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# On Loop Extensions of Groups and M-cohomology Groups

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**1. Introduction.** The extension problem of groups has been investigated by many authors. Especially S. Eilenberg and S. MacLane studied this problem by using the cohomology theory in [8](1) and [9]. R. H. Bruck studied the problem of the Moufang loop-extensions of an abelian group by a Moufang loop, the central Moufang extensions, by using the factor set in [6].

In this paper, we shall investigate such a loop extension $L$ of a group $G$ by a group $\Gamma$ as satisfies the following conditions: i) $L$ is a Bol-Moufang loop (i.e. $a[b(ac)]=[a(ba)]c$ in $L$), ii) $L$ has $G$ as a normal subgroup in its nucleus, iii) $L/G\cong\Gamma$. Let us call the above extension the *BM-extension of G by Γ*. For this purpose of ours, we shall construct a new cohomology group (named *M-cohomology group*). This new $M$-cohomology group will enable us to discuss our extension problem in parallel with the ordinary extension problem of groups, and to make the Bruck's result clearer, when $\Gamma$ is a group. Moreover, in the same way as that used in the group extension, we shall be able to treat the case where the group $G$ is non-abelian.

In §§2 and 3, we shall study: (i) the necessary and sufficient conditions for the existence of the $BM$-extension of $G$ by $\Gamma$; and (ii) the conditions for two $BM$-extensions to be equivalent. Two $BM$-extensions $L_1$ and $L_2$ are equivalent if there exists an isomorphism between them, under which $G$ and each coset of $L_1/G$ and $L_2/G$ are invariant. In order to classify the $BM$-extensions, §4 will be devoted to the construction of the $M$-cohomology group $H^{*n}$ which corresponds to the ordinary cohomology group in the group extensions. By using our $M$-cohomology group, we shall classify all $BM$-extensions of an abelian group $G$ by a group $\Gamma$ in §5. §6 is concerned with the properties of the element of $H^{*3}$ which is determined by the homomorphism $\theta:\Gamma\rightarrow \mathrm{Aut}(G)/\mathrm{In}(G)$ induced by the $BM$-extension. By using these properties, in §7 we shall study the necessary and sufficient condition for the existence of the $BM$-extension of a non-abelian group $G$ by a group $\Gamma$, and when such $BM$-extension exists, we shall show that all non-equivalent $BM$-extensions correspond one-to-one to the elements of the second $M$-cohomology group $H^{*2}$ $(\Gamma, C)$, ($C$ is the center of $G$). The last §8 will contain some properties of the third $M$-cohomology group.

**2. *BM*-extensions of groups.** In this section, we obtain the necessary and sufficient conditions for the existence of the $BM$-extension $L$ of a group

(1) The number in the brackets refers to the references at the end of this paper.

$G$ by a group $\Gamma$. Let $G$ and $\Gamma$ be two given groups whose elements are denoted by the letters $a, b, c, \cdots$ and $\alpha, \beta, \gamma, \cdots$ respectively. Now, let $L$ be a $BM$-extension of $G$ by $\Gamma$ and let us choose the representatives $g_\alpha$ of the cosets of $G$ in $L$ such that the unit element 1 of $L$ is the representative of $G$. Then there are the elements $f(\alpha, \beta)$ of $G$ such that

$$(1) \qquad g_\alpha g_\beta = g_{\alpha\beta} f(\alpha, \beta),$$

where $f(\alpha, \varepsilon) = e = f(\varepsilon, \beta)$, ($e$ and $\varepsilon$ are the unit elements of $G$ and $\Gamma$ respectively).

On the other hand, the product of two arbitrary elements $g_\alpha a$ and $g_\beta b$ of $L$ is:

$$(2) \qquad (g_\alpha a)(g_\beta b) = g_{\alpha\beta} f(\alpha, \beta)(aTg_\beta)b,$$

where $T_x$ is an inner mapping of $L$ defined by $x(yT_x) = yx$, $(x, y \in L)$ and it is an inner automorphism of $G$ when $x$ and $y$ belong to $G$.

Since $G$ is contained in the nucleus of $L$, where the nucleus is the set of all elements $a$ satisfying the conditions $a(xy) = (ax)y$, $x(ay) = (xa)y$ and $x(ya) = (xy)a$; $a, x, y \in L$, it follows that if $T_x$ is restricted in $G$, it is an automorphism of $G$. Further, for an arbitrary element $a$ of $G$ we have:

$$(3) \qquad aTg_{\alpha\beta}T_{f(\alpha,\beta)} = aTg_\alpha Tg_\beta.$$

Since $L$ is a Bol-Moufang loop, so it holds that $g_\alpha[g_\beta(g_\alpha g_\gamma)] = [g_\alpha(g_\beta g_\alpha)]g_\gamma$. From this relation, we have:

$$(4) \qquad f(\alpha, \beta\alpha\gamma)f(\beta, \alpha\gamma)f(\alpha, \gamma) = f(\alpha\beta\alpha, \gamma)\big(f(\alpha, \beta\alpha)Tg_\gamma\big)\big(f(\beta, \alpha)Tg_\gamma\big).$$

Thus, to a given $BM$-extension, there correspond the elements $f(\alpha, \beta)$ of $G$ and a system of automorphisms $Tg_\alpha$ of $G$ which satisfy the conditions (3), (4) and $f(\alpha, \varepsilon) = e = f(\varepsilon, \beta)$. A system of all elements $f(\alpha, \beta)$ of $G$ is called an *M-factor set*.

Conversely, let us assume that in a group $G$ a system of elements $f(\alpha, \beta)$; $\alpha, \beta \in \Gamma$ is chosen and that every element $\alpha$ of $\Gamma$ is associated with an automorphism $T_\alpha$ of $G$ for which the conditions (3), (4) and $f(\alpha, \varepsilon) = e = f(\varepsilon, \beta)$ are satisfied. Then we may show that there exists a $BM$-extension of $G$ by $\Gamma$ for which elements $f(\alpha, \beta)$ are the $M$-factor set and the set $\{T_\alpha\}$ is a system of automorphisms in the above sense.

Now let $\bar{L}$ be the set of all pairs $(\alpha, a)$, $\alpha \in \Gamma$, $a \in G$. In $\bar{L}$, we define the equality and the multiplication as follows: (i) $(\alpha, a) = (\beta, b)$ if and only if $\alpha = \beta$ and $a = b$; (ii) $(\alpha, a)(\beta, b) = \big(\alpha\beta, f(\alpha, \beta)(aT_\beta)b\big)$. Then $\bar{L}$ is a loop with the unit element $(\varepsilon, e)$, and further it is a Bol-Moufang loop by the conditions (3) and (4).

Next, if we denote the set of elements $\bar{a} = (\varepsilon, a)(a \in G)$ by $\bar{G}$, then it holds that $\bar{G} \cong G$. Further the correspondence carrying the element $(\alpha, a)$ of $\bar{L}$ into the element $\alpha$ of $\Gamma$ is a homomorphism of $\bar{L}$ onto $\Gamma$ and its kernel is the subgroup $\bar{G}$. So, $\bar{G}$ is a normal subgroup of $\bar{L}$ and it holds that $\bar{L}/\bar{G}\cong\Gamma$. By the simple calculation, we can see that $\bar{G}$ is contained in the nucleus of $\bar{L}$. Therefore $\bar{L}$ is a $BM$-extension of $\bar{G}$ by $\Gamma$.

Next, if we use the notation $\bar{g}_\alpha=(\alpha, e)$, it holds $\bar{g}_\alpha\bar{a}=(\alpha, a)$. If we choose $\bar{g}_\alpha$ as the representatives of $\bar{G}$ in $\bar{L}$, then it follows that $\bar{g}_\alpha\bar{g}_\beta=\bar{g}_{\alpha\beta}\overline{f(\alpha, \beta)}$, $\bar{a}\bar{g}_\alpha=\bar{g}_\alpha\overline{aT_\alpha}$. So, if we identify $\bar{G}$ with $G$, the $M$-factor set of $\bar{L}$ is $f(\alpha, \beta)$, and a system of automorphisms is $\{T_\alpha\}$. Thus $\bar{L}$ is a required $BM$-extension of $G$ by $\Gamma$.

Let $L$ be the $BM$-extension whose $M$-factor set is $f(\alpha, \beta)$ and the system of automorphisms is $\{T_\alpha\}$. If we construct the above $BM$-extension $\bar{L}$ using this $M$-factor set $f(\alpha, \beta)$ and the system of automorphisms $T_\alpha$, two $BM$-extensions $L$ and $\bar{L}$ are isomorphic by the correspondence which associates $g_\alpha a$ with $\bar{g}_\alpha\bar{a}$. Further, each element of the subgroup $G$ and each coset of $L/G$ and $\bar{L}/G$ are invariant under this isomorphism.

