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A Stroboscopic Method for the Critical Case where the Jacobian Vanishes

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1. Introduction.

In this paper, we are concerned with a real system of n nonlinear differential equations of the form as follows:

$$\frac{dx_i}{dt}=\varepsilon f_i(x, t, \varepsilon) \quad (i=1, 2,\cdots, n), \tag{1.1}$$

where

1° ε is a parameter such that $|\varepsilon|<\delta$ $(\delta>0)$;

2° the functions $f_i(x, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ are periodic in t with period T (>0) and are continuous in the domain

$$D: |x|=\sum_{i=1}^{n}|x_i|<L, \ -\infty<t<+\infty, \ |\varepsilon|<\delta$$

together with $\dfrac{\partial f_i(x, t, \varepsilon)}{\partial x_j}$, $\dfrac{\partial f_i(x, t, \varepsilon)}{\partial \varepsilon}$ $(i, j=1, 2,\cdots, n)$.

Let us consider the functions

$$F_i(x)=\frac{1}{T}\int_0^T f_i(x, t, 0)\,dt \quad (i=1, 2,\cdots, n). \tag{1.2}$$

Then, as is well known, there exists a periodic solution of (1.1) provided there exists a real solution $x_i=c_i$ $(i=1, 2,\cdots, n)$ of the system of equations

$$F_i(x)=0 \quad (i=1, 2,\cdots, n) \tag{1.3}$$

and the Jacobian J of $F_i(x)$ with respect to x_j does not vanish for $x_i=c_i$ $(i=1, 2,\cdots, n)$. In this case, as is well known, the stability of the assured periodic solution of (1.1) is decided according to the signs of the eigenvalues of J.

But, if the Jacobian J vanishes for $x_i=c_i$ $(i=1, 2,\cdots, n)$, the periodic solution of (1.1) does not necessarily exist even if there exists a real solution of (1.3).

In the present paper, we investigate some cases where the Jacobian J vanishes for $x_i=c_i$ $(i=1, 2,\cdots, n)$ but nevertheless the equation (1.1) has a periodic solution.

For our discussions, the assumption 2° is not strong enough, because our investigation needs more minute computation than in the ordinary case, i.e. the case where the Jacobian J does not vanish for $x_i=c_i$ $(i=1, 2,\cdots, n)$. Thus, in the present paper, the condition 2° is replaced by the stronger one as follows:

2°′ the functions $f_i(x, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ are periodic in t with period T (>0) and are continuous in the domain

$$D: |x| = \sum_{i=1}^{n} |x_i| < L, \ -\infty < t < +\infty, \ |\varepsilon| < \delta$$

together with their derivatives with respect to (x, ε) up to the 3rd order.

2. Preliminary calculations.

Let

$$x_i = \varphi_i(u, t, \varepsilon) \quad (i=1, 2,\cdots, n) \tag{2.1}$$

be the solution of (1.1) such that

$$\varphi_i(u, 0, \varepsilon) = u_i \quad (i=1, 2,\cdots, n), \tag{2.2}$$

where $|u| = \sum_{i=1}^{n} |u_i| < L$. From the form of (1.1) and the assumptions on $f_i(x, t, \varepsilon)$ $(i=1, 2,\cdots, n)$, it is readily seen that, if $|\varepsilon|$ is sufficiently small, the functions $\varphi_i(u, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ are expanded as

$$\begin{aligned} \varphi_i(u, t, \varepsilon) = \varphi_i^{(0)}(u, t) + \varepsilon\varphi_i^{(1)}(u, t) + \varepsilon^2\varphi_i^{(2)}(u, t) \\ + \varepsilon^3\varphi^{(3)}(u, t) + q_i(u, t, \varepsilon) \quad (i=1, 2,\cdots, n) \end{aligned} \tag{2.3}$$

for any finite value of t, where $q_i(u, t, \varepsilon) = O(\varepsilon^4)$ as $\varepsilon \to 0$.

Now, by the initial condition (2.2), it is evident that

$$\begin{cases} \varphi_i^{(0)}(u, 0) = u_i, \\ \varphi_i^{(1)}(u, 0) = \varphi_i^{(2)}(u, 0) = \varphi_i^{(3)}(u, 0) = q_i(u, 0, \varepsilon) = 0 \quad (i=1, 2,\cdots, n). \end{cases} \tag{2.4}$$

If we substitute (2.3) into the initial equation (1.1) and compare the coefficients of powers of ε, we have the system of the linear differential equations with respect to $\varphi_i^{(0)}, \varphi_i^{(1)}, \varphi_i^{(2)}, \varphi_i^{(3)}$ $(i=1, 2,\cdots, n)$. These equations are solved successively under the initial conditions (2.4) as follows:

$$\begin{cases} \varphi_i^{(0)}(u, t) = u_i, \\ \varphi_i^{(1)}(u, t) = \int_0^t f_i(u, t_1, 0)dt_1, \\ \varphi_i^{(2)}(u, t) = \int_0^t [\sum_{j=1}^{n} f_{ij}(u, t_1, 0)\int_0^{t_1} f_j(u, t_2, 0)dt_2 + f_i'(u, t_1, 0)]dt_1, \end{cases} \tag{2.5}$$

$$
\left\{
\begin{aligned}
\varphi_i^{(3)}(u, t) = \int_0^t \Big[& \sum_{j=1}^n f_{ij}(u, t_1, 0) \int_0^{t_1} \Big\{ \sum_{k=1}^n f_{jk}(u, t_2, 0) \\
& \times \int_0^{t_2} f_k(u, t_3, 0)\, dt_3 + f'_j(u, t_2, 0) \Big\} dt_2 \\
& + \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n f_{ijk}(u, t_1, 0) \Big(\int_0^{t_1} f_j(u, t_2, 0)\, dt_2 \Big) \Big(\int_0^{t_1} f_k(u, t_2, 0)\, dt_2 \Big) \\
& + \sum_{j=1}^n f'_{ij}(u, t_1, 0) \int_0^{t_1} f_j(u, t_2, 0)\, dt_2 + \frac{1}{2} f''_i(u, t_1, 0) \Big] dt_1
\end{aligned}
\right.
$$

