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Reduction of Group Varieties and Transformation Spaces

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In the paper [3], Koizumi and Shimura solved affirmatively the following problem: *let A and B be abelian varieties defined over a field k with a prime divisor $\mathfrak{p}$. Suppose that there exists a homomorphism of A onto B, defined over k. If A is without defect for $\mathfrak{p}$, then is there an abelian variety which is isomorphic to B over k and without defect for $\mathfrak{p}$?* In this paper we shall generalize this result for the cases of arbitrary group varieties and homogeneous spaces (Theorem 3), and apply it to a problem which concerns compatibility of the reduction process with the process making a coset space of a group variety by a subgroup (Theorem 4). Our generalization is not complete, because we need a ground ring extension in the process of constructing a group $\mathfrak{p}'$-variety (resp. a homogeneous $\mathfrak{p}'$-space) from a pre-group $\mathfrak{p}$-variety (resp. a pre-homogeneous $\mathfrak{p}$-space). However if k is complete with respect to the prime $\mathfrak{p}$, we do not need any ground ring extension. In other words it is possible to generalize completely the result obtained in [3] in this case.

First we shall define a pre-group $\mathfrak{p}$-variety, a pre-transformation $\mathfrak{p}$-space, etc., which corresponds to a pre-group, a pre-transformation space, etc. in [9], and prove some basic results (§1). Next Weil's idea in [11] is adapted to the case of $\mathfrak{p}$-simple $\mathfrak{p}$-varieties. The main result of §2 is stated in Theorem 1, whose applications will be seen in §3. Then we shall apply Weil's method of construction of a group variety (resp. a transformation space) from a pre-group (resp. a pre-transformation space) to the case of $\mathfrak{p}$-simple $\mathfrak{p}$-varieties. Theorem 2 in §3 corresponds to the main theorem in [9]. Theorem 3 is, then, a direct consequence of the basic results in §1 and Theorem 2. In §4 an application of Theorem 3 is given, to which we referred already in the above. §5 is devoted to the study of the reduction of generalized Jacobian varieties under a certain restriction.

Throughout the paper, we shall fix the basic field k and a discrete valuation ring $\mathfrak{o}$ with the maximal ideal $\mathfrak{p}$ and denote by κ the residue class field $\mathfrak{o}/\mathfrak{p}$. The terminologies and the notations in [8] and [13] will be freely used.

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§1. Group $\mathfrak{p}$-varieties and homogeneous $\mathfrak{p}$-spaces.

Let $(V, \bar{V})$ and $(W, \bar{W})$ be two $\mathfrak{p}$-simple $\mathfrak{p}$-varieties[1], and let f be a rational

1) We shall denote $\mathfrak{p}$-varieties by $(V, \bar{V})$ etc.. For the precise notations, see §5 in [13]. A $\mathfrak{p}$-variety is called to be $\mathfrak{p}$-simple, if the corresponding model of a function field is $\mathfrak{p}$-simple.

mapping of V into W defined over k. Let x be a generic point of V over k. Then $y=f(x)$ is a generic point of a $\mathfrak{p}$-subvariety $(W_0, \bar{W}_0)$ of $(W, \bar{W})$. Let M and N be the models $M(V, \bar{V})$ and $M(W_0, \bar{W}_0)$ of the function fields $k(x)$ and $k(y)$ respectively. Let a be a point of $(V, \bar{V})$ such that the spot P of M corresponding to a dominates a spot of N. Then we say that *f is defined at a*. If f is defined at a generic point of $\bar{V}$ over κ, we say that *f is defined modulo $\mathfrak{p}$*. Moreover if the generating spot of M over $\mathfrak{p}$ dominates that of N, f is called a *$\mathfrak{p}$-rational mapping*. Then f defines naturally a rational mapping $\bar{f}$ of $\bar{V}$ into $\bar{W}_0$, which maps a generic point of $\bar{V}$ over κ onto that of $\bar{W}_0$. Let f be a birational correspondence between V and W. If f and f^{-1} are both $\mathfrak{p}$-rational, we say that f is a *$\mathfrak{p}$-birational correspondence between $(V, \bar{V})$ and $(W, \bar{W})$*. Then we say that *f is biregular at a* in $(V, \bar{V})$, if the corresponding spot P to a in M is also a spot of N.

Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety such that V is a pre-group defined over k[2]. Let f be the normal law of composition on V. Then if the birational correspondence of $V\times V$ into itself, which map (x, y) onto $(x, f(x, y))$ and onto $(f(x, y), y)$ respectively, are both $\mathfrak{p}$-birational, we say that $(V, \bar{V})$ is a *pre-group $\mathfrak{p}$-variety*. Let ϕ be the inverse function of the pre-group V. Then if f and ϕ are everywhere defined on $(V\times V, \bar{V}\times\bar{V})$ and $(V, \bar{V})$ respectively, $(V, \bar{V})$ is called a *group $\mathfrak{p}$-variety*[3].

PROPOSITION 1. *Let $(V, \bar{V})$ be a pre-group $\mathfrak{p}$-variety. Then the inverse function ϕ on V is $\mathfrak{p}$-birational.*

PROOF. Let x and y (resp. $\bar{x}$ and $\bar{y}$) be independent generic points of V over k (resp. of $\bar{V}$ over κ) and put $z=f(x, y)$ (resp. $\bar{z}=f(\bar{x}, \bar{y})$). If μ is the rational mapping of $V\times V$ into V which maps (z, y) onto x, μ is $\mathfrak{p}$-rational and we have $\phi(x)=\mu(y, z)$ (cf. the proof of Proposition 4 in [9]). Therefore we have $[(y, z)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{z})]\supset[\mu(y, z)\xleftarrow{\mathfrak{o}}\mu(\bar{y}, \bar{z})]=[\phi(x)\xrightarrow{\mathfrak{o}}\mu(\bar{y}, \bar{z})]$. On the other hand we have $[(y, z)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{z})]\cap k(x)=[(x)\xrightarrow{\mathfrak{o}}(\bar{x})]$. Since μ is $\mathfrak{p}$-rational, $[\phi(x)\xrightarrow{\mathfrak{o}}\mu(\bar{y}, \bar{z})]$ is a discrete valuation ring and hence we have $[(x)\xrightarrow{\mathfrak{o}}(\bar{x})]=[\phi(x)\xrightarrow{\mathfrak{o}}\mu(\bar{y}, \bar{z})]$. This means that ϕ is $\mathfrak{p}$-birational and $\phi(\bar{x})=\mu(\bar{y}, \bar{z})$. q.e.d.

Let $(V, \bar{V})$ be a pre-group $\mathfrak{p}$-variety and $(W, \bar{W})$ a $\mathfrak{p}$-simple $\mathfrak{p}$-variety such that W is a pre-transformation space with respect to V, defined over k[4]. Let g be the normal law of composition on W with respect to V, and let x and u be independent generic points of V and W over k. If the birational correspondence between $V\times W$ and itself, which maps (x, u) onto $(x, g(x, u))$, is $\mathfrak{p}$-birational, $(W, \bar{W})$ is called a *pre-transformation $\mathfrak{p}$-space with respect to $(V, \bar{V})$*. Moreover if W (resp. $\bar{W}$) is a pre-homogeneous space with respect to V (resp. $\bar{V}$)[4], $(W, \bar{W})$ is called a *pre-homogeneous $\mathfrak{p}$-space with respect to $(V, \bar{V})$*.

2) For the definition, see [9].

3) Notice that this definition is different from that of [3]. Our group $\mathfrak{p}$-variety is a group $\mathfrak{p}$-variety without defect for $\mathfrak{p}$ in the sense of [3].

4) For the definition, see [9].

Suppose that $(V, \bar{V})$ is a group $\mathfrak{p}$-variety. If g is defined everywhere on $(V\times W, \bar{V}\times\bar{W})$, we call $(W, \bar{W})$ a *transformation* $\mathfrak{p}$-*space with respect to* $(V, \bar{V})$. Moreover if W and $\bar{W}$ are both homogeneous spaces with respect to V and $\bar{V}$ respectively, $(W, \bar{W})$ is called a *homogeneous* $\mathfrak{p}$-*space with respect to* $(V, \bar{V})$.