Now, we define the equivalence of $BM$-extensions; that is, two $BM$-extensions $L$ and $L'$ are equivalent if there exists an isomorphism between $L$ and $L'$ that on $G$ coincides with identity automorphism and that maps onto each other the cosets of $G$ corresponding to one and the same element of $\Gamma$.

Then, we have the following results:

PROPOSITION 1. *For each BM-extension of a group $G$ by a group $\Gamma$, if we choose the representatives $g_\alpha$ such that $g_\varepsilon=1$ there correspond the elements $f(\alpha,\beta)$ of $G$ and a system of automorphisms $T_\alpha$ which satisfy the conditions*:

$$aT_{\alpha\beta}T_{f(\alpha,\beta)}=aT_\alpha T_\beta, \qquad a\in G,\quad \alpha,\beta\in\Gamma,$$
$$f(\alpha, \beta\alpha\gamma)f(\beta, \alpha\gamma)f(\alpha, \gamma)=f(\alpha\beta\alpha, \gamma)\big(f(\alpha, \beta\alpha)T_\gamma\big)\big(f(\beta, \alpha)T_\gamma\big), \qquad \alpha,\beta,\gamma\in\Gamma,$$
$$f(\alpha, \varepsilon)=e=f(\varepsilon, \beta). \qquad \alpha,\beta\in\Gamma.$$

*Conversely, to every system of elements $f(\alpha, \beta)$ and of automorphisms $T_\alpha$ of $G$ which satisfy the above conditions, there corresponds a BM-extension of $G$ by $\Gamma$ which is uniquely determined up to equivalence.*

**3. The necessary and sufficient conditions for two *BM*-extensions to be equivalent.** As we have seen in the previous section, if an $M$-factor set and a system of automorphisms are given, an equivalent class of $BM$-extension is uniquely determined. But, for an equivalent class of $BM$-extensions, $M$-factor set is not unique. So, we inquire the necessary and sufficient conditions for two $BM$-extensions to be equivalent.

Now, let $L$ and $L'$ be two equivalent $BM$-extensions of $G$ by $\Gamma$ which are given by the $M$-factor sets $f(\alpha, \beta)$ and $f'(\alpha, \beta)$, and the systems of automorphisms $T_\alpha$ and $T'_\alpha$ for the choice of representatives $g_\alpha(g_\varepsilon=1)$ and $g'_\alpha(g'_\varepsilon=1)$ respectively. If $\varphi$ is the isomorphism which gives the equivalence between $L$ and $L'$, there exist the elements $c_\alpha(\alpha\in\Gamma)$ such that $\varphi(g_\alpha')=g_\alpha c_\alpha(c_\varepsilon=e)$. And it holds that $\varphi(g'_\alpha g'_\beta)=g_{\alpha\beta}f(\alpha,\beta)\ (c_\alpha T_\beta)c_\beta$. On the other hand, $\varphi(g'_\alpha g'_\beta)=\varphi(g'_{\alpha\beta}f'(\alpha,\beta))=g_{\alpha\beta}c_{\alpha\beta}f'(\alpha,\beta)$. So, we have

$$f'(\alpha,\beta)=c_{\alpha\beta}^{-1}f(\alpha,\beta)\,(c_\alpha T_\beta)c_\beta. \tag{5}$$

Further, it holds that

$$T_\alpha'=\varphi(aTg'_\alpha)=aT_{\varphi(g'_\alpha)}=aT_\alpha Tc_\alpha. \tag{6}$$

Conversely, if two extensions $L$ and $L'$ are given and if we can find their $M$-factor sets and systems of automorphisms which satisfy the conditions (5) and (6), $L$ and $L'$ are equivalent under the isomorphism $g_\alpha c_\alpha a \leftrightarrow g'_\alpha a$. Thus, we have the proposition:

PROPOSITION 2. *Two BM-extensions $L$ and $L'$ of a group $G$ by a group $\Gamma$ which are given by the M-factor sets $f(\alpha,\beta)$ and $f'(\alpha,\beta)$ and the automorphisms $T_\alpha$ and $T'_\alpha$, respectively, are equivalent if and only if every element $\alpha$ of $\Gamma$ can be associated with an element $c_\alpha (c_\varepsilon=e)$ of $G$ such that the following conditions are satisfied:*

$$f'(\alpha,\beta)=c_{\alpha\beta}^{-1}f(\alpha,\beta)\,(c_\alpha T_\beta)c_\beta,$$
$$T'_\alpha=T_\alpha Tc_\alpha.$$

**4. *M*-cohomology groups.** We proceed to the study of the set of all $BM$-extensions. In the extension problem of groups, the second cohomology group is used in order to classify the set of non-equivalent classes of the extensions. In order to classify the set of all $BM$-extensions, we construct a new cohomology group which corresponds to the ordinary cohomology group. It is named *M-cohomology group.*

Now, let $G$ be an abelian group and $\Gamma$ be a group. And suppose that a function on $G\times\Gamma$ into $G$ is given, written $a\alpha$ for $a\in G$, $\alpha\in\Gamma$, such that

$$(a+b)\alpha=a\alpha+b\alpha,\quad a(\alpha\beta)=(a\alpha)\beta,\quad a\varepsilon=a;\qquad a,b\in G,\quad \alpha,\beta\in\Gamma.$$

Every function $f(\alpha_1,\alpha_2,\cdots,\alpha_n)$ of $n$ elements of $\Gamma$ with values in $G$ is called an $n$-dimensional cochain. In particular, the zero-dimensional cochains are the elements of $G$. The set of all $n$-dimensional cochains is a group $C^n(\Gamma,G)$ in the ordinary sense. With every $n$-dimensional cochain $f$, we associate an $(n+1)$-dimensional cochain $\partial f$ called the *M-coboundary* of the cochain $f$ and defined as follows[2]:

$$\begin{cases}(\partial f)(\alpha)=a-a\alpha,\\ (\partial f)(\alpha_1,\alpha_2)=f(\alpha_2)-f(\alpha_1\alpha_2)+f(\alpha_1)\alpha_2,\end{cases}$$

(2) In the right side of the formula of the case $n\geqq 2$ in this definition, we take the forms $(-1)^{n-1}f(\alpha_1,\alpha_2,\ldots,\alpha_{n-2},\alpha_n,\alpha_{n-1}\cdots\alpha_{n+1})$, $-(-1)^{n-1}f(\alpha_1,\alpha_2,\ldots,\alpha_{n-2},\alpha_n,\alpha_{n-1}\cdots\alpha_1)\alpha_{n+1}$ and $-(-1)^{n-1}f(\alpha_1,\alpha_2,\ldots,\alpha_{n-2},\alpha_{n-1}\cdots\alpha_n\cdots\alpha_{n-1},\varepsilon)\alpha_{n+1}$ respectively when $i$ equals $n-1$ in the 2nd, 7th and 9th terms.

$$
(7)\quad \begin{cases}
(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) = f(\alpha_2, \alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_n\alpha_1, \alpha_1\alpha_{n+1}) \qquad (n \geqq 2) \\
\quad + \sum_{i=2}^{n-1}(-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i, \cdots, \alpha_i\cdots\alpha_n\cdots\alpha_i, \alpha_i\cdots\alpha_{n+1}) \\
\quad + (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_{n+1}) \\
\quad + \sum_{i=1}^{n-1}(-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_n, \alpha_{n+1}) \\
\quad + (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n\cdots\alpha_{n+1}) \\
\quad - f(\alpha_2, \alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_n\alpha_1, \alpha_1)\alpha_{n+1} \\
\quad - \sum_{i=2}^{n-1}(-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i, \cdots, \\
\qquad\qquad\qquad \alpha_i\cdots\alpha_n\cdots\alpha_i, \alpha_i\cdots\alpha_1)\alpha_{n+1} \\
\quad - (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \varepsilon)\alpha_{n+1} \\
\quad - \sum_{i=1}^{n-1}(-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_n, \varepsilon)\alpha_{n+1} \\
\quad - (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n\cdots\alpha_1)\alpha_{n+1}.
\end{cases}
$$

In the above definition (7), the product $\alpha_i\cdots\alpha_j\cdots\alpha_i$ $(i<j)$ means the following product of $\alpha_1, \cdots, \alpha_i, \alpha_j$. In order to define this product, we construct the diagram of $\alpha_k$ by the following processes:

(i) We put $\alpha_1$ on the left end and $\alpha_{n+1}$ on the right end on the same line.