where

$$
\left\{
\begin{aligned}
& f_{ij}(x, t, \varepsilon) = \frac{\partial f_i}{\partial x_j}(x, t, \varepsilon), \quad f'_i(x, t, \varepsilon) = \frac{\partial f_i}{\partial \varepsilon}(x, t, \varepsilon), \\
& f_{ijk}(x, t, \varepsilon) = \frac{\partial^2 f_i}{\partial x_j \partial x_k}(x, t, \varepsilon), \quad f'_{ij}(x, t, \varepsilon) = \frac{\partial^2 f_i}{\partial x_j \partial \varepsilon}(x, t, \varepsilon), \\
& f''_i(x, t, \varepsilon) = \frac{\partial^2 f_i}{\partial \varepsilon^2}(x, t, \varepsilon).
\end{aligned}
\right.
$$

As is readily seen, the necessary and sufficient condition that the solution $x_i = \varphi_i(u, t, \varepsilon)$ $(i = 1, 2, \cdots, n)$ is periodic in t with period T, is that

(2.6) $$\varphi_i(u, 0, \varepsilon) = \varphi_i(u, T, \varepsilon) \quad (i = 1, 2, \cdots, n).$$

This condition can be written by (2.3) and (2.5) as follows:

(2.7) $$\varphi_i^{(1)}(u, T) + \varepsilon \varphi_i^{(2)}(u, T) + \varepsilon^2 \varphi_i^{(3)}(u, T) + O(\varepsilon^3) = 0 \quad (i = 1, 2, \cdots, n).$$

Now, we assume that

1° the equation

(2.8) $$\varphi_i^{(1)}(u, T) = 0 \quad (i = 1, 2, \cdots, n)$$

has a real solution $u_i = c_i$ (i=1, 2, $\cdots$, n) such that $|c| = \sum_{i=1}^n |c_i| < L$;

2° the Jacobian $J_0 = \det\Big(\frac{\partial \varphi_i^{(1)}}{\partial u_j}(c, T) \Big)$ $(i, j = 1, 2, \cdots, n)$ vanishes.

In the present paper, we shall investigate the case where the rank k of the Jacobian matrix $\Big(\frac{\partial \varphi_i^{(1)}}{\partial u_j}(c, T) \Big)$ $(i, j = 1, 2, \cdots, n)$ is not zero.

Let l be the rank of the matrix $\Big(\frac{\partial \varphi_i^{(1)}}{\partial u_j}(c, T), \varphi_i^{(2)}(c, T) \Big)$, then, evidently $l \geqq k \geqq 1$.

The case where $k < l$ and the case where $k = l$ shall be studied separately in the sequel.

3. Existence of a periodic solution: Case I where $k < l$.

By our assumptions, we may assume, without loss of generality, that

$$(3.1)\qquad J_1=\det\Big(\frac{\partial\varphi_\alpha}{\partial u_\beta}(c,\ T)\Big)\neq 0\quad(\alpha,\ \beta=1,\ 2,\cdots,\ k).$$

Then there exist numbers $\xi_{\nu\alpha}$ ($\alpha=1, 2,\cdots, k$; $\nu=k+1,\cdots, n$) such that

$$(3.2)\qquad \frac{\partial\varphi_\nu^{(1)}}{\partial u_j}(c,\ T)+\sum_{\alpha=1}^{k}\xi_{\nu\alpha}\frac{\partial\varphi_\alpha^{(1)}}{\partial u_j}(c,\ T)=0\quad(j=1,\ 2,\cdots,\ n;\ \nu=k+1,\cdots,\ n).$$

Making use of these $\xi_{\nu\alpha}$ ($\alpha=1, 2,\cdots, k$; $\nu=k+1,\cdots, n$), let us rewrite the equations (2.7) as follows:

$$(3.3)\qquad \begin{cases}\varphi_\alpha^{(1)}(u,\ T)+\varepsilon\varphi_\alpha^{(2)}(u,\ T)+o(\varepsilon)=0,\\ \Big\{\varphi_\nu^{(1)}(u,\ T)+\sum_{\beta=1}^{k}\xi_{\nu\beta}\varphi_\beta^{(1)}(u,\ T)\Big\}\\ \qquad+\varepsilon\Big\{\varphi_\nu^{(2)}(u,\ T)+\sum_{\beta=1}^{k}\xi_{\nu\beta}\varphi_\beta^{(2)}(u,\ T)\Big\}+o(\varepsilon)=0,\\ \qquad\qquad(\alpha=1,\ 2,\cdots,\ k;\ \nu=k+1,\ \cdots,\ n).\end{cases}$$

Since the Jacobian J_1 does not vanish, for sufficiently small $|\varepsilon|$, the first k equations of (3.3) can be solved with respect to u_α ($\alpha=1, 2,\cdots, k$) in the neighborhood of $u_i=c_i$ ($i=1, 2,\cdots, n$) as follows:

$$(3.4)\qquad u_\alpha=u_\alpha(u_{k+1},\cdots,\ u_n,\ \varepsilon)\quad(\alpha=1,\ 2,\cdots,\ k),$$

where

$$(3.5)\qquad u_\alpha(c_{k+1},\cdots,\ c_n,\ 0)=c_\alpha\quad(\alpha=1,\ 2,\cdots,\ k).$$

For brevity, let us write the functions (3.4) as $u_\alpha=u_\alpha(u_\nu, \varepsilon)$. Such a notation is used in the sequel without any comment.

