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Positive solutions of singular Emden-Fowler type systems

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1. Introduction

In this paper we will discuss positive solutions of the singular Emden-Fowler type system

$$(1) \qquad \begin{cases} y'' = a(t)z^{-\lambda}, \\ z'' = b(t)y^{-\mu}, \end{cases} \qquad t \ge t_0,$$

where λ, $\mu > 0$ are constants, and a and b are continuous functions on $[t_0, \infty)$. The following conditions are always assumed to hold:

(C_1) a and b have unbounded supports.

(C_2) The improper integrals

$$A(t) \equiv \int_t^\infty a(s)ds \quad \text{and} \quad B(t) \equiv \int_t^\infty b(s)ds$$

converge for $t \ge t_0$, and $A(t)$, $B(t) \ge 0$, $t \ge t_0$.

(C_3) AB has unbounded support.

A vector function $(y, z) \in C^2[t_0, \infty) \times C^2[t_0, \infty)$ is called a positive solution of system (1) when it solves system (1) and $y(t)$, $z(t) > 0$ for $t \ge t_0$.

The singular Emden-Fower type equations of the form

$$y'' = h(t)y^{-\lambda}, \qquad t \ge t_0,$$

with $\lambda > 0$, $h \in C[t_0, \infty)$, have been treated in several papers; see [1-5]. Especially, the author [5] showed under suitable conditions that this equation admits a positive solution $y(t)$ satisfying

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0.$$

Sufficient conditions for the uniqueness of such solutions were also given. However, it seems that very litttle is known about such decaying solutions of system (1). Therefore in this paper we will give sufficient conditions which ensure the existence of positive solutions (y, z) of (1) satisfying

$$\begin{aligned} &\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0, \\ &\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\lim_{t\to\infty} y(t) = \ell, \quad \lim_{t\to\infty} y'(t) = 0, \\ &\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0, \end{aligned} \tag{3_ℓ}$$

with $\ell > 0$. A criterion for the nonexistence of these solutions will also be given.

2. Main Results

In order to derive the existence theorems, we need the next basic lemma which gives a sufficient condition for systen (1) to have a postive solution (y, z) tending to a positive limit as $t \to \infty$.

LEMMA. *Suppose that*

$$\int_{t_0}^{\infty} A(t)dt < \infty, \tag{4}$$

$$\int_{t_0}^{\infty} B(t)dt < \infty, \tag{5}$$

and

$$\int_{t_0}^{\infty} tA(t)B(t)dt < \infty.$$

Then for any ℓ, $m > 0$, *system* (1) *admits a positive solution* (y, z) *satisfying*

$$\begin{aligned} &\lim_{t\to\infty} y(t) = \ell, \quad \lim_{t\to\infty} y'(t) = 0, \\ &\lim_{t\to\infty} z(t) = m, \quad \lim_{t\to\infty} z'(t) = 0, \end{aligned} \tag{6}$$

and $y'(t)$, $z'(t) < 0$ *in* $[t_0, \infty)$.

PROOF. It is easily verified from condition (C_2) that $(y, z) \in C^2[t_0, \infty) \times C^2[t_0, \infty)$ becomes a positive solution of system (1) satisfying (6) if and only if it solves the system

$$y(t) = \ell + \int_t^{\infty} A(s)[z(s)]^{-\lambda} ds - \lambda \int_t^{\infty} \left(\int_s^{\infty} A(r)[z(r)]^{-\lambda-1} z'(r) dr \right) ds, \tag{7}$$

$$z(t) = m + \int_t^\infty B(s)[y(s)]^{-\mu} ds - \mu \int_t^\infty \left(\int_s^\infty B(r)[y(r)]^{-\mu-1} y'(r) dr \right) ds, \tag{8}$$

for $t \geq t_0$. First we solve this system in some neighborhood of infinity, say $t \geq T \geq t_0$. Choose c, $k > 0$ and $T \geq t_0$ so that

$$\lambda m^{-\lambda-1}\left(\ell^{-\mu} + k \int_T^\infty A(s) ds \right) \leq c,$$

$$\mu \ell^{-\mu-1}\left(m^{-\lambda} + c \int_T^\infty B(s) ds \right) \leq k,$$

$$m^{-\lambda}\int_T^\infty A(s) ds + \lambda m^{-\lambda-1}\left(\ell^{-\mu} + k \int_T^\infty A(s) ds \right)\left(\int_T^\infty sA(s)B(s) ds \right) \leq \ell,$$

$$\ell^{-\mu}\int_T^\infty B(s) ds + \mu \ell^{-\mu-1}\left(m^{-\lambda} + c \int_T^\infty B(s) ds \right)\left(\int_T^\infty sA(s)B(s) ds \right) \leq m.$$