PROPOSITION 2. *Let $(G, \bar{G})$ be a group $\mathfrak{p}$-variety, and let $(H, \bar{H})$ and $(T, \bar{T})$ be a homogeneous $\mathfrak{p}$-space and a transformation $\mathfrak{p}$-space with respect to $(G, \bar{G})$ respectively. Let λ be a rational mapping of H into T such that λ is defined modulo $\mathfrak{p}$ and $\lambda(xu)$[5] is equal to $x\lambda(u)$ for independent generic points x and u of G and H over k respectively. Then λ is everywhere defined on $(H, \bar{H})$.*

PROOF. Let $\bar{a}$ be a point of $\bar{H}$ and $\bar{x}$ a generic point of $\bar{G}$ over $\kappa(\bar{a})$. Then $\bar{x}^{-1}\bar{a}$ is a generic point of $\bar{H}$ over $\kappa(\bar{a})$. Then we have, by assumptions, $[(x, u) \xrightarrow{\mathfrak{o}} (\bar{x}, \bar{x}^{-1}\bar{a})] \supset [(x, \lambda(u)) \xrightarrow{\mathfrak{o}} (\bar{x}, \lambda(\bar{x}^{-1}\bar{a}))] \supset [x\lambda(u) \xrightarrow{\mathfrak{o}} \bar{x}\lambda(\bar{x}^{-1}\bar{a})] = [\lambda(xu) \xrightarrow{\mathfrak{o}} \bar{x}\lambda(\bar{x}^{-1}\bar{a})]$. On the other hand we have $[(x, u) \xrightarrow{\mathfrak{o}} (\bar{x}, \bar{x}^{-1}\bar{a})] = [(x, xu) \xrightarrow{\mathfrak{o}} (\bar{x}, \bar{a})]$, and hence, applying Proposition 7 in [8], we have $[(x, u) \xrightarrow{\mathfrak{o}} (\bar{x}, \bar{x}^{-1}\bar{a})] \cap k(xu) = [(xu) \xrightarrow{\mathfrak{o}} (\bar{a})]$. Therefore the spot $[(xu) \xrightarrow{\mathfrak{o}} (\bar{a})]$ dominates the spot $[\lambda(xu) \xrightarrow{\mathfrak{o}} \bar{x}\lambda(\bar{x}^{-1}\bar{a})]$. Since xu is a generic point of H over k, λ is defined at $\bar{a}$.

Similarly it is easily seen that λ is defined at any point of H, applying Proposition 17 of Chap. II in [12] instead of Proposition 7 in [8]. q.e.d.

COROLLARY. *Let $(G, \bar{G})$ and $(F, \bar{F})$ be two group $\mathfrak{p}$-varieties. Let λ be a rational mapping of G into F such that λ is defined modulo $\mathfrak{p}$ and $\lambda(xy)$ is equal to $\lambda(x)\cdot\lambda(y)$ for any independent generic points x and y of G over k. Then λ is everywhere defined on $(G, \bar{G})$.*

PROOF. This is a direct consequence of Proposition 2, if we notice that $(G, \bar{G})$ and $(F, \bar{F})$ are considered naturally as a homogeneous $\mathfrak{p}$-space and a transformation $\mathfrak{p}$-space with respect to $(G, \bar{G})$ respectively. q.e.d.

PROPOSITION 3. *Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and W a variety defined over k such that there is a generically surjective mapping f of V into W defined over k. Then there are a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(T, \bar{T})$ and a birational correspondence h between W and T defined over k such that $f_0 = g\circ f$ is a $\mathfrak{p}$-rational mapping of $(V, \bar{V})$ into $(T, \bar{T})$.*

PROOF. Let x be a generic point of V over k and $\bar{x}$ that of $\bar{V}$ over κ. Then the specialization ring $R = [(x) \xrightarrow{\mathfrak{o}} (\bar{x})]$ is a discrete valuation ring[6]. Let y be the image of x by f. Then $k(x)$ contains $k(y)$. If S is the contraction of R to $k(y)$, S is also a discrete valuation ring of $k(y)$ whose maximal ideal is generated by a prime element π of $\mathfrak{o}$. Then the residue class field K of S is a finitely generated regular extension over κ. Let $z_1, \cdots, z_t$ be the elements of S such

5) For simplicity, we shall often write xa, etc. instead of $g(x, a)$, etc..

6) The generating spot of a $\mathfrak{p}$-simple model is a discrete valuation ring with a prime element which is a prime element of $\mathfrak{o}$.

that their residues $\bar{z}_1, \cdots, \bar{z}_t$ generate K over κ. Since $k(y)$ is separable over k, the integral closure of $\mathfrak{o}[z]$ is also an affine ring over $\mathfrak{o}$ (cf. Proposition 4, Appendix in [4]). Therefore we may assume that $\mathfrak{o}[z]$ is integrally closed. Let $\mathfrak{m}$ be the maximal ideal of S, and put $\mathfrak{n}=\mathfrak{o}[z]\cap\mathfrak{m}$. Then it is easily seen that the rank of $\mathfrak{n}$ is equal to 1, and hence that $\mathfrak{o}[z]_{\mathfrak{n}}$ is a discrete valuation ring contained in S. This means that $\mathfrak{o}[z]_{\mathfrak{n}}$ is equal to S. Let $(T', \bar{T}')$ be the $\mathfrak{p}$-variety which is the locus of z over $\mathfrak{o}$. Then it is easy to see that there is an $\mathfrak{o}$-open subset $(T, \bar{T})$ of $(T', \bar{T}')$ which has only one generating spot S over $\mathfrak{p}$. The multiplicity $\mu(S)$ is equal to 1, since S has π as a prime element (cf. §3 in [13]). $(T, \bar{T})$ and the birational correspondence h between W and T, which maps y onto z, are our solution. q.e.d.

PROPOSITION 4. *Let $(G, \bar{G})$ be a group $\mathfrak{p}$-variety and let T be a homogeneous space defined over k with respect to G. Let g be the normal law on T. Assume that T has a point a rational over k. Then there is a pre-homogeneous $\mathfrak{p}$-space $(T_0, \bar{T}_0)$ with respect to $(G, \bar{G})$ such that T is birationally equivalent to T_0 over k by the birational correspondence h of T into T_0 and such that $h(g(*, h^{-1}(*)))$ is the normal law on $(T_0, \bar{T}_0)$. Moreover the mapping of $(G, \bar{G})$ into $(T_0, \bar{T}_0)$ which maps x onto $h(g(x, a))$ is $\mathfrak{p}$-rational.*

PROOF. The rational mapping g' of G into T, which is obtained from g by putting $g'(x)=g(x, a)=xa$ for a generic point x of G over k, is defined over k. Since T is a homogeneous space over k, g' is a surjective mapping onto T. By Proposition 3, there are a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(T_0, \bar{T}_0)$ and a birational correspondence h between T and T_0 such that $f_0=h\circ g'$ is $\mathfrak{p}$-rational. Let t be the image of x by f_0. Then we have $k(t)=k(xa)$. Now we define a normal law g_0 of composition on T_0 with respect to G by putting $g_0(y, t)=h(yh^{-1}(t))=h(yxa)$, where y is a generic point of G over $k(x)$. Then we have $k(y, t)=k(y, xa)$ and $k(y, yxa)=k(y, h(yxa))=k(y, g_0(y, t))$. On the other hand we have $k(y, xa)=k(y, yxa)$, since T is a homogeneous space with respect to G. Therefore we have $k(y, t)=k(y, g_0(y, t))$. Moreover let z be a generic point of G over $k(x, y)$. Then we have $g_0(z, g_0(y, t))=g_0(zy, t)$. These relations mean that g_0 is a normal law of composition on T_0 with respect to G.

Next let $\bar{x}$ and $\bar{y}$ be two independent generic points of $\bar{G}$ over κ. Then we have $[(x, y)\xrightarrow{\mathfrak{o}}(\bar{x}, \bar{y})]\supset[(yx)\xrightarrow{\mathfrak{o}}(\bar{y}\bar{x})]\supset[f_0(yx)\xrightarrow{\mathfrak{o}}f_0(\bar{y}\bar{x})]=[h(yxa)\xrightarrow{\mathfrak{o}}f_0(\bar{y}\bar{x})]=[g_0(y, t)\xrightarrow{\mathfrak{o}}f_0(\bar{y}\bar{x})]$, since f_0 is $\mathfrak{p}$-rational. On the other hand we have $[(y, x)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{x})]\cap k(y, t)=[(y, t)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{t})]$, where $\bar{t}$ is the image of $\bar{x}$ by f_0. Therefore we have $[(y, t)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{t})]\supset[g_0(y, t)\xrightarrow{\mathfrak{o}}f_0(\bar{y}, \bar{x})]$ and hence g_0 is $\mathfrak{p}$-rational. Since $g_0(y, t)$ is a generic point of T_0 over $k(y)$, we have similarly $[(y, g_0(y, t))\xrightarrow{\mathfrak{o}}(y, g_0(\bar{y}, \bar{t}))]=[(y^{-1}, g_0(y, t))\xrightarrow{\mathfrak{o}}(\bar{y}^{-1}, g_0(\bar{y}, \bar{t}))]\supset[(t)\xrightarrow{\mathfrak{o}}(\bar{t})]$, and hence we have $[(y, t)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{t})]=[(y, g_0(y, t))\xrightarrow{\mathfrak{o}}(\bar{y}, g_0(\bar{y}, \bar{t}))]$. This means that $(T_0, \bar{T}_0)$ is a pre-homogeneous $\mathfrak{p}$-space with respect to $(G, \bar{G})$, since g_0 is $\mathfrak{p}$-rational. q.e.d.

PROPOSITION 5. *Let $(G, \bar{G})$ be a group $\mathfrak{p}$-variety and G' a group variety de-*

fined over k, such that there is a rational homomorphism λ of G onto G' defined over k. Then there are a pre-group $\mathfrak{p}$-variety $(G_0, \bar{G}_0)$ and a birational correspondence h between G' and G_0 defined over k, such that $h(x'y')=h(x')h(y')$ for independent generic points x' and y' of G' over k and such that $h\cdot\lambda$ is $\mathfrak{p}$-rational.

