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On the reflexivity of certain fibered 3-knots

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§ 1. Introduction

Given $A \in GL(n, \boldsymbol{Z})$, $\Lambda^i A$ denotes its i-fold exterior product with itself. Let A satisfy the CS condition: $n \geq 3$, $\det A = 1$ and $\det(\Lambda^i A - I) = \pm 1$ for $1 \leq i \leq [n/2]$. The mapping torus of a diffeomorphism on a torus T^n induced by A is a homology $S^1 \times S^n$ and has a unique loop up to isotopy whose conjugates generate the fundamental group ([2]). Taking surgery on the loop we get a pair of $(n-1)$-knots in homotopy $(n+1)$-spheres according to the two framings. If these knots are equivalent, then the knot is called reflexive. The reflexive knot is determined by its exterior. The criterion for the reflexivity is given by Cappell-Shaneson.

PROPOSITION 1. (Cappell-Shaneson [2]) *Let $A \in GL(n, \boldsymbol{Z})$ satisfy the CS condition. Then the associated knot is reflexive if and only if there is a matrix $B \in GL(n, \boldsymbol{Z})$ such that $AB = BA$ and the restriction of B to the negative eigenspace of A has negative determinant.*

The purpose of this paper is to prove the following theorem by using Cappell-Shaneson criterion and extending the technique of Hillman-Wilson [3].

THEOREM 2. *Let $A \in GL(4, \boldsymbol{Z})$ satisfy the CS condition and* $\det(A - I) = 1$. *Then the characteristic polynomial $f_A(t)$ of A is either* (1) $t^4 + at^3 - 2(a+1)t^2 + (a+1)t + 1$ or (1)′ $t^4 + at^3 - 2at^2 + (a-1)t + 1$ *where $a = -$ trace (A). The associated knot is reflexive if and only if A is of type* (1) *and* $a = 0$, *or A is of type* (1)′ *and* $a = 1$.

REMARK 1. The non-reflexivity of the case (1) with $a \leq -1$ is noted in [3].

REMARK 2. If $\det(A - I) = -1$, then $f_A(t)$ is either (2) $t^4 + at^3 - 2(a+2)t^2 + (a+1)t + 1$ or (2)′ $t^4 + at^3 - 2(a+1)t^2 + (a-1)t + 1$. The reflexivity of the knot in this case is not completely determined yet.

Before closing the introduction we note that the characteristic polynomial of $A \in GL(n, \boldsymbol{Z})$ satisfying the CS condition is irreducible. In fact, the CS condition implies $\det(\Lambda^i A - I) = \pm 1$ for $1 \leq i \leq n-1$. By Newman [5, p. 50] any square matrix A over $\boldsymbol{Z}$ is similar to some block triangular matrix

$\begin{pmatrix} A_{11} & & * \\ & \ddots & \\ 0 & & A_{kk} \end{pmatrix}$, where each A_{ii} $(1 \leq i \leq k)$ is a square matrix with irreducible characteristic polynomial. If A_{11} is an $l \times l$ matrix, we have generators $\{e_1, \dots, e_n\}$ such that $\Lambda^l A(e_1 \wedge \cdots \wedge e_l) = (\det A_{11})(e_1 \wedge \cdots \wedge e_l) = \pm 1(e_1 \wedge \cdots \wedge e_l)$. So, $\det(\Lambda^l A - I)$ is even. This implies that l must be n, that is, $f_A(t)$ itself is irreducible.

§2. Proof of Theorem 2

Let $f_A(t) = t^4 + a_1 t^3 + a_2 t^2 + a_3 t + 1$. Then $\det(\Lambda^2 A - I) = -(a_3 - a_2)^2 = \pm 1$. If $\det(A - I) = f_A(1) = 1$, we get the first part of Theorem 2. If A is of type (1)′ then $f_{A^{-1}}(t) = t^4 + (a-1)t^3 - 2((a-1)+1)t^2 + ((a-1)+1)t + 1$ and A^{-1} is of type (1). Hence, we may assume that A is of type (1) hereafter. It is not difficult to check the following

LEMMA 2.1. (1) *$f_A(t) = t^4 + at^3 - 2(a+1)t^2 + (a+1)t + 1$ has negative roots if and only if $a \geq 0$. If $a \geq 0$, there are two negative roots and the other roots are not real.*
(2) *If $a = 0$, then $B = A + I$ satisfies the reflexivity condition of Proposition* 1.