For $(u_\nu, \varepsilon)=(c_\nu, 0)$, the derivatives of the functions $u_\alpha=u_\alpha(u_\nu, \varepsilon)$ ($\alpha=1, 2, \cdots, k$) are obtained readily as follows:

$$(3.6)\qquad \begin{cases}\dfrac{\partial u_\alpha}{\partial u_\nu}=-\dfrac{1}{J_1}\sum_{\beta=1}^{k}\dfrac{\partial\varphi_\beta^{(1)}}{\partial u_\nu}D_{\beta\alpha},\\[2mm] \dfrac{\partial u_\alpha}{\partial\varepsilon}=-\dfrac{1}{J_1}\sum_{\beta=1}^{k}\varphi_\beta^{(2)}D_{\beta\alpha}\end{cases}\qquad(\alpha=1,\ 2,\cdots,\ k;\ \nu=k+1,\cdots,\ n),$$

where $D_{\beta\alpha}$ are the cofactors of the elements $\dfrac{\partial\varphi_\beta^{(1)}}{\partial u_\alpha}$ ($\alpha, \beta=1, 2,\cdots, k$) in J_1.

In order to solve the equations (3.3), let us substitute $u_\alpha=u_\alpha(u_\nu, \varepsilon)$ ($\alpha=1, 2,\cdots, k$) into the last $n-k$ equations of (3.3). The resulting equations are written as follows:

$$(3.7)\qquad \Psi_\mu(u_\nu,\ \varepsilon)\overset{\text{def}}{=}\psi_\mu^{(1)}(u_\nu,\ \varepsilon)+\varepsilon\psi_\mu^{(2)}(u_\nu,\ \varepsilon)+o(\varepsilon)=0\qquad(\mu=k+1,\cdots,\ n),$$

where

$$\psi_\mu^{(i)}(u_\nu, \varepsilon) = \varphi_\mu^{(i)}(u_\alpha(u_\nu, \varepsilon), u_\nu, T) + \sum_{\beta=1}^{k} \xi_{\mu\beta}\varphi_\beta^{(i)}(u_\alpha(u_\nu, \varepsilon), u_\nu, T) \tag{3.8}$$
$$(i = 1, 2, \cdots; \mu = k+1, \cdots, n).$$

Now, by (3.5) and (3.2), it holds that

$$\begin{cases} \psi_\mu^{(1)}(c_\nu, 0) = 0, \\ \dfrac{\partial \psi_\mu^{(1)}}{\partial u_\lambda}(c_\nu, 0) = 0, \\ \dfrac{\partial \psi_\mu^{(1)}}{\partial \varepsilon}(c_\nu, 0) = 0 \end{cases} \tag{3.9}$$
$$(\mu, \lambda = k+1, \cdots, n).$$

Hence the equations (3.7) are of the forms as follows:

$$\Psi_\mu(u_\nu, \varepsilon) = \frac{1}{2} \sum_{\kappa=k+1}^{n} \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial u_\lambda}(c_\nu, 0)(u_\kappa - c_\kappa)(u_\lambda - c_\lambda)$$
$$+ \tilde{\psi}_\mu^{(1)}(u_\nu, \varepsilon) + \varepsilon[\psi_\mu^{(2)}(c_\nu, 0) + \tilde{\psi}_\mu^{(2)}(u_\nu, \varepsilon)] + o(\varepsilon) = 0$$
$$(\mu = k+1, \cdots, n),$$

where

$$\begin{cases} \tilde{\psi}_\mu^{(1)}(u_\nu, \varepsilon) = \psi_\mu^{(1)}(u_\nu, \varepsilon) - \dfrac{1}{2} \displaystyle\sum_{\kappa=k+1}^{n} \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial u_\lambda}(c_\nu, 0)(u_\kappa - c_\kappa)(u_\lambda - c_\lambda), \\ \tilde{\psi}_\mu^{(2)}(u_\nu, \varepsilon) = \psi_\mu^{(2)}(u_\nu, \varepsilon) - \psi_\mu^{(2)}(c_\nu, 0) \end{cases} \tag{3.10}$$
$$(\mu = k+1, \cdots, n).$$

Here, by the assumption that $k < l$, at least one of $\psi_\mu^{(2)}(c_\nu, 0)$'s $(\mu = k+1, \cdots, n)$ does not vanish.

Let us investigate the case where $\varepsilon > 0$. The case where $\varepsilon < 0$ can be reduced to the former case by the substitution $\varepsilon = -\varepsilon'$.

Put

$$u_\nu - c_\nu = \varepsilon^{1/2} v_\nu \quad (\nu = k+1, \cdots, n), \tag{3.11}$$

then the functions $\Psi_\mu(u_\nu, \varepsilon) = \Psi_\mu(c_\nu + \varepsilon^{1/2} v_\nu, \varepsilon)$ $(\mu = k+1, \cdots, n)$ are of the forms

$$\Psi_\mu(c_\nu + \varepsilon^{1/2} v_\nu, \varepsilon) = \varepsilon \Omega_\mu(v_\nu, \varepsilon) \tag{3.12}$$
$$= \varepsilon \Big[\frac{1}{2} \sum_{\kappa=k+1}^{n} \sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial u_\lambda}(c_\nu, 0) v_\kappa v_\lambda + \psi_\mu^{(2)}(c_\nu, 0) + o(1)\Big]$$

as $\varepsilon \to 0$.

Now, let us consider the quadratic equations

$$\Omega_\mu(v_\nu, 0) = 0 \quad (\mu = k+1, \cdots, n), \tag{3.13}$$

and suppose these equations have a real solution $v_\nu = d_\nu$ $(\nu = k+1, \cdots, n)$.