Proof. Let g be the rational mapping of $G\times G'$ onto G' such that $g(x, x')=\lambda(x)x'$ for independent generic points x and x' of G and G' respectively. Then G' is considered as a homogeneous space with respect to G. Since the unit element e' of G' is a point rational over k, there are a pre-homogeneous $\mathfrak{p}$-space $(G_0, \bar{G}_0)$ and a birational correspondence h between G' and G_0 by Proposition 4. Moreover if x and y are independent generic point of G and if t is the image of x by the $\mathfrak{p}$-rational mapping $h\circ\lambda$ of $(G, \bar{G})$ into $(G_0, \bar{G}_0)$, the rational mapping $g_0(y, t)=h(\lambda(yx))$ is the normal law of composition on $(G_0, \bar{G}_0)$ with respect to $(G, \bar{G})$. Let s be the image of y by $h\circ\lambda$. Then the rational mapping $f_0(s, t)=h(h^{-1}(s)h^{-1}(t))=h(\lambda(y)\lambda(x))$ of $G_0\times G_0$ into G_0 defines the structure of a pre-group on G_0. Then we have $g_0(y, t)=h(\lambda(yx))=h(\lambda(y)\lambda(x))=f_0(s, t)$, and hence $[(y, t)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{t})]\supset[g_0(y, t)\xrightarrow{\mathfrak{o}}(g_0(\bar{y}, \bar{t}))]=[f_0(s, t)\xrightarrow{\mathfrak{o}}g_0(\bar{y}, \bar{t})]$, where $\bar{y}$ and $\bar{t}$ are independent generic points of G and G_0 over κ. On the other hand, if $\bar{s}$ is the image of $\bar{y}$ by $h\circ\lambda$, we have $[(y, t)\xrightarrow{\mathfrak{o}}(\bar{y}, \bar{t})]\cap k(s, t)=[(s, t)\xrightarrow{\mathfrak{o}}(\bar{s}, \bar{t})]$ and hence it is easy to see that $[(s, t)\xrightarrow{\mathfrak{o}}(\bar{s}, \bar{t})]=[(s, f_0(s, t))\xrightarrow{\mathfrak{o}}(\bar{s}, g_0(\bar{y}, \bar{t}))]$. Therefore f_0 is $\mathfrak{p}$-rational and $f_0(\bar{s}, \bar{t})$ is equal to $g_0(\bar{y}, \bar{t})$.

Similarly we have $[(s, t)\xrightarrow{\mathfrak{o}}(\bar{s}, \bar{t})]=[(t, f_0(s, t))\xrightarrow{\mathfrak{o}}(\bar{t}, f_0(\bar{s}, \bar{t}))]$. Therefore $(G_0, \bar{G}_0)$ is a pre-group $\mathfrak{p}$-variety. q.e.d.

§2. Descent of ground rings.

First we assume that $\mathfrak{o}$ is complete. Let k' be a separable extension of k of finite degree n. Let $\mathfrak{J}$ be the set of all distinct isomorphisms of k' over k into the algebraic closure $\bar{k}$ of k. If σ is an element of $\mathfrak{J}$, we denote by k^{σ} the image of k' by σ. Let $\mathfrak{o}^{\sigma}$ be the valuation ring of k^{σ} with the maximal ideal $\mathfrak{p}^{\sigma}$, which is the unique prolongation[7] of $\mathfrak{p}$ in k^{σ}. In particular we put $\mathfrak{o}'=\mathfrak{o}^{\varepsilon}$, and $\mathfrak{p}'=\mathfrak{p}^{\varepsilon}$ where ε is the identity isomorphism of k'. Let $(V, \bar{V})$ be a $\mathfrak{p}'$-simple $\mathfrak{p}'$-variety and σ an element of $\mathfrak{J}$. Then we shall denote by $(V^{\sigma}, \bar{V}^{\sigma})$ the $\mathfrak{p}^{\sigma}$-simple $\mathfrak{p}^{\sigma}$-variety which is the transform of $(V, \bar{V})$ by the isomorphism σ. Similarly if f is a rational mapping of a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(V_0, \bar{V}_0)$ into $(V, \bar{V})$, we denote by f^{σ} the transform of f by σ.

Proposition 6. *Let k' be a separable extension of k of finite degree n and $\mathfrak{J}$ the set of all the isomorphisms of k' into the algebraic closure $\bar{k}$ of k. Assume that $\mathfrak{o}$ is complete and that k' is unramified over k[8]. Let $(V_0, \bar{V}_0)$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and $(V, \bar{V})$ a $\mathfrak{p}'$-simple projective (resp. affine) $\mathfrak{p}'$-variety, such that*

7) Let k' be an extension of k and $\mathfrak{o}'$ a discrete valuation ring of k' such that $\mathfrak{o}'\supset\mathfrak{o}$ and $\mathfrak{o}\cap\mathfrak{p}'=\mathfrak{p}$, where $\mathfrak{p}'$ is the maximal ideal of $\mathfrak{o}'$. Then we say that $\{\mathfrak{o}', \mathfrak{p}'\}$ (or simply $\mathfrak{p}'$) is a prolongation of $\{\mathfrak{o}, \mathfrak{p}\}$ (or $\mathfrak{p}$) in k'.

8) This means that $\mathfrak{p}\mathfrak{o}'=\mathfrak{p}'$ and $\mathfrak{o}'/\mathfrak{p}'$ is separable over $\mathfrak{o}/\mathfrak{p}$.

there is a $\mathfrak{p}'$*-birational correspondence* f *between* $(V_0, \bar{V}_0)$ *and* $(V, \bar{V})$. *Then there is a* $\mathfrak{p}$*-simple projective (resp. affine)* $\mathfrak{p}$*-variety* $(W, \bar{W})$ *and a* $\mathfrak{p}$*-birational correspondence* F *between* $(V_0, \bar{V}_0)$ *and* $(W, \bar{W})$, *such that* $F\circ f^{-1}$ *is biregular at every point of* $(V, \bar{V})$ *where the mappings* $f^{\sigma}\circ f^{-1}$ *are defined for all* $\sigma \in \mathfrak{J}$.

This proposition is a generalization of Proposition 1 in [11], whose proof is also available for our proposition. In fact the compositum K of fields k^{σ} is also unramified over k, since $\mathfrak{o}$ is complete (cf. §1, Chap. 4 in [1]). Therefore any subfield of K containing k is unramified over k. Let K_{ρ} be as in the proof of Proposition 1 in [11], and $(\mathfrak{o}_{\rho}, \mathfrak{p}_{\rho})$ the prolongation of $(\mathfrak{o}, \mathfrak{p})$ in K_{ρ}. Then we can choose a basis $(\alpha_1, \cdots, \alpha_{d_{\rho}})$ of K_{ρ} over k, such that each α_i is in $\mathfrak{o}_{\rho}$ and the residues modulo $\mathfrak{p}_{\rho}$ are a basis of $\mathfrak{o}_{\rho}/\mathfrak{p}_{\rho}$ over $\mathfrak{o}/\mathfrak{p}$, since K_{ρ} is unramified over k. Then $h_{\rho\nu}$ in the proof are expressed as linear combinations of $g_{\omega(\rho)}$ with coefficients in the integral closure of $\mathfrak{o}$ in K. This means that our proposition is proved in the same way as in that of Proposition 1 in [11].

Now we return to the general case, i.e. we do not assume that $\mathfrak{o}$ is complete.

LEMMA 1. *Let* F *and* H *be rational mappings of a* $\mathfrak{p}$*-simple* $\mathfrak{p}$*-variety* $(X, \bar{X})$ *into two* $\mathfrak{p}$*-simple* $\mathfrak{p}$*-varieties* $(W, \bar{W})$ *and* $(T, \bar{T})$, *both defined modulo* $\mathfrak{p}$. x *being a generic point of* X *over* k, *assume that* $t=H(x)$ *is a generic point of* T *over* k *and that* H *is* $\mathfrak{p}$*-rational. If* R_t *is the generating spot*[9] *of* $(T, \bar{T})$ *in* $k(t)$ *over* $\mathfrak{p}$ *with the maximal ideal* $\mathfrak{P}_t$, *we assume that* x *has a locus* $(V_t, \bar{V}_t)$ *over* R_t *which is a* $\mathfrak{P}_t$*-variety. Let* F_t *be the mapping of* $(V_t, \bar{V}_t)$ *into* $(W, \bar{W})$ *induced by* F *on* $(V_t, \bar{V}_t)$. *Then* F *is defined at every point of* $(V_t, \bar{V}_t)$ *where* F_t *and* H *are both defined.*

The proof is very similar to that of Lemma 2 in [11]. Therefore we omit the proof.

Let $(T, \bar{T})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and t a generic point of T over k. If R_t is the generating spot of $(T, \bar{T})$ in $k(t)$ over $\mathfrak{p}$, denote by $\mathfrak{P}_t$ the maximal ideal of R_t. Similarly if t' is also a generic point of T over k, denote by $R_{t'}$ and $\mathfrak{P}_{t'}$ the generating spot of $(T, \bar{T})$ in $k(t')$ over $\mathfrak{p}$ and its maximal ideal. Then if $(V_t, \bar{V}_t)$ is a $\mathfrak{P}_t$-simple $\mathfrak{P}_t$-variety, we shall denote by $(V_{t'}, \bar{V}_{t'})$ the transform of $(V_t, \bar{V}_t)$ by the isomorphism of $k(t)$ onto $k(t')$ which maps t onto t'. Similarly if f_t is a $\mathfrak{P}_t$-rational mapping of $(V_t, \bar{V}_t)$ into a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(V, \bar{V})$, we shall denote by $f_{t'}$ the transform of f_t by the same isomorphism.

