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Representations of solutions, translation formulae and asymptotic behavior in discrete linear systems and periodic continuous linear systems

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Abstract. We give a method for studying of asymptotic behavior of solutions to periodic continuous linear systems and discrete linear systems. It is based on a representation of solutions given in the paper, which is a reformation of the variation of constants formula into the sum of a τ-periodic function and an exponential-like function. By using such representations, the set of initial values is completely classified according to the asymptotic behavior of the solutions to the continuous system. In particular, the set of initial values of bounded solutions is precisely determined. To give the representation for the continuous system, we will establish translation formulae by comparing two representations of solutions to a discrete linear system. These two representations are deeply related to the binomial coefficients, the Bernoulli numbers and the Stirling numbers.

1. Introduction

Let $\mathbf{C}$ be the set of all complex numbers and $\mathbf{R}$ the set of all real numbers. We set $\mathbf{N} = \{1, 2, \dots\}$, $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$ and $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$.

We consider periodic linear inhomogeneous differential equations of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \qquad x(0) = w, \tag{1}$$

and linear difference equations of the form

$$x_{n+1} = Bx_n + b, \qquad x_0 = w, \qquad n \in \mathbf{N}_0, \tag{2}$$

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where $A(t)$ is a periodic continuous $p \times p$ matrix function with period $\tau > 0$, $f : \mathbf{R} \to \mathbf{C}^p$ a τ-periodic continuous function, B a complex $p \times p$ matrix and $b \in \mathbf{C}^p$.

Criteria on the existence of τ-periodic solutions and bounded solutions to the equation (1) have been considered in the literature, e.g., [3, 4, 6, 7, 13, 17, 19, 20, 24]. Among them, a fundamental result on the existence of a τ-periodic solution of (1) is Massera's theorem: the equation (1) has a bounded solution on $\mathbf{R}_+ := [0, \infty)$ if and only if it has a τ-periodic solution. Massera's theorem is rephrased in terms of sets of initial values as follows: $\mathbf{IB} \neq \emptyset$ if and only if $\mathbf{IP} \neq \emptyset$, where $\mathbf{IB}$ and $\mathbf{IP}$, respectively, are the sets of initial values at $t = 0$ for all bounded solutions on $\mathbf{R}_+$ and for all τ-periodic solutions. A more important problem, as we believe, is to explicitly determine the sets $\mathbf{IB}$ and $\mathbf{IP}$; however, this problem has not been throughly resolved. We emphasize that it is generally not so easy to describe the set $\mathbf{IB}$. This is a motivation of the present article.

As a special case, if $A(t) = A$, a constant matrix, in the equation (1), the above problem was studied for the first time and was completely solved by Kato, Naito and Shin [13]. Their approach is based on a new representation (Lemma 16) of solutions to the equation (1) with $A(t) = A$, in which the solutions are expressed as the sum of a τ-periodic function and an exponential-like function. Such a representation of solutions is obtained by transforming a new representation of solutions of the discrete linear system

$$x_{n+1} = e^{\tau A} x_n + b, \qquad x_0 = w, \qquad n \in \mathbf{N}_0 \tag{3}$$

into the continuous linear system (1).

The purpose of the present paper is to give a method for investigating the asymptotic behavior of solutions to the periodic continuous linear system (1), following the lines of arguments in [13]. It is based on a representation of solutions, which is a reformulation of the variation of constants formula into the sum of a τ-periodic function and an exponential-like function. By using the representation, the sets of initial values are completely classified according to the asymptotic behavior of the corresponding solutions to the continuous system (1). In particular, the set $\mathbf{IB}$ of the initial values of bounded solutions is precisely characterized.

Firstly, we give the representation (Theorem 6) of solutions to the discrete linear equation (2) by introducing characteristic quantities described by the initial value w, the inhomogeneous term b and the projection from $\mathbf{C}^p$ to the generalized eigenspace of B. It has a form different from a result (Theorem 9) in [13] for the case where $B = e^{\tau A}$.

Secondly, we establish translation formulae. As mentioned above, there are two different representations of solutions to the equation (2) for the case where $B = e^{\tau A}$; one (Theorem 9) is based on A and the other (Theorem 6) is based on B. They describe the same solution, but they are of different forms in appearance. Therefore it is very important to investigate the mathematical mechanism of translation from the representation in terms of B into the one in terms of A, and vice versa. By comparing the two representations of solutions, we establish translation formulae (Theorem 11), which are deeply related to the binomial coefficients, the Bernoulli numbers and the Stirling numbers. The translation formulae play an essential role in the proof of our representation theorem of solutions for the periodic continuous linear system (1).

Thirdly, we give novel representations (Theorem 1) of solutions to the equation (1). It is well-known that the solution of the equation (1) is given as

$$x(t; w, f) = U(t, 0)w + \int_0^t U(t, s)f(s)ds \qquad (t \in \mathbf{R}), \tag{4}$$

by using the variation of constants formula, where $U(t, s)$ stands for the solution operator of the associated homogeneous equation

$$\frac{d}{dt}x(t) = A(t)x(t). \tag{5}$$

To investigate asymptotic behavior of solutions for equations (1), we have to analyze the integral term in the right side of (4). However the analysis is not easy. Moreover, it seems that the periodicity of the equation is not explicitly reflected in the representation (4). We therefore transform (4) into the sum of a τ-periodic function and an exponential-like function.

The idea of our proof is stated as follows. By Floquet's theorem it is well-known that the equation (1) is reduced to the equation of the form $y'(t) = Ay(t) + g(t)$. For this equation, a representation of solutions has been already obtained in [13]. As a result, a representation of solutions to the equation (1) is immediately obtained, which depends on the characteristic exponents. However, it is not easy to obtain a representation of solutions in terms of the characteristic multipliers. To get over this difficult, we will utilize translation formulae as mentioned earlier.

Finally, as applications of the precceding results, we completely characterize the asymptotic behavior of solutions of (1). The set $\mathbf{C}^p$ of initial values of the solutions is completely classified according to the asymptotic behavior of solutions to the equation (1). In particular, the set **IB** for (1) is exactly determined in the concrete fashion (see Theorem 4).

We give the main results in the first half (section 2) and their proofs in the latter half (sections 3, 4, and 5) of the present paper.

2. Main results

In this section we give main results together with some terminologies and notations. First, we give representations of solutions of the equation (1). Next, we systematically and completely characterize asymptotic behavior, boundedness and periodicity of solutions to the equation (1).

2.1. Representations of solutions. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H, and by $h_H(\eta)$ the index of $\eta \in \sigma(H)$. Let $G_H(\eta) = \mathbf{N}((H - \eta E)^{h_H(\eta)})$ be the generalized eigenspace corresponding to $\eta \in \sigma(H)$, where E is the unit matrix. Let $Q_\eta = Q_\eta(H) : \mathbf{C}^p \to G_H(\eta)$ be the projection corresponding to the direct sum decomposition

$$\mathbf{C}^p = \bigoplus_{\eta \in \sigma(H)} G_H(\eta).$$

In particular, let H be related by $H = e^{\tau A}$, $\tau > 0$. Then by the spectral mapping theorem we see that $\sigma(H) = e^{\tau\sigma(A)}$ and

$$\sigma_\mu(A) := \{\lambda \in \sigma(A) \mid \mu = e^{\tau\lambda}\} \neq \emptyset$$

for every $\mu \in \sigma(H)$. Moreover, the following relations hold:

$$h_H(\mu) = \max\{h_A(\lambda) \mid \lambda \in \sigma_\mu(A)\},$$

$$G_H(\mu) = \bigoplus_{\lambda \in \sigma_\mu(A)} G_A(\lambda) \tag{6}$$

and

$$HQ_\lambda(A) = Q_\lambda(A)H, \qquad Q_\mu = \sum_{\lambda \in \sigma_\mu(A)} Q_\lambda(A),$$

$$Q_\lambda(A)Q_\mu = Q_\lambda(A) \qquad (\lambda \in \sigma_\mu(A)). \tag{7}$$

The k-th derivative $a^{(k)}(z)$ of the function $a(z) = (z-1)^{-1}$ $(z \neq 1)$ is given by

$$a^{(k)}(z) = -k!(1-z)^{-k-1}. \tag{8}$$

For any $\mu \in \sigma(H)$ with $\mu \neq 1$, a matrix $Z_\mu(H)$ is defined by

$$Z_\mu(H) = \sum_{k=0}^{h_H(\mu)-1} \frac{a^{(k)}(\mu)}{k!}(H - \mu E)^k = -\sum_{k=0}^{h_H(\mu)-1} \frac{1}{(1-\mu)^{k+1}}(H - \mu E)^k \quad (\mu \neq 1).$$

Now we will introduce characteristic quantities: for $\mu \in \sigma(H)$ and $w, b \in \mathbf{C}^p$,

$$\gamma_\mu(w, b; H) = Q_\mu w + Z_\mu(H)Q_\mu b \qquad (\mu \neq 1),$$

and

$$\delta(w, b; H) = (H - E)Q_1 w + Q_1 b \qquad (\mu = 1).$$

The solution operator $U(t,s) : \mathbf{C}^p \to \mathbf{C}^p$, $t, s \in \mathbf{R}$ is defined by

$$U(t,s)w = u(t; s, w),$$

where $u(t; s, w)$ is the unique solution of the equation (5) with the initial condition $u(s) = w \in \mathbf{C}^p$. Since $U(\tau, 0)$ is a nonsingular matrix, we can take a matrix A such that $U(\tau, 0) = e^{\tau A}$. Define $P(t) = U(t, 0)e^{-tA}$. Then it is easy to see that $P(t + \tau) = P(t)$. Thus we have the Floquet representation $U(t, 0) = P(t)e^{tA}$. The period map $V(t)$, $t \in \mathbf{R}$ is defined by $V(t) = U(t, t - \tau) = U(t + \tau, t)$, from which it follows that $V(t + \tau) = V(t)$, $t \in \mathbf{R}$ and $V(t)U(t,s) = U(t,s)V(s)$, $t, s \in \mathbf{R}$. Note that $\sigma(V(t)) = \sigma(V(0))$ holds (see Lemma 13). For $\mu \in \sigma(V(0))$ the projection $Q_\mu(t) = Q_\mu(V(t)) : \mathbf{C}^p \to G_{V(t)}(\mu)$ has the property $Q_\mu(t)U(t,s) = U(t,s)Q_\mu(s)$. Set

$$b_f = \int_0^\tau U(\tau, s) f(s) ds$$

in the equation (1). Then characteristic quantities are defined by

$$\gamma_\mu(w, b_f) = \gamma_\mu(w, b_f; V(0)), \qquad \delta(w, b_f) = \delta(w, b_f; V(0)).$$

The eigenvalues of $V(0) = e^{\tau A}$ and A are called the characteristic multiplier and the characteristic exponent of $U(t,s)$, respectively. Now we introduce a matrix $S_\mu(t)$ to change $e^{t\lambda}$ for $\lambda \in \sigma_\mu(A)$ to a form of $\mu \in \sigma(V(0))$. For any characteristic multiplier $\mu \in \sigma(V(0))$ we take a characteristic exponent ρ, $(-\pi < \Im(\tau\rho) \le \pi)$ such that $\mu = e^{\tau\rho}$. Define μ^t by $\mu^t = e^{t\tau\rho}$ and set

$$S_\mu(t) = \mu^{-t/\tau} \sum_{\lambda \in \sigma_\mu(A)} e^{t\lambda} P_\lambda,$$

where $P_\lambda = Q_\lambda(A)$, $\lambda \in \sigma(A)$. Then $S_\mu(t)$ is τ-periodic. In fact, by choosing a ρ $(-\pi < \Im(\tau\rho) \le \pi)$ such that $\mu = e^{\tau\rho}$ for $\lambda \in \sigma_\mu(A)$ we get

$$e^{t\lambda} = e^{(t/\tau)\tau\lambda} = e^{(t/\tau)\tau\rho} e^{(t/\tau)(\tau\lambda - \tau\rho)} = \mu^{t/\tau} e^{(t/\tau)(\tau\lambda - \tau\rho)},$$

which implies that

$$\sum_{\lambda \in \sigma_\mu(A)} e^{t\lambda} P_\lambda = \mu^{t/\tau} \sum_{\lambda \in \sigma_\mu(A)} e^{(t/\tau)(\tau\lambda - \tau\rho)} P_\lambda.$$

Hence we obtain

$$S_\mu(t) = \sum_{\lambda \in \sigma_\mu(A)} e^{(t/\tau)(\tau\lambda - \tau\rho)} P_\lambda.$$

Since $e^{\tau\lambda} = e^{\tau\rho} = \mu$, there is an integer $n(\lambda)$ such that $\tau\lambda - \tau\rho = 2n(\lambda)\pi i$. Therefore $S_\mu(t)$ is τ-periodic.

Put $R_\mu(t) = P(t)S_\mu(t)$. Then $R_\mu(t)$ is also τ-periodic.

Now we state a representation theorem of solutions to the equation (1).

Theorem 1. *Let $\mu \in \sigma(V(0))$. The component $Q_\mu(t)x(t)$ of solutions $x(t)$ of the equation* (1) *satisfying the initial condition $x(0) = w$ is expressed as follows:*

1) *Let $\mu \neq 1$. Then $Q_\mu(t)x(t)$ is expressed as*

$$Q_\mu(t)x(t) = U(t,0)\gamma_\mu(w,b_f) + h_\mu(t,f) \qquad (t \in \mathbf{R}) \tag{9}$$

$$= R_\mu(t)\mu^{t/\tau} \sum_{k=0}^{h_{V(0)}(\mu)-1} \left(\frac{t}{\tau}\right)_k \frac{1}{k!\mu^k}(V(0) - \mu E)^k \gamma_\mu(w,b_f)$$
$$+ h_\mu(t,f) \qquad (t \in \mathbf{R}), \tag{10}$$

and $h_\mu(t,f)$ is a τ-periodic solution of the equation (1) *in $G_{V(t)}(\mu)$, where*

$$h_\mu(t,f) = -U(t,0)Z_\mu(V(0))Q_\mu(0)b_f + \int_0^t U(t,s)Q_\mu(s)f(s)ds.$$

2) *Let $\mu = 1$. Then $Q_\mu(t)x(t)$ is expressed as*

$$Q_1(t)x(t) = R_1(t) \sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!}(V(0) - E)^k \delta(w,b_f)$$
$$+ R_1(t)Q_1(0)w + h_1(t,f) \qquad (t \in \mathbf{R}) \tag{11}$$

and $h_1(t,f)$ is a τ-periodic continuous function, where

$$h_1(t,f) = -R_1(t) \sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!}(V(0) - E)^k Q_1(0)b_f$$
$$+ \int_0^t U(t,s)Q_1(s)f(s)ds.$$

We give a simple example of Theorem 1. Consider the one dimensional periodic linear differential equation of the type

$$\frac{d}{dt}x(t) = a(t)x + f(t), \tag{12}$$

where $a(t)$ and $f(t)$ are τ-periodic and continuous real-valued functions.

Set

$$\alpha(t,s) = \int_s^t a(r)dr \quad (t,s \in \mathbf{R}) \qquad \text{and} \qquad \alpha(t) = \alpha(t,0).$$

The solution operator $U(t,s)$ of the homogeneous equation associated with the equation (12) is given by $U(t,s) = e^{\alpha(t,s)}$. Since $\alpha(t+\tau,t) = \alpha(\tau)$, the period map $V(t)$ has the property $V(t) = V(0) = e^{\alpha(\tau)}$ for all $t \in \mathbf{R}$, and hence $Q_\mu(t) = Q_\mu(0) = 1$. Obviously, $\sigma(V(0)) = \{\mu\}$, $\mu = e^{\alpha(\tau)}$. It is easy to verify that $h_{V(0)}(\mu) = 1$. Setting $m(a) = \frac{\alpha(\tau)}{\tau}$, we have that $V(0) = e^{\tau m(a)}$. Thus its characteristic exponent λ is given by $\lambda = m(a)$, and $P_\lambda = 1$. Hence $\mu^{t/\tau} = e^{tm(a)}$. By Floquet's Theorem $U(t,0)$ is expressed as $U(t,0) = P(t)e^{tm(a)}$. Therefore $S_\mu(t)$ and $R_\mu(t)$ are given as

$$S_\mu(t) = \mu^{-t/\tau}e^{\lambda t}P_\lambda = e^{-tm(a)}e^{tm(a)} = 1,$$

and

$$R_\mu(t) = P(t)S_\mu(t) = P(t) = e^{\alpha(t)-tm(a)} = e^{\alpha(t)-(t/\tau)\alpha(\tau)},$$

respectively. If $\alpha(\tau) \neq 0$, then $\mu \neq 1$ and

$$Z_\mu(V(0)) = \frac{1}{\mu - 1}, \qquad \gamma_\mu(w, b_f) = w + \frac{1}{\mu - 1}b_f.$$

If $\alpha(\tau) = 0$, then $\mu = 1$ and $h_{V(0)}(1) = 1$. Thus $R_1(t) = e^{\alpha(t)}$ and $\delta(w, b_f) = b_f$.

