2 - \alpha^2$. Hence

$$f(xv_1 + yv_2) = \alpha^2(x - y)^2 + 4\beta^2 xy = (x, y)G\begin{pmatrix} x \\ y \end{pmatrix},$$

$$G = \begin{bmatrix} \alpha^2 & 2\beta^2 - \alpha^2 \\ 2\beta^2 - \alpha^2 & \alpha^2 \end{bmatrix} = T^t F T,$$

where T is the integral matrix whose columns are the coordinates of the vectors v_1, v_2. Since these vectors are $\mathbf{Q}$-linearly independent, G is a positive definite integral quadratic form. In particular,

$$\det G = 4\beta^2(\alpha^2 - \beta^2) > 0.$$

Hence $\alpha > \beta > 0$.

Take any integer $z \geq \alpha^2$. Then, $\alpha^2 - \beta^2 < 2z + 1$ and $z > \beta$. Then,

$$z^2 < z^2 + \alpha^2 - \beta^2 < (z + 1)^2.$$

Hence, $z^2 + \alpha^2 - \beta^2 = m$ is not the square of an integer. Setting $x = z - \beta > 0$ and $y = z + \beta > 0$, we have that

$$\alpha^2(x - y)^2 + 4\beta^2 xy = 4\beta^2(z^2 + \alpha^2 - \beta^2) = 4\beta^2 m$$

is not the square of an integer. This completes the proof.

Theorem 2. *Let F be an n-ary indefinite integral quadratic form. Assume $n > 2$. Let H be a finite index subgroup of* $\mathrm{Aut}(F)$. *Then, the union of the limit-points of all the axes of the hyperbolic elements of H is dense in the hyperquadric $\mathbf{Q}_F$.*

Proof. The set of axes of the hyperbolic elements of H coincides with the set of axes of the hyperbolic elements of $\mathrm{Aut}(F)$. Indeed, if $U \in \mathrm{Aut}(F)$ is a hyperbolic automorph, supported by $\mathrm{sp}(U)$, the powers of U have the same support. Since some power of U belongs to H, the axis of U is the axis of a hyperbolic element of H. That some power of U belongs to H is a consequence of H having finite index in Aut f, since this implies that there is a finite index subgroup $K \leq H \leq \mathrm{Aut}(F)$ which is normal in $\mathrm{Aut}(F)$.

Note that the quadratic form F is rationally equivalent to a diagonal integral quadratic form G (see [2, Theorem 1]). Then F and G are projectively equivalent. By Proposition 1, there is $M \in GL(n, \mathbf{R})$ such that $M^{-1} S_F M = S_G$, where S_F and S_G are finite index subgroups of $\mathrm{Aut}(F)$ and $\mathrm{Aut}(G)$, respectively, and also $M^t F M = G$. Hence the union of the limit-points of all the axes of the hyperbolic elements of H is dense in the hyperquadric $\mathbf{Q}_F$ if and only if the union of the limit-points of all the axes of the hyperbolic elements of $\mathrm{Aut}(G)$ is dense in the hyperquadric $\mathbf{Q}_G$. Thus, we may assume that F is diagonal and $H = \mathrm{Aut}(F)$. Since F and $-F$ have the same axes, we may assume also that $F = \langle a_1, \dots, a_r, -b_1, \dots, -b_s \rangle$, where $a_1, \dots, a_r, b_1, \dots, b_s$ are positive square-free integers and $s \leq r < n$.

Let W_+ be the r-dimensional linear space generated by $e_1, \dots, e_r$ and let W_- the s-dimensional linear space generated by $e_{r+1}, \dots, e_n$. The subspaces W_+ and W_- define linear projective subvarieties of $\mathbf{R}P^{n-1}$, denoted by $\mathbf{L}_+$ and $\mathbf{L}_-$. Let $\mathbf{q} = [q]$ be a point in the hyperquadric $\mathbf{Q}_F$. It determines univocally the two points $\mathbf{q}_+ = [q_+] \in \mathbf{L}_+$ and $\mathbf{q}_- = [q_-] \in \mathbf{L}_-$, defined as in the paragraph prior to Proposition 2. Given any neighborhood $\mathbf{Z}$ of $\mathbf{q}$ in $\mathbf{Q}_F$ there are convex neighborhoods $\mathbf{Y}_+$ of $\mathbf{q}_+$ in $\mathbf{L}_+$ and $\mathbf{Y}_-$ of $\mathbf{q}_-$ in $\mathbf{L}_-$ such that $\mathbf{Y} = (\mathbf{Y}_+\mathbf{Y}_-)_q \subset \mathbf{Z}$ is a neighborhood of $\mathbf{q}$ in $\mathbf{Q}_F$.