PROPOSITION 7. *Let* $(T, \bar{T})$ *be a* $\mathfrak{p}$*-simple* $\mathfrak{p}$*-variety and* t *a generic point of* T *over* k. *Let* $(V_t, \bar{V}_t)$ *be a* $\mathfrak{P}_t$*-simple* $\mathfrak{P}_t$*-variety which is an* R_t*-open subset*[10] *of*

9) We shall understand by this the generating spot in $k(t)$ of the model $M(T, \bar{T})$ corresponding to $(T, \bar{T})$.

10) We can naturally define a topology on a $\mathfrak{p}$-variety from the Zariski topology on the model corresponding to this $\mathfrak{p}$-variety. This topology will be called the $\mathfrak{o}$-topology, and an open (resp. closed) subset in this topology is called $\mathfrak{o}$-open (resp. $\mathfrak{o}$-closed).

an affine $\mathfrak{P}_t$-variety, and $(V, \bar{V})$ a $\mathfrak{p}$-simple $\mathfrak{p}$-variety such that there is a $\mathfrak{P}_t$-birational correspondence f_t between $(V, \bar{V})$ and $(V_t, \bar{V}_t)$. Let $\bar{a}$ be a point of $\bar{V}_t$ such that $f_{t'}\circ f_t^{-1}$ is biregular at $\bar{a}$, where t' is a generic point of T over $k(t)$. Then there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension k' of k, and there are a $\mathfrak{p}'$-simple affine $\mathfrak{p}'$-variety $(W, \bar{W})$ and a $\mathfrak{p}'$-birational correspondence F between $(V, \bar{V})$ and $(W, \bar{W})$ such that $F\circ f_t^{-1}$ is biregular at $\bar{a}$. Moreover if $\mathfrak{o}$ is complete, $(W, \bar{W})$ and F are taken as a $\mathfrak{p}$-simple affine $\mathfrak{p}$-variety and a $\mathfrak{p}$-birational correspondence.

This proposition is a generalization of Proposition 2 in [11], whose proof is also available in our case. In fact we can construct a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(X, \bar{X})$ and a $\mathfrak{P}_t$-birational correspondence g_t between $(T\times V_t, \bar{T}\times\bar{V}_t)$ and $(X, \bar{X})$, which correspond to X and g_t in the case of Proposition 2 in [11]. Then by Lemma 1 g_t is biregular at $(\bar{t}', \bar{a})$, if $\bar{t}'$ is a generic point of $\bar{T}$ over $R_t/\mathfrak{P}_t$. Let A_0 be the R_t-closed subset of $(T\times V_t, \bar{T}\times\bar{V}_t)$ where g_t is not biregular and put $\bar{A}_0=A_0\cap(\bar{T}\times\bar{V}_t)$. Then $\bar{T}\times\bar{a}$ is not contained in $\bar{A}_0$. From $\bar{A}_0$ we can obtain a κ-open subset $\bar{T}'$ of $\bar{T}$ such that, if $\bar{t}_1$ is any algebraic point over κ in $\bar{T}'$, g_t is biregular at $(\bar{t}_t', \bar{a})$. Let $\bar{t}_1$ be a simple point on $\bar{T}'$, separably algebraic over κ, and P the spot of $(T, \bar{T})$ in $k(t)$ corresponding to $\bar{t}_1$. Then P is a regular local ring with a system $(\pi, \tau_1, \cdots, \tau_n)$ of parameters containing a prime element π of $\mathfrak{o}$ (cf. Proposition 6 in [13]). Let $\mathfrak{q}$ be the prime ideal $(\tau_1, \cdots, \tau_n)$ of P and put $Q=P_{\mathfrak{q}}$. Let t_1 be a point of T which corresponds to Q. Then t_1 is also a simple point of T, separably algebraic over k, and the specialization ring $\mathfrak{o}'=[t_1\xrightarrow{\mathfrak{o}}\bar{t}_1]$ is isomorphic to $P/\mathfrak{q}$, which is an unramified discrete valuation ring over $\mathfrak{o}$. Let $\mathfrak{p}'$ be the maximal ideal of $\mathfrak{o}'$. Then, in the same way as in the case of Proposition 2 in [11], we easily see that there are a $\mathfrak{p}'$-simple affine $\mathfrak{p}'$-variety $(V_1, \bar{V}_1)$ and a $\mathfrak{p}'$-birational correspondence f_1 between $(V, \bar{V})$ and $(V_1, \bar{V}_1)$ such that $f_1\circ f_t^{-1}$ is biregular at $\bar{a}$. Therefore we may put $(W, \bar{W})=(V_1, \bar{V}_1)$ and $F=f_1$.

If $\mathfrak{o}$ is complete, it is easily seen that we can apply Proposition 6 to $(V_1, \bar{V}_1)$, $(V, \bar{V})$ and f_1. Therefore there are a $\mathfrak{p}$-simple affine variety $(W, \bar{W})$ and a $\mathfrak{p}$-birational correspondence F between $(V, \bar{V})$ and $(W, \bar{W})$ such that, if $f_i (i=1, \cdots, s)$ are the transforms of f_1 by all the isomorphisms of the quotient field k' of $\mathfrak{o}'$ over k, $F\circ f_1^{-1}$ is biregular where all the $f_i\circ f_1^{-1}$ are defined. Then we easily see that $F\circ f_t^{-1}$ is biregular at $\bar{a}$.

Corollary. *Let $(T, \bar{T})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety, and let t and t' be independent generic points of T over k. Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and $(V_t, \bar{V}_t)$ a $\mathfrak{P}_t$-simple $\mathfrak{P}_t$-variety such that there is a $\mathfrak{P}_t$-birational correspondence f_t between $(V, \bar{V})$ and $(V_t, \bar{V}_t)$. Assume that $f_{t'}\circ f_t^{-1}$ is everywhere biregular. Then if $\bar{a}$ is any point of $\bar{V}_t$, there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension of k, and there are a $\mathfrak{p}'$-simple affine $\mathfrak{p}'$-variety $(W, \bar{W})$ and a $\mathfrak{p}'$-birational correspondence F between $(V, \bar{V})$ and $(W, \bar{W})$ such that $F\circ f_t^{-1}$ is biregular at $\bar{a}$. Moreover if $\mathfrak{o}$ is complete, $(W, \bar{W})$ and F are taken as a $\mathfrak{p}$-simple affine $\mathfrak{p}$-*

variety and a $\mathfrak{p}$-birational correspondence.

This corollary corresponds to Corollary of Proposition 2 in [11] and the proofs are quite similar. Therefore we omit the proof.

THEOREM 1. *Let $(T, \bar{T})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety. Let t and t' be two independent generic points of T over k. Denote by R_t and $\mathfrak{P}_t$ (resp. $R_{t'}$ and $\mathfrak{P}_{t'}$) the generating spot of $(T, \bar{T})$ in $k(t)$ (resp. $k(t')$) over $\mathfrak{p}$ and its maximal ideal. Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety, and $(V_t, \bar{V}_t)$ a $\mathfrak{P}_t$-simple $\mathfrak{P}_t$-variety such that there is a $\mathfrak{P}_t$-birational correspondence f_t between $(V, \bar{V})$ and $(V_t, \bar{V}_t)$. Assume that $f_{t'}\circ f_t^{-1}$ is everywhere biregular. Then there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension k' of k, and there are a $\mathfrak{p}'$-simple $\mathfrak{p}'$-variety $(W, \bar{W})$ and a $\mathfrak{p}'$-birational correspondence F between $(V, \bar{V})$ and $(W, \bar{W})$ such that $F\circ f_t^{-1}$ is a $\mathfrak{P}'_t$-birational biregular correspondence between $(V_t, \bar{V}_t)$ and $(W, \bar{W})$, denoting by $\mathfrak{P}'_t$ the maximal ideal of the generating spot of $(T, \bar{T})$ in $k'(t)$ over $\mathfrak{p}'$. Moreover if $\mathfrak{o}$ is complete, $(W, \bar{W})$ and F are taken as a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and a $\mathfrak{p}$-birational correspondence.*

This theorem is a generalization of Theorem 5 in [11]. The proof is also given similarly by using the above corollary and Corollary of Proposition 2 in [11].

§3. Construction of group $\mathfrak{p}$-varieties and transformation $\mathfrak{p}$-spaces.