By Theorem 1, the solution $x(t; w, f)$ of the equation (12) is given as follows.

Proposition 1.

1) *Let* $\mu \neq 1$. *Then the solution* $x(t; w, f)$ *of the equation* (12) *is expressed by*

$$\begin{aligned} x(t; w, f) &= e^{\alpha(t)}\left(w + \frac{1}{\mu - 1}b_f\right) + h_\mu(t, f) \\ &= e^{\alpha(t)-(t/\tau)\alpha(\tau)}e^{(t/\tau)\alpha(\tau)}\left(w + \frac{1}{\mu - 1}b_f\right) + h_\mu(t, f) \end{aligned}$$

and $h_\mu(t, f)$ *is a* τ*-periodic solution to the equation* (12), *where*

$$h_\mu(t, f) = \frac{e^{\alpha(t)}}{1 - \mu}b_f + \int_0^t e^{\alpha(t,s)}f(s)ds = \frac{e^{\alpha(t)}}{1 - \mu}\int_t^{t+\tau} e^{\alpha(\tau,s)}f(s)ds.$$

2) *Let* $\mu = 1$. *Then the solution* $x(t; w, f)$ *of the equation* (12) *is expressed by*

$$x(t; w, f) = \frac{te^{\alpha(t)}}{\tau} b_f + e^{\alpha(t)} w + h_1(t, f)$$

and $h_1(t, f)$ *is a* τ*-periodic function, where*

$$h_1(t, f) = -\frac{te^{\alpha(t)}}{\tau} b_f + \int_0^t e^{\alpha(t,s)} f(s) ds.$$

As a special case of Theorem 1, we consider the case where $|\mu| \neq 1$, $\mu \in \sigma(V(0))$. Set

$$\sigma_+(V(0)) = \{\mu \mid |\mu| > 1\} \qquad \text{and} \qquad \sigma_-(V(0)) = \{\mu \mid |\mu| < 1\}.$$

Then $\sigma(V(0)) = \sigma_+(V(0)) \cup \sigma_-(V(0))$.

Theorem 2. *Let* $\mu \in \sigma(V(0))$.

1) *If* $\mu \in \sigma_+(V(0))$, *then*

$$Z_\mu(V(0)) Q_\mu(0) b_f = \int_0^\infty U(0, s) Q_\mu(s) f(s) ds,$$

$$Q_\mu(t) x(t) = U(t, 0) \gamma_\mu(w, b_f) - \int_t^\infty U(t, s) Q_\mu(s) f(s) ds, \tag{13}$$

and the integral term with minus sign in (13) *is a continuous* τ*-periodic solution of the equation* (1) *in* $G_{V(t)}(\mu)$.

2) *If* $\mu \in \sigma_-(V(0))$, *then*

$$Z_\mu(V(0)) Q_\mu(0) b_f = -\int_{-\infty}^0 U(0, s) Q_\mu(s) f(s) ds,$$

$$Q_\mu(t) x(t) = U(t, 0) \gamma_\mu(w, b_f) + \int_{-\infty}^t U(t, s) Q_\mu(s) f(s) ds, \tag{14}$$

where the integral term in (14) *is a continuous* τ*-periodic solution of the equation* (1) *in* $G_{V(t)}(\mu)$.

2.2. Asymptotic behavior. As applications of Theorem 1 and Theorem 2, we characterize asymptotic behavior, boundedness and periodicity of solutions to the equation (1).

First, we shall state general results on asymptotic behavior of solutions to the equation (1). To do so, we introduce the following concept: an index $d(\mu)$,

$\mu \in \sigma(V(0))$, of growth order for the component $Q_\mu w$ of the initial value w to the equation (1) is defined as follows:

If $\mu \neq 1$, then $d(\mu) = 0$ in the case that $\gamma_\mu(w, b_f) = 0$; otherwise, $d(\mu)$ is a positive integer such that

$$(V(0) - \mu E)^{d(\mu)-1}\gamma_\mu(w, b_f) \neq 0, \qquad (V(0) - \mu E)^{d(\mu)}\gamma_\mu(w, b_f) = 0.$$

If $\mu = 1$, then $d(1) = 0$ in the case that $\delta(w, b_f) = 0$; otherwise, $d(1)$ is a positive integer such that

$$(V(0) - E)^{d(1)-1}\delta(w, b_f) \neq 0, \qquad (V(0) - E)^{d(1)}\delta(w, b_f) = 0.$$

If $\sigma(V(0)) = \{\mu_1, \mu_2, \dots, \mu_s\}$, then we denote by $(d(\mu_1), d(\mu_2), \dots, d(\mu_s))$ the index of growth order for initial value w to the equation (1). Clearly, $d(\mu) \leq h_{V(0)}(\mu)$.

Asymptotic behavior of solutions to the equation (1) is quickly derived in the following theorem.

Theorem 3. *Let $\mu \in \sigma(V(0))$ and let $Q_\mu(t)x(t)$ be the component of the solution $x(t) := x(t; w, f)$ of the equation* (1).

1) *The case where $|\mu| > 1$.*

(1) *If $d(\mu) = 0$, then $Q_\mu(t)x(t)$ is τ-periodic:*

$$Q_\mu(t)x(t) = -\int_t^\infty U(t,s)Q_\mu(s)f(s)ds.$$

(2) *If $d(\mu) = 1$, then $Q_\mu(t)x(t)$ is unbounded on $\mathbf{R}_+$:*

$$Q_\mu(t)x(t) = \mu^{t/\tau}R_\mu(t)\gamma_\mu(w, b_f) + h_\mu(t, f) \to \infty \qquad (t \to +\infty),$$

and $Q_\mu(t)x(t)$ is asymptotically τ-periodic on $\mathbf{R}_-$:

$$Q_\mu(t)x(t) = -\int_t^\infty U(t,s)Q_\mu(s)f(s)ds + o(1) \qquad (t \to -\infty).$$

(3) *If $d(\mu) > 1$, then $Q_\mu(t)x(t)$ is unbounded on $\mathbf{R}_+$:*

$$Q_\mu(t)x(t) = R_\mu(t)\frac{\left(\frac{t}{\tau}\right)^{d(\mu)-1}\mu^{t/\tau}}{(d(\mu)-1)!\mu^{d(\mu)-1}}(V(0) - \mu E)^{d(\mu)-1}\gamma_\mu(w, b_f)$$
$$+ o(t^{d(\mu)-1}\mu^{t/\tau}) \qquad (t \to +\infty),$$

and $Q_\mu(t)x(t)$ is asymptotically τ-periodic on $\mathbf{R}_-$:

$$Q_\mu(t)x(t) = -\int_t^\infty U(t,s)Q_\mu(s)f(s)ds + o(1) \qquad (t \to -\infty).$$

2) *The case where* $|\mu| < 1$.

(1) *If* $d(\mu) = 0$, *then* $Q_\mu(t)x(t)$ *is* τ*-periodic:*

$$Q_\mu(t)x(t) = \int_{-\infty}^{t} U(t,s)Q_\mu(s)f(s)ds.$$

(2) *If* $d(\mu) = 1$, *then* $Q_\mu(t)x(t)$ *is asymptotically* τ*-periodic on* $\mathbf{R}_+$*:*

$$Q_\mu(t)x(t) = \int_{-\infty}^{t} U(t,s)Q_\mu(s)f(s)ds + o(1) \qquad (t \to +\infty),$$

and $Q_\mu(t)x(t)$ *is unbounded on* $\mathbf{R}_-$*:*

$$Q_\mu(t)x(t) = \mu^{t/\tau}R_\mu(t)\gamma_\mu(w, b_f) + h_\mu(t, f) \to \infty \qquad (t \to -\infty).$$

(3) *If* $d(\mu) > 1$, *then* $Q_\mu(t)x(t)$ *is asymptotically* τ*-periodic on* $\mathbf{R}_+$*:*

$$Q_\mu(t)x(t) = \int_{-\infty}^{t} U(t,s)Q_\mu(s)f(s)ds + o(1) \qquad (t \to +\infty),$$

and $Q_\mu(t)x(t)$ *is unbounded on* $\mathbf{R}_-$*:*

$$Q_\mu(t)x(t) = R_\mu(t)\frac{\left(\frac{t}{\tau}\right)^{d(\mu)-1}\mu^{t/\tau}}{(d(\mu)-1)!\mu^{d(\mu)-1}}(V(0) - \mu E)^{d(\mu)-1}\gamma_\mu(w, b_f)$$
$$+ o(t^{d(\mu)-1}\mu^{t/\tau}) \qquad (t \to -\infty).$$

3) *The case where* $|\mu| = 1$, $\mu \neq 1$.

(1) *If* $d(\mu) = 0$, *then* $Q_\mu(t)x(t)$ *is* τ*-periodic:* $Q_\mu(t)x(t) = h_\mu(t, f)$.

(2) *If* $d(\mu) = 1$, *then* $Q_\mu(t)x(t)$ *is bounded on* $\mathbf{R}$*:*

$$Q_\mu(t)x(t) = R_\mu(t)\mu^{t/\tau}\gamma_\mu(w, b_f) + h_\mu(t, f).$$

(3) *If* $d(\mu) > 1$, *then* $Q_\mu(t)x(t)$ *is unbounded on* $\mathbf{R}_+$ *and* $\mathbf{R}_-$*:*

$$Q_\mu(t)x(t) = R_\mu(t)\frac{\left(\frac{t}{\tau}\right)^{d(\mu)-1}\mu^{t/\tau}}{(d(\mu)-1)!\mu^{d(\mu)-1}}(V(0) - \mu E)^{d(\mu)-1}\gamma_\mu(w, b_f)$$
$$+ o(t^{d(\mu)-1}) \qquad (|t| \to \infty).$$

4) *The case where* $\mu = 1$.

(1) *If* $d(1) = 0$, *then* $Q_1(t)x(t)$ *is* τ*-periodic:*

$$Q_1(t)x(t) = R_1(t)Q_1(0)w + h_1(t, f).$$

(2) *If* $d(1) = 1$, *then* $Q_1(t)x(t)$ *is unbounded on* $\mathbf{R}_+$ *and* $\mathbf{R}_-$*:*

$$Q_1(t)x(t) = \frac{t}{\tau}R_1(t)\delta(w, b_f) + R_1(t)Q_1(0)w + h_1(t, f). \tag{15}$$

(3) *If* $d(1) > 1$, *then* $Q_1(t)x(t)$ *is unbounded on* $\mathbf{R}_+$ *and* $\mathbf{R}_-$*:*

$$Q_1(t)x(t) = \left(\frac{t}{\tau}\right)^{d(1)} \frac{1}{d(1)!} R_1(t)(V(0) - E)^{d(1)-1}\delta(w, b_f)$$
$$+ o(t^{d(1)}) \qquad (|t| \to \infty).$$

Proof. The proof is easily derived from Theorem 1 and Theorem 2. □

Next, using Theorem 3, we characterize by initial sets bounded solutions and τ-periodic solutions for the equation (1). The proof immediately follows from Theorem 3.

Theorem 4. *The following statements hold true.*

1 *The solution* $x(t; w, f)$ *of the equation* (1) *is bounded on* $\mathbf{R}_+$ *if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $|\mu| > 1$, *then* $\gamma_\mu(w, b_f) = 0$;
2) *if* $\mu \neq 1$ *and* $|\mu| = 1$, *then* $(V(0) - \mu E)\gamma_\mu(w, b_f) = 0$;
3) *if* $\mu = 1$, *then* $\delta(w, b_f) = 0$.

2 *The solution* $x(t; w, f)$ *of the equation* (1) *is bounded on* $\mathbf{R}$ *if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $|\mu| \neq 1$, *then* $\gamma_\mu(w, b_f) = 0$;
2) *if* $\mu \neq 1$ *and* $|\mu| = 1$, *then* $(V(0) - \mu E)\gamma_\mu(w, b_f) = 0$;
3) *if* $\mu = 1$, *then* $\delta(w, b_f) = 0$.

3 *The solution* $x(t; w, f)$ *of the equation* (1) *is* τ*-periodic if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $\mu \neq 1$, *then* $\gamma_\mu(w, b_f) = 0$;
2) *if* $\mu = 1$, *then* $\delta(w, b_f) = 0$.

As stated in Introduction, the sets **IB** and **IP** are characterized by using Theorem 4. For the case where $A(t) = A$ in Theorem 4, see [13].

If $b_f = 0$ in Theorem 4, then the following result immediately follows from Theorem 4. Clearly, if $f = 0$, then $b_f = 0$.

Corollary 1. *Assume that* $b_f = 0$. *The following statements hold true.*

1 *The solution* $x(t; w, f)$ *of the equation* (1) *is bounded on* $\mathbf{R}_+$ *if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $|\mu| > 1$, *then* $Q_\mu(0)w = 0$;
2) *if* $|\mu| = 1$, *then* $Q_\mu(0)w \in \mathbf{N}(V(0) - \mu E)$.

2 *The solution* $x(t; w, f)$ *of the equation* (1) *is bounded on* $\mathbf{R}$ *if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $|\mu| \neq 1$, *then* $Q_\mu(0)w = 0$;
2) *if* $|\mu| = 1$, *then* $Q_\mu(0)w \in \mathbf{N}(V(0) - \mu E)$.

3 *The solution* $x(t; w, f)$ *of the equation* (1) *is* τ*-periodic if and only if the following conditions hold: For every* $\mu \in \sigma(V(0))$,

1) *if* $\mu \neq 1$, *then* $Q_\mu(0)w = 0$;
2) *if* $\mu = 1$, *then* $Q_\mu(0)w \in \mathbf{N}(V(0) - E)$.

Theorem 4 does not imply the existence of bounded solutions or τ-periodic solutions to the equation (1). The following theorem is concerned with its existence, which is a refined version of Massera's theorem [17]. Its proof is obvious from Theorems 4.

Theorem 5. *The following statements are equivalent.*

1) *The equation* (1) *has a bounded solution on* $\mathbf{R}_+$.
2) $1 \in \sigma(V(0))$ *and there is a* $Q_1(0)w$ *such that* $\delta(w, b_f) = 0$; *or* $1 \notin \sigma(V(0))$.
3) *The equation* (1) *has a* τ*-periodic solution.*

Corollary 2. *Let* $1 \in \sigma(V(0))$ *in the equation* (1). *The following statements are equivalent:*

1) *There is a* $Q_1(0)w$ *such that* $\delta(w, b_f) = 0$;
2) $Q_1(0)b_f \in (V(0) - E)G_{V(0)}(1)$;
3) *There is a* $w \in \mathbf{C}^p$ *such that* $(E - V(0))w = b_f$;

For other well-known conditions concerned with conditions in Corollary 2, refer to [25, pp. 467–469], [19, Theorem 2.2 in Chapter 8] and [6, Theorem 2.3.1, Corollary 2.3.2].

Notice that the results corresponding to Theorem 3, Theorem 4 and Theorem 5 are also proved for the difference equation (2) by using the same argument as above. The descrete version of Theorem 1 is given in the next section.

3. A representation of solutions of discrete linear systems

In this section we give a representation of solutions for the equation (2) with characteristic quantities. To describe representations of solutions, we will state briefly basic facts on the binomial theorem. Let $x \in \mathbf{R}$ and $k \in \mathbf{N}_0$. The well-known factorial function $(x)_k$ is defined by

$$(x)_k = \begin{cases} 1, & (k = 0) \\ x(x-1)(x-2)\dots(x-k+1) & (k \in \mathbf{N}). \end{cases}$$

In particular, if $x = n$ is a positive integer, then

$$\frac{(n)_k}{k!} = \binom{n}{k} := \frac{n!}{k!(n-k)!}, \qquad (n)_k = 0 \quad (k > n).$$

By the binomial theorem the relation

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}=\begin{cases}1 & (n=0)\\ 0 & (n\in\mathbf{N})\end{cases}$$

is easily shown. Moreover, we have

$$\sum_{i=k}^{n-1}\binom{i}{k}=\binom{n}{k+1} \qquad (n\in\mathbf{N}). \tag{16}$$

Indeed, applying the binomial theorem to the relation

$$\sum_{i=0}^{n-1}(1+x)^i=\frac{(1+x)^n-1}{x},$$

and comparing the coefficients of the terms x^k, we obtain the relation (16).