If there are projective points $[u] \in \mathbf{Y}_+$ and $[v] \in \mathbf{Y}_-$ such that the vectors u and v satisfy the hypothesis of Proposition 2, then, there is a hyperbolic automorph U of F supported by the 2-dimensional linear space generated by v and u. Then, the axis of U is the projective line passing through $[u] \in \mathbf{Y}_+$ and $[v] \in \mathbf{Y}_-$. The axis of U intersects the hyperquadric $\mathbf{Q}_F$ in the limit points of U (Proposition 3). One of these limit points belongs to $\mathbf{Z}$. Hence, the proof will be completed by proving that there are projective points $[u] \in \mathbf{Y}_+$ and $[v] \in \mathbf{Y}_-$, such that the vectors u and v satisfy the hypothesis of Proposition 2.

Say that a vector u of $\mathbf{R}^n$ (and the corresponding projective point $[u]$) is *integral* if u has integral coordinates. First, note that the set of integral points of $\mathbf{R}P^{n-1}$ is dense because they correspond to affine points with rational coordinates in the standard affine charts, $x_1 = 1, \dots, x_n = 1$, covering $\mathbf{R}P^{n-1}$.

Take an arbitrary integral point $[v] \in \mathbf{Y}_-$. Write $-f(v) = a$, where a is a positive integer. Take two arbitrary integral points $[u_1] \neq [u_2] \in \mathbf{Y}_+$. Applying Proposition 4 to the positive definite, integral quadratic form af, restricted to the span of u_1 and u_2, there exist integers $x \geq 0$ and $y \geq 0$ such that $af(u) = m$ is not the square of an integer, where $u = xu_1 + yu_2$. Note that $[u] \in \mathbf{Y}_+$, since $\mathbf{Y}_+$ is convex. Then, $u \in W_+$ and $v \in W_-$ satisfy the hypothesis of Proposition 2, because

$$-f(v)f(u) = m$$

is not the square of an integer. This completes the proof.

Though we will not make use of it, we record the following Corollary.

Corollary 1. *Let F be an n-ary indefinite integral quadratic form F, $n > 2$. Then, any finite index subgroup H of* Aut f *is* $\mathbf{R}$*-irreducible. Namely, the only linear subspaces of* $\mathbf{R}^n$ *left globally invariant by all the elements of H are of dimensions zero and n.*

Proof. By Theorem 2, the only linear subspaces of $\mathbf{R}^n$ left globally invariant by all hyperbolic elements of H are of dimensions zero and n. Hence H is $\mathbf{R}$-irreducible.

Now, we are ready to prove that (2) implies (3) in Theorem 1. For $n = 2$, there is an *ad hoc* proof in [3, Proposition 4]. Therefore we assume $n > 2$.

Assume that there are finite index subgroups $S_F \le \mathrm{Aut}(F)$ and $S_G \le \mathrm{Aut}(G)$ which are conjugate in $GL(n, \mathbf{R})$. By Lemma 1, they are conjugate in $GL(n, \mathbf{Q})$. Let $R \in GL(n, \mathbf{Q})$ be such that

$$R^{-1} S_F R = S_G.$$

Write $R = \frac{1}{m} E$, where E is integral and m is an integer. Then

$$R^{-1} S_F R = E^{-1} S_F E = S_G.$$

Let $H = E^t F E$. Then H is an n-ary integral, indefinite quadratic form such that $S_G \le \mathrm{Aut}(H)$. By Proposition 3 and Theorem 2, $\mathbf{Q}_G \subset \mathbf{Q}_H$. Then $H = \lambda G$, where $\lambda \in \mathbf{Q}$ since H and G are rational. Hence $E^t F E = \lambda G$, where $\lambda \in \mathbf{Q}$ and E is integral. That is, $F \overset{P}{\sim} G$. This completes the proof of Theorem 1.

References

[1] J. W. S. Cassels, Rational Quadratic Forms, Academic Press, London, New York, San Francisco, 1978.

[2] B. W. Jones, The Arithmetic Theory of Quadratic Forms, Carus Math. Monographs No. 10, MAA, Baltimore, 1950.

[3] J. M. Montesinos-Amilibia, On integral quadratic forms having commensurable groups of automorphisms, Hiroshima Math. J. 43, 371–411 (2013).

[4] M. Newman, Integral Matrices, Academic Press, London, New York, San Francisco, 1972.

[5] J. Souto, Private communication, November 22, 2013.

José María Montesinos-Amilibia
Facultad de Matemáticas
Universidad Complutense
28040 *Madrid*
Spain
E-mail: montesin@mat.ucm.es