J. Sci. Hiroshima Univ. Ser. A-I
27 (1963), 137–149

On Spectral Representations of Generalized Spectral Operators

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(Received September. 19, 1963)

Introduction. In the previous paper [3], the author introduced a theory of generalized spectral operators based on spectral representations instead of spectral measures. As Foias [2] first indicated, the spectral representation corresponding to a generalized spectral or scalar operator is not uniquely determined. In fact, if we take a spectral representation U and a nilpotent operator Q: $Q^{k+1}=0$, commuting with U and if we define V by

$$V(f)=U(f)+U(Df)Q+\frac{U(D^2f)Q^2}{2}+\cdots+\frac{U(D^kf)Q^k}{k!}$$

($D=\frac{1}{2}\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right), f=f(\xi, \eta)\in C_c^\infty$), then U and V are different C_c^∞-spectral representations corresponding to the same scalar operator.

In the present paper, we shall show that, for two commuting spectral representations U and V corresponding to the same scalar operator, $U(f)-V(f)$ is quasi-nilpotent and in many cases, there is a relation expressed in the above form. (See §3 and §6.)

On the due course of our argument, we shall see (§4) that the operators $S_U=U(\lambda)$ and $S_U^*=U(\bar{\lambda})$ ($\lambda=\xi+i\eta$ and $\bar{\lambda}=\xi-i\eta$) together determine the representation U. Thus, in connection with our result mentioned above, we see that $S_U^*-S_V^*$ is nilpotent in a certain sense when $S_U=S_V$ and S_U^* commutes with S_V^* (§5).

We are able to consider the uniquely determined canonical representation for a scalar opertor S satisfying $S=S_U=S_U^*$ (§7). Such operators can be regarded as a generalization of Hermitian operators and will be called *real* scalar operators.

§ 1. Preliminaries.

1) *The space* $C_c^m(0\leq m\leq\infty)$. In the present paper, the basic function algebra (cf. [3]) is restricted to $C_c^m(0\leq m\leq\infty)$, the space of all complex valued m-times continuously differentiable (infinitely differentiable, if $m=\infty$) functions with compact supports on the two dimensional real space R^2. When we speak of a point of R^2 as a variable of functions, we often identify it with a point in the complex number field C, which is topologically equivalent to R^2. Thus, $f(\lambda)$ and $f(\xi, \eta)$ express the same function, where $\lambda=\xi+i\eta\in C$ and $(\xi, \eta)\in R^2$. Throughout this paper, δ always denotes a compact set and σ an

open set in $R^2=C$. For any $f \in C_c^m$, λf is the function $\lambda f(\lambda)\equiv(\xi+i\eta)f(\xi, \eta)$ and $\bar{\lambda} f$ is the function $\bar{\lambda} f(\lambda)=(\xi-i\eta)f(\xi, \eta)$. The support of f is denoted by *supp f*.

Given a compact set δ, $C_\delta^m=\{f \in C^m;\ supp\ f \subseteq \delta\}$ (m: finite) is a Banach space with the norm

$$\|f\|_{m,\delta}=\sup_{\substack{0\leq m_1+m_2\leq m\\ \lambda\in\delta}}\left|\frac{\partial^{m_1+m_2}f}{\partial\xi^{m_1}\partial\eta^{m_2}}(\lambda)\right|$$

and C_δ^∞ is a Frèchet space with the norms $\{\|f\|_{k,\delta};\ k=1, 2, \cdots\}$. For any m, we introduce the inductive limit topology in C_c^m defined by $\{C_\delta^m\}_\delta$. We know that if $m_1\leq m_2$, then $C_c^{m_2}$ is dense in $C_c^{m_1}$ in the topology of $C_c^{m_1}$.

The following lemma, which is an extension of the Weierstrass approximation theorem, will be used in several places in this paper.

LEMMA 1. (Cf. [5] p. 108, Théorèm III.) *Let $f_0 \in C_c^m$ be fixed. For any $f \in C^m$, there exists a sequence $\{P_n\}$ of polynomials in ξ and η such that $P_n f_0 \to f f_0$ in C_c^m $(0\leq m\leq\infty)$.*

2) *The space E.* The space E on which we consider operators is supposed to be a separated locally convex space such that $L(E)$, the space of all continuous linear operators on E, is quasi-complete with respect to an $\mathfrak{S}$-topology ([1]; $\mathfrak{S}$ is a family of bounded sets in E). We always consider the given $\mathfrak{S}$-topology in $L(E)$ unless otherwise specified. Then $L(E)$ is quasi-complete with respect to the simple convergence topology and E is also quasi-complete. The topology of bounded convergence (the case $\mathfrak{S}=$ all bounded sets in E) will be denoted by τ_b. The strong dual of E is denoted by E'. For $T \in L(E)$, $sp(T)$ is the spectrum of T (see [6]).