Consider the set X of all functions (y, z) in $C^1[T, \infty) \times C^1[T, \infty)$ whose components satisfy the inequalities

$$\ell \leq y(t) \leq 2\ell,$$

$$m \leq z(t) \leq 2m,$$

$$0 \leq -y'(t) \leq m^{-\lambda} A(t) + c \int_t^\infty A(s)B(s) ds,$$

$$0 \leq -z'(t) \leq \ell^{-\mu} B(t) + k \int_t^\infty A(s)B(s) ds$$

for $t \geq T$. Clearly X is a nonempty closed convex subset of the Fréchet space $C^1[T, \infty) \times C^1[T, \infty)$. Define the mapping $\mathscr{F} : X \to C^1[T, \infty) \times C^1[T, \infty)$ by $\mathscr{F}(y, z) = (\bar{y}, \bar{z})$, where

$$\bar{y}(t) = \ell + \int_t^\infty A(s)[z(s)]^{-\lambda} ds - \lambda \int_t^\infty \left(\int_s^\infty A(r)[z(r)]^{-\lambda-1} z'(r) dr \right) ds,$$

$$\bar{y}'(t) = -A(t)[z(t)]^{-\lambda} + \lambda \int_t^\infty A(s)[z(s)]^{-\lambda-1} z'(s) ds,$$

and

$$\bar{z}(t) = m + \int_t^\infty B(s)[y(s)]^{-\mu} ds$$
$$- \mu \int_t^\infty \left(\int_s^\infty B(r)[y(r)]^{-\mu-1} y'(r) dr \right) ds,$$
$$\bar{z}'(t) = - B(t)[y(t)]^{-\mu} + \mu \int_t^\infty B(s)[y(s)]^{-\mu-1} y'(s) ds,$$

for $t \geq T$. We shall show that $\mathscr{F}$ maps X continuously into a relatively compact set of itself.

To prove $\mathscr{F} X \subset X$, let $(y, z) \in X$. Since

$$\int_s^\infty A(r)[z(r)]^{-\lambda-1} |z'(r)| dr$$
$$\leq m^{-\lambda-1} \left[\ell^{-\mu} \int_s^\infty A(r)B(r) dr + k \int_s^\infty A(r) \left(\int_r^\infty A(u)B(u) du \right) dr \right]$$
$$\leq m^{-\lambda-1} \left[\ell^{-\mu} \int_s^\infty A(r)B(r) dr + k \left(\int_s^\infty A(r) dr \right) \left(\int_s^\infty A(r)B(r) dr \right) \right],$$

for $s \geq T$, we obtain

$$|\bar{y}'(t)| \leq m^{-\lambda} A(t) + \lambda m^{-\lambda-1} \left(\ell^{-\mu} + k \int_T^\infty A(s) ds \right) \left(\int_t^\infty A(s)B(s) ds \right)$$
$$\leq m^{-\lambda} A(t) + c \int_t^\infty A(s)B(s) ds, \qquad t \geq T,$$

and

$$0 \leq \bar{y}(t) - \ell \leq m^{-\lambda} \int_T^\infty A(s) ds + \lambda \int_t^\infty \left(\int_s^\infty A(r)[z(r)]^{-\lambda-1} |z'(r)| dr \right) ds$$
$$\leq m^{-\lambda} \int_T^\infty A(s) ds$$
$$+ \lambda m^{-\lambda-1} \left[\ell^{-\mu} \int_t^\infty \left(\int_s^\infty A(r)B(r) dr \right) ds + k \int_t^\infty \left(\int_s^\infty A(r) dr \right) \left(\int_s^\infty A(r)B(r) dr \right) ds \right]$$
$$\leq m^{-\lambda} \int_T^\infty A(s) ds + \lambda m^{-\lambda-1} \left(\ell^{-\mu} + k \int_T^\infty A(s) ds \right) \left(\int_T^\infty sA(s)B(s) ds \right)$$
$$\leq \ell, \qquad t \geq T.$$

