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On the influence of time-periodic dissipation on energy and dispersive estimates

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Abstract. In this short note we discuss the influence of a time-periodic dissipation term on a-priori estimates for solutions to dissipative wave equations. The approach is based on a diagonalisation argument for high frequencies and results from spectral theory of periodic differential equations / Floquet theory for bounded frequencies.

In recent years much attention was paid to hyperbolic equations with time-dependent coefficients and the question of the influence of the precise time-dependence on asymptotic properties of solutions. In particular, we refer to [ReYa00], [HiRe03], [ReSm05], [Wir06], [Wir07] or the survey article [Rei04] for an overview of results. All these papers have in common that they use assumptions on derivatives of the coefficients to avoid the (bad) influence of oscillations. That oscillations may have deteriorating influences was shown in [Yag01] for the example of a wave equation with time-periodic speed of propagation. In this case (some) solutions have exponentially growing energy. The counter-example of [ReSm05] shows that even for the Cauchy problem

$$u_{tt} - a^2(t)\Delta u = 0$$

with $a(t) = 2 + \sin(\log(e+t)^\alpha)$ and $\alpha > 2$ (some) solutions have supra-polynomially increasing energy, while for $\alpha = 1$ polynomial growth may occur and for $\alpha < 1$ the energy can be bounded by t^ε for any $\varepsilon > 0$. Similar results can be obtained for oscillations in mass terms, especially for the case of periodic mass terms we can always find solutions with exponentially increasing energy.

Our aim is to show that the influence of oscillations in the dissipation term is different. To be precise, we want to show that a wave equation with a periodic in time dissipation term,

$$u_{tt} - \Delta u + 2b(t)u_t = 0, \qquad b(t+T) = b(t) \geq 0,$$

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satisfies the same Matsumura-type estimate as obtained for constant dissipation in [Mat76]. We conjecture that the results of [Wir07] (where only very slow oscillations were treated and decay results of the same structure were obtained) can be extented to general dissipation terms with $tb(t) \to \infty$ without further assumptions on derivatives. However, it is an open problem how to achieve such a result.

This note is organised as follows: In Section 1 we give the basic assumptions on the Cauchy problem under consideration and discuss properties of its fundamental solution and the associated monodromy operator. Main result is Theorem 1.1. In Section 2 we discuss applications of Theorem 1.1 to energy and more generally L^p–L^q decay estimates of solutions. Results are given in Theorems 2.1 and 2.2 and resemble whose for the damped wave equation based on [Mat76]. Finally, Theorem 2.3 implies a diffusion phenomenon for periodically damped wave equations generalizing results of [IkNi03] and [ChHa03].

1. Representation of solutions

We consider the Cauchy problem

$$u_{tt} - \Delta u + 2b(t)u_t = 0, \qquad u(0,\cdot) = u_1, \qquad u_t(0,\cdot) = u_2 \tag{1.1}$$

for a wave equation with time-dependent dissipation, where we assume that the coefficient $b(t)$ is continuous, of bounded variation and satisfies $b(t) > 0$ a.e. together with

$$b(t+T) = b(t) \qquad \text{for some } T > 0 \text{ and all } t \in \mathbf{R}.$$

We denote the mean value of $b(t)$ as

$$\beta = \frac{1}{T}\int_0^T b(t)\mathrm{d}t.$$

Using a partial Fourier transform with respect to the spatial variables we reduce the Cauchy problem (1.1) to the ordinary differential equation

$$\hat{u}_{tt} + |\xi|^2\hat{u} + 2b(t)\hat{u}_t = 0, \tag{1.2}$$

which we reformulate as first order system $\mathrm{D}_t V = A(t,\xi)V$ for $V = (|\xi|\hat{u}, \mathrm{D}_t\hat{u})^T$ with coefficient matrix

$$A(t,\xi) = \begin{pmatrix} & |\xi| \\ |\xi| & 2\mathrm{i}b(t) \end{pmatrix}.$$

As usual, $\mathrm{D}_t = -\mathrm{i}\partial_t$ denotes the Fourier derivative. Aim of our investigation is to describe the corresponding fundamental solution $\mathscr{E}(t,s,\xi)$, i.e. the matrix-valued solution to

$$\mathrm{D}_t\mathscr{E}(t,s,\xi) = A(t,\xi)\mathscr{E}(t,s,\xi), \qquad \mathscr{E}(s,s,\xi) = I \in \mathbf{C}^{2\times 2} \tag{1.3}$$

or, exploiting the periodic structure of the problem, the corresponding family of monodromy matrices $\mathscr{M}(t,\xi) = \mathscr{E}(t+T,t,\xi)$. Note, that the T-periodicity of $b(t)$ implies directly the T-translation invariance of the fundamental solution, $\mathscr{E}(t,s,\xi) = \mathscr{E}(t+T,s+T,\xi)$. The monodromy matrix $\mathscr{M}(t,\xi)$ is T-periodic and satisfies the commutator equation

$$\mathrm{D}_t\mathscr{M}(t,\xi) = [A(t,\xi),\mathscr{M}(t,\xi)], \qquad \mathscr{M}(T,\xi) = \mathscr{M}(0,\xi).$$

Indeed, the semigroup property $\mathscr{E}(t,s,\xi)\mathscr{E}(s,t,\xi) = I$ yields $\mathrm{D}_s\mathscr{E}(t,s,\xi) = -\mathscr{E}(t,s,\xi)A(s,\xi)$ and, therefore, $\mathrm{D}_t\mathscr{M}(t,\xi) = A(t+T,\xi)\mathscr{E}(t+T,t,\xi) - \mathscr{E}(t+T,t,\xi)A(t,\xi) = [A(t,\xi),\mathscr{M}(t,\xi)]$.