We collect here the fundamental notions and some important results given in [3].

3) *C_c^m-spectral representations on E* (cf. Def. 1.1 of [3]). A mapping U of C_c^m into $L(E)$ is called a *C_c^m-spectral representation* if it satisfies the follwing two conditions:

a) $f\to U(f)$ is a continuous linear multiplicative mapping of the topological algebra C_c^m into the topological algebra $L(E)$;

b) There exists a net $\{f_\alpha\}$ in C_c^m such that $U(f_\alpha)x\to x$ for all $x \in E$.

If $m_1\leq m_2$, then any $C_c^{m_1}$-spectral representation is a $C_c^{m_2}$-spectral representation.

4) *Spaces $E_{U,\delta}$* (Def. 2.1 of [3]). We define $E_{U,\sigma}=\{U(f)x;\ f \in C_c^m,\ supp\ f \subset \sigma,\ x \in E\}$ for an open set σ and $E_{U,\delta}=\bigcap_{\delta\subset\sigma}E_{U,\sigma}$ for a compact set δ. $E_{U,\delta}$ is a closed subspace of E. It is easy to see that $E_{U,\sigma}=\bigcup_{\sigma\supset\delta}E_{U,\delta}$. Therefore, $E_{U,\sigma}$ is determined by $\{E_{U,\delta}\}_{\delta;\ \text{compact}}$. Let $E_U\equiv E_{U,C}=\bigcup_\delta E_{U,\delta}$ (this is denoted by $E_{U,\infty}$ in [3]). It is a dense subspace of E. If *supp U* is compact, then $E_U=$

$E_{U,\delta} = E$ for all $\delta \supseteq supp\ U$.

LEMMA 2. (Prop. 2.1 and Prop. 2.2 of [3]) *Let δ be a compact set and U be a C_c^m-spectral representation.*

a) *$x \in E_{U,\delta}$ if and only if $U(f)x=0$ for all $f \in C_c^m$ such that supp $f \cap \delta = \phi$;*

b) *$x \in E_{U,\delta}$ if and only if $U(f)x=x$ for all $f \in C_c^m$ such that $f=1$ on a neighborhood of δ.*

5) *C_c^m-scalar operators* (Def. 1.2 and 2.2 of [3]). A linear transformation S on E with domain D_S is called a *C_c^m-scalar operator* if there is a C_c^m-spectral representation U such that $D_S \supseteq E_U$ and $S/E_{U,\delta} = U(\lambda f_\delta)/E_{U,\delta}$ for any compact set δ, where f_δ is any function in C_c^m such that $f_\delta = 1$ on a neighborhood of δ.

If $m_1 \leq m_2$, then any $C_c^{m_1}$-scalar operator is $C_c^{m_2}$-scalar.

Given a C_c^m-spectral representation U, let $S_U x = U(\lambda f_\delta)x$ for $x \in E_{U,\delta}$. Then S_U with domain E_U is a C_c^m-scalar operator such that U is a corresponding spectral representation.

LEMMA 3. (Cf. Th. 1.1 of [3]) *Let U be a C_c^m-spectral representation on E and let $\varphi \in C_c^m$. Then*

a) *The representation U_φ defined by*

$$U_\varphi(f) = U(f \circ \varphi - f(0)) + f(0)I \quad \textit{for} \quad f \in C_c^m$$

is a C_c^m-spectral representation.

b) *$U(\varphi)$ is a C_c^m-scalar operator such that U_φ is a corresponding spectral representation. Furthermore, we have* $\mathrm{sp}(U(\varphi)) \subseteq \varphi(C)$.

6) *C_c^m-spectral operators* (Def. 3.1 of [3]). A linear transformation T on E with domain D_T is called a *C_c^m-spectral operator* if there is a C_c^m-spectral representation U such that $D_T \supseteq E_U$, $TU(f) = U(f)T$ on E_U and $TU(f) \in L(E)$ for all $f \in C_c^m$ and $sp(T/E_{U,\delta}) \subseteq \delta$ for any compact set δ such that $E_{U,\delta} \neq \{0\}$. It is known that a C_c^m-scalar operator is a C_c^m-spectral operator.