Then, if

$$J_2 = \det\Big(\sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial u_\lambda}(c_\nu, 0) d_\lambda \Big) \neq 0 \qquad (\mu, \kappa = k+1, \cdots, n), \tag{3.14}$$

the equations $\Omega_\mu(v_\nu, \varepsilon) = 0$ $(\mu = k+1, \cdots, n)$ have certainly a unique real solution v_ν $(\nu = k+1, \cdots, n)$ which tends to d_ν as $\varepsilon \to 0$. Evidently such a solution is continuously differentiable with respect to $\varepsilon^{1/2}$, consequently it is of the form

$$v_\nu = d_\nu + O(\varepsilon^{1/2}) \quad (\nu = k+1, \cdots, n). \tag{3.15}$$

By (3.11), the solution v_ν of the above form yields the solution u_ν of the equations (3.7) which is of the form

$$u_\nu = c_\nu + \varepsilon^{1/2} d_\nu + o(\varepsilon^{1/2}) \quad (\nu = k+1, \cdots, n). \tag{3.16}$$

If we substitute (3.16) into (3.4) and make use of the first of (3.6), we see that

$$u_\alpha = c_\alpha - \varepsilon^{1/2} \frac{1}{J_1} \sum_{\beta=1}^{k} \sum_{\nu=k+1}^{n} \frac{\partial \varphi_\beta^{(1)}}{\partial u_\nu}(c, T) D_{\beta\alpha} d_\nu + o(\varepsilon^{1/2}) \qquad (\alpha = 1, 2, \cdots, k). \tag{3.17}$$

The results obtained above are stated as

Theorem 1. *In the case where $k < l$, if the quadratic equations* (3.13) *have a real solution $v_\nu = d_\nu$ $(\nu = k+1, \cdots, n)$ and the Jacobian J_2 defined by* (3.14) *does not vanish, then there exists a periodic solution of* (1.1) *corresponding to u_i $(i = 1, 2, \cdots, n)$ given by* (3.17) *and* (3.16).

4. Existence of a periodic solution: Case II where $k = l$.

As in the case I, we may assume (3.1) without loss of generality. In the present case, due to the assumption that $k = l$, the equalities

$$\varphi_\nu^{(2)}(c, T) + \sum_{\alpha=1}^{k} \xi_{\nu\alpha} \varphi_\alpha^{(2)}(c, T) = 0 \quad (\nu = k+1, \cdots, n) \tag{4.1}$$

hold at the same time as (3.2).

As in the case I, we rewrite the equations (2.7) as follows:

$$\begin{cases} \varphi_\alpha^{(1)}(u, T) + \varepsilon \varphi_\alpha^{(2)}(u, T) + \varepsilon^2 \varphi_\alpha^{(3)}(u, T) + o(\varepsilon^2) = 0, \\ \Big\{ \varphi_\nu^{(1)}(u, T) + \sum_{\beta=1}^{k} \xi_{\nu\beta} \varphi_\beta^{(1)}(u, T) \Big\} + \varepsilon \Big\{ \varphi_\nu^{(2)}(u, T) + \sum_{\beta=1}^{k} \xi_{\nu\beta} \varphi_\beta^{(2)}(u, T) \Big\} \end{cases} \tag{4.2}$$

$$+\varepsilon^2\left\{\varphi_\nu^{(3)}(u,\ T)+\sum_{\beta=1}^{k}\xi_{\nu\beta}\varphi_\beta^{(3)}(u,\ T)\right\}+o(\varepsilon^2)=0$$
$$(\alpha=1,\ 2,\cdots,\ k;\ \nu=k+1,\cdots,\ n)$$

and we substitute the solution

$$u_\alpha=u_\alpha(u_\nu,\ \varepsilon)\quad(\alpha=1,\ 2,\cdots,\ k) \tag{4.3}$$

of the first k equations into the last $(n-k)$ equations. Then the resulting equations are of the same form as (3.7), but, in the present case, due to (4.1),

$$\psi_\mu^{(2)}(c_\nu,\ 0)=0\quad(\nu=k+1,\cdots,\ n) \tag{4.4}$$

in addition to (3.9).

Thus the equations (3.7) are written in the present case as follows:

$$\begin{aligned}\Psi_\mu(u_\nu,\ \varepsilon)=&\frac{1}{2}\sum_{\kappa=k+1}^{n}\sum_{\lambda=k+1}^{n}\frac{\partial^2\psi_\mu^{(1)}}{\partial u_\kappa\partial u_\lambda}(c_\nu,\ 0)\,(u_\kappa-c_\kappa)\,(u_\lambda-c_\lambda)\\&+\varepsilon\sum_{\kappa=k+1}^{n}\frac{\partial^2\psi_\mu^{(1)}}{\partial u_\kappa\partial\varepsilon}(c_\nu,\ 0)\,(u_\kappa-c_\kappa)+\frac{1}{2}\varepsilon^2\frac{\partial^2\psi_\mu^{(1)}}{\partial\varepsilon^2}(c_\nu,\ 0)+\tilde{\psi}_\mu^{(1)}(u_\nu,\ \varepsilon)\\&+\varepsilon\left[\sum_{\kappa=k+1}^{n}\frac{\partial\psi_\mu^{(2)}}{\partial u_\kappa}(c_\nu,\ 0)\,(u_\kappa-c_\kappa)+\varepsilon\frac{\partial\psi_\mu^{(2)}}{\partial\varepsilon}(c_\nu,\ 0)+\tilde{\psi}_\mu^{(2)}(u_\nu,\ \varepsilon)\right]\\&+\varepsilon^2[\psi_\mu^{(3)}(c_\nu,\ 0)+\tilde{\psi}_\mu^{(3)}(u_\nu,\ \varepsilon)]+o(\varepsilon^2)\\&\qquad(\mu=k+1,\cdots,\ n),\end{aligned}$$

where $\tilde{\psi}_\mu^{(i)}(u_\nu,\ \varepsilon)$ $(i=1,\ 2,\ 3;\ \mu=k+1,\cdots,\ n)$ are respectively the remainders in $\psi_\mu^{(i)}(u_\nu,\ \varepsilon)$ from which the terms written explicitly are subtracted.