1.1. Considerations for large frequencies. Our aim is to show that the monodromy matrix is contractive for large frequencies, i.e.

$$\|\mathscr{M}(t,\xi)\| < 1 \qquad \text{holds true for } |\xi| > N \tag{1.4}$$

uniformly in $t \in [0,T]$ for a (still to be determined) constant N. Because we consider only large frequencies $|\xi| > N$ the influence of $|\xi|$ in $A(t,\xi)$ seems to be stronger than the influence of the (comparatively small) coefficient $b(t)$. To employ this, we apply two transformations to the system. For the first step we use the unitary matrices

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = M^*$$

and set $V^{(0)} = M^{-1}V$, such that

$$\mathrm{D}_t V^{(0)} = \left(\begin{pmatrix} |\xi| & \\ & -|\xi| \end{pmatrix} + \mathrm{i}b(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) V^{(0)}.$$

We denote the first (diagonal) matrix as $\mathscr{D}(\xi)$ and the remainder term as $R_0(t,\xi)$. For convenience we set $\mathscr{D}_1 = \mathscr{D} + \operatorname{diag} R_0$ and $R_1 = R_0 - \operatorname{diag} R_0$. In the next step we follow an idea from [Eas89, Chapter 1.6] and construct a matrix $N_1(t,\xi)$ subject to

$$\mathrm{D}_t N_1 = [\mathscr{D}_1, N_1] + R_1 \tag{1.5}$$

and $N_1(0,\xi) = I$ (we could use any starting point $N_1(s,\xi) = I$ here without changing much of the calculation). The choice of $N_1(t,\xi)$ implies that

$$(\mathrm{D}_t - \mathscr{D}_1 - R_1)N_1 - N_1(\mathrm{D}_t - \mathscr{D}_1) = \mathrm{D}_t N_1 - [\mathscr{D}_1, N_1] - R_1 N_1 = R_1(I - N_1),$$

such that with $R_2 = -N_1^{-1}R_1(I - N_1)$ the operator equation $(\mathrm{D}_t - \mathscr{D}_1 - R_1)N_1 = N_1(\mathrm{D}_t - \mathscr{D}_1 - R_2)$ holds true. Thus, provided $N_1(t,\xi)$ is invertible, we conclude that $V^{(1)} = N_1^{-1}V^{(0)}$ satisfies

$$\mathrm{D}_t V^{(1)} = (\mathscr{D}_1(t,\xi) + R_2(t,\xi))V^{(1)}. \tag{1.6}$$

It remains to understand in which sense the remainder R_2 is better than the remainder R_1 and that it is indeed possible to choose the zone constant N large enough to guarantee the invertibility of N_1.

For this we solve (1.5). Since $\mathscr{D}_1$ is diagonal, we see that $\mathrm{D}_t \operatorname{diag} N_1 = 0$ and therefore $\operatorname{diag} N_1 = I$. The two off-diagonal entries of $N_1 = \begin{pmatrix} 1 & n^- \\ n^+ & 1 \end{pmatrix}$ satisfy

$$\mathrm{D}_t n^\pm(t,\xi) = \pm 2|\xi| n^\pm(t,\xi) + \mathrm{i}b(t), \qquad n^\pm(0,\xi) = 0,$$

such that

$$n^\pm(t,\xi) = \int_0^t \mathrm{e}^{\pm 2\mathrm{i}|\xi|(t-s)} b(s)\mathrm{d}s = \int_0^t \mathrm{e}^{\pm 2\mathrm{i}s|\xi|} b(t-s)\mathrm{d}s,$$

especially we see that $n^+ = \overline{n^-}$ and as Fourier transforms of $b(t-s)1_{[0,t]}(s)$ the Riemann-Lebesgue lemma implies $n^\pm(t,\xi) \to 0$ as $|\xi| \to \infty$ for any *fixed* t. We show that this is true uniformly in $t \in [0,2T]$, provided that b is of bounded variation. Indeed, integration by parts yields

$$\begin{aligned} |n^\pm(t,\xi)| &= \left| \frac{1}{\pm 2\mathrm{i}|\xi|} \mathrm{e}^{\pm 2\mathrm{i}|\xi|s} b(t-s)\Big|_{s=0+}^{t-} + \frac{1}{\pm 2\mathrm{i}|\xi|} \int_0^t \mathrm{e}^{\pm 2\mathrm{i}|\xi|s} b'(t-s)\mathrm{d}s \right| \\ &\le C(1+t)|\xi|^{-1} \end{aligned}$$

and the assertion follows.