(ii) We put $\alpha_n$ at the middle between $\alpha_1$ and $\alpha_{n+1}$.

(iii) We put $\alpha_{n-1}$ at the middle between $\alpha_1$ and $\alpha_n$, and between $\alpha_n$ and $\alpha_{n+1}$.

(iv) When the diagram of $\alpha_1, \alpha_i, \alpha_{i+1}, \cdots, \alpha_{n+1}$ is obtained by the above processes, we put $\alpha_{i-1}$ at the middle between all elements in this diagram. And, thus we continue those processes until we put $\alpha_1$, (but we do not put $\alpha_1$ between $\alpha_2$ and $\alpha_1$ on the left end of the diagram). For example, in the case $n=4$, the diagram is: $\alpha_1\alpha_2\alpha_1\alpha_3\alpha_1\alpha_2\alpha_1\alpha_4\alpha_1\alpha_2\alpha_1\alpha_3\alpha_1\alpha_2\alpha_1\alpha_5$.

The product $\alpha_i\cdots\alpha_j\cdots\alpha_i$ is the product of the elements $\alpha_k$ in the above diagram, which appear in the part spread from $\alpha_i$, (which is the nearest to $\alpha_j$ on the left side of $\alpha_j$), to $\alpha_i$, (which is the nearest to $\alpha_j$ on the right side of $\alpha_j$). But when $j=n+1$, the product $\alpha_i\cdots\alpha_{n+1}$ is the left half part of the above product. Moreover, the last arguments $\alpha_i\cdots\alpha_1$ and $\alpha_n\cdots\alpha_1$ in the 7th and 10th terms of (7) $(n\geqq 2)$ are the products which are obtained by putting away the last $\alpha_{n+1}$ from the products $\alpha_i\cdots\alpha_{n+1}$ and $\alpha_n\cdots\alpha_{n+1}$, respectively.

It is easy to verify that the mapping $f\to\partial f$ is a homomorphism of $C^n(\Gamma, G)$ into $C^{n+1}(\Gamma, G)$. In the following, we shall prove that:

THEOREM 1. *If $f$ is any cochain, then $\partial(\partial f)=0$.*

PROOF. When $n=0$ and 1, we can prove this relation by a simple calculation. So we assume $n\geqq 2$. If $f$ is an $n$-dimensional cochain, then $\partial(\partial f)$ is an $(n+2)$-dimensional cochain, that is, a function of $\alpha_1, \alpha_2, \cdots, \alpha_{n+2}$. If we calculate it by using (7), we have the following:

$$
(8)\quad \begin{cases}
\partial(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}, \alpha_{n+2}) \\
\quad = (\partial f)(\alpha_2, \alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_{n+1}\alpha_1, \alpha_1\alpha_{n+2}) \\
\quad + \sum_{i=2}^{n}(-1)^i(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i, \cdots, \\
\qquad\qquad\qquad\qquad\qquad\qquad \alpha_i\cdots\alpha_{n+1}\cdots\alpha_i, \alpha_i\cdots\alpha_{n+2}) \\
\quad + (-1)^{n+1}(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_{n+2}) \\
\quad + \sum_{i=1}^{n}(-1)^i(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_{n+2}) \\
\quad + (-1)^{n+1}(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_{n+1}\cdots\alpha_{n+2}) \\
\quad - (\partial f)(\alpha_2, \alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_{n+1}\alpha_1, \alpha_1)\alpha_{n+2} \\
\quad - \sum_{i=1}^{n}(-1)^i(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i, \cdots, \\
\qquad\qquad\qquad\qquad\qquad\qquad \alpha_i\cdots\alpha_{n+1}\cdots\alpha_i, \alpha_i\cdots\alpha_1)\alpha_{n+2} \\
\quad - (-1)^{n+1}(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_n, \varepsilon)\alpha_{n+2} \\
\quad - \sum_{i=1}^{n}(-1)^i(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_{n+1}, \varepsilon)\alpha_{n+2} \\
\quad - (-1)^{n+1}(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_{n+1}\cdots\alpha_1)\alpha_{n+2}.
\end{cases}
$$

In the first term of the right side of (8), if we denote the arguments $\alpha_2$, $\alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_{n+1}\alpha_1, \alpha_1\alpha_{n+2}$ by the letters $\beta_1, \beta_2, \cdots, \beta_{n+1}$, the diagram of $\beta_1$, $\beta_2, \cdots, \beta_{n+1}$ is obtained in the diagram of $\alpha_1, \alpha_2, \cdots, \alpha_{n+2}$ by taking away the letter $\alpha_1$ on the left end. The diagram of the arguments of the second term in (8) is obtained in the diagram of $\alpha_1, \alpha_2, \cdots, \alpha_{n+2}$ by taking away the letters $\alpha_1, \cdots, \alpha_i$ which appear in the part from $\alpha_1$ on the left end to the nearest $\alpha_i$ of the right side of this $\alpha_1$. Further, the diagram of the arguments of the third term is obtained in the diagram of $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ by exchanging the last $\alpha_{n+1}$ for $\alpha_{n+2}$, and the diagrams of the arguments of the 4th and 5th terms coincide with the diagram of $\alpha_1, \alpha_2, \cdots, \alpha_{n+2}$.

Now, we develop the right side of (8) by applying $\partial$ which is cointained in it. First, we consider the sum of the terms involving no operators. To calculate this sum, we assign a number to each term of (8), that is, $[1]$ to the first term; $[2], \cdots, [n]$ to the terms contained in the second term; $[n+1]$ to the third term; $[1]', [2]', \cdots, [n]'$ to the terms contained in the 4th term; and $[n+1]'$ to the 5th term. Further we assign the numbers $[i]_1, [i]_2, \cdots, [i]_n$; $[i]_{1'}, [i]_{2'}, \cdots, [i]_{n'}$ to the terms obtained by developing the $[i]$ term of (8). By the same method, we assign the numbers to the terms obtained by developing the $[i]'$ term of (8). Then the number of all terms involving no operators is equal to $4n(n+1)$. The number of the terms such that $i \leqq j$ among these terms $[i]_j, [i]_{j'}, [i]'_j, [i]'_{j'}$ is equal to $2n(n+1)$ and also the number of the terms such that $i > j$ is equal to $2n(n+1)$, and these terms of two kinds cancel each other as shown in the following table:

| Type of Term | Number | Sign |
|---|---|---|
| $f(\alpha_1\alpha_3\alpha_1,\ \alpha_2\alpha_1\alpha_4\alpha_1\alpha_2,\ \cdots,\ \alpha_2\alpha_1\alpha_{n+1}\alpha_1\alpha_2,\ \alpha_2\alpha_1\alpha_{n+2})$ | $\begin{cases}[1]_1\\ [2]_{1'}\end{cases}$ | $+1$<br>$(-1)^3$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i,\ \alpha_{i+1}\cdots\alpha_{i+3}\cdots\alpha_{i+1},\ \cdots,$<br>$\alpha_{i+1}\cdots\alpha_{n+1}\cdots\alpha_{i+1},\ \alpha_{i+1}\cdots\alpha_{n+2})\qquad (2\leqq i\leqq n-1)$ | $\begin{cases}[i]_i\\ [i+1]_{i'}\end{cases}$ | $(-1)^{2i}$<br>$(-1)^{2i+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{n-1},\ \alpha_n\cdots\alpha_{n+2})$ | $\begin{cases}[n]_n\\ [n+1]_{n'}\end{cases}$ | $(-1)^{2n}$<br>$(-1)^{2n+1}$ |
| $f(\alpha_2, \alpha_1\alpha_3\alpha_1, \cdots, \alpha_1\alpha_j\alpha_1, \alpha_1\alpha_{j+2}\alpha_1, \alpha_1\alpha_{j+1}\cdots\alpha_{j+3}\cdots\alpha_{j+1}\alpha_1,$<br>$\cdots, \alpha_1\alpha_{j+1}\cdots\alpha_{n+1}\cdots\alpha_{j+1}\alpha_1, \alpha_1\alpha_{j+1}\cdots\alpha_{n+2})\ (2\leqq j\leqq n-1)$ | $\begin{cases}[1]_j\\ [j+1]_1\end{cases}$ | $(-1)^j$<br>$(-1)^{j+1}$ |
| $f(\alpha_2,\ \alpha_1\alpha_3\alpha_1,\ \cdots,\ \alpha_1\alpha_n\alpha_1,\ \alpha_1\alpha_{n+2})$ | $\begin{cases}[1]_n\\ [n+1]_1\end{cases}$ | $(-1)^n$<br>$(-1)^{n+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+1},\ \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i,\ \cdots,\ \alpha_i\cdots\alpha_j\cdots\alpha_i,$<br>$\alpha_i\cdots\alpha_{j+2}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{j+1}\cdots\alpha_{j+3}\cdots\alpha_{j+1}\cdots\alpha_i,\ \cdots,$<br>$\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{n+1}\cdots\alpha_{j+1}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{j+1}\cdots\alpha_{n+2})$<br>$(i<j,\ 2\leqq i,\ j\leqq n-1)$ | $\begin{cases}[i]_j\\ [j+1]_i\end{cases}$ | $(-1)^{i+j}$<br>$(-1)^{i+j+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+1},\ \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i,\ \cdots,\ \alpha_i\cdots\alpha_n\cdots\alpha_i,$<br>$\alpha_i\cdots\alpha_{n+2})\qquad (2\leqq i\leqq n-1)$ | $\begin{cases}[i]_n\\ [n+1]_i\end{cases}$ | $(-1)^{i+n}$<br>$(-1)^{i+n+1}$ |
| $f(\alpha_2\alpha_1\alpha_3\alpha_1\alpha_2,\ \alpha_1\alpha_4\alpha_1,\ \cdots,\ \alpha_1\alpha_{n+1}\alpha_1,\ \alpha_1\alpha_{n+2})$ | $\begin{cases}[1]_{1'}\\ [2]'_1\end{cases}$ | $(-1)$<br>$(-1)^2$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+1}\cdots\alpha_{i+2}\cdots\alpha_{i+1},\ \alpha_i\cdots\alpha_{i+3}\cdots\alpha_i,\ \cdots,$<br>$\alpha_i\cdots\alpha_{n+1}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{n+2})\qquad (2\leqq i\leqq n-1)$ | $\begin{cases}[i]_{i'}\\ [i+1]'_i\end{cases}$ | $(-1)^{2i}$<br>$(-1)^{2i+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{n-1},\ \alpha_{n+1}\cdots\alpha_{n+2})$ | $\begin{cases}[n]_{n'}\\ [n+1]'_n\end{cases}$ | $(-1)^{2n}$<br>$(-1)^{2n+1}$ |
| $f(\alpha_2,\ \alpha_1\alpha_3\alpha_1,\ \cdots,\ \alpha_1\alpha_j\alpha_1,\ \alpha_1\alpha_{j+1}\cdots\alpha_{j+2}\cdots\alpha_{j+1}\alpha_1,$<br>$\alpha_1\alpha_{j+3}\alpha_1,\ \cdots,\ \alpha_1\alpha_{n+1}\alpha_1,\ \alpha_1\alpha_{n+2})\qquad (2\leqq j\leqq n-1)$ | $\begin{cases}[1]_{j'}\\ [j+1]'_1\end{cases}$ | $(-1)^j$<br>$(-1)^{j+1}$ |
| $f(\alpha_2,\ \alpha_1\alpha_3\alpha_1,\ \cdots,\ \alpha_1\alpha_n\alpha_1,\ \alpha_1\alpha_{n+1}\cdots\alpha_{n+2})$ | $\begin{cases}[1]_{n'}\\ [n+1]'_1\end{cases}$ | $(-1)^n$<br>$(-1)^{n+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+1},\ \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i,\ \cdots,\ \alpha_i\cdots\alpha_j\cdots\alpha_i,$<br>$\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{j+2}\cdots\alpha_{j+1}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{j+3}\cdots\alpha_i,\ \cdots,$<br>$\alpha_i\cdots\alpha_{n+1}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{n+2})\qquad (i<j,\ 2\leqq i,\ j\leqq n-1)$ | $\begin{cases}[i]_{j'}\\ [j+1]'_i\end{cases}$ | $(-1)^{i+j}$<br>$(-1)^{i+j+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+1},\ \alpha_i\cdots\alpha_{i+2}\cdots\alpha_i,\ \cdots,\ \alpha_i\cdots\alpha_n\cdots\alpha_i,$<br>$\alpha_i\cdots\alpha_{n+1}\cdots\alpha_{n+2})\qquad (2\leqq i\leqq n-1)$ | $\begin{cases}[i]_{n'}\\ [n+1]'_i\end{cases}$ | $(-1)^{i+n}$<br>$(-1)^{i+n+1}$ |
| $f(\alpha_3,\ \alpha_1\alpha_2\alpha_1\alpha_4\alpha_1\alpha_2\alpha_1,\ \cdots,\ \alpha_1\alpha_2\alpha_1\alpha_{n+1}\alpha_1\alpha_2\alpha_1,$<br>$\alpha_1\alpha_2\alpha_1\alpha_{n+2})$ | $\begin{cases}[1]'_1\\ [2]_1\end{cases}$ | $(-1)$<br>$+1$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{i-1},\ \alpha_{i+2},\ \alpha_i\cdots\alpha_{i+1}\cdots\alpha_{i+3}\cdots\alpha_{i+1}\cdots\alpha_i,\ \cdots,$<br>$\alpha_i\cdots\alpha_{i+1}\cdots\alpha_{n+1}\cdots\alpha_{i+1}\cdots\alpha_i,\ \alpha_i\cdots\alpha_{i+1}\cdots\alpha_{n+2})$<br>$(2\leqq i\leqq n-1)$ | $\begin{cases}[i]'_i\\ [i+1]_i\end{cases}$ | $(-1)^{2i}$<br>$(-1)^{2i+1}$ |
| $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{n-1},\ \alpha_{n+2})$ | $\begin{cases}[n]'_n\\ [n+1]_n\end{cases}$ | $(-1)^{2n}$<br>$(-1)^{2n+1}$ |
| $f(\alpha_1\alpha_2\alpha_1,\ \alpha_3,\ \cdots,\ \alpha_j,\ \alpha_{j+2},\ \alpha_{j+1}\cdots\alpha_{j+3}\cdots\alpha_{j+1},\ \cdots,$<br>$\alpha_{j+1}\cdots\alpha_{n+1}\cdots\alpha_{j+1},\ \alpha_{j+1}\cdots\alpha_{n+2})\qquad (2\leqq j\leqq n-1)$ | $\begin{cases}[1]'_j\\ [j+1]_{1'}\end{cases}$ | $(-1)^{j+1}$<br>$(-1)^j$ |
| $f(\alpha_1\alpha_2\alpha_1,\ \alpha_3,\ \alpha_4,\ \cdots,\ \alpha_n,\ \alpha_{n+2})$ | $\begin{cases}[1]'_n\\ [n+1]_{1'}\end{cases}$ | $(-1)^{n+1}$<br>$(-1)^n$ |

| Type of Term | Number | Sign |
|---|---|---|
| $f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_j, \alpha_{j+2}, \alpha_{j+1}\cdots\alpha_{j+3}\cdots\alpha_{j+1}, \cdots, \alpha_{j+1}\cdots\alpha_{n+1}\cdots\alpha_{j+1}, \alpha_{j+1}\cdots\alpha_{n+2})$ $(i<j,\ 2\leqq i,\ j\leqq n-1)$ | $\begin{cases}[i]'_j \\ [j+1]_{i'}\end{cases}$ | $(-1)^{i+j}$ <br> $(-1)^{i+j+1}$ |
| $f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_n, \alpha_{n+2})$ $(2\leqq i\leqq n-1)$ | $\begin{cases}[i]'_n \\ [n+1]_{i'}\end{cases}$ | $(-1)^{i+n}$ <br> $(-1)^{i+n+1}$ |
| $f(\alpha_1\alpha_2\alpha_1\alpha_3\alpha_1\alpha_2\alpha_1, \alpha_4, \alpha_5, \cdots, \alpha_{n+2})$ | $\begin{cases}[1]'_{1'} \\ [2]'_{1'}\end{cases}$ | $(-1)^2$ <br> $(-1)^3$ |
| $f(\alpha_1, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_{i+2}\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+3}, \cdots, \alpha_{n+2})$ $(2\leqq i\leqq n-1)$ | $\begin{cases}[i]'_{i'} \\ [i+1]'_{i'}\end{cases}$ | $(-1)^{2i}$ <br> $(-1)^{2i+1}$ |
| $f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n\cdots\alpha_{n+1}\cdots\alpha_{n+2})$ | $\begin{cases}[n]'_{n'} \\ [n+1]'_{n'}\end{cases}$ | $(-1)^{2n}$ <br> $(-1)^{2n+1}$ |
| $f(\alpha_1\alpha_2\alpha_1, \alpha_3, \cdots, \alpha_j, \alpha_{j+1}\cdots\alpha_{j+2}\cdots\alpha_{j+1}, \alpha_{j+3}, \cdots, \alpha_{n+2})$ $(2\leqq j\leqq n-1)$ | $\begin{cases}[1]'_{j'} \\ [j+1]'_{1'}\end{cases}$ | $(-1)^{j+1}$ <br> $(-1)^j$ |
| $f(\alpha_1\alpha_2\alpha_1, \alpha_3, \cdots, \alpha_n, \alpha_{n+1}\cdots\alpha_{n+2})$ | $\begin{cases}[1]'_{n'} \\ [n+1]'_{1'}\end{cases}$ | $(-1)^{n+1}$ <br> $(-1)^n$ |
| $f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_j, \alpha_{j+1}\cdots\alpha_{j+2}\cdots\alpha_{j+1}, \alpha_{j+3}, \cdots, \alpha_{n+1}, \alpha_{n+2})(i<j,\ 2\leqq i,\ j\leqq n-1)$ | $\begin{cases}[i]'_{j'} \\ [j+1]'_{i'}\end{cases}$ | $(-1)^{i+j}$ <br> $(-1)^{i+j+1}$ |
| $f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i\cdots\alpha_{i+1}\cdots\alpha_i, \alpha_{i+2}, \cdots, \alpha_n, \alpha_{n+1}\cdots\alpha_{n+2})$ $(2\leqq i\leqq n-1)$ | $\begin{cases}[i]'_{n'} \\ [n+1]'_{i'}\end{cases}$ | $(-1)^{i+n}$ <br> $(-1)^{i+n+1}$ |