Let us put

$$u_\nu-c_\nu=\varepsilon v_\nu\quad(\nu=k+1,\cdots,\ n). \tag{4.5}$$

Then the functions $\Psi_\mu(u_\nu,\ \varepsilon)=\Psi_\mu(c_\nu+\varepsilon v_\nu,\ \varepsilon)$ $(\mu=k+1,\cdots,\ n)$ can be written as follows:

$$\begin{aligned}\Psi_\mu(c_\nu+\varepsilon v_\nu,\ \varepsilon)&=\varepsilon^2\Omega_\mu(v_\nu,\ \varepsilon)\\&=\varepsilon^2\Bigg[\frac{1}{2}\sum_{\kappa=k+1}^{n}\sum_{\lambda=k+1}^{n}\frac{\partial^2\psi_\mu^{(1)}}{\partial u_\kappa\partial u_\lambda}(c_\lambda,\ 0)v_\kappa v_\lambda\\&\quad+\sum_{\kappa=k+1}^{n}\left\{\frac{\partial^2\psi_\mu^{(1)}}{\partial u_\kappa\partial\varepsilon}(c_\nu,\ 0)+\frac{\partial\psi_\mu^{(2)}}{\partial u_\kappa}(c_\nu,\ 0)\right\}v_\kappa\\&\quad+\frac{1}{2}\frac{\partial^2\psi_\mu^{(1)}}{\partial\varepsilon^2}(c_\nu,\ 0)+\frac{\partial\psi_\mu^{(2)}}{\partial\varepsilon}(c_\nu,\ 0)+\psi_\mu^{(3)}(c_\nu,\ 0)+o(1)\Bigg]\\&\qquad(\mu=k+1,\cdots,\ n).\end{aligned} \tag{4.6}$$

Therefore, if the quadratic equations

$$\Omega_\mu(v_\nu,\ 0)=0\quad(\mu=k+1,\cdots,\ n) \tag{4.7}$$

have a real solution $v_\nu = d_\nu$ ($\nu = k+1, \cdots, n$), the equations $\Omega_\mu(v_\nu, \varepsilon) = 0$ ($\mu = k+1, \cdots, n$) have a unique real solution such that $v_\nu = d_\nu + O(\varepsilon)$ ($\nu = k+1, \cdots, n$) as $\varepsilon \to 0$, provided the Jacobian

$$J_2 = \det\left(\sum_{\lambda=k+1}^{n} \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial u_\lambda}(c_\nu, 0) d_\lambda + \frac{\partial^2 \psi_\mu^{(1)}}{\partial u_\kappa \partial \varepsilon}(c_\nu, 0) + \frac{\partial \psi_\mu^{(2)}}{\partial u_\kappa}(c_\nu, 0) \right) \neq 0 \quad (\mu, \lambda = k+1, \cdots, n). \tag{4.8}$$

The solution $v_\nu = d_\nu + O(\varepsilon)$ ($\nu = k+1, \cdots, n$) of $\Omega_\mu(v_\nu, \varepsilon) = 0$ ($\mu = k+1, \cdots, n$) yields the solution of (4.2) of the forms as follows:

$$\begin{cases} u_\alpha = c_\alpha - \varepsilon \dfrac{1}{J_1} \sum\limits_{\beta=1}^{n} D_{\beta\alpha} \left(\sum\limits_{\mu=k+1}^{n} \dfrac{\partial \varphi_\beta^{(1)}}{\partial u_\mu}(c_\nu, 0) d_\mu + \varphi_\beta^{(2)}(c_\nu, 0) \right) + O(\varepsilon) \\ \qquad\qquad (\alpha = 1, 2, \cdots, k), \\ u_\nu = c_\nu + \varepsilon d_\nu + O(\varepsilon) \\ \qquad\qquad (\nu = k+1, \cdots, n). \end{cases} \tag{4.9}$$

The results obtained above are stated as

Theorem 2. *In the case* II *where* $k = l$, *if the quadratic equations* (4.7) *have a real solution* $v_\nu = d_\nu$ ($\nu = k+1, \cdots, n$) *and the Jacobian* J_2 *defined by* (4.8) *does not vanish, then there exists a periodic solution of* (1.1) *corresponding to* u_i ($i = 1, 2, \cdots, n$) *given by* (4.9).

5. Stability of the periodic solution.

Let us consider the real transformation

$$r_i' = \varphi_i(\tilde{u} + r, T, \varepsilon) - \tilde{u}_i \quad (i = 1, 2, \cdots, n), \tag{5.1}$$

where $\tilde{u}_i$ is a real solution of (2.7). Then, as is well known, the stability of the periodic solution $x_i = \varphi_i(\tilde{u}, t, \varepsilon)$ is decided according to the convergency of iteration of the transformation (5.1).

In order to simplify the calculation, let us transform r to s by the linear transformation

$$s = Pr. \tag{5.2}$$

Here $P = \begin{pmatrix} E_k & 0 \\ \Xi & E_{n-k} \end{pmatrix}$, where E_k and E_{n-k} are the unit matrices of order k and $n-k$ respectively and $\Xi = (\xi_{\nu\alpha})$. By (5.2), the transformation (5.1) is rewritten in terms of s as follows:

$$\begin{cases} s_\alpha' = \varphi_\alpha(\tilde{u} + P^{-1}s, T, \varepsilon) - \tilde{u}_\alpha \quad (\alpha = 1, 2, \cdots, k), \\ s_\nu' = \varphi_\nu(\tilde{u} + P^{-1}s, T, \varepsilon) + \sum\limits_{\beta=1}^{k} \xi_{\nu\beta} \varphi_\beta(\tilde{u} + P^{-1}s, T, \varepsilon) - \tilde{u}_\nu - \sum\limits_{\beta=1}^{k} \xi_{\nu\beta} \tilde{u}_\beta \\ \qquad\qquad (\nu = k+1, \cdots, n). \end{cases} \tag{5.3}$$