Now we are in a position to show that (1.4) holds true. Note first, that the transformation matrices satisfy $N_1(t,\xi) \to I$ and therefore $N_1^{-1}(t,\xi) \to I$ uniform in $t \in [0,2T]$ as $|\xi| \to \infty$. Furthermore, the remainder term satisfies $R_2(t,\xi) \to 0$ as $|\xi| \to \infty$ uniformly in $t \in [0,2T]$. Thus for sufficiently large zone constant N we can achieve that

$$\sup_{t \in [0,T]} \|N_1(t+T)\| \exp\left(\int_t^{t+T} \|R_2(s,\xi)\|\mathrm{d}s\right) \|N_1^{-1}(t)\| < \exp \beta T \tag{1.7}$$

holds true. We fix this choice of the constant N and construct the fundamental solution $\mathscr{E}(t,s,\xi)$ to $\mathrm{D}_t - A(t,\xi)$. We start from the transformed version of this equation. The fundamental solution to the diagonal part $\mathrm{D}_t - \mathscr{D}_1$ is given by

$$\frac{\lambda(s)}{\lambda(t)}\tilde{\mathscr{E}}_0(t,s,\xi),$$

where $\tilde{\mathscr{E}}_0(t,s,\xi) = \operatorname{diag}(\exp(\mathrm{i}(t-s)|\xi|), \exp(-\mathrm{i}(t-s)|\xi|))$ is unitary and related to the free propagator $\mathscr{E}_0(t,s,\xi) = M\tilde{\mathscr{E}}_0(t,s,\xi)M^{-1}$ corresponding to $b \equiv 0$ and $\lambda(t) = \exp(\int_0^t b(\tau)\mathrm{d}\tau)$ describes the influence of dissipation. Note that $\lambda(t) \approx \exp\beta t$ as $t \to \infty$. For the fundamental solution to $\mathrm{D}_t - \mathscr{D}_1 - R_2$ we make the *ansatz* $\lambda(s)/\lambda(t)\tilde{\mathscr{E}}_0(t,s,\xi)\mathscr{Q}(t,s,\xi)$ such that

$$\mathrm{D}_t\mathscr{Q}(t,s,\xi) = \tilde{\mathscr{E}}_0(s,t,\xi)R_2(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)\mathscr{Q}(t,s,\xi), \qquad \mathscr{Q}(s,s,\xi) = I,$$

which can be solved directly by the Peano-Baker series

$$\mathscr{Q}(t,s,\xi) = I + \sum_{k=1}^{\infty} \mathrm{i}^k \int_s^t \mathscr{R}_2(t_1,s,\xi) \int_s^{t_1} \mathscr{R}_2(t_2,s,\xi) \dots \int_s^{t_{k-1}} \mathscr{R}_2(t_k,s,\xi)\mathrm{d}t_k \dots \mathrm{d}t_2\mathrm{d}t_1,$$

where we used the notation $\mathscr{R}_2(t,s,\xi) = \tilde{\mathscr{E}}_0(s,t,\xi)R_2(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)$. Note that $\|\mathscr{R}_2(t,s,\xi)\| = \|R_2(t,\xi)\|$ and therefore

$$\|\mathscr{Q}(t,s,\xi)\| \le \exp\left(\int_s^t \|R_2(\tau,\xi)\|\mathrm{d}\tau\right). \tag{1.8}$$

Hence, the fundamental solution to (1.6) is constructed as $\lambda(s)/\lambda(t)\tilde{\mathscr{E}}_0(t,s,\xi)\mathscr{Q}(t,s,\xi)$ and transforming back to the original problem yields

$$\mathscr{E}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)} MN_1(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)\mathscr{Q}(t,s,\xi)N_1^{-1}(s,\xi)M^{-1} \tag{1.9}$$

for all $t \in [0,T]$ and $|\xi| > N$. Therefore, the monodromy matrix $\mathscr{M}(t,\xi) = \mathscr{E}(t+T,t,\xi)$ is representable as

$$\mathscr{M}(t,\xi) = \frac{\lambda(t)}{\lambda(t+T)} MN_1(t+T,\xi)\tilde{\mathscr{E}}_0(t+T,t,\xi)\mathscr{Q}(t+T,t,\xi)N_1^{-1}(t,\xi)M^{-1}$$

and in combination with $\lambda(t)/\lambda(t+T) = \exp(-\beta T)$ the desired result $\|\mathscr{M}(t,\xi)\| < 1$ uniformly in $t \in [0,T]$ and $|\xi| \ge N$ follows by (1.7) and (1.8).

1.2. Treatment of bounded frequencies. Our next aim is to show that for any $c > 0$ there exists a natural number k such that

$$\sup_{t\in[0,T]} \|\mathscr{M}^k(t,\xi)\| < 1 \tag{1.10}$$

for all $c \le |\xi| \le N$. Note, that we are only interested in a compact set in t and ξ here. We will combine spectral theory with a compactness argument. First step is to show that the spectrum spec $\mathscr{M}(t,\xi)$ is contained inside the open unit ball $\{\zeta\in\mathbf{C}\,|\,|\zeta|<1\}$.

Note first, that $\mathscr{M}(t,\xi)\mathscr{E}(t,0,\xi)=\mathscr{E}(t+T,0,\xi)=\mathscr{E}(t,0,\xi)\mathscr{M}(0,\xi)$ and therefore $\mathscr{M}(t,\xi)$ is similar to $\mathscr{M}(0,\xi)$. Especially the spectrum spec $\mathscr{M}(t,\xi)$ is independent of t. Furthermore, we assumed $b(t)$ to be real. Therefore (1.2) has real-valued solutions and it follows that $\mathscr{M}(t,\xi)$ is similar to a real matrix. Furthermore, according to Liouville theorem we see that

$$\begin{aligned}(1.11)\qquad \det\mathscr{M}(t,\xi)&=\exp\left(\mathrm{i}\int_t^{t+T}\operatorname{tr}A(\tau,\xi)\mathrm{d}\tau\right)\\&=\exp\left(-2\int_t^{t+T}b(\tau)\mathrm{d}\tau\right)=\exp(-2\beta T),\end{aligned}$$

such that the eigenvalues $\varkappa_1(\xi)$ and $\varkappa_2(\xi)$ of $\mathscr{M}(t,\xi)$ are either real and of the form $\varkappa_2(\xi)=\varkappa_1(\xi)^{-1}\exp(-2\beta T)$ or complex-conjugate $\varkappa_2(\xi)=\overline{\varkappa_1(\xi)}$ and therefore $|\varkappa_1(\xi)|=|\varkappa_2(\xi)|=\exp(-\beta T)$. For the complex case we observe spec $\mathscr{M}(t,\xi)\subseteq\{|\zeta|=\exp(-\beta T)\}$; for the real case we have to look more carefully. Note, that the eigenvalues are continuous in ξ.