LEMMA 4. (Th. 3.1 of [3]) *If U and V correspond to the same C_c^m-spectral operator, then $E_{U,\delta} = E_{V,\delta}$ for all compact sets δ.*

By this lemma, we sometimes write $E_{T,\delta}$ (resp. E_T) instead of $E_{U,\delta}$ (resp. E_U) for a spectral operator T.

LEMMA 5. (Th. 4.1 of [3]) *If T is a C_c^m-spectral operator and U is a corresponding C_c^m-spectral representation, then the transformation $Q = T - S_U$ defined on E_U satisfies $\lim_{n\to\infty} |\langle Q^n x, x' \rangle|^{1/n} = 0$ for all $x \in E_U$ and $x' \in E'$.*

In particular, if $E_U = E$, then Q is a quasi-nilpotent operator.

§ 2. Auxiliary results on spectral representations.

PROPOSITION 1. *Let U be a C_c^m-spectral representation on E and let U_φ be the spectral representation given in* Lemma 3 *for $\varphi \in C_c^m$. Then, for any compact* *set δ, we have*

$$E_{U_\varphi,\delta}=E_{U,\varphi^{-1}(\delta)} \qquad \text{if} \quad 0\notin\delta;$$

$$E_{U_\varphi,\delta}=\bigcap_{\substack{\sigma\supset\delta\\ \sigma:\text{open}}}\overline{E_{U,\varphi^{-1}(\sigma)}} \qquad \text{if} \quad 0\in\delta.$$

PROOF: (i) The case $0\notin\delta$. In this case, $\varphi^{-1}(\delta)$ is compact. For any open set $\sigma\supset\delta$ such that $0\notin\sigma$, we have

$$E_{U_\varphi,\sigma}=\{U(f\circ\varphi)x;\ supp\, f\subset\sigma\}\subseteq E_{U,\varphi^{-1}(\sigma)}.$$

Since $\varphi^{-1}(\sigma)$ is open, it is easy to see that

$$E_{U,\varphi^{-1}(\delta)}=\bigcap_{\sigma\supset\delta}E_{U,\varphi^{-1}(\sigma)}.$$

Hence, we have

$$E_{U_\varphi,\delta}=\bigcap_{\sigma\supset\delta}E_{U_\varphi,\sigma}\subseteq\bigcap_{\sigma\supset\delta}E_{U,\varphi^{-1}(\sigma)}=E_{U,\varphi^{-1}(\delta)}.$$

Suppose now that $x\in E_{U,\varphi^{-1}(\delta)}$. Let $f\in C_c^m$ be equal to 1 on a neighborhood σ of δ and $f(0)=0$. Then $f\circ\varphi=1$ on $\varphi^{-1}(\sigma)$. Hence, by Lemma 2, b), we see that $U_\varphi(f)x=U(f\circ\varphi)x=x$. Using Lemma 2, b) again, we conclude that $x\in E_{U_\varphi,\delta}$. Therefore, $E_{U_\varphi,\delta}\supseteq E_{U,\varphi^{-1}(\delta)}$, so that the equality holds.

(ii) The case $0\in\delta$. Let $x\in E_{U_\varphi,\delta}$. For any open set $\sigma\supset\delta$, we can find a function $f\in C_c^m$ such that $f=1$ on a neighborhood of δ and $supp\, f\subset\sigma$. Then $x=U_\varphi(f)x$ by Lemma 2, b). Since $f(0)=1$, $U_\varphi(f)x=U(f\circ\varphi-1)x+x$. Hence $x=U(f\circ\varphi-1)x+x$. Now, Let $\{f_\alpha\}\subseteq C_c^m$ is a net such that $U(f_\alpha)x\to x$ for all $x\in E$. Then

$$\begin{aligned}x&=\lim_\alpha U(f_\alpha)x=\lim_\alpha U(f_\alpha)\left[U(f\circ\varphi-1)x+x\right]\\&=\lim_\alpha \{U[f_\alpha(f\circ\varphi)-f_\alpha]x+U(f_\alpha)x\}\\&=\lim_\alpha U[f_\alpha(f\circ\varphi)]x.\end{aligned}$$

Since $supp\,[f_\alpha(f\circ\varphi)]\subseteq\varphi^{-1}(\sigma)$, we have $U[f_\alpha(f\circ\varphi)]x\in E_{U,\varphi^{-1}(\sigma)}$. Hence the above equality limplies that $x\in\overline{E_{U,\varphi^{-1}(\sigma)}}$. Therefore, $E_{U_\varphi,\delta}\subseteq\bigcap_{\sigma\supset\delta}\overline{E_{U,\varphi^{-1}(\sigma)}}$.