Here it is evident that $P^{-1}=\begin{pmatrix} E_k & 0 \\ -\Xi & E_{n-k}\end{pmatrix}$. Since $\tilde{u}_i$ $(i=1, 2,\cdots, n)$ is a solution of (2.7), the transformation (5.3) can be rewritten as follows:

$$
(5.4)\quad \begin{cases} s'_\alpha = \sum_{\beta=1}^{k}\left\{\frac{\partial\varphi_\alpha}{\partial u_\beta}(\tilde{u}, T, \varepsilon) - \sum_{\mu=k+1}^{n}\xi_{\mu\beta}\frac{\partial\varphi_\alpha}{\partial u_\mu}(\tilde{u}, T, \varepsilon)\right\}s_\beta \\ \qquad + \sum_{\mu=k+1}^{n}\frac{\partial\varphi_\alpha}{\partial u_\mu}(\tilde{u}, T, \varepsilon)s_\mu + o(|s|) \qquad (\alpha=1, 2,\cdots, k), \\ s'_\nu = \sum_{\beta=1}^{k}\left[\left(\frac{\partial\varphi_\nu}{\partial u_\beta}(\tilde{u}, T, \varepsilon) + \sum_{\gamma=1}^{k}\xi_{\nu\gamma}\frac{\partial\varphi_\gamma}{\partial u_\beta}(\tilde{u}, T, \varepsilon)\right)\right. \\ \qquad \left. - \sum_{\mu=k+1}^{n}\left\{\xi_{\mu\beta}\left(\frac{\partial\varphi_\nu}{\partial u_\mu}(\tilde{u}, T, \varepsilon) + \sum_{\gamma=1}^{k}\xi_{\nu\gamma}\frac{\partial\varphi_\gamma}{\partial u_\mu}(\tilde{u}, T, \varepsilon)\right)\right\}\right]s_\beta \\ \qquad + \sum_{\mu=k+1}^{n}\left\{\frac{\partial\varphi_\nu}{\partial u_\mu}(\tilde{u}, T, \varepsilon) + \sum_{\beta=1}^{k}\xi_{\nu\beta}\frac{\partial\varphi_\beta}{\partial u_\mu}(\tilde{u}, T, \varepsilon)\right\}s_\mu + o(|s|) \\ \qquad\qquad (\nu=k+1,\cdots, n), \end{cases}
$$

where $|s| = \sum_{i=1}^{n}|s_i|$.

In the sequel, the case I where $k<l$ and the case II where $k=l$ are investigated separately.

Case I. In this case, by (3.16) and (3.17), the partial derivatives $\frac{\partial\varphi_i}{\partial u_j}(\tilde{u}, T, \varepsilon)$ $(i, j,=1, 2,\cdots, n)$ can be written as follows:

$$
(5.5)\quad \begin{aligned} \frac{\partial\varphi_i}{\partial u_j}(\tilde{u}, T, \varepsilon) &= \delta_{ij} + \varepsilon\frac{\partial\varphi_i^{(1)}}{\partial u_j}(c, T) \\ &\quad - \varepsilon^{3/2}\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\sum_{\nu=k+1}^{n}\frac{\partial^2\varphi_i^{(1)}}{\partial u_j\partial u_\alpha}\frac{\partial\varphi_\beta^{(1)}}{\partial u_\nu}D_{\beta\alpha}d_\nu \\ &\quad + \varepsilon^{3/2}\sum_{\mu=k+1}^{n}\frac{\partial^2\varphi_i^{(1)}}{\partial u_j\partial u_\mu}d_\mu + o(\varepsilon^{3/2}) \\ &\qquad\qquad (i, j=1, 2,\cdots, n). \end{aligned}
$$

Let A be the matrix of the coefficients of the linear parts in the right members of (5.4). Then, by (5.5), A is of the form as follows:

$$
(5.6)\quad \begin{aligned} A &= E + \varepsilon A_1 + \varepsilon^{3/2}A_2 + o(\varepsilon^{3/2}) \\ &= E + \varepsilon\begin{pmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)}\end{pmatrix} + \varepsilon^{3/2}\begin{pmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)}\end{pmatrix} + o(\varepsilon^{3/2}), \end{aligned}
$$

where $A_{11}^{(i)}$ and $A_{22}^{(i)}$ $(i=1, 2)$ are respectively $k\times k$-and $(n-k)\times(n-k)$-matrices. As is seen from (5.4) and (5.5), the elements of $A_{11}^{(1)}$, $A_{12}^{(1)}$,$\cdots$are as follows:

$$[A_{11}^{(1)}]_{\alpha\beta}=\frac{\partial\varphi_\alpha^{(1)}}{\partial u_\beta}(c,\ T)-\sum_{\mu=k+1}^{n}\xi_{\mu\beta}\frac{\partial\varphi_\alpha^{(1)}}{\partial u_\mu}(c,\ T),$$

$$[A_{12}^{(1)}]_{\alpha\mu}=\frac{\partial\varphi_\alpha^{(1)}}{\partial u_\mu}(c,\ T),$$