Assume that for a certain frequency $\bar{\xi}\neq 0$ the monodromy matrix $\mathscr{M}(0,\bar{\xi})$ has an eigenvalue with modulus 1. Since it must be real, it is either 1 or -1. Let $\vec{c}=(c_1,c_2)$ be a corresponding eigenvector. Then we can find a domain $\Omega_R=\{x\in\mathbf{R}^n\,|\,|x|\le R\}$ such that $-|\bar{\xi}|^2$ is an eigenvalue of the Dirichlet Laplacian on Ω_R with normalised eigenfunction $\phi(x)$. Let us consider the initial boundary value problem $\Box u+2b(t)u_t=0$ with Dirichlet boundary condition and $u(0,\cdot)=c_1|\bar{\xi}|^{-1}\phi(x)$ and $u_t(0,\cdot)=\mathrm{i}c_2\phi(x)$. Now we look for the solution in the form

$$u(t,x)=f(t)\phi(x)$$

and we show that $f(t)$ is T-periodic (or $2T$-periodic). Indeed, as a consequence of $\Delta\phi=-|\bar{\xi}|^2\phi$, the partial differential equation $\Box u+2b(t)u_t=0$ turns into the ordinary differential equation $u_{tt}+|\bar{\xi}|^2u+2b(t)u_t=0$ (with x regarded as parameter), and hence, the corresponding solution satisfies

$$\mathrm{D}_t\begin{pmatrix}|\bar{\xi}|u(t,x)\\ \mathrm{D}_tu(t,x)\end{pmatrix}=\begin{pmatrix}&|\bar{\xi}|\\|\bar{\xi}|&2\mathrm{i}b(t)\end{pmatrix}\begin{pmatrix}|\bar{\xi}|u(t,x)\\ \mathrm{D}_tu(t,x)\end{pmatrix},\qquad\left.\begin{pmatrix}|\bar{\xi}|u(t,x)\\ \mathrm{D}_tu(t,x)\end{pmatrix}\right|_{t=0}=\vec{c}\phi(x).$$

But this system can be solved by the fundamental solution $\mathscr{E}(t,s,\bar{\xi})$ and especially for $t=T$ we obtain

$$\left.\begin{pmatrix}|\bar{\xi}|u(t,x)\\ \mathrm{D}_tu(t,x)\end{pmatrix}\right|_{t=T}=\mathscr{M}(T,\bar{\xi})\vec{c}\phi(x)=\pm\vec{c}\phi(x)$$

(sign according to the eigenvalue). Thus we conclude that $f(t)$ is a T-periodic (or $2T$-periodic) function with $f(0) = c_1|\bar{\xi}|^{-1}$. However, this is not possible. If we denote the energy of this solution as $E(u;t) = \|\nabla u(t,\cdot)\|_2^2 + \|\mathrm{D}_t u(t,\cdot)\|_2^2$, the standard integration by parts argument gives

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u;t) = -2b(t)\|u_t\|_2^2 = -2b(t)|f'(t)|^2,$$

such that after integration $0 = -2\int_0^{(2)T} b(t)|f'(t)|^2 \mathrm{d}t$. The a.e. positivity of $b(t)$ implies that f is constant and this contradicts $\bar{\xi} \neq 0$. Thus, $\pm 1 \notin \operatorname{spec} \mathscr{M}(t,\xi)$ for $\xi \neq 0$ and therefore (using that the spectrum is inside the unit ball for $|\xi| > N$) the spectral radius satisfies $\rho(\mathscr{M}(t,\xi)) < 1$ for all $\xi \neq 0$. Thus, the spectral radius formula $\|\mathscr{M}^k(t,\xi)\|^{1/k} \to \rho(\mathscr{M}(t,\xi)) < 1$ implies that for any t and ξ we find a number k such that $\|\mathscr{M}^k(t,\xi)\| < 1$.

Next, we want to show that we can find such a number k uniform on any compact frequency interval $|\xi| \in [c,N]$. Set for this $\mathscr{U}_k = \{(t,\xi) \mid \|\mathscr{M}^{(2^k)}(t,\xi)\| < 1\}$. The sets $\mathscr{U}_k$ are clearly open (by the continuity of the monodromy matrix) and satisfy $\mathscr{U}_k \subseteq \mathscr{U}_\ell$ for $k \leq \ell$. The above reasoning shows that the compact set $\mathscr{C} = \{(t,\xi) \mid 0 \leq t \leq T,\ c \leq |\xi| \leq N\}$ is contained in $\bigcup_k \mathscr{U}_k$ and by compactness we find one k such that $\mathscr{C} \subset \mathscr{U}_k$. Hence, the estimate (1.10) is proven.