In this section we shall construct a group $\mathfrak{p}'$-variety or a transformation $\mathfrak{p}'$-space attached to a pre-group $\mathfrak{p}$-variety or a pre-transformation $\mathfrak{p}$-space, where $\mathfrak{p}'$ is a prolongation of $\mathfrak{p}$. For this purpose we define a $\mathfrak{p}$-*chunk*. Let $(W, \bar{W})$ be a pre-transformation $\mathfrak{p}$-space with respect to a pre-group $\mathfrak{p}$-variety $(V, \bar{V})$. Then $(W, \bar{W})$ is called a $\mathfrak{p}$-*chunk of transformation space*, if, any point a of W (resp. $\bar{a}$ of $\bar{W}$) and a generic point x of V over $k(a)$ (resp. $\bar{x}$ of $\bar{V}$ over $\kappa(\bar{a})$), xa and $x^{-1}(xa)$ (resp. $\bar{x}\bar{a}$ and $\bar{x}^{-1}(\bar{x}\bar{a})$) are defined. Moreover if xa (resp. $\bar{x}\bar{a}$) is a generic point of W over $k(a)$ (resp. of $\bar{W}$ over $(\bar{a})$), $(W, \bar{W})$ is called a *homogeneous* $\mathfrak{p}$-*chunk*. A pre-group $\mathfrak{p}$-variety $(V, \bar{V})$ is called a *group* $\mathfrak{p}$-*chunk* if $(V, \bar{V})$ is a homogeneous $\mathfrak{p}$-chunk considered as a pre-transformation $\mathfrak{p}$-space with respect to itself, and if the inverse function ϕ is everywhere defined on $(V, \bar{V})$.

We first give the following

PROPOSITION 8. *Let $(W, \bar{W})$ be a pre-transformation $\mathfrak{p}$-space with respect to a pre-group $\mathfrak{p}$-variety $(V, \bar{V})$. Let Ω be the set of those points a on W or $\bar{a}$ on $\bar{W}$ such that xa and $x^{-1}(xa)$ (resp. $\bar{x}\bar{a}$ and $\bar{x}^{-1}(\bar{x}\bar{a})$) are defined for x generic over $k(a)$ on V (resp. $\bar{x}$ generic over $\kappa(\bar{a})$ on $\bar{V}$). Then Ω is an $\mathfrak{o}$-open subset of $(W, \bar{W})$, and all $\mathfrak{o}$-open subsets of Ω not disjoint with $\bar{W}$ are $\mathfrak{p}$-chunks. If $a\in\Omega\cap W$, we have $x^{-1}(xa)=a$, $k(x, a)=k(x, xa)$, and a is a point of the locus of xa over $k(a)$ on W. If $\bar{a}\in\Omega\cap\bar{W}$, we have $\bar{x}^{-1}(\bar{x}\bar{a})=\bar{a}$, $[(x, u)\xrightarrow{\mathfrak{o}}(\bar{x}, \bar{a})]=[(x, xa)\xrightarrow{\mathfrak{o}}(\bar{x}, \bar{x}\bar{a})]$, where*

x and u are independent generic points of V and W over k respectively, and $\bar{a}$ is a point of the locus of $\bar{x}\bar{a}$ over $\kappa(\bar{a})$ on $\bar{W}$.

This proposition is a generalization of Proposition 3 in [9], whose proof is also available in our case. Therefore we omit the proof.

COROLLARY. *Notations being as in Proposition 8, let Ω_h be the set of all the points a or $\bar{a}$ in Ω such that W (resp. $\bar{W}$) is the locus of xa over $k(a)$ (resp. $\bar{x}\bar{a}$ over $\kappa(\bar{a})$). Then Ω_h is an $\mathfrak{r}$-open subset of Ω, which is not empty if $(W, \bar{W})$ is pre-homogeneous and empty if $(W, \bar{W})$ is not pre-homogeneous. In the former case Ω_h and all $\mathfrak{r}$-open subsets of Ω_h not disjoint with $\bar{W}$ are homogeneous $\mathfrak{p}$-chunks, and if a, b (resp. $\bar{a}$, $\bar{b}$) are any two points of $\Omega_h \cap W$ (resp. $\Omega_h \cap \bar{W}$), there are two generic points x, y of V over $k(a, b)$ (resp. $\bar{x}$, $\bar{y}$ of $\bar{V}$ over $\kappa(\bar{a}, \bar{b})$) such that $xa = yb$ (resp. $\bar{x}\bar{a} = \bar{y}\bar{b}$).*

PROOF. If we show that Ω_h is $\mathfrak{r}$-open, the others are easily seen by the corollary of Proposition 3 in [9]. Since $\Omega_h \cap W$ is k-open and $\Omega_h \cap \bar{W}$ is κ-open, we have to show that the closure of $W - (\Omega_h \cap W)$ in $(W, \bar{W})$ is disjoint with $\Omega_h \cap \bar{W}$. Let a be a point of $W - (\Omega_h \cap W)$ and $\bar{a}$ a specialization of a over $\mathfrak{o}$. Let x be a generic point of V over $k(a)$ and $\bar{x}$ that of $\bar{V}$ over $\kappa(\bar{a})$. Then $\bar{x}\bar{a}$ is a specialization of xa over $R = [(a) \xrightarrow{\mathfrak{o}} (\bar{a})]$. Let S be a valuation ring of $k(a)$ dominating R such that the residue class field of S is algebraic over that of R (cf. Corollary 3 of Theorem 5 in p. 14 of [14]). Then $\bar{x}\bar{a}$ is a specialization of xa over S and hence we have $\dim_{k(a)}(xa) \geqslant \dim_{\kappa(\bar{a})}(\bar{x}\bar{a})$ (cf. Proposition 2 in [8]). This means that the locus of $\bar{x}\bar{a}$ over $\kappa(\bar{a})$ is different from $\bar{W}$. Therefore $\bar{a}$ is not in $\Omega_h \cap \bar{W}$. q.e.d.

From the above Proposition 8 and Corollary, we obtain easily the following proposition, which corresponds to Proposition 4 in [9].

PROPOSITION 9. *To every pre-transformation $\mathfrak{p}$-space (resp. pre-homogeneous $\mathfrak{p}$-space or pre-group $\mathfrak{p}$-variety), there is a $\mathfrak{p}$-birationally equivalent $\mathfrak{p}$-chunk (resp. homogeneous $\mathfrak{p}$-chunk or group $\mathfrak{p}$-chunk) which is an affine $\mathfrak{p}$-variety.*

PROPOSITION 10. *Let $(V, \bar{V})$ be a group $\mathfrak{p}$-chunk and $(W, \bar{W})$ a $\mathfrak{p}$-chunk of transformation $\mathfrak{p}$-space with respect to $(V, \bar{V})$. Let $\bar{s}$ be any point of $\bar{V}$ and $\bar{u}$ a generic point of $\bar{W}$ over $\kappa(\bar{s})$. Then the mapping $\bar{u} \rightarrow \bar{s}\bar{u}$ is a birational correspondence between $\bar{W}$ and itself. Moreover if $(\bar{a}, \bar{b})$ is a point of the graph of this mapping, $\bar{s}\bar{a}$ and $\bar{s}^{-1}\bar{b}$ are defined, and we have $[(x, u) \xrightarrow{\mathfrak{o}} (\bar{s}, \bar{a})] = [(x, xu) \xrightarrow{\mathfrak{o}} (\bar{s}, \bar{b})]$, where x and u are independent generic points of V and W over k.*

The proof is an adaptation of those of Propositions 5 and 6 in [9], and we omit it.

Now we construct a group $\mathfrak{p}'$-variety and a transformation $\mathfrak{p}'$-space attached to a group $\mathfrak{p}$-chunk and a $\mathfrak{p}$-chunk of transformation $\mathfrak{p}$-space, where $\mathfrak{p}'$ is a prolongation of $\mathfrak{p}$ in an extension of k. Let $(V, \bar{V})$ be a group $\mathfrak{p}$-chunk and

$(W, \bar{W})$ a $\mathfrak{p}$-chunk of transformation $\mathfrak{p}$-space with respect to $(V, \bar{V})$, both being assumed to be affine $\mathfrak{p}$-varieties. Let n, n' be the dimension of V, W, and take $N>4n$ and $>3n+n'$. Let $t_1, \cdots, t_N$ (resp. $\bar{t}_1, \cdots, \bar{t}_N$) be independent generic points of V over k (resp. of $\bar{V}$ over κ) and put $\mathfrak{O}_t = [(t_1, \cdots, t_N) \xrightarrow{\mathfrak{o}} (\bar{t}_1, \cdots, \bar{t}_N)]$. Let K_t and $\mathfrak{P}_t$ be the quotient field and the maximal ideal of $\mathfrak{O}_t$ respectively. Then $\mathfrak{O}_t$ is a discrete valuation ring and $\mathfrak{p}\mathfrak{O}_t = \mathfrak{P}_t$. Let u be a generic point of W over K_t and put $u_\alpha = t_\alpha u$ and $(S_\alpha, \bar{S}_\alpha) = (W, \bar{W})$ for each $\alpha = 1, \cdots, N$. Let $(T_{\alpha\beta}, \bar{T}_{\alpha\beta})$ be the locus of (u_α, u_β) over $\mathfrak{O}_t$. Then it is easy to see, by Proposition 6 in [9] and Proposition 10, that $(S_\alpha, \bar{S}_\alpha)$ and $(T_{\alpha\beta}, \bar{T}_{\alpha\beta})$ define a $\mathfrak{P}_t$-simple $\mathfrak{P}_t$-variety $(S_t, \bar{S}_t)$, which is $\mathfrak{P}_t$-birationally equivalent to $(W, \bar{W})$. In the same way we can construct a $\mathfrak{P}_t$-simple $\mathfrak{P}_t$-variety $(G_t, \bar{G}_t)$, which is $\mathfrak{P}_t$-birationally equivalent to $(V, \bar{V})$. Then we can see in the same way as in [9] that $(G_t, \bar{G}_t)$ is a group $\mathfrak{P}_t$-variety and $(S_t, \bar{S}_t)$ is a transformation $\mathfrak{P}_t$-space with respect to $(G_t, \bar{G}_t)$. Moreover if $(W, \bar{W})$ is homogeneous, $(S_t, \bar{S}_t)$ is a homogeneous $\mathfrak{P}_t$-space.