It is well-known that the solution $x_n(w,b)$ of the equation (2) with the initial condition $x_0=w$ is given as

$$x_n(w,b)=B^n w+S_n(B)b, \tag{17}$$

where

$$S_n(B)=\sum_{k=0}^{n-1}B^k \quad (n\in\mathbf{N}), \qquad S_0(B)=O.$$

Noticing that

$$(B-E)x_n(w,b)=B^n[(B-E)w+b]-b,$$

we have that if $1\notin\sigma(B)$, then

$$x_n(w,b)=B^n[w+(B-E)^{-1}b]-(B-E)^{-1}b.$$

The interesting problem is how to deal with the case where $1\in\sigma(B)$. We will employ spectral decomposition methods for this problem. Set $Q_\mu=Q_\mu(B)$ for $\mu\in\sigma(B)$. Applying Q_μ to (17), we have

$$Q_\mu x_n(w,b)=B^n Q_\mu w+S_n(B)Q_\mu b. \tag{18}$$

We will rearrange the right side of the representation (18) by collecting the terms which are the same order with respect to n.

Now we are in a position to state the main theorem in this section.

Theorem 6. *Let $\mu\in\sigma(B)$ and $n\in\mathbf{N}_0$. The component $Q_\mu x_n(w,b)$ of the solution $x_n(w,b)$ of the equation* (2) *is expressed as follows:*

1) *If* $\mu \neq 1$, *then*

$$Q_\mu x_n(w,b) = B^n \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

(1) *If* $\mu \neq 0$, *then*

$$Q_\mu x_n(w,b) = \mu^n \sum_{k=0}^{h_B(\mu)-1} \frac{(n)_k}{k!\mu^k} (B - \mu E)^k \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

(2) *If* $\mu = 0$, *then*

$$Q_0 x_n(w,b) = \begin{cases} -Z_0(B) Q_0 b & (n \geq h_B(0)) \\ B^n \gamma_0(w,b) - Z_0(B) Q_0 b & (n \leq h_B(0) - 1). \end{cases}$$

2) *If* $\mu = 1$, *then*

$$Q_1 x_n(w,b) = \sum_{k=0}^{h_B(1)-1} \frac{(n)_{k+1}}{(k+1)!} (B - E)^k \delta(w,b) + Q_1 w.$$

Corollary 3. *Let* $\mu \in \sigma(B)$ *and* $h_B(\mu) = 1$. *Then the component* $Q_\mu x_n(w,b)$, $n \in \mathbf{N}_0$ *of the solution* $x_n(w,b)$ *of the equation* (2) *is expressed as follows:*

1) *If* $\mu \neq 1$, *then*

$$Q_\mu x_n(w,b) = B^n \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

(1) *If* $\mu \neq 0$, *then*

$$Q_\mu x_n(w,b) = \mu^n \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

(2) *If* $\mu = 0$, *then*

$$Q_0 x_n(w,b) = \begin{cases} -Z_0(B) Q_0 b & (n \geq 1) \\ Q_0 w & (n = 0). \end{cases}$$

2) *If* $\mu = 1$, *then*

$$Q_1 x_n(w,b) = n Q_1 b + Q_1 w. \tag{19}$$

To prove Theorem 6, we will calculate the first term $B^n Q_\mu w$ and the second term $S_n(B) Q_\mu b$ in the right side of (18). Notice that

$$(B - \mu E)^{h_B(\mu)-1} Q_\mu \neq O \quad \text{and} \quad (B - \mu E)^{h_B(\mu)} Q_\mu = O \qquad \text{for } \mu \in \sigma(B).$$

First, the first term $B^n Q_\mu w$ is given in the following lemma. Its proof is easy.

LEMMA 1. *Let* $\mu \in \sigma(B)$.
1) *If* $\mu \neq 0$, *then*

$$B^n Q_\mu = \mu^n \sum_{k=0}^{h_B(\mu)-1} \frac{(n)_k}{k!} \mu^{-k} (B - \mu E)^k Q_\mu, \qquad n = 0, 1, 2, \dots. \tag{20}$$

2) *If* $\mu = 0$, *then*

$$B^n Q_0 = \begin{cases} O & (n \geq h_B(0)) \\ B^n Q_0 & (n \leq h_B(0) - 1). \end{cases}$$

Next, we will calculate the second term $S_n(B) Q_\mu b$.

LEMMA 2. *Let* $\mu \in \sigma(B)$.
1) *If* $\mu \neq 1$, *then*

$$S_n(B) Q_\mu = B^n Z_\mu(B) Q_\mu - Z_\mu(B) Q_\mu.$$

(1) *If* $\mu \neq 0$, *then*

$$S_n(B) Q_\mu = \sum_{k=0}^{h_B(\mu)-1} \frac{(n)_k}{k!} \mu^{n-k} (B - \mu E)^k Z_\mu(B) Q_\mu - Z_\mu(B) Q_\mu.$$

(2) *If* $\mu = 0$, *then*

$$S_n(B) Q_0 = \begin{cases} -Z_0(B) Q_0 & (n \geq h_B(0)) \\ B^n Z_0(B) Q_0 - Z_0(B) Q_0 & (n \leq h_B(0) - 1). \end{cases}$$

2) *If* $\mu = 1$, *then*

$$S_n(B) Q_1 = \sum_{k=0}^{h_B(1)-1} \binom{n}{k+1} (B - E)^k Q_1.$$

PROOF. Set $h = h_B(\mu)$ and $B_\mu = B - \mu E$. It follows from Lemma 1 that

$$S_n(B) Q_\mu = \sum_{i=0}^{n-1} B^i Q_\mu = \sum_{i=0}^{n-1} \sum_{k=0}^{h-1} \frac{(i)_k}{k!} \mu^{i-k} B_\mu^k Q_\mu. \tag{21}$$

First, we consider the case where $\mu \neq 1$, $\mu \neq 0$. Exchanging the order of summation in (21), we have

$$S_n(B) Q_\mu = \sum_{k=0}^{h-1} \frac{1}{k!} \sum_{i=0}^{n-1} (i)_k \mu^{i-k} B_\mu^k Q_\mu = \sum_{k=0}^{h-1} \frac{1}{k!} \sum_{i=0}^{n-1} \left[\frac{d^k}{dz^k} z^i \right]_{z=\mu} B_\mu^k Q_\mu$$
$$= \sum_{k=0}^{h-1} \frac{1}{k!} \left[\frac{d^k}{dz^k} \frac{z^n - 1}{z - 1} \right]_{z=\mu} B_\mu^k Q_\mu.$$

Moreover, using the notation $b(z) = z^n - 1$, we get

$$\begin{aligned}
&\sum_{k=0}^{h-1} \frac{1}{k!} \left[\frac{d^k}{dz^k} \frac{z^n - 1}{z - 1} \right]_{z=\mu} B_\mu^k Q_\mu \\
&\qquad = \sum_{k=0}^{h-1} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu \\
&\qquad = \sum_{i=0}^{h-1} \sum_{k=i}^{h-1} \frac{1}{k!} \binom{k}{i} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu \\
&\qquad = \sum_{i=0}^{h-1} \sum_{k=i}^{h-1} \frac{1}{i!(k-i)!} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu.
\end{aligned}$$

Since $B_\mu^k Q_\mu = O,\ k \geq h$, the following relation holds:

$$\begin{aligned}
&\sum_{k=i}^{h-1} \frac{1}{i!(k-i)!} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu \\
&\qquad = \sum_{k=i}^{h-1+i} \frac{1}{i!(k-i)!} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu \\
&\qquad = \sum_{j=0}^{h-1} \frac{1}{i!j!} b^{(i)}(\mu) a^{(j)}(\mu) B_\mu^{i+j} Q_\mu.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\sum_{i=0}^{h-1} \sum_{k=i}^{h-1} \frac{1}{i!(k-i)!} b^{(i)}(\mu) a^{(k-i)}(\mu) B_\mu^k Q_\mu \\
&\qquad = \sum_{i=0}^{h-1} \frac{b^{(i)}(\mu)}{i!} B_\mu^i \sum_{j=0}^{h-1} \frac{1}{j!} a^{(j)}(\mu) B_\mu^j Q_\mu = \left(\sum_{i=0}^{h-1} \frac{b^{(i)}(\mu)}{i!} B_\mu^i \right) Z_\mu(B) Q_\mu \\
&\qquad = \left((\mu^n - 1)E + \sum_{i=1}^{h-1} \frac{(n)_i}{i!} \mu^{n-i} B_\mu^i \right) Z_\mu(B) Q_\mu \\
&\qquad = \sum_{i=0}^{h-1} \frac{(n)_i}{i!} \mu^{n-i} B_\mu^i Z_\mu(B) Q_\mu - Z_\mu(B) Q_\mu \\
&\qquad = B^n Z_\mu(B) Q_\mu - Z_\mu(B) Q_\mu,
\end{aligned}$$

where in the last step we have used Lemma 1.

Next, we consider the case where $\mu = 0$. Since $Z_0(B) = -\sum_{k=0}^{h-1} B^k$, we see that if $n \geq h$, then $Z_0(B)Q_0 = -\sum_{k=0}^{n-1} B^k Q_0 = -S_n(B)Q_0$; if $n \leq h-1$, then

$$B^n Z_0(B)Q_0 - Z_0(B)Q_0 = -\sum_{k=0}^{h-1} B^{k+n} Q_0 + \sum_{k=0}^{h-1} B^k Q_0$$

$$= -\sum_{k=n}^{h-1} B^k Q_0 + \sum_{k=0}^{h-1} B^k Q_0$$

$$= \sum_{k=0}^{n-1} B^k Q_0 = S_n(B)Q_0.$$

Finally, we consider the case where $\mu = 1$. Using the relation (16), we have

$$S_n(B)Q_1 = \sum_{i=0}^{n-1} \sum_{k=0}^{h-1} \frac{(i)_k}{k!} \mu^{i-k} B_1^k Q_1 = \sum_{i=0}^{n-1} \sum_{k=0}^{h-1} \binom{i}{k} B_1^k Q_1$$

$$= \sum_{k=0}^{h-1} \left(\sum_{i=k}^{n-1} \binom{i}{k} \right) B_1^k Q_1 = \sum_{k=0}^{h-1} \binom{n}{k+1} B_1^k Q_1.$$

Therefore the proof of the lemma is complete. □

Proof of Theorem 6. Put $h = h_B(\mu)$, $B_\mu = B - \mu E$. The case where $\mu = 0$ is obvious from (18) and Lemma 2.

Let $\mu \neq 1$, $\mu \neq 0$. Using (18), (20) and Lemma 2, we have

$$Q_\mu x_n(w,b) = \sum_{k=0}^{h-1} \frac{(n)_k}{k!} \mu^{n-k} B_\mu^k Q_\mu w + \sum_{k=0}^{h-1} \frac{(n)_k}{k!} \mu^{n-k} B_\mu^k Z_\mu(B) Q_\mu b - Z_\mu(B) Q_\mu b,$$

from which it follows that

$$Q_\mu x_n(w,b) = \sum_{k=0}^{h-1} \frac{(n)_k}{k!} \mu^{n-k} B_\mu^k (Q_\mu w + Z_\mu(B) Q_\mu b) - Z_\mu(B) Q_\mu b$$

$$= B^n \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

Let $\mu = 1$. Using (18), (20) and Lemma 2 again, we have

$$Q_1 x_n(w,b) = \sum_{k=0}^{h-1} \binom{n}{k} B_1^k Q_1 w + \sum_{k=0}^{h-1} \binom{n}{k+1} B_1^k Q_1 b$$

$$= Q_1 w + \sum_{k=0}^{h-1} \binom{n}{k+1} B_1^{k+1} Q_1 w + \sum_{k=0}^{h-1} \binom{n}{k+1} B_1^k Q_1 b$$

$$= \sum_{k=0}^{h-1} \frac{(n)_{k+1}}{(k+1)!} B_1^k (B_1 Q_1 w + Q_1 b) + Q_1 w$$

$$= \left(\sum_{k=0}^{h-1} \frac{(n)_{k+1}}{(k+1)!} B_1^k \right) \delta(w, b) + Q_1 w.$$

This proves the theorem. □

Next, we state properties of $Z_\mu(B)$ and $\gamma_\mu(w, b)$ for the equation (2).

Theorem 7. *If* $\mu \neq 1$, $\mu \in \sigma(B)$, *then*

$$(B - E) Z_\mu(B) Q_\mu = Q_\mu. \tag{22}$$

Proof. Putting $n = 1$ in 1) in Lemma 2, we have

$$S_1(B) Q_\mu = B Z_\mu(B) Q_\mu - Z_\mu(B) Q_\mu = (B - E) Z_\mu(B) Q_\mu.$$

Since $S_1(B) = E$, the assertion (22) holds. □

Corollary 4. *Assume that* $1 \notin \sigma(B)$. *Then*

$$(B - E)^{-1} = \sum_{\mu \in \sigma(B)} Z_\mu(B) Q_\mu. \tag{23}$$

Proof. Since $B - E$ is nonsingular and the relation $E = \sum_{\mu \in \sigma(B)} Q_\mu$ holds, (23) is easily obtained from (22). □

Lemma 3. $\gamma_\mu(w, b) = 0$ *if and only if* $(B - E)\gamma_\mu(w, b) = 0$.

Proof. Assume that $(B - E)\gamma_\mu(w, b) = 0$. Then $\gamma_\mu(w, b) \in G_B(1)$. Moreover, since $\gamma_\mu(w, b) = Q_\mu(w + Z_\mu(B) b)$, we see that $\gamma_\mu(w, b) \in G_B(\mu)$. If $\mu \neq 1$, then $G_B(1) \cap G_B(\mu) = \{0\}$. Therefore we obtain $\gamma_\mu(w, b) = 0$ and vice versa. □

4. Translation formulae

In this section we establish translation formulae.

4.1. Formulation of translation formulae. Let us consider the case where B in the equation (2) is nonsingular. Then $0 \notin \sigma(B)$. With the notation

$$B_{[k,\mu]} = \frac{1}{k!\mu^k} (B - \mu E)^k \qquad (\mu \in \sigma(B)),$$

Theorem 6 may be rewritten as follows.

Theorem 8. *Assume that B is nonsingular. Then the component $Q_\mu x_n(w,b)$ of the solution $x_n(w,b)$ of the equation* (2) *is expressed as follows:*

1) *If $\mu \neq 1$, then*

$$Q_\mu x_n(w,b) = \mu^n \sum_{k=0}^{h_B(\mu)-1} (n)_k B_{[k,\mu]} \gamma_\mu(w,b) - Z_\mu(B) Q_\mu b.$$

2) *If $\mu = 1$, then*

$$Q_1 x_n(w,b) = \sum_{k=0}^{h_B(1)-1} (n)_{k+1} \frac{1}{k+1} B_{[k,1]} \delta(w,b) + Q_1 w.$$

On the other hand, for the equation (3) a representation of solutions was already given in the previous papers [13], [20]. It is based on A. To describe the representation of solutions, let us introduce some notations. Set $P_\lambda = Q_\lambda(A)$ and

$$A_{k,\lambda} = \frac{\tau^k}{k!}(A - \lambda E)^k \qquad (\lambda \in \sigma(A)).$$

Let $\omega = 2\pi/\tau$. Define two matrix functions for any $\lambda \in \sigma(A)$ as

$$X_\lambda(A) = \sum_{k=0}^{h_A(\lambda)-1} \varepsilon^{(k)}(\tau\lambda) A_{k,\lambda} \qquad (\lambda \notin i\omega\mathbf{Z})$$

and

$$Y_\lambda(A) = \sum_{k=0}^{h_A(\lambda)-1} B_k A_{k,\lambda} \qquad (\lambda \in i\omega\mathbf{Z}),$$

where $\varepsilon(z) = (e^z - 1)^{-1}$ and B_k, $k = 0, 1, 2, \ldots$, are Bernoulli's numbers (refer to [18]). For the equation (3) we will introduce characteristic quantities: for $\lambda \in \sigma(A)$

$$\alpha_\lambda(w,b) := \alpha_\lambda(w,b;A) = P_\lambda w + X_\lambda(A) P_\lambda b \qquad (\lambda \notin i\omega\mathbf{Z})$$

and

$$\beta_\lambda(w,b) := \beta_\lambda(w,b;A) = \tau(A - \lambda E) P_\lambda w + Y_\lambda(A) P_\lambda b \qquad (\lambda \in i\omega\mathbf{Z}).$$

Theorem 9 [13], [20]. *Let $\lambda \in \sigma(A)$. The component $P_\lambda x_n(w,b)$ of the solution $x_n(w,b)$ of the equation* (3) *is given as follows:*

1) *If $\lambda \notin i\omega\mathbf{Z}$, then*

$$\begin{aligned} P_\lambda x_n(w,b) &= e^{n\tau\lambda} \sum_{k=0}^{h_A(\lambda)-1} n^k A_{k,\lambda} \alpha_\lambda(w,b) - X_\lambda(A) P_\lambda b \\ &= e^{n\tau A} \alpha_\lambda(w,b) - X_\lambda(A) P_\lambda b. \end{aligned}$$

2) *If* $\lambda \in i\omega\mathbf{Z}$, *then*

$$P_\lambda x_n(w,b) = \sum_{k=0}^{h_A(\lambda)-1} n^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b) + P_\lambda w.$$

$X_\lambda(A)$ and $Y_\lambda(A)$ for the equation (3) are characterized as follows. Since the proofs are similar to Theorem 7 and Corollary 4, they are omitted.