Therefore, the sum of the terms involving no operators is equal to zero.(3)

Next, we consider the sum of the terms involving the operators. From the definition of $\partial f$, we have $(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_n, \varepsilon)=0$, so the 8th and 9th terms of (8) vanish. And we easily see that the sum of the terms which involve $r(\geqq 2)$ operators is equal to zero, so it is sufficient to consider the sum of the terms involving only one operator $\alpha_{n+2}$. If we take away the operator $\alpha_{n+2}$ in each term of this sum, we obtain the sum which is obtained by exchanging $\alpha_{n+2}$ for $\varepsilon$ in the above sum of the terms which involve no operators. Therefore, in the same way as the above, we can prove that the sum of the terms which involve only one operator is equal to zero. So, the sum of all terms involving the operators also equals zero. Thus we have $\partial(\partial f)=0$.

Now, we call an $n$-dimensional cochain $f$ an $n$-dimensional *M-cocycle* if $\partial f=0$. All $n$-dimensional $M$-cocycles form a subgroup of $C^n(\Gamma, G)$, which we denote by $Z^{*n}(\Gamma, G)$. On the other hand, for $n>0$ the $n$-dimensional cochains that are $M$-coboundaries of some $(n-1)$-dimensional cochains form also a subgroup of $C^n(\Gamma, G)$, which we denote by $B^{*n}(\Gamma, G)$. Since $\partial(\partial f)=0$, every $M$-coboundary is an $M$-cocycle. So, we have $B^{*n}(\Gamma, G)\subset Z^{*n}(\Gamma, G)$. For $n=0$, if we put $B^{*0}(\Gamma, G)=0$, this relation remains valid.

The factor group $H^{*n}(\Gamma, G)=Z^{*n}(\Gamma, G)/B^{*n}(\Gamma, G)$ is called the $n$-th $M$-

(3) The forms of the following terms are obtained by the slight modifications of the types in this table: $[i]_{(i+1)}$, $[i]_{(i+1)'}$, $[i]'_{(i+1)}$, $[i]'_{(i+1)'}$ $(i=1, 2, \ldots, n-1)$; $[i]_{(n-1)}$, $[i]_{(n-1)'}$, $[i]'_{(n-1)}$ $(i=1, 2, \ldots, n-3, n-1)$.

*cohomology group* of $\Gamma$ over $G$.

In the following sections, we assume that $C^1(\Gamma, G)$ and $C^2(\Gamma, G)$ are the groups of the normalized cochains $f$, that is, $f(\varepsilon)=0$ and $f(\alpha, \varepsilon)=0=f(\varepsilon, \beta)$.

**5. *BM*-extensions and *M*-cohomology groups (the case where *G* is abelian).** We consider the set of all $BM$-extensions of a group $G$ by a group $\Gamma$. Let $L$ be a $BM$-extension of $G$ by $\Gamma$. Then, by (3) in §2, a homomorphism $\theta$ of $\Gamma$ into $\mathrm{Aut}(G)/\mathrm{In}(G)$ is determined, where $\mathrm{Aut}(G)$ is a group of automorphisms of $G$ and $\mathrm{In}(G)$ is the group of all inner automorphisms of $G$. This homomorphism is called the homomorphism associated with this $BM$-extension. Further, by (6) in §3, with all equivalent $BM$-extensions the same homomorphism $\theta$ is associated. In the following, we give a survey of all $BM$-extensions of $G$ by $\Gamma$ associated with given homomorphism $\theta$.

In this section, we consider the case where $G$ is an abelian group. In this case, the homomorphism $\theta$ becomes a homomorphism of $\Gamma$ into $\mathrm{Aut}(G)$, so the group $\Gamma$ is an operator group of $G$. Since the 2-dimensional cochains are normalized, we have the following result comparing (4) with (7) (the case $n=2$); and (5) with (7) (the case $n=1$):

Proposition 3. *All non-equivalent BM-extensions of an abelian group $G$ by a group $\Gamma$ associated with given homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(G)$ correspond one-to-one to the elements of the second M-cohomology group $H^{*2}(\Gamma, G)$.*

**6. Abstract kernels and 3-dimensional *M*-cocycles.** In this section, we make the preparations for the consideration of all $BM$-extensions in the case where $G$ is non-abelian. To a $BM$-extension $L$ of $G$ by $\Gamma$, there corresponds a homomorphism of $L$ onto $\Gamma$, and its kernel is $G$. Further, as we see in the previous section, with this $BM$-extension $L$, a homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(G)/\mathrm{In}(G)$ is associated. After S. MacLane, we call a pair of groups $\Gamma$, $G$ together with a homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(G)/\mathrm{In}(G)$ an *abstract kernel*, and denote it by $(\Gamma, G, \theta)$. Then the extension problem is that of constructing all $BM$-extensions to given abstract kernel.

Now, we note that since every automorphism of $G$ induces an automorphism of the center $C$ of $G$ and since automorphisms in the same automorphism class of $\mathrm{Aut}(G)/\mathrm{In}(G)$ induce the same automorphism of $C$, the homomorphism $\theta\colon \Gamma \to \mathrm{Aut}(G)/\mathrm{In}(G)$ induces a homomorphism $\theta_0\colon \Gamma \to \mathrm{Aut}(C)$. So, we can regard $\Gamma$ as an operator group of the center $C$ of $G$. Therefore, we can construct the $M$-cohomology group $H^{*n}(\Gamma, C)$.

Let $(\Gamma, G, \theta)$ be a given abstract kernel. In every coset $\theta(\alpha)$ of $\mathrm{In}(G)$ in $\mathrm{Aut}(G)$, we choose a representative $\varphi_\alpha$, where $\varphi_\varepsilon$ is the identity automorphism, then there correspond the elements $h(\alpha, \beta)$ of $G$ such that

$$\varphi_\alpha \varphi_\beta = \varphi_{\alpha\beta} T_{h(\alpha,\beta)}, \tag{9}$$

where $h(\alpha, \varepsilon)=e=h(\varepsilon, \beta)$. On the other hand, for $a \in G$, $\varphi \in \mathrm{Aut}(G)$ it holds that

$$\varphi^{-1} T_a \varphi = T_{(a\varphi)}. \tag{10}$$

Calculating $\varphi_\alpha[\varphi_\beta(\varphi_\alpha\varphi_\gamma)] = [\varphi_\alpha(\varphi_\beta\varphi_\alpha)]\varphi_\gamma$, using (9) and (10) we have the relation:

$$T_{h(\alpha,\beta\alpha\gamma)\,h(\beta,\alpha\gamma)\,h(\alpha,\gamma)} = T_{h(\alpha\beta\alpha,\,\gamma)\,([h(\alpha,\,\beta\alpha)\,h(\beta,\,\alpha)]\varphi_\gamma)}.$$

So, there exists an element $z^*(\alpha, \beta, \gamma)$ in the center $C$ of $G$ such that

$$h(\alpha, \beta\alpha\gamma)h(\beta, \alpha\gamma)h(\alpha, \gamma) = z^*(\alpha, \beta, \gamma)h(\alpha\beta\alpha, \gamma)\,\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\gamma\big). \tag{11}$$

We have obtained a 3-dimensional cochain $z^*(\alpha, \beta, \gamma)$ of $\Gamma$ in $C$, associated with the abstract kernel $(\Gamma, G, \theta)$. This cochain $z^*(\alpha, \beta, \gamma)$ is called an *obstruction* of the abstrat kernel $(\Gamma, G, \theta)$.