The estimates for $\bar{z}(t)$ and $\bar{z}'(t)$ are similarly obtained. Thus $\mathscr{F}X \subset X$. The continuity of $\mathscr{F}$ is a simple consequence of the dominated convergence theorem, and the Ascoli-Arzelà theorem asserts that $\mathscr{F}X$ is compact. Therefore the Schauder-Tychonoff fixed point theorem shows that $\mathscr{F}$ has a fixed element (y, z) in X. Hence system (7)–(8) admits a positive solution (y, z) in $[T, \infty)$. Note that the derivative (y', z') is given by

$$y'(t) = -A(t)[z(t)]^{-\lambda} + \lambda \int_t^\infty A(s)[z(s)]^{-\lambda-1} z'(s)ds, \tag{9}$$

and

$$z'(t) = -B(t)[y(t)]^{-\mu} + \mu \int_t^\infty B(s)[y(s)]^{-\mu-1} y'(s)ds \tag{10}$$

for $t \geq T$.

The rest of the proof proceeds as in the proof of [5, Lemma 1]. First we notice that $y'(t), z'(t) < 0$ in $[T, \infty)$. In fact, if it does not hold, then $y'(\tau) = 0$ for some $\tau \geq T$. In view of the sign conditions for $A(t)$ and $z'(t)$, by putting $t = \tau$ in (9), it is found that

$$\lambda \int_\tau^\infty A(s)[z(s)]^{-\lambda-1} z'(s)ds = 0,$$

namely, $A(t)z'(t) \equiv 0$ in $[\tau, \infty)$. Multiplying (10) by $A(t)$, we see that $A(t)B(t) \equiv 0$ in $[\tau, \infty)$. This contradicts our assumption (C_3). Hence $y'(t) < 0$ in $[T, \infty)$, and the same procedure shows that $z'(t) < 0$ in $[T, \infty)$.

Next let us prolong (y, z) to the left as a solution of (1). Let $I \subset [t_0, \infty)$ be the maximal interval of existence of (y, z). It is clear that (7), (8), (9) and (10) are still valid for $t \in I$. We claim again that $y'(t), z'(t) < 0$, $t \in I$. In fact, if this is not true, we, can find $\tau \in I$, $\tau < T$ such that $y'(t), z'(t) < 0$, $t > \tau$ and either $y'(t)$ or $z'(t)$ vanishes at τ. We may suppose $y'(\tau) = 0$. Then, putting $t = \tau$ in (9), we have $A(t)z'(t) \equiv 0$ in $[\tau, \infty)$ and therefore multiplying (10) by $A(t)$, we immediately reach a contradiction as before. Hence $y'(t), z'(t) < 0$ for $t \in I$.

From the above observation it can be shown easily that I coincides with the whole interval $[t_0, \infty)$, and therefore $y'(t), z'(t) < 0$ for $t \in I \equiv [t_0, \infty)$. This completes the proof.

When $\lambda\mu < 1$, by applying our Lemma, we can obtain the following results which ensure the existence of positive solutions of system (1) satisfying (2) or $(3)_\ell$, $\ell > 0$.

THEOREM 1. *Let* $\lambda\mu < 1$. *Suppose that* (4) *and* (5) *hold. Furthermore suppose that*

$$\int_{t_0}^{\infty} t|a(t)|\left(\int_t^{\infty} B(s)ds\right)^{-\lambda} dt < \infty, \tag{11}$$

$$\int_{t_0}^{\infty} t|b(t)|\left(\int_t^{\infty} A(s)ds\right)^{-\mu} dt < \infty. \tag{12}$$

Then system (1) *has a positive solution* (y, z) *satisfying* (2).

THEOREM 2. *Let* $\lambda\mu < 1$. *Suppose that*

$$\int_{t_0}^{\infty} t|b(t)|dt < \infty$$

and (11). *Then, for any* $\ell > 0$, *system* (1) *admits a positive solution* (y, z) *satisfying* $(3)_\ell$.