Theorem 10.

1) *If* $\lambda \in \sigma(A)\backslash i\omega\mathbf{Z}$, *then*

$$(e^{\tau A} - E)X_\lambda(A)P_\lambda = P_\lambda.$$

2) *If* $\lambda \in \sigma(A) \cap i\omega\mathbf{Z}$, *then*

$$(e^{\tau A} - E)Y_\lambda(A)P_\lambda = \tau(A - \lambda E)P_\lambda.$$

Corollary 5. *Assume that* $\sigma(A) \cap i\omega\mathbf{Z} = \varnothing$. *Then*

$$(e^{\tau A} - E)^{-1} = \sum_{\lambda \in \sigma(A)} X_\lambda(A)P_\lambda.$$

Throughout this section we assume that two complex $p \times p$ matrices B and A in the equation (2) and the equation (3), respectively, are related by $B = e^{\tau A}$, $\tau > 0$. Then it is trivial that the equation (3) coincides with the equation (2). The representation of the component $Q_\mu x_n(w,b)$ in Theorem 8 is refined by using subcomponents $P_\lambda x_n(w,b)$, $\lambda \in \sigma_\mu(A)$. This representation and Theorem 9 are different representations of the same component of the solution. To go back and forth between these representations, we have to find some translation formulae referred to in Introduction. Therefore we will turn to find such translation formulae. Comparing the above two representations of solutions, the relations below hold.

If $\mu = e^{\tau\lambda} \neq 1$, then

$$B^n P_\lambda \gamma_\mu(w,b) - Z_\mu(B)P_\lambda b = e^{n\tau A}\alpha_\lambda(w,b) - X_\lambda(A)P_\lambda b, \tag{24}$$

that is,

$$\mu^n \sum_{k=0}^{h_B(\mu)-1} (n)_k B_{[k,\mu]} P_\lambda \gamma_\mu(w,b) - Z_\mu(B)P_\lambda b$$

$$= e^{n\tau\lambda}\sum_{k=0}^{h_A(\lambda)-1} n^k A_{k,\lambda}\alpha_\lambda(w,b) - X_\lambda(A)P_\lambda b. \tag{25}$$

If $\mu = 1$, then

$$\sum_{k=0}^{h_B(1)-1} (n)_{k+1} \frac{1}{k+1} B_{[k,1]} P_\lambda \delta(w,b) = \sum_{k=0}^{h_A(\lambda)-1} n^{k+1} \frac{1}{k+1} A_{k,\lambda} \beta_\lambda(w,b). \tag{26}$$

Since the definition of $h_A(\lambda)$ implies that

$$A_{k,\lambda} P_\lambda \neq O \quad (0 \le k \le h_A(\lambda) - 1), \qquad A_{k,\lambda} P_\lambda = O \quad (k \ge h_A(\lambda)),$$

and since $h_B(\mu) \ge h_A(\lambda)$, $\lambda \in \sigma_\mu(A)$, the range of k in the sums of the right sides of (25) and (26) can be extended over $0 \le k \le h_B(\mu) - 1$. Furthermore, the products $A_{j,\lambda} A_{k,\lambda}$ and $B_{[j,\mu]} B_{[k,\mu]}$ are obtained from the following rules.

Lemma 4.

$$A_{j,\lambda} A_{k,\lambda} = \frac{(j+k)!}{j!k!} A_{j+k,\lambda} = \binom{j+k}{j} A_{j+k,\lambda} = \binom{j+k}{k} A_{j+k,\lambda}.$$

In particular, $A_{1,\lambda} A_{k,\lambda} = (k+1) A_{k+1,\lambda}$ *holds.*

$$B_{[j,\mu]} B_{[k,\mu]} = \frac{(j+k)!}{j!k!} B_{[j+k,\mu]} = \binom{j+k}{j} B_{[j+k,\mu]} = \binom{j+k}{k} B_{[j+k,\mu]}.$$

In particular, $B_{[1,\mu]} B_{[k,\mu]} = (k+1) B_{[k+1,\mu]}$ *holds.*

Now the following statements A) and B) hold.

A) $B^n = e^{n\tau A}$ $(n \in \mathbf{N}_0)$ if and only if for all $\mu \in \sigma(B)$ and $\lambda \in \sigma_\mu(A)$ the relation

$$\sum_{k=0}^{h_B(\mu)-1} (n)_k B_{[k,\mu]} P_\lambda = \sum_{k=0}^{h_B(\mu)-1} n^k A_{k,\lambda} P_\lambda \qquad (n \in N_0) \tag{27}$$

holds. Indeed, if $\mu \neq 1$, then, taking $b = 0$ in (25), the relation (27) holds, because w is arbitrary; if $\mu = 1$, then, taking $b = 0$ in (26) and applying Lemma 4, we obtain the relation (27).

B) $S_n(B) = S_n(e^{\tau A})$ $(n \in \mathbf{N}_0)$ if and only if for all $\mu \in \sigma(B)$ and $\lambda \in \sigma_\mu(A)$ the following relations hold:

(1) If $\mu \neq 1$, then

$$Z_\mu(B) P_\lambda = X_\lambda(A) P_\lambda. \tag{28}$$

(2) If $\mu = 1$, then

$$\sum_{k=0}^{h_B(1)-1} (n)_{k+1} \frac{1}{k+1} B_{[k,1]} P_\lambda = \sum_{k=0}^{h_B(1)-1} n^{k+1} \frac{1}{k+1} A_{k,\lambda} Y_\lambda(A) P_\lambda. \tag{29}$$

Indeed, if $\mu \neq 1$, then, taking $w = 0$ in (24), we have that

$$B^n P_\lambda (Z_\mu(B) P_\lambda b - X_\lambda(A) P_\lambda b) = Z_\mu(B) P_\lambda b - X_\lambda(A) P_\lambda b.$$

Put $n = 1$ and $v = Z_\mu(B) P_\lambda b - X_\lambda(A) P_\lambda b$. Since $P_\lambda v = v$, we have $(B - E)v = 0$, that is, $v \in N(B - E)$. Since $\mu \neq 1$, we get $v = 0$, and hence (28) holds. If $\mu = 1$, then, taking $w = 0$ in (26), we obtain (29).

The relations (27) and (29) depend on n. However, if we regards these relations as the expression of polynomials of n with degree $h_B(\mu) - 1$ and $h_B(1)$, we can derive new relations, independent of n, between coefficient matrices.

The Stirling numbers of the first kind $\begin{bmatrix} j \\ k \end{bmatrix}$ $(= s_{j,k})$ and the Stirling numbers of the second kind $\begin{Bmatrix} k \\ j \end{Bmatrix}$ $(= S_{k,j})$ are introduced as the coefficients of the transform of bases of polynomials as follows:

$$(x)_j = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} x^k, \quad j \in \mathbf{N}_0, \qquad x^k = \sum_{j=0}^{k} \begin{Bmatrix} k \\ j \end{Bmatrix} (x)_j, \quad k \in \mathbf{N}_0.$$

By definition of the Stirling number of the second kind, (27) may be rewritten as

$$\begin{aligned} \sum_{k=0}^{h_B(\mu)-1} (n)_k B_{[k,\mu]} P_\lambda &= \sum_{j=0}^{h_B(\mu)-1} \sum_{k=0}^{j} \begin{Bmatrix} j \\ k \end{Bmatrix} (n)_k A_{j,\lambda} P_\lambda \\ &= \sum_{k=0}^{h_B(\mu)-1} (n)_k \sum_{j=k}^{h_B(\mu)-1} \begin{Bmatrix} j \\ k \end{Bmatrix} A_{j,\lambda} P_\lambda. \end{aligned}$$

Hence if $0 \leq k \leq h_B(\mu) - 1$, then

$$B_{[k,\mu]} P_\lambda = \sum_{j=k}^{h_B(\mu)-1} \begin{Bmatrix} j \\ k \end{Bmatrix} A_{j,\lambda} P_\lambda.$$

Also, by definition of the Stirling number of the first kind we have that, for $0 \leq j \leq h_B(\mu) - 1$,

$$A_{j,\lambda} P_\lambda = \sum_{k=j}^{h_B(\mu)-1} \begin{bmatrix} k \\ j \end{bmatrix} B_{[k,\mu]} P_\lambda.$$

The relation (29) is translated as

$$\sum_{k=0}^{h_B(1)-1}(n)_{k+1}\frac{1}{k+1}B_{[k,1]}P_\lambda = \sum_{j=0}^{h_B(1)-1}\frac{1}{j+1}\sum_{k=0}^{j+1}\left\{\begin{matrix} j+1 \\ k \end{matrix}\right\}(n)_k A_{j,\lambda}Y_\lambda(A)P_\lambda$$

$$= \sum_{j=0}^{h_B(1)-1}\frac{1}{j+1}\sum_{k=0}^{j}\left\{\begin{matrix} j+1 \\ k+1 \end{matrix}\right\}(n)_{k+1}A_{j,\lambda}Y_\lambda(A)P_\lambda$$

$$= \sum_{k=0}^{h_B(1)-1}(n)_{k+1}\sum_{j=k}^{h_B(1)-1}\left\{\begin{matrix} j+1 \\ k+1 \end{matrix}\right\}\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda.$$

Thus, if $0 \le k \le h_B(1)-1$, then

$$\frac{1}{k+1}B_{[k,1]}P_\lambda = \sum_{j=k}^{h_B(1)-1}\left\{\begin{matrix} j+1 \\ k+1 \end{matrix}\right\}\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda. \tag{30}$$

Also, (30) is equivalent to the following relation for $0 \le j \le h_B(1)-1$:

$$\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda = \sum_{k=j}^{h_B(1)-1}\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}\frac{1}{k+1}B_{[k,1]}P_\lambda.$$

Summarizing these results, we arrive at the translation formulae.

Theorem 11. *Let $B = e^{\tau A}$, $\tau > 0$ and $\lambda \in \sigma_\mu(A)$.*

1) **(Translation formula I)** *If $0 \le k \le h_B(\mu)-1$, then*

$$B_{[k,\mu]}P_\lambda = \sum_{j=k}^{h_B(\mu)-1}\left\{\begin{matrix} j \\ k \end{matrix}\right\}A_{j,\lambda}P_\lambda, \tag{31}$$

or equivalently, if $0 \le j \le h_B(\mu)-1$, then

$$A_{j,\lambda}P_\lambda = \sum_{k=j}^{h_B(\mu)-1}\begin{bmatrix} k \\ j \end{bmatrix}B_{[k,\mu]}P_\lambda. \tag{32}$$

2) **(Translation formula II)** *If $\mu \ne 1$, then*

$$Z_\mu(B)P_\lambda = X_\lambda(A)P_\lambda.$$

3) **(Translation formula III)** *Let $\mu = 1$. If $0 \le k \le h_B(1)-1$, then*

$$\frac{1}{k+1}B_{[k,1]}P_\lambda = \sum_{j=k}^{h_B(1)-1}\left\{\begin{matrix} j+1 \\ k+1 \end{matrix}\right\}\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda, \tag{33}$$

or equivalently, if $0 \le j \le h_B(1) - 1$, *then*

$$\frac{1}{j+1} A_{j,\lambda} Y_\lambda(A) P_\lambda = \sum_{k=j}^{h_B(1)-1} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} \frac{1}{k+1} B_{[k,1]} P_\lambda. \tag{34}$$

Remark 1. *We note that* $n \in \mathbf{N}_0$ *in* (27) *and* (29) *may be replaced by any real number* $t \in \mathbf{R}$, *that is, the relations*

$$\sum_{k=0}^{h_B(\mu)-1} (t)_k B_{[k,\mu]} P_\lambda = \sum_{k=0}^{h_B(\mu)-1} t^k A_{k,\lambda} P_\lambda \qquad (t \in R, \lambda \in \sigma_\mu(A)) \tag{35}$$

$$\sum_{k=0}^{h_B(1)-1} (t)_{k+1} \frac{1}{k+1} B_{[k,1]} P_\lambda = \sum_{k=0}^{h_B(1)-1} t^{k+1} \frac{1}{k+1} A_{k,\lambda} Y_\lambda(A) P_\lambda$$
$$(t \in R, \lambda \in \sigma_1(A)) \tag{36}$$

hold true.

Remark 2. *If* $k \ge h_A(\lambda)$, *the right side in the relation* (31) *is the zero matrix. Thus we have the following fact: If* $\lambda \in \sigma_\mu(A)$ *and if* $k \ge h_A(\lambda)$, *then* $B_{[k,\mu]} P_\lambda = O$.

Applying Translation formulae, we give relationships between $\alpha_\lambda(w,b)$ and $\gamma_\mu(w,b)$ for $\lambda \in \sigma_\mu(A)$, $\mu \ne 1$ and between $\beta_\lambda(w,b)$ and $\delta(w,b)$ for $\lambda \in \sigma_1(A)$, which are used in later sections.

Theorem 12. *Let* $\lambda \in \sigma_\mu(A)$.
1) *If* $\mu \ne 1$, *then* $P_\lambda \gamma_\mu(w,b) = \alpha_\lambda(w,b)$ *or* $\gamma_\mu(w,b) = \sum_{\lambda \in \sigma_\mu(A)} \alpha_\lambda(w,b)$.
2) *If* $\mu = 1$, *then*

$$\sum_{k=0}^{h_B(1)-1} \frac{(-1)^k}{k+1} (B-E)^k P_\lambda \delta(w,b) = \beta_\lambda(w,b).$$

Proof. The assertion 1) is obvious from Translation formula II. To prove the assertion 2), it is sufficient to verify the following two relations:

$$\sum_{k=0}^{h_B(1)-1} \frac{(-1)^k}{k+1} (B-E)^{k+1} P_\lambda = A_{1,\lambda} P_\lambda \tag{37}$$

and

$$\sum_{k=0}^{h_B(1)-1} \frac{(-1)^k}{k+1} (B-E)^k P_\lambda = Y_\lambda(A) P_\lambda. \tag{38}$$

By the relation (32) in Translation formula I we get

$$A_{1,\lambda}P_\lambda = \sum_{k=1}^{h_B(1)-1}\begin{bmatrix}k\\1\end{bmatrix}B_{[k,1]}P_\lambda = \sum_{k=0}^{h_B(1)-1}\begin{bmatrix}k+1\\1\end{bmatrix}B_{[k+1,1]}P_\lambda$$

$$= \sum_{k=0}^{h_B(1)-1}(-)^k k! B_{k+1,1}P_\lambda = \sum_{k=0}^{h_B(1)-1}\frac{(-1)^k}{k+1}(B-E)^{k+1}P_\lambda.$$

By definition of the Staring numbers of the first kind we have

$$\begin{bmatrix}k+1\\1\end{bmatrix} = (-1)^k k!.$$

Take $j=0$ in the left side of (34) of Translation formula III. Then we see that the relation

$$Y_\lambda(A)P_\lambda = \sum_{k=0}^{h_B(1)-1}\begin{bmatrix}k+1\\1\end{bmatrix}\frac{1}{k+1}B_{[k,1]}P_\lambda$$

$$= \sum_{k=0}^{h_B(1)-1}(-1)^k k!\frac{1}{k+1}B_{[k,1]}P_\lambda$$

$$= \sum_{k=0}^{h_B(1)-1}\frac{(-1)^k}{k+1}(B-E)^k P_\lambda$$

holds. This proves the theorem. □

Theorem 13. *Let* $\lambda \in \sigma_1(A)$. *Then*

$$\sum_{k=0}^{h_A(\lambda)-1} t^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b) = \sum_{k=0}^{h_B(1)-1}(t)_{k+1}\frac{1}{k+1}B_{[k,1]}P_\lambda\delta(w,b) \quad (t \in \mathbf{R}). \tag{39}$$

Proof. By (36) in Remark 1 we have that

$$\sum_{k=0}^{h_B(1)-1}(t)_{k+1}\frac{1}{k+1}B_{[k,1]}P_\lambda = \sum_{k=0}^{h_A(\lambda)-1} t^{k+1}\frac{1}{k+1}A_{k,\lambda}Y_\lambda(A)P_\lambda, \tag{40}$$

from which it follows that

$$\sum_{k=0}^{h_B(1)-1}(t)_{k+1}\frac{1}{k+1}B_{[k,1]}P_\lambda B_{[1,1]} = \sum_{k=0}^{h_A(\lambda)-1} t^{k+1}\frac{1}{k+1}A_{k,\lambda}Y_\lambda(A)P_\lambda B_{[1,1]}$$

$$= \sum_{k=0}^{h_A(\lambda)-1} t^{k+1}\frac{1}{k+1}A_{k,\lambda}B_{[1,1]}Y_\lambda(A)P_\lambda.$$

In view of (37) and (38) we notice that, if $\lambda \in \sigma_1(A)$, then

$$(B - E)Y_\lambda(A)P_\lambda = A_{1,\lambda}P_\lambda. \tag{41}$$

Hence we obtain

$$\sum_{k=0}^{h_B(1)-1} (t)_{k+1} \frac{1}{k+1} B_{[k,1]} P_\lambda B_{[1,1]} = \sum_{k=0}^{h_A(\lambda)-1} t^{k+1} \frac{1}{k+1} A_{k,\lambda} A_{1,\lambda} P_\lambda. \tag{42}$$

By adding two relations (40) and (42), the relation (39) easily follows. □

As an application of Theorem 12, we shall consider Lyapunov exponents of solutions to the equation (1).