Lemma 1. *Any obstruction of an abstract kernel $(\Gamma, G, \theta)$ is a 3-dimensional M-cocycle of $\Gamma$ in $C$, where $C$ is the center of $G$.*

Proof. We calculate the expression:

$$J = h(\alpha, \beta\alpha\gamma\alpha\beta\alpha\delta)h(\beta, \alpha\gamma\alpha\beta\alpha\delta)h(\alpha, \gamma\alpha\beta\alpha\delta)h(\gamma, \alpha\beta\alpha\delta)h(\alpha, \beta\alpha\delta)h(\beta, \alpha\delta)h(\alpha, \delta)$$

in two ways. In the first way, we begin with the calculations of the first three factors and the last three factors by using (11). Then we have:

$$\begin{aligned}
J &= z^*(\alpha, \beta, \gamma\alpha\beta\alpha\delta)z^*(\alpha, \beta, \delta)h(\alpha\beta\alpha, \gamma\alpha\beta\alpha\delta)h(\gamma, \alpha\beta\alpha\delta)h(\alpha\beta\alpha, \delta)\cdot \\
&\quad \cdot\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_{\gamma\alpha\beta\alpha\delta}\, T_{h(\gamma,\,\alpha\beta\alpha\delta)\,h(\alpha\beta\alpha,\,\delta)}\big)\,\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\delta\big) \\
&= z^*(\alpha, \beta, \gamma\alpha\beta\alpha\delta)z^*(\alpha, \beta, \delta)z^*(\alpha\beta\alpha, \gamma, \delta)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)\,\big([h(\alpha\beta\alpha, \gamma\alpha\beta\alpha)\cdot \\
&\quad \cdot h(\gamma, \alpha\beta\alpha)]\varphi_\delta\big)\,\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\gamma\varphi_{\alpha\beta\alpha}\varphi_\delta\big)\,\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\delta\big) \\
&= z^*(\alpha, \beta, \gamma\alpha\beta\alpha\delta)z^*(\alpha, \beta, \delta)z^*(\alpha\beta\alpha, \gamma, \delta)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)\,\big(h(\alpha\beta\alpha, \gamma\alpha\beta\alpha)\varphi_\delta\big)\cdot \\
&\quad \cdot\big[\{([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_{\gamma\alpha\beta\alpha})h(\gamma, \alpha\beta\alpha)\}\,\varphi_\delta\big]\,\big([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\delta\big).
\end{aligned}$$

In the second way, we begin with the calculation of the middle three factors by applying (11). Then we have:

$$\begin{aligned}
J &= z^*(\alpha, \gamma, \beta\alpha\delta)h(\alpha, \beta\alpha\gamma\alpha\beta\alpha\delta)h(\beta, \alpha\gamma\alpha\beta\alpha\delta)h(\alpha\gamma\alpha, \beta\alpha\delta)h(\beta, \alpha\delta)\cdot \\
&\quad \cdot\big([h(\alpha, \gamma\alpha)h(\gamma, \alpha)]\varphi_\beta\varphi_{\alpha\delta}\big)h(\alpha, \delta) \\
&= z^*(\alpha, \gamma, \beta\alpha\delta)z^*(\beta, \alpha\gamma\alpha, \alpha\delta)h(\alpha, \beta\alpha\gamma\alpha\beta\alpha\delta)h(\beta\alpha\gamma\alpha\beta, \alpha\delta)h(\alpha, \delta)\cdot \\
&\quad \cdot\big([h(\beta, \alpha\gamma\alpha\beta)h(\alpha\gamma\alpha, \beta)]\varphi_\alpha\varphi_\delta\big)\,\big([h(\alpha, \gamma\alpha)h(\gamma, \alpha)]\varphi_\beta\varphi_\alpha\varphi_\delta\big) \\
&= z^*(\alpha, \gamma, \beta\alpha\delta)z^*(\beta, \alpha\beta\alpha, \alpha\delta)z^*(\alpha, \beta\alpha\gamma\alpha\beta, \delta)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)\,\big(h(\alpha, \beta\alpha\gamma\alpha\beta\alpha)\varphi_\delta\big)\cdot \\
&\quad \cdot\big([h(\beta\alpha\gamma\alpha\beta, \alpha)\,\{h(\beta, \alpha\gamma\alpha\beta)h(\alpha\gamma\alpha, \beta)\}\,\varphi_\alpha]\varphi_\delta\big)\,\big([h(\alpha, \gamma\alpha)h(\gamma, \alpha)]\varphi_\beta\varphi_\alpha\varphi_\delta\big) \\
&= z^*(\alpha, \gamma, \beta\alpha\delta)z^*(\beta, \alpha\gamma\alpha, \alpha\delta)z^*(\alpha, \beta\alpha\gamma\alpha\beta, \delta)\,\big(z^{*-1}(\beta, \alpha\gamma\alpha, \alpha)\varphi_\delta\big)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)\cdot \\
&\quad \cdot\big([h(\alpha, \beta\alpha\gamma\alpha\beta\alpha)h(\beta, \alpha\gamma\alpha\beta\alpha)]\varphi_\delta\big)\,\big([h(\alpha\gamma\alpha, \beta\alpha)\,(\{h(\alpha, \gamma\alpha)h(\gamma, \alpha)\}\,\varphi_{\beta\alpha})]\varphi_\delta\big)\cdot \\
&\quad \cdot\big(h(\beta, \alpha)\varphi_\delta\big) \\
&= z^*(\alpha, \gamma, \beta\alpha\delta)z^*(\beta, \alpha\gamma\alpha, \alpha\delta)z^*(\alpha, \beta\alpha\gamma\alpha\beta, \delta)\,\big(z^{*-1}(\beta, \alpha\gamma\alpha, \alpha)\varphi_\delta\big)\cdot
\end{aligned}$$

$$\cdot(z^{*-1}(\alpha, \gamma, \beta\alpha)\varphi_\delta)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)([h(\alpha, \beta\alpha\gamma\alpha\beta\alpha)h(\beta, \alpha\gamma\alpha\beta\alpha)h(\alpha, \gamma\alpha\beta\alpha)\cdot$$
$$\cdot h(\gamma, \alpha\beta\alpha)h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\delta)$$
$$= z^*(\alpha, \gamma, \beta\alpha\delta)z^*(\beta, \alpha\gamma\alpha, \alpha\delta)z^*(\alpha, \beta\alpha\gamma\alpha\beta, \delta)(z^{*-1}(\beta, \alpha\gamma\alpha, \alpha)\varphi_\delta)\cdot$$
$$\cdot(z^{*-1}(\alpha, \gamma, \beta\alpha)\varphi_\delta)(z^*(\alpha, \beta, \gamma\alpha\beta\alpha)\varphi_\delta)h(\alpha\beta\alpha\gamma\alpha\beta\alpha, \delta)\cdot$$
$$\cdot(\{h(\alpha\beta\alpha, \gamma\alpha\beta\alpha)([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_{\gamma\alpha\beta\alpha})h(\gamma, \alpha\beta\alpha)\}\varphi_\delta)([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\delta).$$

Comparing the above two calculations, we have $\partial z^*(\alpha, \beta, \gamma, \delta)=0$.

The cochain $z^*(\alpha, \beta, \gamma)$ depends on the choice of the representatives $\varphi_\alpha$ and on the elements $h(\alpha, \beta)$. In the following we investigate the change of $z^*(\alpha, \beta, \gamma)$ for the choice of $h(\alpha, \beta)$ and $\varphi_\alpha$.

LEMMA 2. *If the choice of $h(\alpha, \beta)$ in* (11) *is changed, then the obstruction $z^*(\alpha, \beta, \gamma)$ is changed to a cohomologous M-cocycle. By suitably changing the choice of $h(\alpha, \beta)$, $z^*(\alpha, \beta, \gamma)$ may be changed to any M-cohomologous cocycle.*

PROOF. Suppose that the elements $h(\alpha, \beta)$ in (11) are replaced by the elements $h'(\alpha, \beta)$. Then there exist the elements $g(\alpha, \beta)$ such that $h'(\alpha, \beta)= h(\alpha, \beta)\, g(\alpha, \beta)$, $g(\alpha, \beta) \in C$, where $g(\alpha, \varepsilon)=e=g(\varepsilon, \beta)$. If we calculate by (11) the 3-dimensional cocycle $z^{*\prime}(\alpha, \beta, \gamma)$ corresponding to the elements $h'(\alpha, \beta)$ we obtain:

$$z^{*\prime}(\alpha, \beta, \gamma)=z^*(\alpha, \beta, \gamma)(g(\alpha, \beta\alpha\gamma)g(\beta, \alpha\gamma)g(\alpha, \gamma)\{g(\alpha\beta\alpha, \gamma)([g(\alpha, \beta\alpha)g(\beta, \alpha)]\varphi_\gamma)\}^{-1})$$
$$= z^*(\alpha, \beta, \gamma)+\partial g(\alpha, \beta, \gamma).$$

And, since $g(\alpha, \beta)$ is an arbitrary normalized 2-dimensional cochain of $\Gamma$ in $C$, so we can obtain as $z^{*\prime}(\alpha, \beta, \gamma)$ every $M$-cohomologous cocycle to $z^*(\alpha, \beta, \gamma)$.