Let $(T, \bar{T})$ be the locus of $(t_1, \cdots, t_N)$ over $\mathfrak{o}$. Then R_t is no other than the generating spot over $\mathfrak{p}$ of the $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(T, \bar{T})$ in $k(t_1, \cdots, t_N)$. Then we can easily see, by the definitions and Corollary of Proposition 2, that $(G_t, \bar{G}_t)$ satisfies the conditions of Theorem 1, and that if $(S_t, \bar{S}_t)$ is a homogeneous $\mathfrak{P}_t$-space, $(S_t, \bar{S}_t)$ also satisfies the same conditions by Proposition 2. Therefore we have the following theorem, applying Proposition 9 and Theorem 1.

THEOREM 2. *(i) Let $(V, \bar{V})$ be a pre-group $\mathfrak{p}$-variety. Then there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension of k, and there is a $\mathfrak{p}'$-birationally equivalent group $\mathfrak{p}'$-variety $(G, \bar{G})$.*

(ii) Let $(W, \bar{W})$ be a pre-transformation $\mathfrak{p}$-space with respect to $(V, \bar{V})$. Then there is a prolongation $\mathfrak{p}''$ of $\mathfrak{p}$ in a separable extension of k, and there is a $\mathfrak{p}''$-birationally equivalent transformation $\mathfrak{p}''$-space $(S, \bar{S})$ with respect to $(G, \bar{G})$. If $(W, \bar{W})$ is a pre-homogeneous $\mathfrak{p}$-space, $(S, \bar{S})$ is a homogeneous $\mathfrak{p}''$-space and $\mathfrak{p}''$ is taken in a finite separable extension of k.

(iii) If $\mathfrak{o}$ is complete, $(G, \bar{G})$ is taken as a group $\mathfrak{p}$-variety, and $(S, \bar{S})$ is taken as a homogeneous $\mathfrak{p}$-space in the case where $(W, \bar{W})$ is pre-homogeneous.

THEOREM 3. *(i) Let $(G, \bar{G})$ be a group $\mathfrak{p}$-variety and G' a group variety defined over k such that there is a rational homomorphism λ of G onto G' defined over k. Then there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension of k, and there is a group $\mathfrak{p}'$-variety $(G_0, \bar{G}_0)$ such that G_0 is biregularly equivalent to G' by the rational isomorphism μ of G' onto G_0 and such that $\mu \cdot \lambda$ is a $\mathfrak{p}'$-rational homomorphism of $(G, \bar{G})$ onto $(G_0, \bar{G}_0)$. $(G_0, \bar{G}_0)$ is uniquely determined up to $\mathfrak{p}'$-birationally biregular isomorphism. If $\mathfrak{o}$ is complete, $(G_0, \bar{G}_0)$ is taken as a group $\mathfrak{p}$-variety.*

(ii) Let S be a homogeneous space, defined over k, with respect to G and $g(*, *)$ the normal law of composition on S. Then there is a prolongation $\mathfrak{p}''$ in a finite separable extension of k, and there is a homogeneous $\mathfrak{p}''$-space $(S_0, \bar{S}_0)$ with respect to $(G, \bar{G})$ such that S_0 is biregularly equivalent to S by the rational

mapping ν *of* S *onto* S_0 *and such that* $\nu(g(*, \nu^{-1}(*)))$ *is the normal law of composition on* $(S_0, \bar{S}_0)$. $(S_0, \bar{S}_0)$ *is uniquely determined up to* $\mathfrak{p}''$*-birationally biregular equivalence. Moreover if* a *is a point of* S *rational over* k, $(S_0, \bar{S}_0)$ *and* ν *can be taken such that* $\nu(g(*, a))$ *is a* $\mathfrak{p}''$*-rational mapping of* $(G, \bar{G})$ *onto* $(S_0, \bar{S}_0)$. *If* $\mathfrak{o}$ *is complete and if* S *has a point rational over* k, $(S_0, \bar{S}_0)$ *is taken as a homogeneous* $\mathfrak{p}$*-space.*

Proof. The existence is seen by Propositions 4, 5 and Theorem 2. Assume that $(G_1, \bar{G}_1)$ and μ_1 satisfy also the same conditions as $(G_0, \bar{G}_0)$ and μ. Let k' be the field in which $\mathfrak{p}'$ is defined. Let x be a generic point of G over k' and put $y_0=\mu\lambda(x)$ and $y_1=\mu_1\lambda(x)$. Then we have $y_1=\mu_1\cdot\mu^{-1}(y_0)$ and $k'(y_0)=k'(y_1)$. Let R be the generating spot over $\mathfrak{p}'$ of $(G, \bar{G})$ in $k'(x)$. Then by assumptions $R\cap k'(y_i)$ is the generating spot of $(G_i, \bar{G}_i)$ in $k'(y_i)$ for $i=0, 1$. This means that $\mu_1\cdot\mu^{-1}$ is $\mathfrak{p}'$-birational, since $R\cap k'(y_0)=R\cap k'(y_1)$. Therefore $(G_0, \bar{G}_0)$ is $\mathfrak{p}'$-birationally isomorphic to $(G_1, \bar{G}_1)$ by Corollary of Proposition 2. Similarly we see the uniqueness in the case of $(S_0, \bar{S}_0)$ by Proposition 2. q.e.d.

§4. Reduction of coset spaces of group varieties.

Now we give an application of Theorem 3 to the reduction of coset spaces of group varieties modulo $\mathfrak{p}$.

Proposition 11. *Let* $(G, \bar{G})$ *be a group* $\mathfrak{p}$*-variety, and* Z *a positive cycle rational over* k *on* G *such that its support* $|Z|$ *is a subgroup of* G. *Then the support* $|\bar{Z}|$ *of the cycle* $\bar{Z}$, *which is obtained from* Z *by the reduction modulo* $\mathfrak{p}$, *is also a subgroup of* $\bar{G}$.

Proof. Let $\bar{a}, \bar{b}$ be two points of $|\bar{Z}|$. Then it is easy to see that there are two points a, b of $|Z|$ such that $(\bar{a}, \bar{b})$ is a specialization of (a, b) over $\mathfrak{o}$, and that $\bar{a}\bar{b}^{-1}$ is a specialization of ab^{-1} over $\mathfrak{o}$. Therefore $\bar{a}\bar{b}^{-1}$ is in $|\bar{Z}|$. q.e.d.

Theorem 4. *Let* $(G, \bar{G})$ *be a group* $\mathfrak{p}$*-variety and* Z *a rational cycle over* k, *consisting of components with coefficients* 1, *such that its support* $|Z|$ *is a subgroup of* G. *Let* $\bar{Z}$ *be the cycle on* $\bar{G}$ *obtained from* Z *by the reduction modulo* $\mathfrak{p}$ *and* $\bar{Z}_1$ *the cycle on* $\bar{G}$ *with coefficients* 1 *consisting of all components of* $\bar{Z}$. *Then there is a prolongation* $\mathfrak{p}'$ *of* $\mathfrak{p}$ *in a finite separable extension of* k, *and there is a homogeneous* $\mathfrak{p}'$*-space* $(H, \bar{H})$ *such that* H *is biregularly equivalent to the coset space* G/Z, *and such that there is a purely inseparable mapping* λ *of* $\bar{G}/\bar{Z}_1$ *onto* $\bar{H}$. $\bar{G}/\bar{Z}_1$ *is biregularly equivalent to* $\bar{H}$ *by* λ, *if and only if* $\bar{Z}=\bar{Z}_1$. *Moreover if* $\mathfrak{o}$ *is complete,* $(H, \bar{H})$ *is taken as a homogeneous* $\mathfrak{p}$*-space.*