By Proposition 1 and Lemma 2.1 it suffices to consider under the following condition hereafter:

(*) $f_A(t)$ has two negative roots θ_1 and θ_2, and the other roots θ_3 and θ_4 are complex conjugate.

PROPOSITION 2.2. *Assume that A satisfies the condition (*) and let $B \in GL(n, \boldsymbol{Z})$ satisfy the condition of Proposition* 1. *Then $f_B(t)$ is irreducible and B has just two real eigenvalues μ_1 and μ_2 corresponding to θ_1 and θ_2. In particular* $\det B = -1$

PROOF. Since $f_A(t)$ is irreducible, all eigenvalues of A are distinct. So, there is a matrix $P \in GL(4, \boldsymbol{C})$ such that $P^{-1}AP$ is a diagonal matrix $\mathrm{diag}(\theta_1, \theta_2, \theta_3, \theta_4)$. Because $AB = BA$, $P^{-1}BP$ is a diagonal matrix $\mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4)$. Note that each eigenvalue μ_i belongs to the extension field $\boldsymbol{Q}(\theta_i)$ of a corresponding eigenvalue θ_i for $1 \leq i \leq 4$. In fact, if x_i be an eigenvector of θ_i then $Bx_i = \mu_i x_i$. We can choose x_i in $\boldsymbol{Q}(\theta_i)^4 \subset \boldsymbol{C}^4$. Because $B((\boldsymbol{Q}(\theta_i)^4) \subset \boldsymbol{Q}(\theta_i)^4$, μ_i must belong to $\boldsymbol{Q}(\theta_i)$. In particular μ_1 and μ_2 are real. Note that $\mu_1 \neq \mu_2$ because $\mu_1 \mu_2 < 0$. If $f_B(t)$ is irreducible, θ_i belongs to $\boldsymbol{Q}(\mu_i)$ and μ_3 and μ_4 can not be real. Assume now that $f_B(t)$ is not irreducible. If $f_B(t)$ can be written $f_B(t) = g_1(t) \cdot g_2(t)$ in the rational number field and $g_1(t)$ and $g_2(t)$ are coprime, then $\boldsymbol{Q}^4$ decomposes into $V_1 \oplus V_2$ where

$V_i = \operatorname{Ker} g_i(B)$. Since $g_i(B)(A_i(V_i)) = A(g_i(B)(V_i)) = 0$, we have $A(V_i) \subset V_i$. This contradicts the irreducibility of $f_A(t)$. Using this it is easy to exclude the case that $f_B(t)$ decomposes into the product of two polynomials which have a degree one common factor. So, $f_B(t)$ must be a square of an irreducible polynomial of degree 2. In this case not only μ_1, μ_2 and trace$(AB) = \theta_1\mu_1 + \theta_2\mu_2 + \theta_3\mu_3 + \theta_4\mu_4$ but also μ_3 and μ_4 are real numbers. Hence μ_3 must be equal to μ_4, which implies $\mu_1 = \mu_2$. This contradicts the reflexivity condition.

q.e.d.

Let θ denote an eigenvalue of A and μ an eigenvalue of B corresponding to θ. We denote R the ring of all integral elements in $\boldsymbol{Q}(\theta)$ and U the group of all units in R. The following lemma comes directly from Dirichlet's theorem (cf. [1, p. 112]) and the condition $(*)$.

LEMMA 2.3. *U is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z} \times \boldsymbol{Z}$ as an abelian group.*

The norm of an element $\varphi \in \boldsymbol{Q}(\theta)$ is the determinant of the linear map of the four-dimensional $\boldsymbol{Q}$-vector space $\boldsymbol{Q}(\theta)$ to itself which maps x to φx. Since the linear map corresponding to θ, $\theta - 1$ and μ are represented by A, $A - I$ and B respectively, we see that θ and $\theta - 1$ are unit of norm 1 in R and μ is a unit of norm -1 in R by Proposition 2.2. Hence the following lemma concludes a proof of Theorem 2.