Remark. We know even a little bit more about the structure of the eigenvalues $\varkappa_1(\xi)$ and $\varkappa_2(\xi)$. We can apply a Liouville type transform to equation (1.2) to deduce Hill's equation

$$v_{tt} + (|\xi|^2 - b^2(t) - b'(t))v = 0, \qquad v = \lambda(t)\hat{u}, \tag{1.12}$$

such that Floquet theory, see e.g. [Eas73], may be applied. This implies that if $b^2(t) + b'(t)$ is not constant (which is equivalent to $b(t)$ not constant) then there exist infinitely many (closed) intervals $I_0 = (-\infty, \tau_0]$ and $I_k = [\tau_k^-, \tau_k^+]$, $k = 1, 2, \ldots$, such that for $|\xi| \in I_j$, $j = 0, 1, \ldots$, the spectrum $\operatorname{spec} \mathscr{M}(t,\xi)$ is real (*intervals of instability* for (1.12)), while for all other ξ the eigenvalues are complex and conjugate to each other (*intervals of stability* for (1.12)). The numbers $\tau_k^{\pm}$ are the eigenvalues of the corresponding periodic eigenvalue problem $-v'' + (b^2(t) + b'(t))v = \lambda^2 v$ with periodic boundary conditions $v(0) = v(2T)$, $v'(0) = v'(2T)$.

1.3. The neighbourhood of $\xi = 0$. The frequency $\xi = 0$ is the only exceptional point of our reasoning, since $\operatorname{spec} \mathscr{M}(t,0) = \{1, \exp(-2\beta T)\}$ contains the eigenvalue 1. This follows directly by solving (1.2); a fundamental system of solutions is given by 1 and $\int_0^t \exp(-2\int_0^s b(\tau)\mathrm{d}\tau)\mathrm{d}s$. We will use ideas from the theory of Hill's equation to understand the structure of $\mathscr{E}(t,s,\xi)$ near

$\xi = 0$. From the remarks of Section 1.2 it is clear that 0 is interior point of I_0 and, choosing c small enough allows to write down a fundamental system of solutions to (1.2) as

$$e^{-\nu_\pm(\xi)t} f_\pm(t,\xi) \tag{1.13}$$

with T-periodic functions $f_\pm(t,\xi)$ and exponents $\nu_\pm(\xi)$ such that $\exp(-\nu_\pm(\xi)T) \in \operatorname{spec} \mathscr{M}(t,\xi)$. It is clear that $\nu_\pm(\xi) > 0$ for $\xi \neq 0$ and we denote them in such a way that $\nu_+(\xi) \to 0$ and $\nu_-(\xi) \to 2\beta$ as $\xi \to 0$. Any solution to (1.2) is a combination of these two solutions. The part corresponding to $\nu_-(\xi)$ is not of interest for us (because it leads to an exponential decay as $t \to \infty$) and we can concentrate on the $\nu_+(\xi)$ part.

We know that $\nu_+(\xi)$ is an analytic function of $|\xi|$ (as long as $\mathscr{M}(t,\xi)$ has no multiple eigenvalues) and can therefore be expanded into a MacLaurin series

$$\nu_+(\xi) = \sum_{k=1}^{\infty} \alpha_k |\xi|^k. \tag{1.14}$$

We want to show that $\alpha_1 = 0$ and $\alpha_2 > 0$. For this we use $\mathscr{M}(0,\xi) = \mathscr{E}(T,0,\xi)$ and calculate the derivatives of $\operatorname{tr} \mathscr{M}(0,\xi)$ with respect to $|\xi|$ at $\xi = 0$. Note, that $\partial_{|\xi|} A(t,\xi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathscr{J}$ and $\partial_{|\xi|}^2 A(t,\xi) = 0$, such that $D_t \partial_{|\xi|} \mathscr{E}(t,s,\xi) = \mathscr{J}\mathscr{E}(t,s,\xi) + A(t,\xi)\partial_{|\xi|}\mathscr{E}(t,s,\xi)$ and $\partial_{|\xi|}\mathscr{E}(s,s,\xi) = 0$. Therefore we obtain the representation

$$\partial_{|\xi|}\mathscr{E}(t,s,\xi) = \int_s^t \mathscr{E}(t,\tau,\xi)\mathscr{J}\mathscr{E}(\tau,s,\xi)\mathrm{d}\tau.$$

Using that

$$\mathscr{E}(t,s,0) = \exp\left(\mathrm{i}\int_s^t A(\tau,0)\mathrm{d}\tau\right) = \operatorname{diag}\left(1, \exp(-2\int_s^t b(\tau)\mathrm{d}\tau\right) = \operatorname{diag}\left(1, \frac{\lambda^2(s)}{\lambda^2(t)}\right)$$

is diagonal, we immediately see that the above integrand has zeros as diagonal entries for $\xi = 0$. This implies $\partial_{|\xi|} \operatorname{tr} \mathscr{M}(t,0) = 0$. For the second derivative we use in analogy that $D_t \partial_{|\xi|}^2 \mathscr{E}(t,s,\xi) = A(t,\xi)\partial_{|\xi|}^2\mathscr{E}(t,s,\xi) + 2\mathscr{J}\partial_{|\xi|}\mathscr{E}(t,s,\xi)$ and $\partial_{|\xi|}^2\mathscr{E}(s,s,\xi) = 0$, such that after integration

$$\begin{aligned}\partial_{|\xi|}^2\mathscr{E}(t,s,\xi) &= 2\int_s^t \mathscr{E}(t,\tau,\xi)\mathscr{J}\partial_{|\xi|}\mathscr{E}(\tau,s,\xi)\mathrm{d}\tau \\ &= 2\int_s^t \mathscr{E}(t,\tau,\xi)\mathscr{J}\int_s^\tau \mathscr{E}(\tau,\theta,\xi)\mathscr{J}\mathscr{E}(\theta,s,\xi)\mathrm{d}\theta\mathrm{d}\tau.\end{aligned}$$