Proof. Let F be the rational mapping of G onto G/Z defined over k such that $F(x)=xa$ for a generic point x of G over k and a rational point a of G/Z over k(cf. Proposition 2 in [10]). Then, by Theorem 3, there is a homogeneous $\mathfrak{p}'$-space $(H, \bar{H})$ and a biregular birational mapping ν of G/Z onto H, where $\mathfrak{p}'$ is a prolongation of $\mathfrak{p}$ in a finite separable extension k' of k. Then H may be considered as the coset space G/Z defined over k' with the natural mapping $F_0=\nu\cdot F$, which can be assumed to be a $\mathfrak{p}'$-rational mapping of $(G, \bar{G})$ onto $(H, \bar{H})$. Let x be a generic point of G over k' and t the image of x by F_0. Similarly let $\bar{x}$ be a generic point of $\bar{G}$ over κ' and $\bar{t}$ the image of $\bar{x}$ by F_0. Then the locus $(\Gamma, \bar{\Gamma})$ of (x, t) over $\mathfrak{o}'$ on $(G\times H, \bar{G}\times\bar{H})$ is the graph of F_0 and is a $\mathfrak{p}'$-simple subvariety of $(G\times H, \bar{G}\times\bar{H})$. The intersection cycle $\Gamma\cdot(G\times t)$ is defined on $G\times H$ and equal to $xZ\times t$ (cf. the proof of Proposition 2 in [10]). Since F_0 is $\mathfrak{p}'$-rational, $\bar{\Gamma}$ is the locus of $(\bar{x}, \bar{t})$ over κ' and hence $\bar{\Gamma}\cdot(\bar{G}\times\bar{t})$ is also defined on $\bar{G}\times\bar{H}$. By Theorems 17 and 18 in [8] we easily see $\bar{x}\bar{Z}\times\bar{t}=\bar{\Gamma}\cdot(\bar{G}\times\bar{t})$. This means that $\bar{x}\bar{Z}$ is a prime rational cycle over $\kappa'(\bar{t})$ and for any point $\bar{s}$ in $|\bar{Z}|=|\bar{Z}_1|$ and a generic point $\bar{x}'$ of $\bar{G}$ over $\kappa'(\bar{s})$, $F_0(\bar{x}')=F_0(\bar{x}'\bar{s})$. Therefore there is a rational mapping λ of $\bar{G}/\bar{Z}_1$ onto $\bar{H}$, which is everywhere defined on $\bar{G}/\bar{Z}_1$. Let F_1 be the natural mapping of $\bar{G}$ onto $\bar{G}/\bar{Z}_1$. Then F_0 is equal to $\lambda\cdot F_1$ on $\bar{G}$ and if Γ_1 is the graph of F_1, $\Gamma_1\cdot(\bar{G}\times\bar{t}_1)$ is equal to $\bar{x}\bar{Z}_1\times\bar{t}_1$, where $\bar{t}_1=F_1(\bar{x})$. Let $\bar{t}_2$ be a point of $\bar{G}/\bar{Z}_1$, whose image by λ is $\bar{t}$. Then there is a point $\bar{x}'$ in $\bar{G}$, such that $\bar{x}'\bar{Z}_1\times\bar{t}_2=\Gamma_1\cdot(\bar{G}\times\bar{t}_2)$. Since $F_0(\bar{x}')=\lambda\cdot F_1(\bar{x}')=\lambda(\bar{t}_2)=\bar{t}$, $\bar{x}'\bar{Z}$ must be $\bar{x}\bar{Z}$ and hence $\bar{x}'\bar{Z}_1$ is equal to $\bar{x}\bar{Z}_1$. This means that $\bar{t}_1=\bar{t}_2$. Therefore λ is purely inseparable. The assertion on biregularity is easily seen from this fact. The last assertion is seen by Theorem 3. q.e.d.

COROLLARY. *Notations being as in Theorem* 4, *assume that the characteristic of* κ *is zero. Then the cycle* $\bar{Z}$ *obtained from* Z *by the reduction modulo* $\mathfrak{p}$ *consists of components with coefficients* 1.

§5. Reduction of generalized Jacobian varieties.

First we shall consider the reduction of a quotient variety of a variety V by a finite group of automorphisms on V.

Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety and f a $\mathfrak{p}$-birational biregular mapping of $(V, \bar{V})$ onto itself. Then we say that f is a $\mathfrak{p}$-automorphism on $(V, \bar{V})$, and f defines naturally an automorphism on $\bar{V}$.

PROPOSITION 12. *Let* $(V, \bar{V})$ *be a* $\mathfrak{p}$-*simple affine* $\mathfrak{p}$-*variey and* $\mathfrak{g}$ *a finite group of* $\mathfrak{p}$-*automorphisms on* $(V, \bar{V})$. *Let* $\bar{\mathfrak{g}}$ *be the finite group of automorphisms on* $\bar{V}$ *which are defined by elements of* $\mathfrak{g}$. *Then there is a* $\mathfrak{p}$-*simple* $\mathfrak{p}$-*variety* $(W, \bar{W})$ *such that* W *is the quotient variety* $V/\mathfrak{g}$ *of* V *by* $\mathfrak{g}$ *and such that* $\bar{W}$ *is the image of the quotient variety* $\bar{V}/\bar{\mathfrak{g}}$ *of* $\bar{V}$ *by* $\bar{\mathfrak{g}}$. *Moreover* $\bar{W}$ *is identified with* $\bar{V}/\bar{\mathfrak{g}}$ *if and only if the order of* $\mathfrak{g}$ *is equal to that of* $\bar{\mathfrak{g}}$.

PROOF. Let x be a generic point of V over k and A the affine ring $\mathfrak{o}[x]$ over $\mathfrak{o}$. Let $A^{\mathfrak{g}}$ be the subring of A which consists of the elements of A fixed by $\mathfrak{g}$. Then it is easy to see that $A^{\mathfrak{g}}$ is also an affine ring over $\mathfrak{o}$ (cf. the proof of Proposition 18, Chap. III in [7]). Let $(W, \bar{W})$ be the affine $\mathfrak{p}$-variety defined by $A^{\mathfrak{g}}$, which is $\mathfrak{p}$-simple, since $\mathfrak{p}A^{\mathfrak{g}}$ is a prime ideal of $A^{\mathfrak{g}}$. The above cited proposition also shows that W is no other than $V/\mathfrak{g}$. On the other hand $\bar{V}$ and

$\bar{W}$ are defined by the affine rings $A/\mathfrak{p}A$ and $A^{\mathfrak{g}}/\mathfrak{p}A^{\mathfrak{g}}$ over κ respectively, and $A^{\mathfrak{g}}/\mathfrak{p}A^{\mathfrak{g}}$ is contained in $(A/\mathfrak{p}A)^{\bar{\mathfrak{g}}}$. Therefore there is an everywhere regular mapping of $\bar{V}/\bar{\mathfrak{g}}$ onto $\bar{W}$. Moreover we have $[V: W]\cdot\mu(W; \bar{W})=\mu(V; \bar{V})[\bar{V}: \bar{W}]$ by Theorem 12 in [8]. Since $(V, \bar{V})$ and $(W, \bar{W})$ are both $\mathfrak{p}$-simple, this means $[V: W]=[\bar{V}: \bar{W}]$. Therefore we have the last assertion. q.e.d.

Proposition 13. *Let $(V, \bar{V})$ be a $\mathfrak{p}$-simple $\mathfrak{p}$-variety such that every finite subset of $(V, \bar{V})$ is contained in an affine $\mathfrak{o}$-open subset. Then there is a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(W_n, \bar{W}_n)$, for any positive integer n, such that W_n (resp. $\bar{W}_n$) is identified with the n-fold symmetric product $V^{(n)}$ of V (resp. $\bar{V}^{(n)}$ of $\bar{V}$).*

Proof. We can easily generalize Proposition 12 for the case where $(V, \bar{V})$ satisfies the same condition as in Proposition 13 (cf. Proposition 19, Chap. III in [7]). Then our assertion is a direct consequence of this fact and the definition of the symmetric products of a variety. q.e.d.

Let $(C, \bar{C})$ be a $\mathfrak{p}$-simple projective $\mathfrak{p}$-variety of dimension 1. We assume that all the singular points of C are rational over k, and that those of $\bar{C}$ are rational over κ. Let x be a generic point of C over k and $\bar{x}$ that of C over κ. Let $\mathfrak{A}$ be a semilocal ring in $k(x)$ in the sense of Rosenlicht [5] such that the places of $\mathfrak{A}$ include all the places of $k(x)$ which are not absolutely simple. Similarly let $\mathfrak{B}$ be a semilocal ring in $\kappa(\bar{x})$ such that the places of $\mathfrak{B}$ include all the places of $\kappa(\bar{x})$ which are not absolutely simple. Then we shall say that *$\mathfrak{A}$-linear equivalence is preserved into $\mathfrak{B}$-linear equivalence under the reduction modulo* $\mathfrak{p}$, if any rational divisor on $\bar{C}$ over κ, which is obtained from a rational divisor on C over k linearly equivalent to zero in the sense of $\mathfrak{A}$-equivalence, is linearly equivalent to zero in the sense of $\mathfrak{B}$-equivalence (cf. §2 in [5]). Let $(\mathfrak{o}', \mathfrak{p}')$ be a prolongation of $(\mathfrak{o}, \mathfrak{p})$ in an extension k' of k. Then we may assume that k' and $k(x)$ (resp. $\kappa'=\mathfrak{o}'/\mathfrak{p}'$ and $\kappa(\bar{x})$) are free over k (resp. κ). We denote by $k'\mathfrak{A}$ (resp. $\kappa'\mathfrak{B}$) the extension of $\mathfrak{A}$ to $k'(x)$ (resp. $\mathfrak{B}$ to $\kappa'(\bar{x})$)[11]. Then we shall say that *$\mathfrak{A}$-linear equivalence is preserved separably into $\mathfrak{B}$-linear equivalence under the reduction modulo* $\mathfrak{p}$, if the following conditions are satisfied; let k' be any separable extension of k and $(\mathfrak{o}', \mathfrak{p}')$ any prolongation of $(\mathfrak{o}, \mathfrak{p})$ in k'. Then $k'\cdot\mathfrak{A}$-linear equivalence is preserved into $\kappa'\mathfrak{B}$-linear equivalence under the reduction modulo $\mathfrak{p}'$.