Definition 1. For a function $\varphi : [t_0, \infty) \to \mathbf{C}^p$ we define the Lyapunov exponent $\chi(\varphi(t))$ by

$$\chi(\varphi(t)) = \limsup_{t \to \infty} \frac{\log\|\varphi(t)\|}{t}$$

if the support of φ is not compact, and $\chi(\varphi(t)) = -\infty$ if the support of φ is compact.

Lemma 5 [21]. *Let $x(t; w, h)$ be any solution of the equation $x'(t) = Ax(t) + h(t)$, $h \in P_\tau(\mathbf{C}^p)$. If there exists a $\lambda \in \sigma(A)$ such that $\Re\lambda > 0$ and that $\alpha_\lambda(w, a_h) \neq 0$, then*

$$\chi(x(t; w, h)) = \max\{\Re\lambda \mid \lambda \in \sigma(A), \Re\lambda > 0, \alpha_\lambda(w, a_h) \neq 0\},$$

otherwise $\chi(x(t; w, h)) = 0$.

The following result easily follows from Lemma 5, Floquet's Theorem and Theorem 12.

Theorem 14. *Let $x(t; w, f)$ be any solution of the equation (1). If there exists a $\mu \in \sigma(V(0))$ such that $|\mu| > 1$ and that $\gamma_\mu(w, b_f) \neq 0$, then*

$$\chi(x(t; w, f)) = \frac{1}{\tau} \max\{\log|\mu| \mid \mu \in \sigma(V(0)), |\mu| > 1, \gamma_\mu(w, b_f) \neq 0\},$$

otherwise $\chi(x(t; w, f)) = 0$.

If Translation formulae are directly proved under the only condition $B = e^{\tau A}$, $\tau > 0$, then the representation (Theorem 9) of solutions based on A is immediately deduced from the one (Theorem 8) based on B, and vice versa.

Notice that the establishment of Translation formulae plays an essential role in proving a main theorem (Theorem 1) in the present paper.

So, we will give direct proofs of Translation formulae in the next subsection by combinatorial computations.

4.2. Proofs of translation formulae. First, we give a proof of Translation formula I. The relations (31) and (32) in Translation formula I are equivalent to each other. So, we will prove (31). To do so, we will characterize the Stirling numbers as the form of the sum. For $n, m, k \in \mathbf{N}$, $p(n, m, k)$ stands for the set of all finite sequences $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbf{N}_0$, $i = 1, 2, \dots, n$, which satisfy

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = m, \qquad \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = k.$$

Moreover, put

$$q(n, m) = \{(\alpha_1, \alpha_2, \dots, \alpha_n) : \alpha_1 + \alpha_2 + \cdots + \alpha_n = m, \alpha_i \in \mathbf{N}_0\}.$$

We make use the product notation such that $\prod_{i=1}^n \alpha_i = \alpha_1 \alpha_2 \dots \alpha_n$.

Lemma 6 [8], [14]. *For $1 \le m \le n$, the following relation holds true:*

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_{\alpha \in p(n,m,n)} n! \prod_{i=1}^{n} \frac{1}{(\alpha_i!)(i!)^{\alpha_i}}.$$

Lemma 7.

$$\left(\sum_{i=1}^{h_A(\lambda)-1} A_{i,\lambda} \right)^k = k! \sum_{i=k}^{h_A(\lambda)-1} \begin{Bmatrix} i \\ k \end{Bmatrix} A_{i,\lambda} \qquad (k \le h_A(\lambda) - 1).$$

Proof. For the simplicity, we set $A_i = A_{i,\lambda}$, $n = h_A(\lambda) - 1$. Applying Newton's polynomial formula, we have

$$\begin{aligned}
\left(\sum_{i=1}^{n} A_i \right)^k &= \sum_{j \in q(n,k)} \frac{k!}{\prod_{i=1}^n (j_i!)} A_1^{j_1} A_2^{j_2} \dots A_n^{j_n} \\
&= \sum_{j \in q(n,k)} \frac{k!}{\prod_{i=1}^n (j_i!)} \left(\frac{\tau}{1!} \right)^{j_1} \left(\frac{\tau^2}{2!} \right)^{j_2} \cdots \left(\frac{\tau^n}{n!} \right)^{j_n} (A - \lambda E)^{j_1 + 2j_2 + \cdots + nj_n} \\
&= \sum_{j \in q(n,k)} \frac{k! \tau^{j_1 + 2j_2 + \cdots + nj_n}}{\prod_{i=1}^n (j_i!)(i!)^{j_i}} (A - \lambda E)^{j_1 + 2j_2 + \cdots + nj_n}.
\end{aligned}$$

Here $q(n,k)$ is classified with the values of $i = j_1 + 2j_2 + \cdots + nj_n$. Indeed, since $(A - \lambda E)^i = 0$ for $i > n$, it is classified with the sum of i such that $k \le i \le n$. Clearly, if $k \le i \le n$ and if $(j_1, j_2, \dots, j_n) \in p(n,k,i)$, then $j_{i+1} = j_{i+2} = \cdots = j_n = 0$, that is, $(j_1, j_2, \dots, j_i) \in p(i,k,i)$. Using Lemma 6, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{n} A_i\right)^k &= k! \sum_{i=k}^{n} \sum_{j \in p(n,k,i)} \frac{1}{\prod_{m=1}^{n} (j_m!)(m!)^{j_m}} \tau^i (A - \lambda E)^i \\ &= k! \sum_{i=k}^{n} \sum_{j \in p(i,k,i)} \frac{i!}{\prod_{m=1}^{i} (j_m!)(m!)^{j_m}} \frac{\tau^i}{i!} (A - \lambda E)^i \\ &= k! \sum_{i=k}^{n} \left\{ \begin{matrix} i \\ k \end{matrix} \right\} A_i.\end{aligned}$$

This proves the lemma. □

The proof of Translation formula I. Using the spectral decomposition theorem of $e^{\tau A}$, we have

$$e^{\tau A} P_\lambda = \mu \left(\sum_{j=0}^{h_A(\lambda)-1} A_{j,\lambda} \right) P_\lambda. \tag{43}$$

Let $k \le h_A(\lambda) - 1$. By Lemma 7 and the relation (43), we get

$$\begin{aligned}(B - \mu E)^k P_\lambda = (e^{\tau A} P_\lambda - e^{\tau\lambda} P_\lambda)^k &= \left(\mu \sum_{j=0}^{h_A(\lambda)-1} A_{j,\lambda} P_\lambda - \mu P_\lambda \right)^k \\ &= \mu^k \left(\sum_{j=1}^{h_A(\lambda)-1} A_{j,\lambda} \right)^k P_\lambda \\ &= \mu^k k! \left(\sum_{j=k}^{h_A(\lambda)-1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} A_{j,\lambda} \right) P_\lambda.\end{aligned}$$

This proves (31). □

Next, we give a proof of Translation formula III. Since the relations (33) and (34) in Translation formula III are equivalent to each other, we will prove (34) only. The following lemmas are needed in the proof of (34).

LEMMA 8 [2], [9]. *If m and j are positive integers, then*

$$\sum_{k=0}^{m} \frac{1}{k+1} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ j \end{matrix} \right] = \frac{1}{m+1} \binom{m+1}{j} B_{m+1-j}.$$

LEMMA 9. *$\lambda \in i\omega\mathbf{Z}$, then*

$$A_{j,\lambda} Y_\lambda(A) P_\lambda = \sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j} A_{i,\lambda} P_\lambda.$$

PROOF. By definition of $Y_\lambda(A)$ and Lemma 4, we have

$$\begin{aligned} A_{j,\lambda} Y_\lambda(A) P_\lambda &= A_{j,\lambda} \sum_{k=0}^{h_A(\lambda)-1} B_k A_{k,\lambda} P_\lambda \\ &= \sum_{k=0}^{h_B(1)-1} \binom{j+k}{j} B_k A_{j+k,\lambda} P_\lambda \\ &= \sum_{i=j}^{h_B(1)-1} \binom{i}{j} B_{i-j} A_{i,\lambda} P_\lambda. \end{aligned}$$

This completes the proof of the lemma. □

The proof of Translation formula III. Notice that if $k < j$, then

$$\left[\begin{matrix} k \\ j \end{matrix} \right] = 0, \qquad \left\{ \begin{matrix} k \\ j \end{matrix} \right\} = 0.$$

Using Translation formula I, we have

$$\begin{aligned} \sum_{k=j}^{h_B(1)-1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \frac{1}{k+1} B_{k,1} P_\lambda &= \sum_{k=0}^{h_B(1)-1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \frac{1}{k+1} B_{k,1} P_\lambda \\ &= \sum_{k=0}^{h_B(1)-1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \frac{1}{k+1} \sum_{i=k}^{h_B(1)-1} \left\{ \begin{matrix} i \\ k \end{matrix} \right\} A_{i,\lambda} P_\lambda \\ &= \sum_{i=0}^{h_B(1)-1} \sum_{k=0}^{i} \frac{1}{k+1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \left\{ \begin{matrix} i \\ k \end{matrix} \right\} A_{i,\lambda} P_\lambda \\ &= \sum_{i=j}^{h_B(1)-1} \sum_{k=0}^{i} \frac{1}{k+1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \left\{ \begin{matrix} i \\ k \end{matrix} \right\} A_{i,\lambda} P_\lambda. \end{aligned}$$

Moreover, combining Lemma 8 and Lemma 9, we get

$$\begin{aligned}
&\sum_{i=j}^{h_B(1)-1}\sum_{k=0}^{i}\frac{1}{k+1}\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}\begin{Bmatrix} i \\ k \end{Bmatrix}A_{i,\lambda}P_\lambda \\
&\quad = \sum_{i=j}^{h_B(1)-1}\frac{1}{i+1}\binom{i+1}{j+1}B_{i+1-(j+1)}A_{i,\lambda}P_\lambda \\
&\quad = \sum_{i=j}^{h_B(1)-1}\frac{1}{i+1}\frac{(i+1)!}{(j+1)!(i-j)!}B_{i-j}A_{i,\lambda}P_\lambda \\
&\quad = \frac{1}{j+1}\sum_{i=j}^{h_B(1)-1}\binom{i}{j}B_{i-j}A_{i,\lambda}P_\lambda \\
&\quad = \frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda.
\end{aligned}$$

Therefore the relation (34) in Translation formula III holds true. □

Finally, we give a proof of Translation formula II.

The left side and right side in Translation formula II contain the derivatives of $a(z) = 1/(z-1)$ and $\varepsilon(z) = \frac{1}{e^z-1}$, respectively. Since $\varepsilon(z) = a(e^z)$, we need the higher derivatives of a composition of two functions. So, we state Fáa di Bruno's formula below.

Lemma 10 [8], [14]. *Let $y = f(u)$, $u = g(x)$ be infinitely differentiable. Put $h(x) = f(g(x))$. Then, for $n \geq 1$, the following relation holds true:*

$$\frac{h^{(n)}(x)}{n!} = \sum_{k=1}^{n} f^{(k)}(g(x)) \sum_{\alpha \in p(n,k,n)} \prod_{i=1}^{n} \frac{1}{(\alpha_i!)}\left(\frac{g^{(i)}(x)}{i!}\right)^{\alpha_i}.$$

Now we will apply the above formula to the functions:

$$f(u) = \frac{1}{1-u} = (1-u)^{-1}, \qquad g(x) = e^x, \qquad h(x) = f(g(x)).$$

Since

$$f^{(k)}(u) = k!(1-u)^{-(k+1)} = k!f^{k+1}(u), \qquad (k = 1, 2, \ldots,),$$

$h^{(n)}(x)$ is translated by Lemma 10 as follows.

$$\begin{aligned}
\frac{h^{(n)}(x)}{n!} &= \sum_{k=1}^{n} k!f^{k+1}(e^x) \sum_{\alpha \in p(n,k,n)} \prod_{i=1}^{n} \frac{1}{(\alpha_i!)}\left(\frac{e^x}{i!}\right)^{\alpha_i} \\
&= \sum_{k=1}^{n} \sum_{\alpha \in p(n,k,n)} \frac{k!}{\prod_{i=1}^{n}(\alpha_i!)(i!)^{\alpha_i}} e^{kx} f^{k+1}(e^x).
\end{aligned}$$

Setting $\mu = e^{\tau\lambda}$, we have

$$\frac{h^{(n)}(\tau\lambda)}{n!} = \sum_{k=1}^{n} \sum_{\alpha \in p(n,k,n)} \frac{k!}{\prod_{i=1}^{n}(\alpha_i!)(i!)^{\alpha_i}} e^{k\tau\lambda} f^{k+1}(e^{\tau\lambda})$$
$$= \sum_{k=1}^{n} \sum_{\alpha \in p(n,k,n)} \frac{k!}{\prod_{i=1}^{n}(\alpha_i!)(i!)^{\alpha_i}} \frac{\mu^k}{(1-\mu)^{k+1}}.$$

By utilizing Lemma 6, $h^{(n)}(\tau\lambda)$ becomes

$$h^{(n)}(\tau\lambda) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{k!\mu^k}{(1-\mu)^{k+1}} \qquad n \geq 1.$$

Set $\varepsilon(x) = -h(x)$. Then in view of (8), we have the following result.

Lemma 11. *If $e^{\tau\lambda} = \mu \neq 1$, then*

$$\varepsilon^{(n)}(\tau\lambda) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{(-1)^k k!\mu^k}{(\mu-1)^{k+1}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \mu^k a^{(k)}(\mu), \qquad n \in \mathbf{N}_0.$$

The proof of Translation formula II. Using Translation formula I, Corollary 2 and Lemma 11, we obtain

$$Z_\mu(B)P_\lambda = \sum_{k=0}^{h_A(\lambda)-1} \mu^k a^{(k)}(\mu) B_{k,\mu} P_\lambda$$
$$= \sum_{k=0}^{h_A(\lambda)-1} \mu^k a^{(k)}(\mu) \sum_{j=k}^{h_A(\lambda)-1} \left\{ {j \atop k} \right\} A_{j,\lambda} P_\lambda$$
$$= \sum_{j=0}^{h_A(\lambda)-1} \left(\sum_{k=0}^{j} \left\{ {j \atop k} \right\} \mu^k a^{(k)}(\mu) \right) A_{j,\lambda} P_\lambda$$
$$= \sum_{j=0}^{h_A(\lambda)-1} \varepsilon^{(j)}(\tau\lambda) A_{j,\lambda} P_\lambda = X_\lambda(A) P_\lambda.$$

This proves Translation formula II. □

5. The proofs of Theorem 1 and Theorem 2 and related results

In this section, we give the proof of the representation theoems of solutions to the equation (1) by combining translation formulae, Floquet's theorem and the representation of solutions given in [13] for the equation (1) with $A(t) = A$.

5.1. Period map. Now we state the properties of the solution operator $U(t,s)$ of the homogeneous equation (5) and the period map $V(t)$.

Lemma 12 [11, 16]. *The solution operator $U(t,s)$ has the following properties:*

1) $U(t,t) = E$ *for all* $t \in \mathbf{R}$.
2) $U(t,s)U(s,r) = U(t,r)$.
3) *The map* $(t,s,x) \mapsto U(t,s)x$ *is continuous for* $(t,s,x) \in \mathbf{R} \times \mathbf{R} \times \mathbf{C}^p$.
4) $U(t+\tau, s+\tau) = U(t,s)$.
5) $U^n(s+\tau, s) = U(s+n\tau, s)$ $(n \in \mathbf{N})$.
6) $U(t+n\tau, s) = U^n(t+\tau, t)U(t,s) = U(t,s)U^n(s+\tau, s)$ $(n \in \mathbf{N}_0)$.
7) $U(t,s)$ *is a nonsingular matrix and* $U(t,s)^{-1} = U(s,t)$.
8) $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$, $\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s)$

It follows from the Floquet representation that

$$U(t,s) = P(t)e^{(t-s)A}P^{-1}(s) \qquad \text{and} \qquad V(t) = P(t)V(0)P(t)^{-1}. \tag{44}$$

Here we recall elementary results in linear algebra. Let C and D be square matrices with the same size. Assume that there exists a nonsingular matrix T such that $CT = TD$. Then $\sigma(C) = \sigma(D)$ and $Q_\gamma(C)T = TQ_\gamma(D)$ $(\gamma \in \sigma(C))$.