PROOF OF THEOREM 1. First notice that all assumptions of our Lemma are fulfilled. Thus for $n \in N$, there exists a positive solution $(y_n, z_n) \in C^2[t_0, \infty) \times C^2[t_0, \infty)$ of system (1) satisfying

$$y_n'(t),\ z_n'(t) < 0, \qquad t \geq t_0, \tag{13}$$

and

$$\lim_{t\to\infty} y_n(t) = \lim_{t\to\infty} z_n(t) = \frac{1}{n},$$

$$\lim_{t\to\infty} y_n'(t) = \lim_{t\to\infty} z_n'(t) = 0.$$

Moreover we recall that it has the form

$$\begin{aligned} y_n(t) &= \frac{1}{n} + \int_t^{\infty}\left(\int_s^{\infty} a(r)[z_n(r)]^{-\lambda}dr\right)ds \\ &= \frac{1}{n} + \int_t^{\infty} A(s)[z_n(s)]^{-\lambda}ds \\ &\quad - \lambda\int_t^{\infty}\left(\int_s^{\infty} A(r)[z_n(r)]^{-\lambda-1}z_n'(r)dr\right)ds, \qquad t \geq t_0, \end{aligned} \tag{14}$$

$$\begin{aligned} z_n(t) &= \frac{1}{n} + \int_t^{\infty}\left(\int_s^{\infty} b(r)[y_n(r)]^{-\mu}dr\right)ds \\ &= \frac{1}{n} + \int_t^{\infty} B(s)[y_n(s)]^{-\mu}ds \\ &\quad - \mu\int_t^{\infty}\left(\int_s^{\infty} B(r)[y_n(r)]^{-\mu-1}y_n'(r)dr\right)ds, \qquad t \geq t_0. \end{aligned} \tag{15}$$

Hence (13), (14) and (15) show that

$$y_n(t) \geq [z_n(t)]^{-\lambda} \int_t^\infty A(s)ds, \qquad t \geq t_0, \tag{16}$$

and

$$z_n(t) \geq [y_n(t)]^{-\mu} \int_t^\infty B(s)ds, \qquad t \geq t_0. \tag{17}$$

Thus it follows from (14) and (17) that

$$\begin{aligned} -y_n'(t) &= \int_t^\infty a(s)[z_n(s)]^{-\lambda} ds \\ &\leq \int_t^\infty |a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} [y_n(s)]^{\lambda\mu} ds \\ &\leq [y_n(t)]^{\lambda\mu} \int_t^\infty |a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} ds, \qquad t \geq t_0, \end{aligned}$$

where the decreasing nature of $y_n(t)$ is also used. The above inequality can be rewritten as

$$-\left(\frac{[y_n(t)]^{1-\lambda\mu}}{1-\lambda\mu} \right)' \leq \int_t^\infty |a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} ds, \qquad t \geq t_0,$$

and the integration over $[t, \infty)$ gives

$$\frac{[y_n(t)]^{1-\lambda\mu}}{1-\lambda\mu} \leq \frac{(1/n)^{1-\lambda\mu}}{1-\lambda\mu} + \int_t^\infty s|a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} ds, \qquad t \geq t_0.$$

Thus the sequence $\{y_n\}$ is uniformly bounded on each compact subset of $[t_0, \infty)$. Moreover we see that the sequences $\{y_n'\}$, $\{z_n\}$ and $\{z_n'\}$ are also uniformly bounded on each compact subset of $[t_0, \infty)$ by the same computation. Hence by the Ascoli-Arzelà theorem we can find a subsequence $\{(y_{n_i}, z_{n_i})\}$ of $\{(y_n, z_n)\}$ and a function $(\tilde{y}, \tilde{z}) \in C[t_0, \infty) \times C[t_0, \infty)$ to which $\{(y_{n_i}, z_{n_i})\}$ converges uniformly on each compact subset of $[t_0, \infty)$. Inequalities (16) and (17) show that $\tilde{y}(t), \tilde{z}(t) > 0$ in $[t_0, \infty)$. Let $n_i \to \infty$ in the equations

$$y_{n_i}(t) = \frac{1}{n_i} + \int_t^\infty \left(\int_s^\infty a(r)[z_{n_i}(r)]^{-\lambda} dr \right) ds, \qquad t \geq t_0,$$

and

$$z_{n_i}(t) = \frac{1}{n_i} + \int_t^\infty \left(\int_s^\infty b(r)[y_{n_i}(r)]^{-\mu} dr \right) ds, \qquad t \geq t_0.$$

Then the dominated convergence theorem asserts that $(\tilde{y}, \tilde{z})$ is a positive solution of (1) satisfying (2). This finishes the proof.

PROOF OF THEOREM 2. The proof is similar to the proof of Theorem 1 above. Therefore we will give only a sketch here.