In the following we shall assume that $\mathfrak{A}$-linear equivalence is preserved separably into $\mathfrak{B}$-linear equivalence under the reduction modulo $\mathfrak{p}$ and that $\mathfrak{A}$-genus g of C is equal to $\mathfrak{B}$-genus of $\bar{C}$. Moreover we assume that there is a simple point x_0 on C, rational over k, whose specialization over $\mathfrak{o}$ is a simple point $\bar{x}_0$ on $\bar{C}$.

By Proposition 13 there is a $\mathfrak{p}$-simple $\mathfrak{p}$-variety $(W, \bar{W})$ such that W (resp. $\bar{W}$) is identified with the g-fold symmetric product of C (resp. $\bar{C}$). A positive divisor of degree g on C (resp. C) is naturally identified with a point of W

11) For the definition, see §3 in [5].

(resp. $\bar{W}$). Let $x_1, \cdots, x_g, y_1, \cdots, y_g$ be independent generic points of C over k and put $X=\sum_{i=1}^{g}(x_i)$ and $Y=\sum_{i=1}^{g}(y_i)$. Then it is known that there is only one positive divisor Z of degree g such that Z is equivalent to $X+Y-g\cdot(x_0)$ in the sense of $\mathfrak{A}$-linear equivalence, and that the rational mapping f of $W\times W$ onto W, which maps (X, Y) onto Z, defines a structure of a pregroup variety on W (cf. [5] and [6]). Let $\bar{x}_1, \cdots, \bar{x}_g, \bar{y}_1, \cdots, \bar{y}_g$ be independent generic points of $\bar{C}$ over κ, and put $\bar{X}=\sum_{i=1}^{g}(\bar{x}_i)$ and $\bar{Y}=\sum_{i=1}^{g}(\bar{y}_i)$. Then $(\bar{X}, \bar{Y})$ is a specialization of (X, Y) over $\mathfrak{o}$, whose specialization ring will be denoted by S. Let $\bar{Z}$ be a specialization of Z over S. Then $\bar{Z}$ is equivalent to $\bar{X}+\bar{Y}-g\cdot(\bar{x}_0)$ in the sense of $\mathfrak{B}$-linear equivalence from the assumptions on $\mathfrak{A}$ and $\mathfrak{B}$. On the other hand $\mathfrak{A}$-genus g on C is equal to $\mathfrak{B}$-genus of $\bar{C}$ and hence $\bar{Z}$ is uniquely determined. We have $\kappa(\bar{X}, \bar{Y})=\kappa(\bar{X}, \bar{Z})=\kappa(\bar{Y}, \bar{Z})$. From these facts we easily see that S dominates the specialization ring $[Z\xrightarrow{\mathfrak{o}}\bar{Z}]$. Similarly we see that $[(X, Z)\xrightarrow{\mathfrak{o}}(\bar{X}, \bar{Z})]$ dominates $[Y\xrightarrow{\mathfrak{o}}\bar{Y}]$ and that $[(Y, Z)\xrightarrow{\mathfrak{o}}(\bar{Y}, \bar{Z})]$ dominates $[Z\xrightarrow{\mathfrak{o}}\bar{X}]$. This means that f defines on $(W, \bar{W})$ a structure of a pre-group $\mathfrak{p}$-variety. Then by Theorem 2 there is a group $\mathfrak{p}'$-variety $(J, \bar{J})$ such that J (resp. $\bar{J}$) is biregularly isomorphic to the generalized Jacobian variety[12] of C (resp. $\bar{C}$) corresponding to $\mathfrak{A}$-linear (resp. $\mathfrak{B}$-linear) equivalence relation, where $\mathfrak{p}'$ is a prolongation of $\mathfrak{p}$ in a finite separable extension of k. Let F be the $\mathfrak{p}'$-birational correspondence of $(W, \bar{W})$ into $(J, \bar{J})$, which transforms the structure of the pre-group $\mathfrak{p}$-variety on $(W, \bar{W})$ into that of $(J, \bar{J})$.

Let $x_1, \cdots, x_g$ (resp. $\bar{x}_1, \cdots, \bar{x}_g$) be independent generic points of C over $k(x)$ (resp. $\bar{C}$ over $\kappa(\bar{x})$) and put $X=\sum_{i=1}^{g}(x_i)$ (resp. $\bar{X}=\sum_{i=1}^{g}(\bar{x})$). Then there is only one positive divisor Y on C (resp. $\bar{Y}$ on $\bar{C}$) such that Y (resp. $\bar{Y}$) is a generic point of W over $k'(x)$ (resp. $\bar{W}$ over $\kappa'(\bar{x})$) and is equivalent to $X+(x)-(x_0)$ (resp. $\bar{X}+(\bar{x})-(\bar{x}_0)$) in the sense of $\mathfrak{A}$-linear (resp. $\mathfrak{B}$-linear) equivalence. Then it is known that $F(Y)-F(X)$ (resp. $F(\bar{Y})-F(\bar{X})$) is a rational point of J over $k'(x)$ (resp. $\bar{J}$ over $\kappa'(\bar{X})$), and that the rational mapping ϕ of C into J (resp. $\bar{\phi}$ of $\bar{C}$ into $\bar{J}$), which maps x to $F(Y)-F(X)$ (resp. $\bar{x}$ onto $F(\bar{Y})-F(\bar{X})$), is a canonical mapping of C into J (resp. $\bar{C}$ into $\bar{J}$). Let $\mathfrak{D}'$ be the generating spot of $(C, \bar{C})$ in $k'(x)$. Since $(\bar{X}, \bar{Y})$ is a specialization of (X, Y) over $\mathfrak{D}'$, $(F(\bar{X}), F(\bar{Y}))$ is a specialization of $(F(X), F(Y))$ over $\mathfrak{D}'$ and hence $F(\bar{Y})-F(\bar{X})$ is a specialization of $F(Y)-F(X)$ over $\mathfrak{D}'$. This means that ϕ is a $\mathfrak{p}'$-rational mapping of $(C, \bar{C})$ into $(J, \bar{J})$ and that $\bar{\phi}$ is defined from ϕ by the reduction modulo $\mathfrak{p}'$. Therefore we have the following

THEOREM 5. *Let $(C, \bar{C})$ be a $\mathfrak{p}$-simple projective $\mathfrak{p}$-variety of dimension 1 such that all the singular points on C (resp. on $\bar{C}$) are rational over k (resp. κ). Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be a semilocal ring of a function field $k(x)$ of C over k (resp. $\kappa(\bar{x})$ of $\bar{C}$ over κ), whose places include all the places in $k(x)$ (resp. $\kappa(\bar{x})$) which are not*

12) For the definition, see [6].

absolutely simple. Assume that $\mathfrak{A}$-linear equivalence is preserved separably into $\overline{\mathfrak{B}}$-linear equivalence under the reduction modulo $\mathfrak{p}$, and that $\mathfrak{A}$-genus of C is equal to $\overline{\mathfrak{B}}$-genus of $\bar{C}$. Then there is a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ in a finite separable extension of k, and there are a group $\mathfrak{p}'$-variety $(J, \bar{J})$ and a $\mathfrak{p}'$-rational mapping ϕ of $(C, \bar{C})$ into $(J, \bar{J})$, such that J (resp. $\bar{J}$) is the generalized Jacobian variety of C (resp. $\bar{C}$) corresponding to $\mathfrak{A}$-linear (resp. $\overline{\mathfrak{B}}$-linear) equivalence relation and such that ϕ (resp. $\bar{\phi}$) is a canonical mapping of C into J (resp. $\bar{C}$ into $\bar{J}$).

COROLLARY. *Let $(C, \bar{C})$ be a $\mathfrak{p}$-simple projective $\mathfrak{p}$-variety of dimension 1 such that $\bar{C}$ is non-singular. Then C is also non-singular, and there are a group $\mathfrak{p}'$-variety $(J, \bar{J})$ and a $\mathfrak{p}'$-rational mapping ϕ of $(C, \bar{C})$ into $(J, \bar{J})$ such that J (resp. $\bar{J}$) is the Jacobian variety of C (resp. $\bar{C}$) and such that ϕ (resp. $\bar{\phi}$) is a canonical mapping of C into J (resp. $\bar{C}$ into $\bar{J}$), where $\mathfrak{p}'$ is a prolongation of $\mathfrak{p}$ in a finite separable extension k.*

PROOF. If a is a singular point on C, any specialization of a on $\bar{C}$ over $\mathfrak{o}$ is also singular (cf. e.g. Proposition 6 in [13]). Since $(C, \bar{C})$ is $\mathfrak{p}$-complete, this means that C is non-singular. Now we apply Theorem 5. The condition on linear equivalence is clearly satisfied (cf. Theorem 20 in [8]). On the other hand the genus of C is equal to that of $\bar{C}$ by Theorem 3 in [2]. Therefore we have our assertion. q.e.d.

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