LEMMA 3. *If the automorphism $\varphi_\alpha$ is changed, then with a suitable new choice of $h(\alpha, \beta)$ the* 3*-dimensional M-cocycle $z^*(\alpha, \beta, \gamma)$ remains unchanged.*

PROOF. Suppose that the automorphisms $\varphi_\alpha$ are replaced by automorphisms $\varphi'_\alpha$ lying in the same automorphism classes $\theta(\alpha)$. Then there exist the elements $c(\alpha)$ in $G$ such that $\varphi'_\alpha=\varphi_\alpha T_{c(\alpha)}$ and $c(\varepsilon)=e$. By (9) and (10),

$$\varphi'_\alpha\varphi'_\beta=\varphi_\alpha\varphi_\beta T_{(c(\alpha)\varphi_\beta)}T_{c(\beta)}=\varphi'_{\alpha\beta}T_{c(\alpha\beta)^{-1}h(\alpha,\beta)(c(\alpha)\varphi_\beta)c(\beta)}.$$

Now, we choose new normalized cochain $h'(\alpha, \beta)$ as follows:

$$h'(\alpha, \beta)=c(\alpha\beta)^{-1}h(\alpha, \beta)\,(c(\alpha)\varphi_\beta)c(\beta). \tag{12}$$

Then we have

$$c(\alpha\beta)h'(\alpha, \beta)=h(\alpha, \beta)c(\beta)\,(c(\alpha)\varphi'_\beta). \tag{13}$$

Using (12) and (13), we can show that the 3-dimensional $M$-cocycle corresponding to $h'(\alpha, \beta)$ is $z^*(\alpha, \beta, \gamma)$ which corresponds to $h(\alpha, \beta)$. In order to show the above, we consider the expression:

$$M = c(\alpha\beta\alpha\gamma)z^*(\alpha, \beta, \gamma)h'(\alpha\beta\alpha, \gamma)\,([h'(\alpha, \beta\alpha)h'(\beta, \alpha)]\varphi'_\gamma).$$

Using (13), if we change $h'(\alpha, \beta)$ into $h(\alpha, \beta)$ one by one from the left in the expression $M$, and using the definition (11) of $z^*(\alpha, \beta, \gamma)$ we have

$$M = z^*(\alpha, \beta, \gamma)h(\alpha\beta\alpha, \gamma)\,([h(\alpha, \beta\alpha)h(\beta, \alpha)]\varphi_\gamma)\,([(c(\alpha)\varphi_\beta\varphi_\alpha)\,(c(\beta)\varphi_\alpha)c(\alpha)]\varphi_\gamma)c(\gamma)$$
$$= h(\alpha, \beta\alpha\gamma)h(\beta, \alpha\gamma)h(\alpha, \gamma)\,([(c(\alpha)\varphi_\beta\varphi_\alpha)\,(c(\beta)\varphi_\alpha)c(\alpha)]\varphi_\gamma)c(\gamma).$$

Further, if we change $h(\alpha, \beta)$ into $h'(\alpha, \beta)$ one by one from the right in the above last expression by (13) and the definition of $\varphi'_\alpha$, we have

$$M = c(\alpha\beta\alpha\gamma)h'(\alpha, \beta\alpha\gamma)h'(\beta, \alpha\gamma)h'(\alpha, \gamma).$$

Comparing the first and the last expressions, we have

$$h'(\alpha, \beta\alpha\gamma)h'(\beta, \alpha\gamma)h'(\alpha, \gamma) = z^*(\alpha, \beta, \gamma)h'(\alpha\beta\alpha, \gamma)\,([h'(\alpha, \beta\alpha)h'(\beta, \alpha)]\varphi'_\gamma).$$

Therefore, the 3-dimensional $M$-cocycle corresponding to $h'(\alpha, \beta)$ is also $z^*(\alpha, \beta, \gamma)$.

Thus, we have proved that only one element $\{z^*(\alpha, \beta, \gamma)\}$ of $H^{*3}(\Gamma, C)$ corresponds to a given abstract kernel $(\Gamma, G, \theta)$, and we denote it by the notation Obs $(\Gamma, G, \theta)$.

**7. *BM*-extensions and *M*-cohomology groups (the case *G* is non-abelian).** In this section, we proceed to the study of the set of all $BM$-extensions of a non-abelian group $G$ by a group $\Gamma$.

First, we seek for the necessary and sufficient condition for the existence of the $BM$-extension corresponding to a given abstract kernel $(\Gamma, G, \theta)$. From the result in the previous section, we have:

THEOREM 2. *The abstract kernel $(\Gamma, G, \theta)$ has a BM-extension if and only if Obs $(\Gamma, G, \theta) = 0$.*

PROOF. Let $L$ be a $BM$-extension corresponding to the kernel $(\Gamma, G, \theta)$. For a definite choice of the coset representatives of $G$, this extension $L$ is defined by a factor set and a system of automorphisms such that:

$$f(\alpha, \varepsilon) = e = f(\varepsilon, \beta),$$
$$a T_\alpha T_\beta = a T_{\alpha\beta} T_{f(\alpha, \beta)}, \tag{14}$$

$$f(\alpha, \beta\alpha\gamma)f(\beta, \alpha\gamma)f(\alpha, \gamma) = f(\alpha\beta\alpha, \gamma)\,(f(\alpha, \beta\alpha)T_\gamma)\,(f(\beta, \alpha)T_\gamma). \tag{15}$$

So, we can take automorphism $T_\alpha$ for the automorphism $\varphi_\alpha$ and $M$-factor set $f(\alpha, \beta)$ for the elements $h(\alpha, \beta)$. From (14), it follows that (9) holds, and (15) shows that we must put $z^*(\alpha, \beta, \gamma) = 0$.

Conversely, assume that Obs $(\Gamma, G, \theta) = 0$. By the lemma 2, for suitable selection of a normalized cochain $h(\alpha, \beta)$ we may obtain the obstruction

$z^*(\alpha, \beta, \gamma)=0$. In this case, for these $\varphi_\alpha$ and $h(\alpha, \beta)$, the conditions (14) and (15) hold. So, from the result in §2, there exists a $BM$-extension of $G$ by $\Gamma$ corresponding to the abstract kernel $(\Gamma, G, \theta)$.

Now, we classify all $BM$-extensions of $G$ by $\Gamma$. In this case, in the same way as used in the case of group extension, we have the following result (cf. e.g. [11] pp. 142–145):

THEOREM 3. *If the abstract kernel $(\Gamma, G, \theta)$ has a BM-extension, then all non-equivalent BM-extensions of a non-abelian group $G$ by a group $\Gamma$ are in one-to-one correspondence with the elements of the second M-cohomology group $H^{*2}(\Gamma, C)$, where $C$ is the center of $G$.*

PROOF. We give only an outline of the proof. Let $L$ be a $BM$-extension of the abstract kernel $(\Gamma, G, \theta)$ and $S$ be a $BM$-extension of the abstract kernel $(\Gamma, C, \theta_0)$. We consider all the possible pairs $(l, m)$ $l \in L$, $m \in S$, subject to the condition that the cosets $lG$ and $mC$ correspond to one and the same element of $\Gamma$. By the multiplication $(l, m)(l', m')=(ll', mm')$, the set of all such pairs becomes a Bol-Moufang loop which we denote $\bar{L}$. The set of all pairs of the form $(z, z^{-1})$ $z \in C$, is a normal subgroup $N$ of $\bar{L}$ and the set of all pairs of the form $(a, z)$, $a \in G$, $z \in C$, also forms a normal subgroup $\bar{G}$ of $\bar{L}$. The loop $L'=\bar{L}/N$ has a normal subgroup $G'=\bar{G}/N$ which is isomorphic to $G$, and $L'$ is a $BM$-extension of the kernel $(\Gamma, G, \theta)$. We call $L'$ the product of $L$ and $S$ and denote by $L\otimes S$. Further, if the extension $L$ of $G$ is given by an $M$-factor set $f(\alpha, \beta)$ and $S$ is given by $t(\alpha, \beta)$, then we may take $f(\alpha, \beta)\ t(\alpha, \beta)$ as an $M$-factor set of $L\otimes S$.