For $\xi = 0$ we can evaluate these integrals and obtain for the trace

$$\partial_{|\xi|}^2 \operatorname{tr} \mathscr{M}(t,0) = 2\int_0^T \int_0^\tau \left(\frac{\lambda^2(\theta)}{\lambda^2(\tau)} + \frac{\lambda^2(\tau)}{\lambda^2(T)\lambda^2(\theta)} \right) \mathrm{d}\theta \mathrm{d}\tau > 0.$$

On the other hand, $\operatorname{tr} \mathscr{M}(t,\xi) = \exp(-\nu_+(\xi)T) + \exp(-\nu_-(\xi)T)$ with $\nu_+(\xi) + \nu_-(\xi) = 2\beta$, such that $\partial_{|\xi|} \operatorname{tr} \mathscr{M}(t,0) = \alpha_1 T(1 - \exp(-2\beta T)) = 0$ implies $\alpha_1 = 0$ and $\partial_{|\xi|}^2 \operatorname{tr} \mathscr{M}(t,0) = 2\alpha_2 T(1 - \exp(-2\beta T)) > 0$ implies $\alpha_2 > 0$.

Hence, we have shown that as $\xi \to 0$ the exponent behaves like $\nu_+(\xi) = \alpha_2|\xi|^2 + \mathcal{O}(|\xi|^3)$ (and, if we look carefully at the representations, we see that all odd coefficients vanish and thus the remainder term is $\mathcal{O}(|\xi|^4)$). This will be enough to obtain energy and dispersive estimates for solutions to our Cauchy problem in Section 2.

1.4. Collection of results. What have we obtained so far? The main results are concerned with the monodromy operator $\mathscr{M}(t,\xi) = \mathscr{E}(t+T,t,\xi)$ and its spectral properties.

Theorem 1.1. (1) *There exists a (large) number $N > 0$ such that for all $|\xi| \geq N$ the monodromy matrix $\mathscr{M}(t,\xi)$ is a contraction (uniform in t), i.e.*

$$\sup_t \|\mathscr{M}(t,\xi)\| < 1.$$

(2) *For any (small) number $c > 0$ there exists an exponent $k \in \mathbf{N}$, such that for $c \leq |\xi| \leq N$ the matrix $\mathscr{M}^k(t,\xi)$ is a contraction (uniform in t), i.e. $\sup_t \|\mathscr{M}^k(t,\xi)\| < 1$.*

(3) *As $\xi \to 0$ the eigenvalues of $\mathscr{M}(t,\xi)$ satisfy*

$$\log \varkappa_1(\xi) = -\alpha_2 T|\xi|^2 + \mathcal{O}(|\xi|^4), \qquad \log \varkappa_2(\xi) = -2\beta T + \alpha_2 T|\xi|^2 + \mathcal{O}(|\xi|^4)$$

with a positive coefficient $\alpha_2 > 0$.

2. Estimates for solutions

The repesentations from Section 1 allow us to estimate the Fourier transform of solutions, in combination with Plancherel's theorem this gives estimates in L^2-spaces, combined with Hölder inequality and mapping properties of the Fourier transform dispersive estimates follow.

2.1. Energy estimates. We distinguish between small and large frequencies. If $|\xi| \geq N$, the monodromy matrix $\mathscr{M}(0,\xi)$ is a contraction and therefore $\|\mathscr{E}(t,0,\xi)\| = \|\mathscr{M}^\ell(s,\xi)\mathscr{E}(s,0,\xi)\| \leq c^\ell \|\mathscr{E}(s,0,\xi)\|$ for $t = \ell T + s$, $s \in [0,T]$ and $c = \sup_t \|\mathscr{M}(t,\xi)\| < 1$. Furthermore, since $b(t) \geq 0$ we know that $\|\mathscr{E}(s,0,\xi)\| \leq 1$ and therefore

$$\|\mathscr{E}(t,0,\xi)\| \le \mathrm{e}^{-\delta(t-T)} \tag{2.1}$$

with $\delta = T^{-1}\log c^{-1} > 0$. Thus, high frequencies lead to an exponential decay. For the intermediate frequencies we obtain similarly $\|\mathscr{E}(t,0,\xi)\| = \|\mathscr{M}^{\ell}(s,\xi)\mathscr{E}(s,0,\xi)\| \le c^{\ell}\|\mathscr{E}(s,0,\xi)\|$ for $t = \ell kT + s$, $s \in [0,kT]$ and $c = \sup_t\|\mathscr{M}(t,\xi)\| < 1$. Again this yields exponential decay, but now of the form

$$\|\mathscr{E}(t,0,\xi)\| \le \mathrm{e}^{-\delta(t-kT)} \tag{2.2}$$

with $\delta = (kT)^{-1}\log c^{-1} > 0$. Hence, the only non-exponential contribution may come from the neighbourhood of $\xi = 0$. For the treatment of small frequencies we have to specify the structure of the estimate we have in mind. While estimating the energy of the solution at time t in terms of the initial energy brings (due to $1 \in \operatorname{spec}\mathscr{M}(t,0)$) only the trivial uniform bound and no decay, an estimate in terms of $\|u_1\|_{H^1}$ and $\|u_2\|_{L^2}$ brings decay. Reason for this is that we can use an additional factor $|\xi|$ for small frequencies.