Conversely, suppose $x\in\bigcap_{\sigma\supset\delta}\overline{E_{U,\varphi^{-1}(\sigma)}}$. Let $f\in C_c^m$ be equal to 1 on a neighborhood σ of δ. Then, we can find a net $\{x_\alpha\}\subseteq E_{U,\varphi^{-1}(\sigma)}$ such that $x_\alpha\to x$. Since $f\circ\varphi-1=0$ on $\varphi^{-1}(\sigma)$, $U(f\circ\varphi-1)x_\alpha=0$ for all α. Therefore, $U_\varphi(f)x=x$. It follows then that $x\in E_{U_\varphi,\delta}$ by Lemma 2, b).

COROLLARY. *If U and V are two C_c^m-spectral representations such that $E_{U,\delta}=E_{V,\delta}$ for all compact sets δ, then $E_{U_\varphi,\delta}=E_{V_\varphi,\delta}$ for any $\varphi\in C_c^m$ and for any compact set δ.*

Lemma 6. *Let U and V be two C_c^m-spectral representations such that $E_{U,\delta}=E_{V,\delta}$ for all compact sets δ. If $f, f_0 \in C_c^m$ and $f_0=1$ on a neighborhood of* supp f, *then* $U(f)V(f_0)=U(f)$.

Proof: Let δ be any compact set containing *supp* f_0 and let $f_\delta \in C_c^m$ be equal to 1 on a neighborhood of δ. Then for any $x \in E_{U,\delta}$,

$$\begin{aligned}[U(f)V(f_0)-U(f)]x &= U(f)V(f_0)x - U(f)V(f_\delta)x \\ &= U(f)V(f_0-f_\delta)x.\end{aligned}$$

Since *supp* $(f_0-f_\delta)\cap$ *supp* $f=\phi$, we have $U(f)V(f_0-f_\delta)x=0$ by Lemma 2, a). Therefore, $U(f)V(f_0)x=U(f)x$ for all $x \in E_{U,\delta}$. Since δ is arbitrary and since E_U is dense in E, we have the lemma.

§ 3. Difference of two spectral representations (I).

The previous proposition, together with Lemma 5, yields one of our main results:

Theorem 1. *If U and V are two commuting C_c^m-spectral representations such that $E_{U,\delta}=E_{V,\delta}$ for all compact sets δ, then $U(\varphi)-V(\varphi)$ is quasi-nilpotent for any $\varphi \in C_c^m$.*

Proof: We consider the C_c^m-spectral representations U_φ and V_φ constructed in Lemma 3 from U and V respectively. Let $T=U(\varphi)$. Then $sp(T)$ is compact and $T \in L(E)$, hence $E_T=E$. Since U_φ is a C_c^m-spectral representation, we have (see Prop. 2.3 of [3])

$$sp\left(U_\varphi(\lambda f_\delta)/E_{U_\varphi,\delta}\right) \subseteq \delta$$

for any compact set δ such that $E_{U_\varphi,\delta} \neq \{0\}$, where $f_\delta \in C_c^m$ is equal to 1 on a neighborhood of δ.

On the other hand, by the definition of U_φ, we have

$$\begin{aligned}U_\varphi(\lambda f_\delta) &= U(\lambda f_\delta \circ \varphi) = U[\varphi(f_\delta\circ\varphi - f_\delta(0))] + f_\delta(0)U(\varphi) \\ &= U(\varphi)[U(f_\delta\circ\varphi - f_\delta(0)) + f_\delta(0)I] \\ &= U(\varphi)U_\varphi(f_\delta).\end{aligned}$$

Hence, $U_\varphi(\lambda f_\delta)/E_{U_\varphi,\delta}=T/E_{U_\varphi,\delta}$. From the corollary to Proposition 1, it follows that $U_\varphi(\lambda f_\delta)/E_{U_\varphi,\delta}=T/E_{V_\varphi,\delta}$. Therefore, we obtain $sp\,(T/E_{V_\varphi,\delta})\subseteq\delta$. This implies that T is C_c^m-spectral with respect to the representation V_φ. Since $V(\varphi)=S_{V_\varphi}$, Lemma 5 implies that $T-V(\varphi)=U(\varphi)-V(\varphi)$ is quasi-nilpotent.