Take positive solutions (y_n, z_n), $n \in N$, of system (1) such that

$$y_n'(t),\ z_n'(t) < 0, \qquad t \geq t_0,$$

and

$$\lim_{t\to\infty} y_n(t) = \ell, \quad \lim_{t\to\infty} z_n(t) = \frac{1}{n},$$

$$\lim_{t\to\infty} y_n'(t) = \lim_{t\to\infty} z_n'(t) = 0.$$

Note that inequality (17) remains valid. Thus as in the proof of Theorem 1 we have

$$\frac{[y_n(t)]^{1-\lambda\mu}}{1-\lambda\mu} \leq \frac{\ell^{1-\lambda\mu}}{1-\lambda\mu} + \int_t^\infty s|a(s)|\left(\int_s^\infty B(r)dr\right)^{-\lambda} ds, \qquad t \geq t_0.$$

Using the inequality $y_n(t) \geq \ell$, $t \geq t_0$, which can be easily obtained, we get the following estimate for $z_n(t)$:

$$z_n(t) \leq \frac{1}{n} + \ell^{-\mu}\int_t^\infty s|b(s)|ds, \qquad t \geq t_0.$$

Hence the argument used in the proof of Theorem 1 leads us to the desired conclusion. The proof is complete.

When $\lambda\mu \geq 1$, it is unknow for the author whether or not system (1) has such positive solutions. However, the same manipulation as in the proof of Theorem 1 gives a nonexistence criterion.

THEOREM 3. *Let* $\lambda\mu \geq 1$.

(i) *If* (5) *and* (11) *hold, then system* (1) *admits no positive solution* (y, z) *satisfying*

$$\begin{aligned} &\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} z'(t) = 0, \\ &\lim_{t\to\infty} z(t) = const \in [0, \infty), \qquad (18) \\ &y'(t),\ z'(t) < 0 \quad \textit{for all large } t. \end{aligned}$$

(ii) *If* (4) *and* (12) *hold, then system* (1) *admits no positive solution* (y, z) *satisfying*

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = \lim_{t\to\infty} y'(t) = 0,$$

$$\text{(19)} \qquad \lim_{t\to\infty} y(t) = const \in [0, \infty),$$

$$y'(t),\ z'(t) < 0 \quad \textit{for all large } t.$$

REMARK. In the case of $a(t) \geq 0$, $t \geq t_0$, the negativity of $y'(t)$ and $z'(t)$ required in (18) and (19) is superfluous. In fact, the boundedness of $y(t)$ and $z(t)$ automatically implies that $y'(t)$ and $z'(t)$ are eventually negative.

PROOF OF THEOREM 3. We only consider (i), because (ii) can be treated similarly.

Let (y, z) be a positive solution of (1) with the required property (18). As in the proof of our Lemma, we see that it satisfies (7) and (8), with $\ell = 0$ and $m = z(\infty) \in [0, \infty)$. Now choose $T \geq t_0$ so large that $y'(t),\ z'(t) < 0$ for $t \geq T$. Then the same manipulation as in the proof of Theorem 1 yields

$$\begin{aligned} y(t) &= \int_t^\infty \left(\int_s^\infty a(r)[z(r)]^{-\lambda} dr \right) ds \\ &\leq \int_t^\infty \left[\int_s^\infty |a(r)| \left(\int_r^\infty B(u)du \right)^{-\lambda} [y(r)]^{\lambda\mu} dr \right] ds \\ &\leq [y(t)]^{\lambda\mu} \int_t^\infty (s-t)|a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} ds, \end{aligned}$$

that is,

$$[y(t)]^{1-\lambda\mu} \leq \int_t^\infty s|a(s)| \left(\int_s^\infty B(r)dr \right)^{-\lambda} ds, \qquad t \geq T,$$

where the monotonicity of $y(t)$ are used. Hence we reach a contradiction by letting $t \to \infty$. The proof is finished.

3. Examples

In this last section we illustrate some examples which are derived from our existence theorems developed above.

EXAMPLE 1. Consider system (1) where a and b satisfy

$$\text{(20)} \qquad \begin{aligned} C_1 t^{-2-\alpha} &\leq a(t) \leq C_2 t^{-2-\alpha}, \\ D_1 t^{-2-\beta} &\leq b(t) \leq D_2 t^{-2-\beta}, \end{aligned}$$

for $t \geq 1$ where α, β, C_i, $D_i > 0$, $i = 1, 2$.