The following lemma is well-known.

Lemma 13 [11, 16]. *For $t, s \in \mathbf{R}$ the following relations hold:*

1) $\sigma(V(t)) = \sigma(V(0))$, $t \in \mathbf{R}$.

2) *Let* $\mu \in \sigma(V(0))$. *Then* $h_{V(s)}(\mu) = h_{V(t)}(\mu)$ *and* $U(t,s)G_{V(s)}(\mu) = G_{V(t)}(\mu)$.

Lemma 14. *The following results hold true:*

1)

$$Q_\mu(t) = P(t)Q_\mu(0)P^{-1}(t) = \sum_{\lambda \in \sigma_\mu(A)} P(t)P_\lambda P^{-1}(t), \qquad Q_\mu(0) = \sum_{\lambda \in \sigma_\mu(A)} P_\lambda.$$

2)

$$G_{V(t)}(\mu) = P(t)G_{V(0)}(\mu) = P(t) \bigoplus_{\lambda \in \sigma_\mu(A)} G_A(\lambda).$$

Proof. Since $V(t)P(t) = P(t)V(0)$, we have that $Q_\mu(t)P(t) = P(t)Q_\mu(0)$ and $G_{V(t)}(\mu) = P(t)G_{V(0)}(\mu)$. In view of (6) and (7), the remainder of the proof is obvious. □

Corollary 6. *Let* $\mu \in \sigma(V(0))$. *Then*

$$U(t,s)Q_\mu(s) = P(t)e^{(t-s)A}Q_\mu(0)P^{-1}(s),$$

and

$$U(t,s)Q_\mu(s) = U(t,0)Q_\mu(0)U^{-1}(s,0) = U(t,0)Q_\mu(0)U(0,s)$$

hold.

Proof. (44) and Lemma 14 imply that

$$\begin{aligned} U(t,s)Q_\mu(s) &= P(t)e^{(t-s)A}P^{-1}(s)P(s)Q_\mu(0)P^{-1}(s) \\ &= P(t)e^{(t-s)A}Q_\mu(0)P^{-1}(s). \end{aligned}$$

Since $Q_\mu(0) = \sum_{\lambda \in \sigma_\mu(A)} P_\lambda$ and e^{tA} and P_λ are commutative, e^{tA} and $Q_\mu(0)$ are also commutative. Therefore we have that

$$\begin{aligned} U(t,s)Q_\mu(s) &= P(t)e^{(t-s)A}Q_\mu(0)P^{-1}(s) = P(t)Q_\mu(0)e^{(t-s)A}P^{-1}(s) \\ &= P(t)e^{tA}Q_\mu(0)e^{-sA}P^{-1}(s) = U(t,0)Q_\mu(0)U^{-1}(s,0) \end{aligned}$$

holds. □

Remark 3. *We note that*

$$G_{V(t)}(\mu) = U(t,0)G_{V(0)}(\mu) = P(t)G_{V(0)}(\mu) \qquad (t \in \mathbf{R}),$$

because of 3) *in Lemma* 13 *and* 2) *in Lemma* 14.

5.2. Representations of solutions: characteristic exponents. First, we give the representation of solutions of equation (1) which is based on characteristic exponents. By the transformation $x = P(t)y$, the equation (1) is reduced to the following equation

$$\frac{d}{dt}y(t) = Ay(t) + h(t), \qquad y(0) = w, \tag{45}$$

where $h(t) = P^{-1}(t)f(t)$. Since $P^{-1}(t)$ is τ-periodic, $h(t)$ is also τ-periodic. Put

$$a_h = \int_0^\tau e^{(\tau-s)A}h(s)ds$$

in the equation (45). The relationship between a_h and b_f is given in the following lemma.

Lemma 15. $a_h = b_f$.

Proof. Since $P(\tau) = E$, we have

$$b_f = \int_0^\tau P(\tau)e^{(\tau-s)A}P^{-1}(s)f(s)ds = \int_0^\tau e^{(\tau-s)A}h(s)ds = a_h.$$

This proves the lemma. □

Let $\lambda \in \sigma(A)$. If a $G_A(\lambda)$-valued function $y(t)$ satisfies the equation

$$\frac{dy}{dt} = Ay(t) + P_\lambda h(t),$$

we say that $y(t)$ is a solution of the equation (45) in $G_A(\lambda)$. Clearly, if $y(t)$ is a solution of the equation (45), then $P_\lambda y(t)$ is a solution of the equation (45) in $G_A(\lambda)$. The following representation of $P_\lambda y(t)$ follows from our papers [13], [20].

Lemma 16. *Let $y(t)$ be a solution of the equation* (45).

1) *If $\lambda \notin i\omega\mathbf{Z}$, then $P_\lambda y(t)$ is expressed as*

$$P_\lambda y(t) = e^{tA}\alpha_\lambda(w, a_h) + u_\lambda(t, h),$$

and $u_\lambda(t, h)$ is a τ-periodic solution of the equation (45) *in $G_A(\lambda)$, where*

$$u_\lambda(t, h) = -e^{tA}X_\lambda(A)P_\lambda a_h + \int_0^t e^{(t-s)A}P_\lambda h(s)ds.$$

2) *If $\lambda \in i\omega\mathbf{Z}$, then $P_\lambda y(t)$ is expressed as*

$$P_\lambda y(t) = \frac{e^{\lambda t}}{\tau}\sum_{k=0}^{h_A(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A - \lambda E)^k\beta_\lambda(w, a_h) + e^{\lambda t}P_\lambda w + u_\lambda(t, h),$$

and $u_\lambda(t, h)$ is a τ-periodic continuous function, which is not necessarily a solution of the equation (45) *in $G_A(\lambda)$, where*

$$u_\lambda(t, h) = -\frac{e^{\lambda t}}{\tau}\sum_{k=0}^{h_A(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A - \lambda E)^k Y_\lambda(A)P_\lambda a_h + \int_0^t e^{(t-s)A}P_\lambda h(s)ds.$$

By using a solution $y(t)$ of the equation (45), the solution $x(t) = P(t)y(t)$ of the equation (1) is expressed as

$$x(t) = \sum_{\lambda \in \sigma(A)} P(t)P_\lambda y(t).$$

Using Lemma 15 and Lemma 16, we can obtain a representation of each component $x_\lambda(t) = P(t)P_\lambda y(t) = P(t)P_\lambda P^{-1}(t)x(t)$. Set

$$f_\lambda(t) = P(t)P_\lambda h(t) = P(t)P_\lambda P^{-1}(t)f(t).$$

THEOREM 15. *Each component $x_\lambda(t)$ of the solution $x(t)$ of the equation* (1) *is expressed as follows:*

1) *If $\lambda \notin i\omega\mathbf{Z}$, then $x_\lambda(t)$ is expressed as*

$$\begin{aligned} x_\lambda(t) &= U(t,0)\alpha_\lambda(w,b_f) + v_\lambda(t,f) \\ &= e^{\lambda t}P(t)\sum_{k=0}^{h_A(\lambda)-1}\frac{t^k}{k!}(A-\lambda E)^k\alpha_\lambda(w,b_f) + v_\lambda(t,f), \end{aligned} \tag{46}$$

and $v_\lambda(t,f)$ is a τ-periodic solution of the equation $y'(t) = A(t)y + f_\lambda(t)$, where

$$v_\lambda(t,f) = -U(t,0)X_\lambda(A)P_\lambda b_f + \int_0^t U(t,s)f_\lambda(s)ds.$$

2) *If $\lambda \in i\omega\mathbf{Z}$, then $x_\lambda(t)$ is expressed as*

$$x_\lambda(t) = \frac{e^{\lambda t}}{\tau}P(t)\sum_{k=0}^{h_A(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A-\lambda E)^k\beta_\lambda(w,b_f) + e^{\lambda t}P(t)P_\lambda w + v_\lambda(t,f)$$

and $v_\lambda(t,f)$ is a τ-periodic continuous function, where

$$v_\lambda(t,f) = -\frac{e^{\lambda t}}{\tau}P(t)\sum_{k=0}^{h_A(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A-\lambda E)^k Y_\lambda(A)P_\lambda b_f + \int_0^t U(t,s)f_\lambda(s)ds.$$

5.3. Representations of solutions: characteristic multipliers. Next, we give the proof of Theorem 1. Our approach is to translate the representation of solutions in Theorem 15 into the representation based on characteristic multipliers of $V(0)$ by combining Translation formulae with Floquet's theorem. To do so, we will decompose the solution (4) into generalized eigenspace of period map $V(t)$.

Multiplying $Q_\mu(t)$ to the solution (4), we have

$$Q_\mu(t)x(t;w,f) = U(t,0)Q_\mu(0)w + \int_0^t U(t,s)Q_\mu(s)f(s)ds \qquad \mu \in \sigma(V(0)).$$

Since $Q_\mu(0) = \sum_{\lambda\in\sigma_\mu(A)} P_\lambda$, it follows from Corollary 6 that

$$\begin{aligned} U(t,s)Q_\mu(s) &= P(t)e^{(t-s)A}Q_\mu(0)P(s)^{-1} \\ &= P(t)\sum_{\lambda\in\sigma_\mu(A)} e^{(t-s)A}P_\lambda P(s)^{-1}. \end{aligned} \tag{47}$$

For a $\mu \in \sigma(V(0))$ we set

$$V(0)_{[k,\mu]} = \frac{1}{\mu^k k!}(V(0) - \mu E)^k.$$

Moreover, since $V(0) = e^{\tau A}$, it follows from Translation formula I and the relation (35) that

$$A_{j,\lambda} P_\lambda = \sum_{k=j}^{h_{V(0)}(\mu)-1} \begin{bmatrix} k \\ j \end{bmatrix} V(0)_{[k,\mu]} P_\lambda \tag{48}$$

and

$$\sum_{k=0}^{h_{V(0)}(\mu)-1} t^k A_{k,\lambda} P_\lambda = \sum_{k=0}^{h_{V(0)}(\mu)-1} (t)_k V(0)_{[k,\mu]} P_\lambda \tag{49}$$

hold for $\lambda \in \sigma_\mu(A)$. Set

$$W(\mu) = \sum_{k=1}^{h_{V(0)}(\mu)-1} \begin{bmatrix} k \\ 1 \end{bmatrix} V(0)_{[k,\mu]} = \sum_{k=1}^{h_{V(0)}(\mu)-1} \frac{(-1)^{k-1}}{\mu^k k}(V(0) - \mu E)^k.$$

Notice that if $h_{V(0)}(\mu) = 1$, then $W(\mu) = O$. If $j = 1$, then (48) is reduced to

$$A_{1,\lambda} P_\lambda = W(\mu) P_\lambda.$$

Lemma 17. *Let $\lambda \in \sigma_\mu(A)$. Then*

$$e^{t(A-\lambda E)} P_\lambda = e^{(t/\tau)W(\mu)} P_\lambda = \sum_{k=0}^{h_{V(0)}(\mu)-1} \left(\frac{t}{\tau}\right)_k V(0)_{[k,\mu]} P_\lambda. \tag{50}$$

Proof. Since $(A - \lambda E)P_\lambda = \frac{1}{\tau} A_{1,\lambda} P_\lambda = \frac{1}{\tau} W(\mu) P_\lambda$, we have that

$$e^{t(A-\lambda E)P_\lambda} = e^{(t/\tau)W(\mu)P_\lambda} = e^{(t/\tau)W(\mu)} P_\lambda$$

holds. Moreover, using (49), we can obtain

$$\begin{aligned} e^{t(A-\lambda E)} P_\lambda &= \sum_{k=0}^{h_{V(0)}(\mu)-1} \frac{1}{k!} t^k (A - \lambda E)^k P_\lambda = \sum_{k=0}^{h_{V(0)}(\mu)-1} \left(\frac{t}{\tau}\right)^k A_{k,\lambda} P_\lambda \\ &= \sum_{k=0}^{h_{V(0)}(\mu)-1} \left(\frac{t}{\tau}\right)_k V(0)_{[k,\mu]} P_\lambda, \end{aligned}$$

which means (50). □

Theorem 16.

$$U(t,0)Q_\mu(0) = \mu^{t/\tau} R_\mu(t) e^{(t/\tau)W(\mu)}$$
$$= \mu^{t/\tau} R_\mu(t) \sum_{k=0}^{h_{V(0)}(\mu)-1} \binom{t}{\tau}_k V(0)_{[k,\mu]}. \tag{51}$$

In particular, if $h_{V(0)}(\mu) = 1$, *then*

$$U(t,0)Q_\mu(0) = \mu^{t/\tau} R_\mu(t) Q_\mu(0) \qquad (\mu \neq 1),$$
$$U(t,0)Q_1(0) = R_1(t)Q_1(0) \qquad (\mu = 1). \tag{52}$$

Proof. Take $s = 0$ in (47). Since $P(0) = E$, (47) is reduced to

$$U(t,0)Q_\mu(0) = P(t) \sum_{\lambda \in \sigma_\mu(A)} e^{tA} P_\lambda.$$

Using Lemma 17, we have

$$\sum_{\lambda \in \sigma_\mu(A)} e^{tA} P_\lambda = \sum_{\lambda \in \sigma_\mu(A)} e^{\lambda t} e^{t(A-\lambda E)} P_\lambda = e^{(t/\tau)W(\mu)} \sum_{\lambda \in \sigma_\mu(A)} e^{\lambda t} P_\lambda$$
$$= e^{(t/\tau)W(\mu)} \mu^{t/\tau} S_\mu(t) = \mu^{t/\tau} S_\mu(t) e^{(t/\tau)W(\mu)},$$

and hence

$$U(t,0)Q_\mu(0) = \mu^{t/\tau} P(t) S_\mu(t) e^{(t/\tau)W(\mu)}.$$

Hence, the definition of $R_\mu(t)$ shows (51). The remainder is obvious. □

Remark 4. *Since* $Q_\mu(0)^2 = Q_\mu(0)$, $U(t,0)Q_\mu(0)$ *in Theorem* 16 *may be expressed as*

$$U(t,0)Q_\mu(0) = \mu^{t/\tau} R_\mu(t) \sum_{k=0}^{h_{V(0)}(\mu)-1} \binom{t}{\tau}_k V(0)_{[k,\mu]} Q_\mu(0).$$

The component $Q_\mu(t)x(t)$ of solutions $x(t)$ of the homogeneous periodic linear differential equation (5) satisfying the initial condition $x(0) = w$ is expressed as follows:

$$Q_\mu(t)x(t) = U(t,0)Q_\mu(0)w \qquad (t \in \mathbf{R})$$
$$= R_\mu(t)\mu^{t/\tau} \sum_{k=0}^{h_{V(0)}(\mu)-1} \binom{t}{\tau}_k \frac{1}{k!\mu^k} (V(0) - \mu E)^k Q_\mu(0)w \qquad (\mu \in \sigma(V(0))).$$

If $x(t; w, f)$ is a solution of the equation (1), then $Q_\mu(t)x(t; w, f)$ is a solution of the equation

$$\frac{d}{dt} y(t) = A(t)y(t) + Q_\mu(t)f(t). \tag{53}$$

In general, if a $G_{V(t)}(\mu)$-valued function $y(t)$ satisfies the equation (53), then $y(t)$ is called a solution of the equation (1) in $G_{V(t)}(\mu)$. If $x(t)$ is a solution of the equation (1), then $Q_\mu(t)x(t) \in G_{V(t)}(\mu)$, $t \in \mathbf{R}$. Hence $Q_\mu(t)x(t)$ is a solution of the equation (1) in $G_{V(t)}(\mu)$.

Lemma 18. *Let $y(t)$ be a solution of the equation* (53). *Then $y(t)$ is a solution of the equation* (1) *in $G_{V(t)}(\mu)$ if and only if $y(0) \in G_{V(0)}(\mu)$.*

Since $\sum_{\mu \in \sigma(V(0))} Q_\mu(t) = E$, we have

$$x(t) = \sum_{\mu \in \sigma(V(0))} Q_\mu(t)x(t), \qquad f(t) = \sum_{\mu \in \sigma(V(0))} Q_\mu(t)f(t),$$

where

$$Q_\mu(t)x(t) = \sum_{\lambda \in \sigma_\mu(A)} x_\lambda(t), \qquad Q_\mu(t)f(t) = \sum_{\lambda \in \sigma_\mu(A)} f_\lambda(t),$$

because of Lemma 14. Set

$$\gamma_\mu(w, b_f) = \gamma_\mu(w, b_f; V(0)), \qquad \delta(w, b_f) = \delta(w, b_f; V(0)).$$

Now we are in a position to prove Theorem 1.

The proof of Theorem 1. The proof follows from the representation of solutions in Theorem 15 and Translation formulae.