Corollary. *If U and V are two commuting C_c^m-spectral representations corresponding to a C_c^m-spectral operator T, then $U(f)-V(f)$ is quasi-nilpotent for any $f \in C_c^m$.*

Proof: This is an immediate consequence of Lemma 4 and the above theorem.

§ 4. The operator S_U^*.

Given a C_c^m-spectral representation U, we define the operator S_U^* by

$$S_U^* x = U(\bar{\lambda} f_\delta)x \quad \text{for} \quad x \in E_{U,\delta},$$

where $f_\delta \in C_c^m$ is equal to 1 on a neighborhood of δ.

LEMMA 7. *The operator S_U^* with the domain E_U is a C_c^m-scalar operator.*

PROOF: If we define $U^*(f) = U(f^*)$ for $f \in C_c^m$, where $f^*(\lambda) = f(\bar{\lambda})$ (i.e., $f^*(\xi, \eta) = f(\xi, -\eta)$), then U^* is a C_c^m-spectral representation and $S_U^* = S_{U^*}$.

PROPOSITION 2. *Let U and V be two spectral representations. Then $U = V$ if and only if $S_U = S_V$ and $S_U^* = S_V^*$.*

PROOF: The "only if" part is trivial. Suppose now that $S_U = S_V$ and $S_U^* = S_V^*$. It follows that $E_U = E_V$. For any $f \in C_c^m$ and for any $x \in E_U = E_V$, there exists a compact sed δ such that $\delta \supseteq supp\, f$ and $x \in E_{U,\delta} \cap E_{V,\delta}$. Let $f_\delta \in C_c^m$ be equal to 1 on a neighborhood of δ. By Lemma 1, we can find a sequence $\{P_n\}$ of polynomials such that $P_n f_\delta \to f f_\delta = f$ in C_c^m. Now, each P_n can be written in the form $\sum b_{jk} \lambda^j \bar{\lambda}^k$, so that

$$U(P_n f_\delta)x = \sum b_{jk} S_U{}^j S_U^{*k} x,$$

$$V(P_n f_\delta)x = \sum b_{jk} S_V{}^j S_V^{*k} x.$$

Hence, $U(P_n f_\delta)x = V(P_n f_\delta)x$ by assumption. Hence, by the continuity of U and V, we have $U(f) = V(f)$ on E_U, hence on E.

COROLLARY. *Let U and V be two C_c^m-spectral representations corresponding to a scalar operator S. Then, $U = V$ if and only if $S_U^* = S_V^*$.*

PROPOSITION 3. *Let U* (resp. V) *be a C_c^m-*(resp. $C_c^{m'}$-) *spectral representation and suppose that $E_U = E_V$. Then U and V are commuting if and only if S_U, S_V, S_U^* and S_V^* commute each other.*

PROOF: Given $f \in C_c^m$, $g \in C_c^{m'}$ and $x \in E_U = E_V$, there is a compact set δ such that $\delta \supseteq (supp\, f) \cup (supp\, g)$ and $x \in E_{U,\delta} \cap E_{V,\delta}$. Again by Lemma 1, we can find sequences $\{P_n\}$ and $\{Q_n\}$ of polynomials in ξ and η such that $P_n f_\delta \to f$ in C_c^m and $Q_n f_\delta \to g$ in $C_c^{m'}$, where $f_\delta \in C_c^\infty$ is equal to 1 on a neighborhood of δ. Then, as in the proof of the previous proposition, we obtain $U(f)V(g) = V(g)U(f)$.

COROLLARY. *Let U* (resp. V) *be a C_c^m-* (resp. $C_c^{m'}$-) *spectral representation corresponding to a given scalar operator S. Then U and V are commuting if and only if S_U^* and S_V^* commute.*

5. Difference of two spectral representations (II).

In the case the spectral representations U and V correspond to the same *scalar* operator, we are able to discuss in more details using the operators S_U^* and S_V^*.