(i) Let $\lambda\mu < 1$. From Theorem 1, we see that if

$$\alpha - \lambda\beta > 0 \quad \text{and} \quad \beta - \mu\alpha > 0, \tag{21}$$

then system (1) has a positive solution (y, z) satisfying (2). Especially, the singular Emden-Fowler system

$$\begin{cases} y'' = t^{-2-\alpha} z^{-\lambda}, \\ z'' = t^{-2-\beta} y^{-\mu}, \end{cases} \quad t \geq 1, \quad \alpha, \beta > 0, \tag{22}$$

has a positive solution (y, z) of the form

$$\begin{cases} y(t) = Ct^{-p}, \\ z(t) = Dt^{-q}, \end{cases} \quad \text{where } C, D, p, q > 0, \tag{23}$$

if and only if (21) holds. In fact, the function (y, z) of the form (23) becomes a positive solution of (22) if and only if the simultaneous algebraic equation

$$\begin{cases} Cp(p+1) = D^{-\lambda}, & -p-2 = \lambda q - 2 - \alpha, \\ Dq(q+1) = C^{-\mu}, & -q-2 = \mu p - 2 - \beta \end{cases}$$

has a solution (C, D, p, q).

(ii) Let $\lambda\mu > 1$. It is easy to see that if

$$\alpha - \lambda\beta > 0 \quad \text{or} \quad \beta - \mu\alpha > 0, \tag{24}$$

then, the particular system (22) never has a positive solution (y, z) of the form (23). On the other hand, our Theorem 3 asserts more strongly that if (24) holds, then system (1) with (20) does not have any positive solution (y, z) satisfying (2).

Example 2. Let $\lambda\mu < 1$. Consider system (1) with a and b satisfying

$$C_1 t^{-\alpha_1 - 2} \leq a(t) \leq C_2 t^{-\alpha_2 - 2},$$

$$D_1 t^{-\beta_1 - 2} \leq b(t) \leq D_2 t^{-\beta_2 - 2},$$

for $t \geq 1$, where C_i, $D_i > 0$, $i = 1, 2$; $\alpha_1 > \alpha_2 > 0$, and $\beta_1 > \beta_2 > 0$ are parameters. Then, it follows from Theorem 1 that there exist $\alpha_i = \alpha_i(\lambda, \mu) > 0$ and $\beta_i = \beta_i(\lambda, \mu) > 0$ such that system (1), with this $\alpha_i(\lambda, \mu)$ and $\beta_i(\lambda, \mu)$, has a positive solutin (y, z) fulfilling (2). To see this it suffices to notice the fact that, for any given λ, μ satisfying $\lambda\mu < 1$, the linear algebraic inequality

$$\begin{cases} \alpha_2 - \lambda\beta_1 > 0, & \alpha_1 - \alpha_2 > 0, \\ \beta_2 - \mu\alpha_1 > 0, & \beta_1 - \beta_2 > 0 \end{cases}$$

admits a positive solution α_i, β_i. Here we may as well adapt the next result which is well-known in convex analysis: For each $n \times n$ matrix Lexactly one of the following two cases holds.

(i) There exists n-vector ξ satisfying $L\xi > 0$ and $\xi > 0$.

(ii) There exists n-vector $\eta \neq 0$ satisfying ${}^tL\eta \leq 0$ and $\eta \geq 0$.

(The order relation $v \geq w[v > w]$ for vectors $v = (v_i)$, $w = (w_i)$ is defined as $v_i \geq w_i$ $[v_i > w_i]$ for all i.)

EXAMPLE 3. Finally we present an example in which $a(t)$ and $b(t)$ oscillate. Let $\lambda\mu < 1$. Consider system (1) with

$$a(t) = -\left(\frac{1 + \sin t}{t^{2+\varepsilon}}\right)',$$

$$b(t) = -\left(\frac{1 + \sin t}{t^{2+\delta}}\right)',$$

for $t \geq 1$, where ε, $\delta > 0$. We then see that

$$a(t) = O(t^{-2-\varepsilon}), \qquad \int_t^\infty A(s)ds \geq C_1 t^{-1-\varepsilon},$$

$$b(t) = O(t^{-2-\delta}), \qquad \int_t^\infty B(s)ds \geq C_2 t^{-1-\delta},$$

as $t \to \infty$ for some C_1, $C_2 > 0$. Therefore Theorems 1 and 2 assert that system (1) has a positive solution (y, z) satisfying (2) if

$$\lambda(1 + \delta) < \varepsilon \qquad \text{and} \qquad \mu(1 + \varepsilon) < \delta,$$

and that system (1) has a positive solution (y, z) satisfying $(3)_\ell$, $\ell > 0$, if

$$\lambda(1 + \delta) < \varepsilon.$$

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