1) Let $\mu \neq 1$ and $\lambda \in \sigma_\mu(A)$. Since the representation (46) of solutions in Theorem 15 is given as

$$x_\lambda(t) = U(t,0)\alpha_\lambda(w, b_f) + v_\lambda(t, f),$$

we obtain

$$Q_\mu(t)x(t) = \sum_{\lambda \in \sigma_\mu(A)} x_\lambda(t) = U(t,0) \sum_{\lambda \in \sigma_\mu(A)} \alpha_\lambda(w, b_f) + \sum_{\lambda \in \sigma_\mu(A)} v_\lambda(t, f).$$

Using Theorem 12 and Lemma 14, we have

$$\alpha_\lambda(w, b_f) = P_\lambda \gamma_\mu(w, b_f) \qquad \text{and} \qquad Q_\mu(0) = \sum_{\lambda \in \sigma_\mu(A)} P_\lambda,$$

so that

$$\sum_{\lambda \in \sigma_\mu(A)} \alpha_\lambda(w, b_f) = Q_\mu(0)\gamma_\mu(w, b_f) = \gamma_\mu(w, b_f).$$

This shows that

$$Q_\mu(t)x(t) = U(t,0)\gamma_\mu(w,b_f) + \sum_{\lambda\in\sigma_\mu(A)} v_\lambda(t,f).$$

Using Translation formula II, we get

$$\sum_{\lambda\in\sigma_\mu(A)} X_\lambda(A)P_\lambda = \sum_{\lambda\in\sigma_\mu(A)} Z_\mu(V(0))P_\lambda = Z_\mu(V(0))Q_\mu(0),$$

and hence, we see that $h_\mu(t,f) = \sum_{\lambda\in\sigma_\mu(A)} v_\lambda(t,f)$ holds. Since $v_\lambda(t,f)$ is a τ-periodic solution of the equation $y'(t) = A(t)y + f_\lambda(t)$, $h_\mu(t,f)$ is also a τ-periodic solution of $y'(t) = A(t)y + Q_\mu(t)f(t)$. Since $-Z_\mu(V(0))Q_\mu(0)b_f \in G_{V(0)}(\mu)$, it follows from Lemma 18 that $h_\mu(t,f)$ is a τ-periodic solution of the equation (1) in $G_{V(t)}(\mu)$. This implies that (9) holds. (10) follows from Theorem 16.

2) Since $V(0) = e^{\tau A}$, it follows from Theorem 13 that, for any $t\in\mathbf{R}$,

$$\begin{aligned}\frac{1}{\tau}\sum_{k=0}^{h(\lambda)-1}&\frac{t^{k+1}}{(k+1)!}(A-\lambda E)^k\beta_\lambda(w,b_f)\\ &=\sum_{k=0}^{h_B(1)-1}\left(\frac{t}{\tau}\right)_{k+1}\frac{1}{(k+1)!}(V(0)-E)^kP_\lambda\delta(w,b_f).\end{aligned}\tag{54}$$

On the other hand, we have that

$$P(t)\sum_{\lambda\in\sigma_1(A)} e^{\lambda t}P_\lambda = R_1(t) = R_1(t)Q_1(0).\tag{55}$$

Using the above relation (54), (55) and Theorem 15, we obtain

$$\begin{aligned}Q_1(t)x(t) &= \sum_{\lambda\in\sigma_1(A)} x_\lambda(t)\\ &= P(t)\sum_{\lambda\in\sigma_1(A)}\frac{e^{\lambda t}}{\tau}\sum_{k=0}^{h(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A-\lambda E)^k\beta_\lambda(w,b_f)\\ &\quad+\sum_{\lambda\in\sigma_1(A)} e^{\lambda t}P(t)P_\lambda w + \sum_{\lambda\in\sigma_1(A)} v_\lambda(t,f)\\ &= P(t)\sum_{\lambda\in\sigma_1(A)} e^{\lambda t}P_\lambda\sum_{k=0}^{h_B(1)-1}\left(\frac{t}{\tau}\right)_{k+1}\frac{1}{(k+1)!}(V(0)-E)^k\delta(w,b_f)\\ &\quad+R_1(t)Q_1(0)w + \sum_{\lambda\in\sigma_1(A)} v_\lambda(t,f)\end{aligned}$$

$$= R_1(t) \sum_{k=0}^{h_B(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!} (V(0)-E)^k \delta(w, b_f)$$

$$+ R_1(t)Q_1(0)w + \sum_{\lambda \in \sigma_1(A)} v_\lambda(t, f). \tag{56}$$

It is easy to show that $h_1(t, f) = \sum_{\lambda \in \sigma_1(A)} v_\lambda(t, f)$ holds and that $h_1(t, f)$ is τ-periodic. This completes the proof. □

The following result is derived from Theorem 1 and Theorem 16.

Corollary 7. *Let* $\mu \in \sigma(V(0))$, $h_{V(0)}(\mu) = 1$ *in Theorem* 1. *Then the component* $Q_\mu(t)x(t)$ *of solutions* $x(t)$ *of the equation* (1) *satisfying the initial condition* $x(0) = w$ *is expressed as follows:*

1) *If* $\mu \neq 1$, *then*

$$Q_\mu(t)x(t) = U(t,0)\gamma_\mu(w, b_f) + h_\mu(t, f)$$
$$= \mu^{t/\tau} R_\mu(t)\gamma_\mu(w, b_f) + h_\mu(t, f) \qquad (t \in \mathbf{R}).$$

2) *If* $\mu = 1$, *then*

$$Q_1(t)x(t) = \frac{t}{\tau} U(t,0)Q_1(0)b_f + U(t,0)Q_1(0)w + h_1(t, f)$$
$$= \frac{t}{\tau} R_1(t)Q_1(0)b_f + R_1(t)Q_1(0)w + h_1(t, f) \qquad (t \in \mathbf{R}).$$

Proof. If $\mu \neq 1$, the assertion 1) immediately follows from Theorem 1. Let $\mu = 1$. Since $h_{V(0)}(1) = 1$, it follows from (52) that $U(t,0)Q_1(0) = R_1(t)Q_1(0)$ holds. Thus (11) in Theorem 1 is reduced to

$$Q_1(t)x(t) = \frac{t}{\tau} R_1(t)\delta(w, b_f) + R_1(t)Q_1(0)w + h_1(t, f) \qquad (t \in \mathbf{R}).$$

Since $h_{V(0)}(1) = 1$, we have that $V(0)Q_1(0) = Q_1(0)$, and hence $\delta(w, b_f) = Q_1(0)b_f$. Those facts imply the assertion 2). □

We state some remarks, which are concerned with Theorem 1.

1) We define $z_\mu(t)$ as

$$z_\mu(t) = \begin{cases} U(t,0)Z_\mu(V(0))Q_\mu(0)b_f, & \mu \neq 1 \\ R_1(t) \displaystyle\sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!} (V(0)-E)^k Q_1(0)b_f, & \mu = 1. \end{cases} \tag{57}$$

Then

$$Q_\mu(t)x(t) = (U(t,0)Q_\mu(0)w + z_\mu(t)) + \left(-z_\mu(t) + \int_0^t U(t,s)Q_\mu(s)f(s)ds\right).$$

The second term of the right side coincides with $h_\mu(t,f)$ in Theorem 1. The function $z_\mu(t)$ is called a periodicizing function for the equation (1) (cf. [20]). Setting

$$F(t) = \int_0^t U(t,s)f(s)ds,$$

we see that $F(t+\tau) - F(t) = U(t,0)b_f$ holds. Note that $z_\mu(t)$ is the $G_{V(t)}(\mu)$-component of a continuous solution (indefinite sum) $z(t) = \Delta_\tau^{-1}(U(t,0)b_f)$ of the equation

$$\Delta_\tau z(t) := z(t+\tau) - z(t) = U(t,0)b_f, \qquad t \in \mathbf{R}. \tag{58}$$

2) Let us consider the case where $1 \notin \sigma(V(0))$. Then

$$z(t,f) = U(t,0)(V(0)-E)^{-1}b_f$$

is a periodicizing function for the equation (1). Take $z(t,f)$ as $z(t)$ in (58). Then the solution $x(t;w,f)$ of the equation (1) may be rewritten

$$x(t) = U(t,0)(w + (V(0)-E)^{-1}b_f) + h(t,f) \qquad (t \in \mathbf{R})$$

where $h(t,f) = -z(t,f) + F(t)$. $U(t,0)(w + (V(0)-E)^{-1}b_f)$ is a solution of the homogeneous equation (5) associated with the equation (1) and the function $h(t,f)$ is a τ-periodic solution of the equation (1). Moreover, $h(t,f)$ is expressed as

$$h(t,f) = (E - V(0))^{-1}\int_t^{t+\tau} U(t+\tau,s)f(s)ds.$$

3) As a general case, the following relation holds for the solution $x(t;w,f)$ to the equation (1):

$$(V(t)-E)x(t;w,f) = U(t,0)((V(0)-E)w + b_f) + v(t,f) \qquad (t \in \mathbf{R}). \tag{59}$$

where

$$v(t,f) = -U(t,0)b_f + \int_0^t U(t,s)(V(s)-E)f(s)ds = \int_{t+\tau}^t U(t+\tau,s)f(s)ds$$

is a τ-periodic solution of the equation $y' = A(t)y + (V(t)-E)f(t)$.

Using (59) and Theorem 1, we have the following result.

Proposition 2. *Let $\mu = 1 \in \sigma(V(0))$ and let $Q_\mu(t)x(t)$ be the component of the solution $x(t) := x(t;w,f)$ of the equation* (1).

1) *If* $d(1) = 0$, *then* $(V(t) - E)Q_1(t)x(t)$ *is* τ*-periodic:*

$$(V(t) - E)Q_1(t)x(t) = (V(t) - E)h_1(t, f) - R_1(t)Q_1(0)b_f.$$

2) *If* $d(1) = 1$, *then* $(V(t) - E)Q_1(t)x(t)$ *is* τ*-periodic:*

$$(V(t) - E)Q_1(t)x(t) = R_1(t)\delta(w, b_f) + (V(t) - E)h_1(t, f) - R_1(t)Q_1(0)b_f.$$

3) *If* $d(1) > 1$, *then* $(V(t) - E)Q_1(t)x(t)$ *is unbounded on* $\mathbf{R}_+$ *and* $\mathbf{R}_-$*:*

$$(V(t) - E)Q_1(t)x(t) = \frac{\left(\frac{t}{\tau}\right)^{d(1)-1}}{(d(1) - 1)!} R_1(t)(V(0) - E)^{d(1)-1}\delta(w, b_f)$$
$$+ o(t^{d(1)-1}) \qquad (|t| \to +\infty).$$

5.4. A special case. We give the proof of Theorem 2. The following lemma of dichotomy type is proved in many literatures, e.g. [11], [16]. We now give a proof in our situation. Hereafter, we denote by $\|M\|$ any norm of a $p \times p$ matrix M.

Lemma 19. *Let* $\mu \in \sigma(V(0))$.

1) *If* $|\mu| > 1$, *there exist* $M > 0$ *and* $\delta > 0$ *such that*

$$\|U(s, t)Q_\mu(t)\| < Me^{-\delta(t-s)}, \qquad t \geq s.$$

2) *If* $|\mu| < 1$, *there exist* $N > 0$ *and* $\varepsilon > 0$ *such that*

$$\|U(t, s)Q_\mu(s)\| < Ne^{-\varepsilon(t-s)}, \qquad t \geq s.$$

Proof. It follows from Corollary 6 that

$$U(t, s)Q_\mu(s) = P(t)e^{(t-s)A}Q_\mu(0)P(s)^{-1}$$

holds for $\mu \in \sigma(V(0))$. Since $P(t)$, $P^{-1}(t)$ are τ-periodic, there is a constant $H > 0$ such that $\|P(t)\| < H$, $\|P^{-1}(t)\| < H$. Hence we have

$$\|U(t, s)Q_\mu(s)\| \leq H^2\|e^{(t-s)A}Q_\mu(0)\| = H^2\left\|\sum_{\lambda \in \sigma_\mu(A)} e^{(t-s)A}P_\lambda\right\|$$
$$\leq H^2 \sum_{\lambda \in \sigma_\mu(A)} \|e^{(t-s)A}P_\lambda\|.$$

Since $\sigma_\mu(A)$ is a finite set, the assertions in the lemma are easily proved. □

The proof of Theorem 2. By Lemma 19, the improper integrals in the theorem is convergent.

1) Let $\mu \in \sigma_+(V(0))$. Since

$$\begin{aligned}\int_{(k-1)\tau}^{k\tau} U(0,s)Q_\mu(s)f(s)ds &= \int_0^\tau U(0,(k-1)\tau+s)Q_\mu(s)f(s)ds \\ &= U(0,k\tau)\int_0^\tau U(k\tau,(k-1)\tau+s)Q_\mu(s)f(s)ds \\ &= U(0,k\tau)\int_0^\tau U(\tau,s)Q_\mu(s)f(s)ds \\ &= U(0,k\tau)Q_\mu(0)b_f,\end{aligned}$$

we have

$$\begin{aligned}\int_0^{n\tau} U(0,s)Q_\mu(s)f(s)ds &= \sum_{k=1}^n \int_{(k-1)\tau}^{k\tau} U(0,s)Q_\mu(s)f(s)ds \\ &= \sum_{k=1}^n U(0,k\tau)Q_\mu(0)b_f \\ &= U(0,n\tau)S_n(V(0))Q_\mu(0)b_f.\end{aligned}$$

From $U(0,n\tau) = (V(0)^{-1})^n$ it follows that

$$\begin{aligned}&U(0,n\tau)S_n(V(0))Q_\mu(0)b_f \\ &\quad = U(0,n\tau)[V^n(0)Z_\mu(V(0))Q_\mu(0)b_f - Z_\mu(V(0))Q_\mu(0)b_f] \\ &\quad = Z_\mu(V(0))Q_\mu(0)b_f - U(0,n\tau)Z_\mu(V(0))Q_\mu(0)b_f.\end{aligned}$$

Since $\lim_{n\to\infty} U(0,n\tau)Z_\mu(V(0))Q_\mu(0)b_f = 0$ by Lemma 19, we obtain

$$\int_0^\infty U(0,s)Q_\mu(s)f(s)ds = Z_\mu(V(0))Q_\mu(0)b_f.$$

Hence $h_\mu(t,f)$ in Theorem 1 is expressed as

$$\begin{aligned}h_\mu(t,f) &= -U(t,0)Z_\mu(V(0))Q_\mu(0)b_f + \int_0^t U(t,s)Q_\mu(s)f(s)ds \\ &= -U(t,0)\int_0^\infty U(0,s)Q_\mu(s)f(s)ds + \int_0^t U(t,s)Q_\mu(s)f(s)ds \\ &= -\int_0^\infty U(t,s)Q_\mu(s)f(s)ds + \int_0^t U(t,s)Q_\mu(s)f(s)ds \\ &= -\int_t^\infty U(t,s)Q_\mu(s)f(s)ds.\end{aligned}$$

Therefore we obtain the assertion 1) in the theorem.

2) Let $\mu \in \sigma_-(V(0))$. Since

$$\int_{-k\tau}^{-(k-1)\tau} U(0,s)Q_\mu(s)f(s)ds = U((k-1)\tau, 0)Q_\mu(0)b_f,$$

we have

$$\begin{aligned}\int_{-n\tau}^{0} U(0,s)Q_\mu(s)f(s)ds &= S_n(V(0))Q_\mu(0)b_f \\ &= V^n(0)Z_\mu(V(0))Q_\mu(0)b_f - Z_\mu(V(0))Q_\mu(0)b_f.\end{aligned}$$

On the other hand, we have $\lim_{n\to\infty} V^n(0)Z_\mu(V(0))Q_\mu(0)b_f = 0$ by Lemma 19, we obtain

$$\int_{-\infty}^{0} U(0,s)Q_\mu(s)f(s)ds = -Z_\mu(V(0))Q_\mu(0)b_f.$$

Hence $h_\mu(t,f)$ in Theorem 1 is expressed as

$$\begin{aligned}h_\mu(t,f) &= -U(t,0)Z_\mu(V(0))Q_\mu(0)b_f + \int_0^t U(t,s)Q_\mu(s)f(s)ds \\ &= U(t,0)\int_{-\infty}^{0} U(0,s)Q_\mu(s)f(s)ds + \int_0^t U(t,s)Q_\mu(s)f(s)ds \\ &= \int_{-\infty}^{t} U(t,s)Q_\mu(s)f(s)ds.\end{aligned}$$

Therefore we obtain the assertion 2) in the theorem. □

Using Theorem 2, we have the following result, cf. [4].