PROPOSITION 4. *Let U and V be two C_c^m-spectral representations corresponding to a scalar operator S. Suppose that S_U^* and S_V^* commute and let $Q=S_U^*-S_V^*$.*

a) *If m is finite, then $Q^{2m+1}x=0$ for all $x \in E_U$;*

b) *If $m=\infty$ and if the given topology of $L(E)$ is τ_b, then for any bounded set B in $E_{U,\delta}$* (δ is a fixed compact set) *and for any equi-continuous part A' in E' there exists a positive integer $k_0=k_0(B, A')$ such that $\langle Q^k x, x'\rangle=0$ for all $k \geq k_0$, $x \in B$ and $x' \in A'$.*

PROOF: Let δ be any compact set and let $x \in E_{U,\delta}$. We consider the functions

$$f_z(\xi, \eta)=e^{2i(v\xi-u\eta)} \cdot f_\delta(\xi, \eta) \equiv e^{\bar{\lambda}z-\bar{z}\lambda} \cdot f_\delta(\lambda),$$
$$f_z^{-1}(\xi, \eta)=e^{2i(u\eta-v\xi)} \cdot f_\delta(\xi, \eta) \equiv e^{\bar{z}\lambda-z\bar{\lambda}} \cdot f_\delta(\lambda),$$

where $\lambda=\xi+i\eta$ and $z=u+iv$ are complex numbers and $f_\delta \in C_c^m$ is equal to 1 on a neighborhood of δ. Then, obviously $f_z, f_z^{-1} \in C_c^m$. By considering the power series expansions of the exponential functions and the convergence of the series in the space C_c^m, we can see that[1)]

$$U(f_z)x=exp[zU(\bar{\lambda}f_\delta)-\bar{z}U(\lambda f_\delta)]x=exp(zS_U^*-\bar{z}S_U)x,$$
$$V(f_z^{-1})x=exp[\bar{z}V(\lambda f_\delta)-zV(\bar{\lambda}f_\delta)]x=exp(\bar{z}S_V-zS_V^*)x.$$

Since $S_U=S_V$ and $V(f_z^{-1}) \in E_{U,\delta}$, we have

$$U(f_z)V(f_z^{-1})x=exp[z(S_U^*-S_V^*)]x=exp(zQ)x.$$

Let $\delta_0=supp\, f_\delta$ and let $\|f\|_k \equiv \|f\|_{k,\delta_0}$ for $f \in C_{\delta_0}^m$ (cf. §1,1). Then it is easy to see that for any z with $|z| \geq 1$,

$$\|f_z\|_k \leq M_k|z|^k, \quad \|f_z^{-1}\|_k \leq M'_k|z|^k,$$

where M_k and M'_k are positive numbers independent of z.

a) Now, let m be finite. Then U is a continuous mapping of the Banach space $C_{\delta_0}^m$ into $L(E)$. Therefore, $\{U(f); f \in C_{\delta_0}^m, \|f\|_m \leq 1\}$ is a bounded set in $L(E)$. Since E is quasi-complete, it follows that the set $\{U(f)x; f \in C_{\delta_0}^m, \|f\|_m \leq 1, x \in B\}$ is bounded in E for any bounded set B in E. (See [1], Corollary 1 in p. 22.) Hence there is a positive number $M_{B,x'}(x' \in E')$ such that

$$|\langle U(f_z)x, x'\rangle| \leq M_{B,x'}|z|^m$$

1) For an operator T, $exp\ Tx$ is defined by the series $exp\ Tx=\sum_{n=0}^{\infty}\frac{T^n x}{n!}$. Since E is quasi-complete, all the series of exponentials appearing here converge in E.

for all $x \in B$ and $|z| \geq 1$. Similarly, there exists a positive number $N_{x,x'}(x \in E_{U,\delta}$, $x' \in E')$ such that

$$|\langle V(f_z^{-1})x, x'\rangle| \leq N_{x,x'}|z|^m$$

for all $|z| \geq 1$. Therefore, the set $B_x = \{V(f_z^{-1})x/|z|^m;\ |z| \geq 1\}$ is bounded in E, so that we have

$$|\langle U(f_z)V(f_z^{-1})x, x'\rangle| \leq M_{B_x,x'}|z|^{2m}$$

for all $|z| \geq 1$, or

$$|\langle exp(zQ)x, x'\rangle| \leq M_{B_x,x'}|z|^{2m}$$

for all $|z| \geq 1$. Since $\langle exp(zQ)x, x'\rangle$ is an entire function of z, it follows then that it is a polynomial of degree at most $2m$. Therefore, $Q^{2m+1} = 0$.