Taking $L$ as fixed, we now show that every $BM$-extension $L'$ of the kernel $(\Gamma, G, \theta)$ is equivalent to a $BM$-extension $L\otimes S$ for suitable choice of the $BM$-extension $S$ of $(\Gamma, C, \theta_0)$. We choose the representatives $g'_\alpha$ of $G$ in $L'$ such that the automorphisms $T'_\alpha$ are the same automorphisms induced by the representatives $g_\alpha$ of $G$ in $L$. Assume that for this choice of the representatives $g'_\alpha$, the $M$-factor set of $L'$ is $f'(\alpha, \beta)$. Since $T_{f(\alpha,\beta)}=T_{f'(\alpha,\beta)}$ there exist the elements $t'(\alpha, \beta)$ of the center $C$ of $G$ such that $f'(\alpha, \beta)=f(\alpha, \beta)t'(\alpha, \beta)$. Both the $M$-factor sets $f'(\alpha, \beta)$ and $f(\alpha, \beta)$ satisfy the condition (15) and are normalized, so $t'(\alpha, \beta)$ also satisfies (15) and is normalized. Therefore, there exists a $BM$-extension $S'$ of the abstract kernel $(\Gamma, C, \theta_0)$. Then we can show that $L'$ is equivalent to the product $L\otimes S'$. Further, we can show that if $S_1$ and $S_2$ are two $BM$-extensions of $(\Gamma, C, \theta_0)$ then the $BM$-extensions $L_1=L\otimes S_1$ and $L_2=L\otimes S_2$ are equivalent if and only if $S_1$ and $S_2$ are equivalent. Thus, we have proved the theorem.

**8. Some properties of $H^{*3}$.** In this section, we make clear the relation between the abstract kernel $(\Gamma, G, \theta)$ and its obstruction, and we obtain some properties of the third $M$-cohomology group $H^{*3}$.

Now, let $(\Gamma, G, \theta)$ be an abstract kernel. Then, as we see in §6, there exists an element $\{z^*(\alpha, \beta, \gamma)\}$ of $H^{*3}(\Gamma, C)$ corresponding to $(\Gamma, G, \theta)$. On the other hand, it is known that in the theory of the group extension, to 2-dimensional normalized cochain $h(\alpha, \beta)$ which is defined by (9), there exists an element $z(\alpha, \beta, \gamma)$ of $Z^3(\Gamma, C)$ such that

$$(16) \qquad h(\alpha, \beta\gamma)h(\beta, \gamma) = z(\alpha, \beta, \gamma)h(\alpha\beta, \gamma) \left(h(\alpha, \beta)\varphi_\gamma\right).$$

From (11) and (16), we have

$$(17) \qquad z^*(\alpha, \beta, \gamma) = z(\alpha, \beta\alpha, \gamma) + z(\beta, \alpha, \gamma).$$

That is, with an obstruction $z^*(\alpha, \beta, \gamma)$, there corresponds an element $z(\alpha, \beta, \gamma)$ of $Z^3(\Gamma, C)$ which satisfies (17).

Conversely, we consider a mapping $\rho$ of $Z^3(\Gamma, C)$ into $C^3(\Gamma, C)$ carrying $f(\alpha, \beta, \gamma)$ into $f^*(\alpha, \beta, \gamma) = f(\alpha, \beta\alpha, \gamma) + f(\beta, \alpha, \gamma)$. Then, by the simple calculation, we can show that $f^*(\alpha, \beta, \gamma) \in Z^{*3}(\Gamma, C)$ and $\rho$ is a homomorphism of $Z^3$ into $Z^{*3}$.

Further, we show the following:

PROPOSITION 4. *For any element $f^*$ of $\rho(Z^3)$, there exists an abstract kernel $(\Gamma, G, \theta)$ with center $C$ and with the obstruction $f^*$.*

PROOF. It is known that for any element $f$ of $Z^3(\Gamma, C)$, there exists an abstract kernel $(\Gamma, G, \theta)$ with center $C$ and with 3-dimensional cocycle $f$ (cf. [9] p. 334; [12] p. 129). Let $f^*(\alpha, \beta, \gamma)$ be an arbitrary element of $\rho(Z^3)$. Then there exists an element $f(\alpha, \beta, \gamma)$ of $Z^3(\Gamma, C)$ such that $f^*(\alpha, \beta, \gamma) = f(\alpha, \beta\alpha, \gamma) + f(\beta, \alpha, \gamma)$. If we construct an abstract kernel $(\Gamma, G, \theta)$ with certer $C$ and with $f(\alpha, \beta, \gamma)$ as its obstruction of the group extension, it is easily seen that $(\Gamma, G, \theta)$ is the required kernel.

It is clear that $H^2(\Gamma, C)$ is imbedded isomorphically into $H^{*2}(\Gamma, C)$. Concerning the relation between $H^3$ and $H^{*3}$, it holds as follows:

PROPOSITION 5. $\rho(H^3) = \rho(Z^3)/\rho(B^3) \cong$ *subgroup of* $H^{*3}$.

PROOF. It is sufficient to show that $\rho$ is a homomorphism of $B^3(\Gamma, C)$ onto $B^{*3}(\Gamma, C)$. In fact, to any element $f(\alpha, \beta, \gamma)$ of $B^3(\Gamma, C)$ there exists 2-dimensional normalized cochain $f'$ in $C$ such that $f(\alpha, \beta, \gamma) = \delta f'(\alpha, \beta, \gamma)$, where $\delta$ is the group-coboundary operation. Then, it holds for $f^*(\alpha, \beta, \gamma) = \rho\left(f(\alpha, \beta, \gamma)\right)$ that $f^*(\alpha, \beta, \gamma) = \partial f'(\alpha, \beta, \gamma)$, that is, $f^* \in B^{*3}(\Gamma, C)$. Conversely, for any element $f^*(\alpha, \beta, \gamma)$ of $B^{*3}(\Gamma, C)$, there exists a 2-dimensional normalized cochain $f'(\alpha, \beta)$ in $C$ such that $f^*(\alpha, \beta, \gamma) = \partial f'(\alpha, \beta, \gamma)$. Using this cochain $f'(\alpha, \beta)$, we define an element of $B^3(\Gamma, C)$ as follows:

$$f(\alpha, \beta, \gamma) = f'(\beta, \gamma) - f'(\alpha\beta, \gamma) + f'(\alpha, \beta\gamma) - f'(\alpha, \beta)\gamma.$$

Then we have $\rho(f) = f^*$, that is, any element of $B^{*3}(\Gamma, C)$ has its inverse image in $Z^3(\Gamma, C)$. So $\rho$ is a homomorphism of $B^3$ onto $B^{*3}$. Thus, we have proved the proposition.

COROLLARY. *Let $N$ be the kernel of the homomorphism $\rho$. If $N$ is contained in $B^3$, then it holds that*

$$H^3 \cong \text{subgroup of } H^{*3}.$$

RRMARK. We can easily see that if $N$ is contained in $B^3(\Gamma, C)$ each element of $N$ is the group-coboundary of an element of $Z^{*2}(\Gamma, C)$.

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## References

[1] A. A. Albert, *Qusigroups* I, Trans. Amer. Math. Soc., **54** (1943), 507–519.
[2] ———, ——— II, ibid., **55** (1944), 401–419.
[3] R. Bear. *Erweiterung von Gruppen und ihren Isomorphismen*, Math. Zeit., **38** (1934), 375–416.
[4] R. H. Bruck, *Contribution to the theory of loops*, Trans. Amer. Math. Soc., **60** (1946), 245–354.
[5] ———, *An extension theory for Moufang loop*, Bull. Amer. Math. Soc., **53** (1947), Abstract. 1–11.
[6] ———, *An extension theory for a certain class of loops*, Bull. Amer. Math. Soc., **57** (1951), 11–26.
[7] ———, *A survey of binary systems*, Springer-Verlag., 1958.
[8] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups* I, Ann. of Math., **48** (1947), 51–78.
[9] ——— ———, ——— II, ibid., **48** (1947). 326–341.
[10] ——— ———, *Algebraic cohomology groups and loops*, Duke Math. J., **14** (1947), 435–463.
[11] A. G. Kurosh, *The theory of groups* I *and* II, New York, 1956.
[12] S. MacLane, *Homology*, Springer-Verlag., 1963.

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