Corollary 8. *Let $\mu \in \sigma(V(0))$, $|\mu| \neq 1$. Let $x(t)$ be a solution of the equation* (1)*, whose initial condition satisfies $\gamma_\mu(w, b_f) = 0$ for all $\mu \in \sigma(V(0))$. Then $Q_\mu(t)x(t)$ is expressed as*

$$Q_\mu(t)x(t) = \begin{cases} -\displaystyle\int_t^{\infty} U(t,s)Q_\mu(s)f(s)ds, & (\mu \in \sigma_+(V(0))) \\ \displaystyle\int_{-\infty}^{t} U(t,s)Q_\mu(s)f(s)ds, & (\mu \in \sigma_-(V(0))), \end{cases} \tag{60}$$

which is a τ-periodic function.

Corollary 9. *$z_\mu(t)$ in (57) is rewritten as*

$$z_\mu(t) = \begin{cases} \displaystyle\int_0^\infty U(t,s)Q_\mu(s)f(s)ds, & (\mu \in \sigma_+(V(0))) \\ \displaystyle -\int_{-\infty}^0 U(t,s)Q_\mu(s)f(s)ds, & (\mu \in \sigma_-(V(0))). \end{cases}$$

5.5. Examples. In this subsection, we will illustrate Theorem 1 or Corollary 7 through examples. Let us consider representations of solutions of the initial value problem of 2-dimensional periodic linear differential equation of the type

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \qquad x(0) = w. \tag{61}$$

Assume that

$$A(t) = \begin{pmatrix} a(t) & 0 \\ b(t) & a(t) \end{pmatrix},$$

where $a(t)$ and $b(t)$ are continuous τ-periodic real-valued functions, and that $f(t)$ is a 2-dimensional continuous τ-periodic function.

Set

$$\alpha(t,s) = \int_s^t a(s)ds, \qquad \alpha(t) = \alpha(t,0), \qquad \beta(t,s) = \int_s^t b(s)ds, \qquad \beta(t) = \beta(t,0).$$

The solution operator $U(t,s)$ of the homogeneous equation $x' = A(t)x$ associated with the equation (61) is given by

$$U(t,s) = e^{\alpha(t,s)}e^{\beta(t,s)B}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

(cf. [15]), that is,

$$U(t,s) = e^{\alpha(t,s)}\begin{pmatrix} 1 & 0 \\ \beta(t,s) & 1 \end{pmatrix}.$$

Since $a(t)$ and $b(t)$ are τ-periodic, we have

$$\alpha(t+\tau,t) = \alpha(\tau,0) = \alpha(\tau), \qquad \beta(t+\tau,t) = \beta(\tau,0) = \beta(\tau),$$

and

$$V(t) = e^{\alpha(t+\tau,t)}e^{\beta(t+\tau,t)B} = e^{\alpha(\tau)E+\beta(\tau)B} = V(0) = e^{\alpha(\tau)}\begin{pmatrix} 1 & 0 \\ \beta(\tau) & 1 \end{pmatrix}.$$

Set $A = \frac{\alpha(\tau)}{\tau}E + \frac{\beta(\tau)}{\tau}B$. Since $V(0)$ and A is related to $V(0) = e^{\tau A}$, we see that by Floquet's theorem $U(t,0) = P(t)e^{tA}$ holds. Hence the τ-periodic function $P(t)$ is expressed as

$$P(t) = U(t,0)e^{-tA} = e^{(\alpha(t)-(t/\tau)\alpha(\tau))}e^{(\beta(t)-(t/\tau)\alpha(\tau))B}.$$

Moreover, since $\sigma(A) = \{\frac{\alpha(\tau)}{\tau}\}$ and $\sigma(V(0)) = e^{\tau\sigma(A)}$ by the spectral mapping theorem, we have $\sigma(V(0)) = \{\mu\}$, $\mu = e^{\alpha(\tau)}$. Furthermore, it is easy to verify that the relation $Q_\mu(t) = Q_\mu(0) = E$ holds for the projection $Q_\mu(t) : \mathbf{C}^2 \to G_{V(0)}(\mu)$, $\mu \in \sigma(V(0))$. By easy calculations, we have the following result.

Lemma 20. *If $\beta(\tau) \neq 0$, then $h_{V(0)}(\mu) = 2$; If $\beta(\tau) = 0$, then $h_{V(0)}(\mu) = 1$.*

Applying Theorem 1 to the equation (61), the following result holds, in which the forms of the τ-periodic function $h_\mu(t,f)$ are omitted.

Proposition 3.

1) *Let $\mu \neq 1$. Then the solution $x(t;w,f)$ of the equation* (61) *is expressed by*

$$x(t;w,f) = e^{\alpha(t)}\begin{pmatrix} 1 & 0 \\ \beta(t) & 1 \end{pmatrix}\gamma_\mu(w,b_f) + h_\mu(t,f), \tag{62}$$

where

$$\gamma_\mu(w,b_f) = \left(w + \frac{1}{\mu-1}\begin{pmatrix} 1 & 0 \\ \frac{\mu}{1-\mu}\beta(\tau) & 1 \end{pmatrix}b_f\right).$$

2) *Let $\mu = 1$. Then the solution $x(t;w,f)$ of the equation* (61) *is expressed by*

$$\begin{aligned} x(t;w,f) = {}& \frac{te^{\alpha(t)}}{\tau}\begin{pmatrix} 1 & 0 \\ \beta(t) - \frac{t+\tau}{2\tau}\beta(\tau) & 1 \end{pmatrix}\delta(w,b_f) \\ & + e^{\alpha(t)}\begin{pmatrix} 1 & 0 \\ \beta(t) - \frac{t}{\tau}\beta(\tau) & 1 \end{pmatrix}w + h_1(t,f), \end{aligned} \tag{63}$$

where

$$\delta(w,b_f) = \begin{pmatrix} 0 & 0 \\ \beta(\tau) & 0 \end{pmatrix}w + b_f.$$

Proof. We will apply Theorem 1 to the equation (61). By easy calculations, we have

$$\mu^{t/\tau} = e^{\lambda t} = e^{(\alpha(\tau)/\tau)t}, \qquad S_\mu(t) = (\mu)^{-t/\tau}e^{(\alpha(\tau)/\tau)t}E = E,$$

and

$$R_\mu(t) = P(t) = e^{\alpha(t)-(t/\tau)\alpha(\tau)}\begin{pmatrix} 1 & 0 \\ \beta(t) - \frac{t}{\tau}\beta(\tau) & 1 \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned}
&\sum_{k=0}^{1}\left(\frac{t}{\tau}\right)_k \frac{1}{k!\mu^k}(V(0)-\mu E)^k Q_\mu(0) \\
&\quad = E + \frac{t}{\tau}e^{-\alpha(\tau)}(e^{\tau A} - \mu E) \\
&\quad = E + \frac{t}{\tau}e^{-\alpha(\tau)}(e^{\alpha(\tau)E+\beta(\tau)B} - e^{\alpha(\tau)}E) \\
&\quad = E + \frac{t}{\tau}(e^{\beta(\tau)B} - E) = E + \frac{t}{\tau}\beta(\tau)B \\
&\quad = \begin{pmatrix} 1 & 0 \\ \frac{t}{\tau}\beta(\tau) & 1 \end{pmatrix}.
\end{aligned}$$

Let $\mu \neq 1$. Then $\alpha(\tau) \neq 0$. Since

$$Z_\mu(V(0)) = -\frac{1}{1-\mu}E - \frac{\mu}{(1-\mu)^2}\begin{pmatrix} 0 & 0 \\ \beta(\tau) & 0 \end{pmatrix} = \frac{1}{\mu-1}\begin{pmatrix} 1 & 0 \\ \frac{\mu}{1-\mu}\beta(\tau) & 1 \end{pmatrix},$$

we obtain

$$\gamma_\mu(w, b_f) = w + \frac{1}{\mu-1}\begin{pmatrix} 1 & 0 \\ \frac{\mu}{1-\mu}\beta(\tau) & 1 \end{pmatrix} b_f.$$

Therefore (10) in Theorem 1 becomes (62).

Let $\mu = 1$. Then $\alpha(\tau) = 0$. Thus we have $R_1(t) = P(t) = e^{\alpha(t)E+\beta(t)B} \cdot e^{-(t/\tau)\beta(\tau)B}$, and hence

$$R_1(t) = e^{\alpha(t)}\begin{pmatrix} 1 & 0 \\ \beta(t) - \frac{t}{\tau}\beta(\tau) & 1 \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned}
&\sum_{k=0}^{1}\left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!}(V(0)-E)^k Q_1(0) \\
&\quad = \frac{t}{\tau}E + \frac{t}{2\tau}\left(\frac{t}{\tau} - 1\right)(e^{\tau A} - E)
\end{aligned}$$

$$= \frac{t}{\tau}\left(E + \frac{1}{2}\left(\frac{t}{\tau} - 1\right)(e^{\beta(\tau)B} - E)\right)$$

$$= \frac{t}{\tau}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{t-\tau}{2\tau}\begin{pmatrix} 0 & 0 \\ \beta(\tau) & 0 \end{pmatrix}\right)$$

$$= \frac{t}{\tau}\begin{pmatrix} 1 & 0 \\ \frac{t-\tau}{2\tau}\beta(\tau) & 1 \end{pmatrix}.$$

Therefore the solution $x(t; w, f)$ of the equation (61) has the form of (63).

5.6. Backward solutions of the difference equation (2). If B is nonsingular, the solution $x_n(w, b)$ of the equation (2) exists uniquely for negative integers n. Let $P_\tau(\mathbf{C}^p)$ denote the set of all τ-periodic and continuous $\mathbf{C}^p$-valued functions defined on $\mathbf{R}$. If there exists a $f(t) \in P_\tau(\mathbf{C}^p)$ such that $b = b_f$, we may regard $x_n(w, b)$ as the value of the solution at $t = n\tau$ to some τ-periodic linear differential equation

$$x'(t) = A(t)x(t) + f(t), \qquad x(0) = w$$

such that $V(0) = B$. We infer that the representation of the solution $x_n(w, b)$ for $n \leq 0$ to the equation (2) can be obtained by substituting $t = n\tau$ into the formulae in Theorem 1.

To carry out this plan, we see that for any $b \in \mathbf{C}^p$ there exists a τ-periodic function $f(t)$ such that $b = b_h$.

Lemma 21. *The mapping from $f \in P_\tau(\mathbf{C}^p)$ to $b_f \in \mathbf{C}^p$ is surjective.*

Proof. For any $n \in \mathbf{N}$ such that $\frac{2}{n} \leq \tau$, we define a τ-periodic continuous function $\varphi_n \in P_\tau(\mathbf{R})$ as follows: the restriction of $\varphi_n(s)$ to $[0, \tau]$

$$\varphi_n(s)|_{[0,\tau]} = \begin{cases} n^2 s, & 0 \leq s \leq 1/n \\ -n^2(s - 2/n), & 1/n \leq s \leq 2/n \\ 0, & 2/n \leq s \leq \tau. \end{cases}$$

Clearly, we have

$$\int_0^\tau \varphi_n(s)ds = 1.$$

Moreover, we get

$$\int_0^\tau U(\tau, s)\varphi_n(s)ds = \int_0^\tau (U(\tau, s) - U(\tau, 0))\varphi_n(s)ds + U(\tau, 0)\int_0^\tau \varphi_n(s)ds$$

$$= \int_0^\tau (U(\tau, s) - U(\tau, 0))\varphi_n(s)ds + U(\tau, 0).$$

For any $\varepsilon > 0$ there is an $n \in \mathbf{N}$ such that $\|U(\tau,s) - U(\tau,0)\| \le \varepsilon$ for $s \in [0, 2/n]$. Since $\varphi_n(s) = 0$ on $\left[\frac{2}{n}, \tau\right]$, we have

$$\left\| \int_0^\tau U(\tau,s)\varphi_n(s)ds - U(\tau,0) \right\| \le \int_0^{2/n} \|U(\tau,s) - U(\tau,0)\| \, |\varphi_n(s)| ds$$
$$\le \varepsilon \int_0^{2/n} \varphi_n(s)ds = \varepsilon,$$

that is,

$$\lim_{n\to\infty} \int_0^\tau U(\tau,s)\varphi_n(s)ds = U(\tau,0).$$

Since $U(\tau,0)$ is nonsingular, $G := \int_0^\tau U(\tau,s)\varphi_{n_0}(s)ds$ is also nonsingular for sufficiently large $n_0 \in \mathbf{N}$. For any $c \in \mathbf{C}^p$, we put $f_c(s) = \varphi_{n_0}(s)c$. Then $f_c(s) \in P_\tau(\mathbf{C}^p)$ and $b_{f_c} = Gc$, which implies that the mapping from $f \in P_\tau(\mathbf{C}^p)$ to $b_f \in \mathbf{C}^p$ is surjective. □

Theorem 17. *Assume that B is nonsingular in the equation* (2). *The representations of $Q_\mu x_n(w,b)$ in Theorem* 8 *are also valid for negative integers n.*

Proof. First, we shall show that the equation (2) corresponds to a periodic system, which is the same type as the equation (1). Since B is nonsingular, there is a τ-periodic equation $z'(t) = A(t)z$ such that $B = U(\tau,0)$, where $U(t,s)$ stands for the solution operator. Moreover, it follows from Lemma 21 that there is a $f \in P_\tau(\mathbf{C}^p)$ such that $b = b_f$. Hence the equation (2) corresponds to the equation $x'(t) = A(t)x + f(t) \cdots (PE)$. Denote by $x(t) := x(t; w, f)$ the solution of the equation (PE) satisfying the initial condition $x(0) = w$. Set $Q_\mu = Q_\mu(0)$. Now we calculate the values $Q_\mu x(n\tau)$ at $t = n\tau$, $n \in \mathbf{Z}$ in Theorem 1.

Since $S_\mu(t)$ and $R_\mu(t)$ are τ-periodic, $R_\mu(0) = P(0)S_\mu(0) = Q_\mu$. Thus we have the following assertion:

1) Let $\mu \ne 1$. Since $h_\mu(0,f) = -Z_\mu(V(0))Q_\mu b$, we see that

$$Q_\mu x(n\tau) = \mu^n \sum_{k=0}^{h_B(\mu)-1} (n)_k B_{k,\mu}\gamma_\mu(w,b) - Z_\mu(B)Q_\mu b.$$

2) Let $\mu = 1$. Since $h_1(0,f) = 0$, we have

$$Q_1 x(n\tau) = \sum_{k=0}^{h_B(1)-1} (n)_{k+1} \frac{1}{k+1} B_{k,1}\delta(w,b) + Q_1 w.$$

This proves the theorem. □

5.7. Remarks on bounded solutions to the equation (1). Finally, as an application of Lemma 21, we will give a result concerned with Coppel's result. For a general continuous linear system, Coppel gives in section 1 of chapter V in the book [3] the dichotomy conditions for the fundamental matrix as the necessary and sufficient condition for the existence of bounded solutions corresponding to any bounded, continuous forcing function $f(t)$. For the periodic continuous system (1) we discuss the necessary and sufficient condition for the existence of bounded solutions on $\mathbf{R}_+$ for every function $f \in P_\tau(\mathbf{C}^p)$. To do so, the following lemma is needed.

Lemma 22. *The following statements are equivalent.*

1) $(E - V(0))w = b_f$ *has at least one solution for every function* $f \in P_\tau(\mathbf{C}^p)$.

2) $1 \notin \sigma(V(0))$.

3) *There exists a function* $f \in P_\tau(\mathbf{C}^p)$ *such that* $(E - V(0))w = b_f$ *has a unique solution.*

Proof. Since $\{b_f \mid f \in P_\tau(\mathbf{C}^p)\} = \mathbf{C}^p$ by Lemma 21, we see that the condition 1) holds if and only if $(V(0) - E)$ is invertible. The remainder of the proof is obvious. □

The following theorem is based on the spectrum of $V(0)$ instead of the dichotomy condition.

Theorem 18. *The following statements are equivalent.*

1) *The equation* (1) *has at least one bounded solution on* $\mathbf{R}_+$ *for every function* $f \in P_\tau(\mathbf{C}^p)$.

2) $1 \notin \sigma(V(0))$.

3) *There exists a function* $f \in P_\tau(\mathbf{C}^p)$ *such that the equation* (1) *has a unique* τ*-periodic solution.*

4) *The equation* (1) *has a unique* τ*-periodic solution for every function* $f \in P_\tau(\mathbf{C}^p)$.

Proof. We shall show the equivalence of the assertions 1) and 2). By Massera's theorem the assertion 1) holds if and only if the equation (1) has a τ-periodic solution for every function $f \in P_\tau(\mathbf{C}^p)$. The latter condition holds if and only if $(V(0) - E)w = -b_f$ has a solution w for every $f \in P_\tau(\mathbf{C}^p)$. Therefore the equivalence of the assertions 1) and 2) follows from Lemma 22. The remainder of the proof is obvious. □

Farkas [6, Corollary 2.3.2] points out that if the condition 2) holds in above, then the condition 4) holds, but says nothing for its converse.

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