b) Next, suppose that $m = \infty$ and the given topology of $L(E)$ is τ_b. Let B be a bounded set in $E_{U,\delta}$. Since V is a continuous mapping of $C_{\delta_0}^\infty$ into $L(E)$, there is a positive integer $k_1 = k_1(B, x')$ $(x' \in E')$ such that

$$|\langle V(f)x, x'\rangle| \leq N_{B,x'}\|f\|_{k_1}$$

for all $x \in B$ and $f \in C_{\delta_0}^\infty$. Therefore,

$$|\langle V(f_z^{-1})x, x'\rangle| \leq N'_{B,x'}|z|^{k_1}$$

for all $x \in B$ and $|z| \geq 1$. Hence the set $B_1 = \{V(f_z^{-1})x/|z|^{k_1};\ x \in B,\ |z| \geq 1\}$ is bounded in E. Similarly there is another integer $k_2 = k_2(B_1, A')$ for an equicontinuous part A' in E' such that

$$|\langle U(f)x, x'\rangle| \leq M_{B_1,A'}\|f\|_{k_2}$$

for all $x \in B_1$, $x' \in A'$ and $f \in C_{\delta_0}^\infty$. Hence

$$|\langle U(f_z)V(f_z^{-1})x, x'\rangle| \leq M'_{B,A'}|z|^{k_1+k_2}$$

for all $x \in B$, $x' \in A'$ and $|z| \geq 1$. Therefore, by taking $k_0 = k_1 + k_2 + 1$, we have $\langle Q^k x, x'\rangle = 0$ for all $x \in B$, $x' \in A'$ and $k \geq k_0$ by a similar argument as in a).

COROLLARY 1. *If E is a Banach space, then we can choose k_0 independent of B and A' in the statement* b) *of the above proposition.*

PROOF: If E is a Banach space, then the space $L(E)$ with the topology τ_b is also a Banach space. Since U (resp. V) is continuous on $C_{\delta_0}^\infty$, there exists a positive integer k_1 (resp. k_2) such that $\|U(f)\| \leq M\|f\|_{k_1}$ (resp. $\|V(f)\| \leq M'\|f\|_{k_2}$) for $f \in C_{\delta_0}^\infty$. Hence, we have

$$\|U(f_z)V(f_z^{-1})\| \leq M''|z|^{k_1+k_2}.$$

Therefore, we conclude that $Q^{k_1+k_2+1}x = 0$ for all $x \in E_{U,\delta}$ and $k_0 = k_1 + k_2 + 1$ depends only on δ.

COROLLARY 2. *Let S be a C_c^m-scalar operator with compact spectrum and let U and V be two C_c^m-spectral representations corresponding to S such that S_U^* and S_V^* commute and let $Q=S_U^*-S_V^*$.*

a) *If m is finite, then $Q^{2m+1}=0$.*

b) *If $m=\infty$ and if the given topology of $L(E)$ is τ_b, then Q is a quasi-nilpotent operator such that for any bounded set B in E and any equi-continuous part A' in E', there exists a positive integer $k_0=k_0(B, A')$ such that $\langle Q^k x, x'\rangle=0$ for all $x \in B$, $x' \in A'$ and $k \geq k_0$. If, in particular, E is a Banach space, then Q is a nilpotent operator.*

PROOF: It is enough to take $\delta=sp(S)$ in the above proposition and corollary.

REMARK. In general, the condition that S_U^* and S_V^* commute cannot be removed to obtain the nilpotency of Q. In fact, there is an example of C_c^m-spectral representations U and V corresponding to the same scalar operator such that Q is not even quasi-nilpotent:

Let E be the two dimensional complex linear space and let

$$U(f)(\alpha, \beta)=(f(1)\alpha+(Df)(1)\beta, f(1)\beta),$$
$$V(f)(\alpha, \beta)=(f(1)\alpha, f(1)\beta+(Df)(1)\alpha)$$

for $f \in C_c^1$ and $(\alpha, \beta) \in E$, where $D=\frac{1}{2}\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right)$. Then U and V are C_c^1-spectral representations on E corresponding to the identity I. Since $Q(\alpha, \beta)=[U(\lambda)-V(\lambda)](\alpha, \beta)=(\beta, -\alpha)$, Q is not quasi-nilpotent.