LEMMA 2.4. *Assume $a \geq 1$. There are no units in R of norm -1.*

To prove this lemma, we may take a negative θ, say θ_1. We shall prove the following

LEMMA 2.5. *Assume $a \geq 1$. Then $-\theta$, $-(\theta - 1)$, and $\theta(\theta - 1)$ are not square in R.*

PROOF OF LEMMA 2.4 ASSUMING LEMMA 2.5. First note that θ and $\theta - 1$ are independent in U. In fact, otherwise $\theta^k(\theta - 1)^l = \pm 1$ for some integers k, l with $|k| + |l| \neq 0$. We may assume that $|k| + |l|$ is the minimum of such integers. Put $k = 2k_0 - \varepsilon$ and $l = 2l_0 - \varepsilon'$ where k_0, l_0 are integers and ε, ε' are 0 or 1. If $\varepsilon = \varepsilon' = 0$ then $\theta^{k_0}(\theta - 1)^{l_0} = \pm 1$, which contradicts the minimality. So, we can write $\pm \theta^{\varepsilon}(\theta - 1)^{\varepsilon'} = (\theta^{k_0}(\theta - 1)^{l_0})^2$. Since the right-hand side is positive, the left-hand side must be either $-\theta$, $-(\theta - 1)$ or $\theta(\theta - 1)$ and this contradicts the Lemma 2.5. Hence θ and $\theta - 1$ are independent in U. Let G be the subgroup of U generated by ± 1, θ and $\theta - 1$. Then G is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z} \times \boldsymbol{Z}$ as a group and all the elements of G have norm 1. Assume that there is $\mu \in U$ of norm -1. Because U is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z} \times \boldsymbol{Z}$, there exists a nonzero integer m such that $\mu^{2m} \in G$. We can assume $|m|$ is the minimum of such integers. Then

$\pm\,\theta^k(\theta-1)^l=\mu^{2m}$ for some integers k, l. Put $k=2k_0-\varepsilon$ and $l=2l_0-\varepsilon'$ as before. If $\varepsilon=\varepsilon'=0$, then $\theta^{k_0}(\theta-1)^{l_0}=\pm\,\mu^m$. But since the left-hand side is of norm 1 and μ is of norm -1, m must be $2m'$ for some integer m'. This contradicts the minimality of $|m|$. Hence, $\pm\,\theta^{\varepsilon}(\theta-1)^{\varepsilon'}=(\theta^{k_0}(\theta-1)^{l_0}\mu^{-m})^2$. Since $\mu\in\boldsymbol{Q}(\theta)\subset\boldsymbol{R}$, the right-hand side is positive. This leads to a contradiction as before. q.e.d.

PROOF OF LEMMA 2.5. We have to prove three cases (i.e. $-\theta$, $-(\theta-1)$ and $\theta(\theta-1)$) independently. Since we can prove these cases by the same kind of argument, we shall give a proof only for $-\theta$ here. Let $f(t)$ be the minimal polynomial of $-\theta$, then $f(t)=t^4-at^3-2(a+1)t^2-(a+1)t+1$. Let φ be a unit in R and assume $\varphi^2=-\theta$, then φ is a root of the polynomial $g(t)=t^8-at^6-2(a+1)t^4-(a+1)t^2+1$. Because $\boldsymbol{Q}(\varphi)=\boldsymbol{Q}(\theta)$, the minimal polynomial of φ has degree 4. Since all the terms of $g(t)$ have even degree and $f(t)$ is irreducible, $g(t)$ must be written as $g(t)=(t^4+\alpha t^3+\beta t^2+\gamma t+\delta)(t^4-\alpha t^3+\beta t^2-\gamma t+\delta)$, where $\alpha, \beta, \gamma\in\boldsymbol{Z}$ and $\delta=-1$ or 1. By comparing coefficients, we have $2\beta-\alpha^2=-a$, $2\delta-2\alpha\gamma+\beta^2=-2(a+1)$ and $2\delta\beta-\gamma^2=-(a+1)$. Hence, $(\gamma-\alpha)^2+\beta^2=2(\delta+1)(\beta-1)+1$ and β is even. Since $\gamma^2-\alpha^2=1+2(\delta-1)\beta$ and $\delta=\pm\,1$, we have $\gamma^2-\alpha^2=1$ and $\alpha=0$. This leads to a contradiction: $0\le\beta^2\le\beta^2+2(\delta+1)=-2a\le-2$. q.e.d.

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