If $|\xi| \le c$ is sufficiently small, we know that a fundamental system of solutions to (1.2) is given by $\exp(-\nu_\pm(\xi)t)f_\pm(t,\xi)$ with real T-periodic functions $f_\pm(t,\xi)$ and exponents $\nu_+(\xi) = \alpha_2|\xi|^2 + \mathcal{O}(|\xi|^4)$, $\nu_-(\xi) = 2\beta - \alpha_2|\xi|^2 + \mathcal{O}(|\xi|^4)$. Furthermore, $f_\pm(t,\xi)$ are non-zero for all t and ξ. This follows from the fact that they are periodic and non-zero for $\xi = 0$. Thus, if they would have a zero for some t and ξ we could find a smallest value of ξ where the zero occurs. By differentiability of $f_\pm$ it follows that for this fixed ξ we would obtain a zero of order at least 2, which contradicts the fact that $\exp(-\nu_\pm(\xi)t)f_\pm(t,\xi)$ is a not identically vanishing solution of the second order equation (1.2). Hence, we may assume that $f_\pm(0,\xi) = 1$ for all $|\xi| \le c$. This allows to express the special fundamental system of solutions $\Phi_1(t,\xi)$ and $\Phi_2(t,\xi)$ with $\Phi_1(0,\xi) = 1$, $\partial_t\Phi_1(0,\xi) = 0$, $\Phi_2(0,\xi) = 0$ and $\partial_t\Phi_2(t,\xi) = 1$, i.e. the fundamental system representing solutions to (1.2) as

$$\hat{u}(t,\xi) = \sum_{j=1,2}\Phi_j(t,\xi)\hat{u}_j(\xi), \qquad |\xi| \le c, \tag{2.3}$$

in terms of $f_\pm(t,\xi)$ and the exponents $\nu_\pm(\xi)$. A simple calculation shows

$$\Phi_2(t,\xi) = \frac{\mathrm{e}^{-\nu_+(\xi)t}f_+(t,\xi) - \mathrm{e}^{-\nu_-(\xi)t}f_-(t,\xi)}{\nu_-(\xi) - \nu_+(\xi) + \partial_t f_+(0,\xi) - \partial_t f_-(0,\xi)},$$

$$\Phi_1(t,\xi) = \frac{\mathrm{e}^{-\nu_+(\xi)t}f_+(t,\xi) + \mathrm{e}^{-\nu_-(\xi)t}f_-(t,\xi)}{2} - \left(\frac{\partial_t f_+(0,\xi) + \partial_t f_-(0,\xi)}{2} - \beta\right)\Phi_2(t,\xi).$$

Both functions are smooth in ξ and differentiable in t, especially it follows that the denominator in the first expression is non-zero. Since we are interested in polynomial decay rates, we can forget about all the f_--terms, which immediately lead to exponential decay. Thus, to estimate $|\xi|\hat{u}(t,\xi)$ in terms of $\hat{u}_1$ and $\hat{u}_2$, the typical term to estimate is $|\xi|e^{-\nu_+(\xi)t}f_+(t,\xi)$ (multiplied by a ξ-dependent function). Now $\nu_+(\xi) \sim \alpha_2|\xi|^2$ implies the uniform decay rate $t^{-1/2}$ for this term. To estimate $\partial_t\hat{u}(t,\xi)$ in terms of $\hat{u}_1$ and $\hat{u}_2$ we have to consider the typical term $e^{-\nu_+(\xi)t}\partial_t f_+(t,\xi) - \nu_+(\xi)e^{-\nu_+(\xi)t}f_+(t,\xi)$. The second addend gives t^{-1}, while the first one has to be considered in detail. Note, that $\Phi_1(t,0) = 1$, such that comparing representations implies $f_+(t,0) = 1$. Since the equation was parametrised by $|\xi|^2$, smoothness in $|\xi|$ and periodicity in t imply $\partial_t f_+(t,\xi) = |\xi|^2\tilde{h}_+(t,\xi)$ with a bounded T-periodic function $\tilde{h}_+(t,\xi)$. Therefore we see, that the first addend gives the same decay rate t^{-1}.

We collect our results in the following theorem.

Theorem 2.1. *The solution $u(t,x)$ of the Cauchy problem* (1.1) *satisfies the a priori estimates*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_1\|_{L^2} + \|u_2\|_{H^{-1}},$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1/2}(\|u_1\|_{H^1} + \|u_2\|_{L^2}),$$

$$\|\partial_t u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}(\|u_1\|_{H^1} + \|u_2\|_{L^2})$$

Furthermore, for any cut-off function $\chi \in C^\infty(\mathbf{R})$ with $\chi(s) = 0$ near $s = 0$ and $\chi(s) = 1$ for large s there exists a constant $\delta > 0$ such that the exponential estimate

$$\|\chi(\mathrm{D})u(t,\cdot)\|_{L^2} + \|\chi(\mathrm{D})\nabla u(t,\cdot)\|_{L^2} + \|\chi(\mathrm{D})\partial_t u(t,\cdot)\|_{L^2} \lesssim e^{-\delta t}(\|u_1\|_{H^1} + \|u_2\|_{L^2})$$

holds true.