$$[A_{21}^{(1)}]_{\nu\beta}=0,$$

$$[A_{22}^{(1)}]_{\nu\mu}=0;$$

$$\begin{aligned}[A_{21}^{(2)}]_{\nu\beta}=&-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k}\sum_{\mu=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\beta\partial u_\alpha}\frac{\partial\varphi_\gamma^{(1)}}{\partial u_\mu}D_{\gamma\alpha}d_\mu+\sum_{\mu=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\beta\partial u_\mu}d_\mu\\
&+\sum_{\gamma=1}^{k}\xi_{\nu\gamma}\left\{-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\delta=1}^{k}\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\beta\partial u_\alpha}\frac{\partial\varphi_\delta^{(1)}}{\partial u_\lambda}D_{\delta\alpha}d_\lambda+\sum_{\mu=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\beta\partial u_\mu}d_\mu\right\}\\
&-\sum_{\mu=k+1}^{n}\xi_{\mu\beta}\left\{-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{\partial\varphi_\gamma^{(1)}}{\partial u_\lambda}D_{\gamma\alpha}d_\lambda+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda\right\}\\
&-\sum_{\mu=k+1}^{n}\sum_{\nu=1}^{k}\xi_{\mu\beta}\xi_{\nu\gamma}\left\{-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\delta=1}^{k}\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{\partial\varphi_\delta^{(1)}}{\partial u_\lambda}D_{\delta\alpha}d_\lambda\right.\\
&\left.+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda\right\},\end{aligned}$$

$$\begin{aligned}[A_{22}^{(2)}]_{\nu\mu}=&-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{\partial\varphi_\gamma^{(1)}}{\partial u_\lambda}D_{\gamma\alpha}d_\lambda+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda\\
&+\sum_{\beta=1}^{k}\xi_{\nu\beta}\left\{-\frac{1}{J_1}\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k}\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\beta^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{\partial\varphi_\gamma^{(1)}}{\partial u_\lambda}D_{\gamma\alpha}d_\lambda+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\beta^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda\right\}.\end{aligned}$$

Since A is of the form (5.6), A can be written in the exponential form

$$A=\exp(\varepsilon B),$$

where B is of the form as follows:

$$\begin{aligned}B&=A_1+\varepsilon^{1/2}A_2+o(\varepsilon^{1/2})\\&=\begin{pmatrix}A_{11}^{(1)}+o(1) & A_{12}^{(1)}+o(1)\\ \varepsilon^{1/2}A_{21}^{(2)}+o(\varepsilon^{1/2}) & \varepsilon^{1/2}A_{22}^{(2)}+o(\varepsilon^{1/2})\end{pmatrix}.\end{aligned}\tag{5.7}$$

If $\det A_{11}^{(1)}\neq 0$, the characteristic roots of B are given by Urabe's lemma [1] as follws:

$$\mu_\alpha+o(1)\quad(\alpha=1,2,\cdots,k)\quad\text{and}\quad\varepsilon^{1/2}[\lambda_\nu+o(1)]\quad(\nu=k+1,\cdots,n),$$

where μ_α and λ_ν are respectively the characteristic roots of the matrices

$$A_{11}^{(1)}\quad\text{and}\quad A_{22}^{(2)}-A_{21}^{(2)}A_{11}^{(1)-1}A_{12}^{(1)}.\tag{5.8}$$

But, as is well known, if $R\mu_\alpha < 0$, $R\lambda_\nu < 0$ $(\alpha = 1, 2, \dots, k;\ \nu = k+1, \dots, n)$, the periodic solution assured in Theorem 1 is stable since $\varepsilon > 0$ (§3).

Thus we have

Theorem 3. *The periodic solution whose existence is guaranteed by Theorem* 1 *is stable if the real parts of the characteristic roots of the matrices* (5.8) *are all negative.*

Case II. In this case, by (4.9), the partial derivatives $\dfrac{\partial \varphi_i}{\partial u_j}(\tilde{u}, T, \varepsilon)$ $(i, j = 1, 2, \dots, n)$ can be written as follows:

$$
(5.9)\quad \frac{\partial \varphi_i}{\partial u_j}(\tilde{u}, T, \varepsilon) = \delta_{ij} + \varepsilon \frac{\partial \varphi_i^{(1)}}{\partial u_j}(c, T)
+ \varepsilon^2 \Big\{ -\sum_{\alpha=1}^{k} \frac{\partial^2 \varphi_i^{(1)}}{\partial u_j \partial u_\alpha} \frac{1}{J_1} \sum_{\beta=1}^{k} D_{\beta\alpha} \Big(\sum_{\nu=k+1}^{n} \frac{\partial \varphi_\beta^{(1)}}{\partial u_\nu} d_\nu + \varphi_\beta^{(2)} \Big)
+ \sum_{\nu=k+1}^{n} \frac{\partial^2 \varphi_i^{(1)}}{\partial u_j \partial u_\nu} d_\nu + \frac{\partial \varphi_i^{(2)}}{\partial u_j} \Big\} + o(\varepsilon^2) \quad (i, j = 1, 2, \dots, n).
$$

Then, substituting these into the right members of (5.4), we see that the matrix A of the coefficients of their linear parts can be written as follows:

$$
(5.10)\quad A = E + \varepsilon A_1 + \varepsilon^2 A_2 + o(\varepsilon^2)
= E + \varepsilon \begin{pmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ A_{21}^{(2)} & A_{22}^{(2)} \end{pmatrix} + o(\varepsilon^2),
$$

where

$$
[A_{11}^{(1)}]_{\alpha\beta} = \frac{\partial \varphi_\alpha^{(1)}}{\partial u_\beta}(c, T) - \sum_{\mu=k+1}^{n} \xi_{\mu\beta} \frac{\partial \varphi_\alpha^{(1)}}{\partial u_\mu}(c, T),
$$

$$
[A_{12}^{(1)}]_{\alpha\mu} = \frac{\partial \varphi_\alpha^{(1)}}{\partial u_\mu}(c, T),
$$