2.2. Dispersive estimates. We will continue this short note with some remarks on dispersive and more generally L^p–L^q decay estimates. Again only the small frequencies are of interest, since by Sobolev embedding the previous theorem implies

$$\begin{aligned}&\|\chi(\mathrm{D})u(t,\cdot)\|_{L^q} + \|\chi(\mathrm{D})\nabla u(t,\cdot)\|_{L^q} + \|\chi(\mathrm{D})\partial_t u(t,\cdot)\|_{L^q}\\ &\quad\lesssim e^{-\delta t}(\|u_1\|_{H^{p,r_p+1}} + \|u_2\|_{H^{p,r_p}})\end{aligned}$$

for any choice of indices $1 \le p \le 2 \le q \le \infty$ and regularity $r_p > n(1/p - 1/q)$. Thus it remains to consider the typical terms from the previous section near $\xi = 0$. Instead of Plancherel's theorem we use Hölder inequality together with the L^p–$L^{p'}$ boundedness of the Fourier transform for $pp' = p + p'$ to deduce

$$\|\chi(\mathrm{D})|\mathrm{D}|e^{-\nu_+(\mathrm{D})t}f_+(t,\mathrm{D})\|_{L^p\to L^q} \le \|\chi(\xi)|\xi|e^{-\nu_+(\xi)t}f_+(t,\xi)\|_{L^r}$$

for any $1 \le p \le 2 \le q \le \infty$ and with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. The L^r-norm can be calculated directly using $v_+(\xi) \sim \alpha_2|\xi|^2$, which implies the decay rate $t^{-1/2-n/2r}$. Similarly we obtain for the derivative terms the rate $t^{-1-n/2r}$ and for the solution itself $t^{-n/2r}$.

THEOREM 2.2. *The solution* $u(t,x)$ *of the Cauchy problem* (1.1) *satisfies the a priori estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-(n/2)(1/p-1/q)}(\|u_1\|_{H^{p,r_p}} + \|u_2\|_{H^{p,r_p-1}}),$$

$$\|\nabla u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-1/2-(n/2)(1/p-1/q)}(\|u_1\|_{H^{p,r_p+1}} + \|u_2\|_{H^{p,r_p}}),$$

$$\|\partial_t u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-1-(n/2)(1/p-1/q)}(\|u_1\|_{H^{p,r_p+1}} + \|u_2\|_{H^{p,r_p}}).$$

for all $1 \le p \le 2 \le q \le \infty$ *and* $r_p > n(1/p - 1/q)$.

2.3. Diffusion phenomenon. For proving estimates for the solution u we used that the only bad term in the representation of solutions was $\mathrm{e}^{-v_+(\xi)t} f_+(t,\xi) \sim \mathrm{e}^{-\alpha_2|\xi|^2 t}$, which corresponds to the Fourier multiplier for a corresponding heat equation

$$w_t = \alpha_2 \Delta w, \qquad w(0,\cdot) = w_0. \tag{2.4}$$

Choosing w_0 in dependence of u_1 and u_2 allows to cancel the corresponding terms in the representation of solutions such that the norm of the difference $||u - w||_{L^2}$ decays. For constant b and with $\alpha_2 = (2b)^{-1}$ this was observed in [Nis97] and [YaMi01] for lower dimensions. The general abstract result is due to [IkNi03] and [ChHa03], which we are now able to extend to periodic dissipation terms. If we choose

$$w_0 = u_1 + \frac{1}{2\beta - \gamma} u_2, \qquad \gamma = \partial_t f_-(0,0) = 2\beta - (1 - \mathrm{e}^{-2\beta T})\left(\int_0^T \frac{\mathrm{d}\tau}{\lambda^2(\tau)}\right)^{-1} \tag{2.5}$$

and use the α_2 from Section 1.3,

$$\alpha_2 = \frac{1}{T(1 - \mathrm{e}^{-2\beta T})} \int_0^T \int_0^\tau \left(\frac{\lambda^2(\theta)}{\lambda^2(\tau)} + \frac{\lambda^2(\tau)}{\lambda^2(T)\lambda^2(\theta)}\right) \mathrm{d}\theta \mathrm{d}\tau, \tag{2.6}$$

$$\lambda(t) = \exp\left(\int_0^t b(s)\mathrm{d}s\right),$$

then the following result holds true:

THEOREM 2.3. *The solutions* $u(t,x)$ *of* (1.1) *and* $w(t,x)$ *of* (2.4) *satisfy under the relation* (2.5) *the a-priori estimate*

$$||u(t,\cdot) - w(t,\cdot)||_{L^2} \lesssim (1+t)^{-1}(||u_1||_{H^1} + ||u_2||_{L^2}).$$

To prove this result we first note that we can forget about all terms in the representation which give a faster decay. The choice of the initial datum (2.5) implies that the only term of interest is $(\mathrm{e}^{-\nu_+(\xi)t} f_+(t,\xi) - \mathrm{e}^{-\alpha_2|\xi|^2 t})\hat{w}_0$ and $f_+(t,\xi) = 1 + \mathcal{O}(|\xi|^2)$ together with $\nu_+(\xi) = \alpha_2|\xi|^2 + \mathcal{O}(|\xi|^4)$ localised near $|\xi| = 0$. But this multiplier can be estimated by a combination of $\mathrm{e}^{-\nu_+(\xi)t}|\xi|^2$ and $\mathrm{e}^{-\min(\nu_+(\xi),\alpha_2|\xi|^2)t}|\xi|^4 t$. Both terms decay uniformly like $(1+t)^{-1}$ and the assertion follows.

A similar statement with improvement of one decay order holds for dispersive and L^p–L^q estimates as well as for estimates of higher order spatial derivatives. The reasoning is analogous.

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