$$
[A_{21}^{(1)}]_{\nu\beta} = 0,
$$

$$
[A_{22}^{(1)}]_{\nu\mu} = 0;
$$

$$
[A_{21}^{(2)}]_{\nu\beta} = -\sum_{\alpha=1}^{k} \frac{\partial^2 \varphi_\nu^{(1)}}{\partial u_\beta \partial u_\alpha} \frac{1}{J_1} \sum_{\gamma=1}^{k} D_{\gamma\alpha} \Big(\sum_{\mu=k+1}^{n} \frac{\partial \varphi_\gamma^{(1)}}{\partial u_\mu} d_\mu + \varphi_\gamma^{(2)} \Big)
+ \sum_{\mu=k+1}^{n} \frac{\partial^2 \varphi_\nu^{(1)}}{\partial u_\beta \partial u_\mu} d_\mu + \frac{\partial \varphi_\nu^{(2)}}{\partial u_\beta}
+ \sum_{\gamma=1}^{k} \xi_{\nu\gamma} \Big\{ -\sum_{\alpha=1}^{k} \frac{\partial^2 \varphi_\gamma^{(1)}}{\partial u_\beta \partial u_\alpha} \frac{1}{J_1} \sum_{\delta=1}^{k} D_{\delta\alpha} \Big(\sum_{\mu=k+1}^{n} \frac{\partial \varphi_\delta^{(1)}}{\partial u_\mu} d_\mu + \varphi_\delta^{(2)} \Big)
$$

$$+\sum_{\mu=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\beta\partial u_\mu}d_\mu+\frac{\partial\varphi_\gamma^{(2)}}{\partial u_\beta}\Big\}$$

$$-\sum_{\mu=k+1}^{n}\xi_{\mu\beta}\Big\{-\sum_{\alpha=1}^{k}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{1}{J_1}\sum_{\gamma=1}^{k}D_{\gamma\alpha}\Big(\sum_{\lambda=k+1}^{n}\frac{\partial\varphi_\gamma^{(1)}}{\partial u_\lambda}d_\lambda+\varphi_\gamma^{(2)}\Big)$$

$$+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda+\frac{\partial\varphi_\nu^{(2)}}{\partial u_\mu}\Big\}$$

$$-\sum_{\mu=k+1}^{n}\sum_{\gamma=1}^{k}\xi_{\mu\beta}\xi_{\nu\gamma}\Big\{-\sum_{\alpha=1}^{k}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{1}{J_1}\sum_{\delta=1}^{k}D_{\delta\alpha}\Big(\sum_{\lambda=k+1}^{n}\frac{\partial\varphi_\delta^{(1)}}{\partial u_\lambda}d_\lambda+\varphi_\delta^{(2)}\Big)$$

$$+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda+\frac{\partial\varphi_\gamma^{(2)}}{\partial u_\mu}\Big\},$$

$$[A_{22}^{(2)}]_{\nu\mu}=-\sum_{\alpha=1}^{k}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{1}{J_1}\sum_{\beta=1}^{k}D_{\beta\alpha}\Big(\sum_{\lambda=k+1}^{n}\frac{\partial\varphi_\beta^{(1)}}{\partial u_\lambda}d_\lambda+\varphi_\beta^{(2)}\Big)$$

$$+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\nu^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda+\frac{\partial\varphi_\nu^{(2)}}{\partial u_\mu}$$

$$+\sum_{\beta=1}^{k}\xi_{\nu\beta}\Big\{-\sum_{\alpha=1}^{k}\frac{\partial^2\varphi_\beta^{(1)}}{\partial u_\mu\partial u_\alpha}\frac{1}{J_1}\sum_{\gamma=1}^{k}D_{\gamma\alpha}\Big(\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\gamma^{(2)}}{\partial u_\lambda}d_\lambda+\varphi_\gamma^{(2)}\Big)$$

$$+\sum_{\lambda=k+1}^{n}\frac{\partial^2\varphi_\beta^{(1)}}{\partial u_\mu\partial u_\lambda}d_\lambda+\frac{\partial\varphi_\beta^{(2)}}{\partial u_\mu}\Big\}.$$

Since A is of the form (5.10), A can be written in the exponential form

$$A=\exp(\varepsilon B),$$

where B is of the form as follows:

$$B=A_1+\varepsilon\Big(A_2-\frac{1}{2}A_1^2\Big)+o(\varepsilon) \tag{5.11}$$

$$=\begin{pmatrix}A_{11}^{(1)}+o(1) & A_{12}^{(1)}+o(1)\\ \varepsilon A_{21}^{(2)}+o(\varepsilon) & \varepsilon A_{22}^{(2)}+o(\varepsilon)\end{pmatrix}.$$

If det $A_{11}^{(1)}\neq 0$, the characteristic roots of the matrix B are given by Urabe's lemma [1] as follows:

$$\mu_\alpha+o(1)\quad(\alpha=1,2,\cdots,k)\quad\text{and}\quad\varepsilon[\lambda_\nu+o(1)]\quad(\nu=k+1,\cdots,n),$$

where μ_α and λ_ν are respectively the characteristic roots of the matrices

$$A_{11}^{(1)}\quad\text{and}\quad A_{22}^{(2)}-A_{21}^{(2)}A_{11}^{(1)-1}A_{12}^{(1)}. \tag{5.12}$$

But, as is well known, if $R\mu_\alpha<0$, $R\lambda_\nu<0$ ($\alpha=1,2,\cdots,k$; $\nu=k+1,\cdots,n$), the periodic solution assured in Theorem 2, is stable since $\varepsilon>0$ (§3).

Thus we have

Theorem 4. *The periodic solution whose existence is guaranteed by Theorem* 2 *is stable, if the real parts of the characteristic roots of the matrices* (5.12) *are all negative.*

In conclusion, the author whishes to express his hearty gratitude to Prof. Urabe for his kind guidance and constant advice.

Reference

[1] M. Urabe: *On the Nonlinear Autonomous System Admitting of a Family of Periodic Solutions near its Certain Periodic Solution.* J. Sci. Hiroshima Univ., Ser. A., 22 (1958), 153–173.

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