§ 6. Difference of two spectral representations (III).

THEOREM 2. *Let U and V be two C_c^m-spectral representations corresponding to a scalar operator S such that S_U^* commutes with S_V^*. Let $Q=S_U^*-S_V^*$ and $D=\frac{1}{2}\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right)$.*

a) *If m is finite, then*

$$U(f)=\sum_{k=0}^{2m}\frac{1}{k!}Q^k V(D^k f)$$

for $f \in C_c^{3m}$.

b) *If $m=\infty$, if E is a Banach space and if the given topology of $L(E)$ is τ_b, then*

$$U(f)=\sum_{k=0}^{k_0}\frac{1}{k!}Q^k V(D^k f)$$

for $f \in C_c^\infty$, where k_0 is a positive integer depending on f. If, in addition, sp(S) *is compact, then k_0 can be chosen independent of f.*

PROOF: a) Let $f \in C_c^{3m}$, let δ be a compact neighborhood of $supp\, f$ and let $f_\delta \in C_c^m$ be equal to 1 on a neighborhood of δ. By Lemma 1, we can find a sequence $\{P_n\}$ of polynomials in ξ and η such that

$$(D^k P_n) f_\delta \to (D^k f) f_\delta = D^k f \quad (n \to \infty)$$

in C_c^m for all $k=0, 1, \cdots, 2m$, since $f \in C_c^{3m}$.

If P is a polynomial in ξ and η, then it can be written in the form $\sum b_{\mu\nu} \lambda^\mu \bar{\lambda}^\nu$. Hence, for $x \in E_{U,\delta}$,

$$\begin{aligned} U(Pf_\delta)x &= \sum b_{\mu\nu} [U(\lambda f_\delta)]^\mu [U(\bar{\lambda} f_\delta)]^\nu x \\ &= \sum b_{\mu\nu} [V(\lambda f_\delta)]^\mu [V(\bar{\lambda} f_\delta) + Q]^\nu x \\ &= \sum_{\mu,\nu} \sum_k \binom{\nu}{k} b_{\mu\nu} Q^k V(\lambda^\mu \bar{\lambda}^{\nu-k} f_\delta) x \\ &= \sum_k \frac{1}{k!} Q^k V[(D^k P) f_\delta] x. \end{aligned}$$

By Proposition 4, we know that $Q^{2m+1}=0$. Hence,

$$U(P_n f_\delta)x = \sum_{k=0}^{2m} \frac{1}{k!} Q^k V[(D^k P_n) f_\delta] x$$

for $x \in E_{U,\delta}$. Letting $n \to \infty$, we have

$$U(f)x = \sum_{k=0}^{2m} \frac{1}{k!} Q^k V(D^k f) x$$

for $x \in E_{U,\delta}$. Let $f_0 \in C_c^m$ be equal to 1 on a neighborhood of $supp\, f$ and $supp\, f_0 \subseteq \delta$. Then $U(f_0)x \in E_{U,\delta}$ for any $x \in E$ and $U(ff_0) = U(f)$. Also we have $V(D^k f) U(f_0) = V(D^k f)$ by Lemma 6. Hence we have the required formula.

b) In this case, there is $k_0 = k_0(\delta)$ ($\delta = supp\, f$) such that $Q^{k_0+1}=0$ by Corollary 1 to Proposition 4. Hence we obtain the expression of $U(f)$ by an argument similar to a). Here, we should remark that, given $f \in C_c^\infty$, we can find a sequence $\{P_n\}$ of polynomials such that $(D^k P_n) f_\delta \to D^k f$ $(n \to \infty)$ in C_c^∞ for all $k=0, 1, \cdots, k_0$.

COROLLARY. *Let U and V be two C_c^m-spectral representations corresponding to a scalar operator S and suppose S_U^* commutes with S_V^*.*

a) *If m is finite, then $[U(f) - V(f)]^{2m+1} = 0$ for any $f \in C_c^m$.*

b) *If $m = \infty$ and if E is a Banach space with the given topology τ_b in $L(E)$, then $U(f) - V(f)$ is a nilpotent operator for any $f \in C_c^\infty$.*

PROOF: a) If $f \in C_c^{3m}$, then Theorem 2, a) implies that $[U(f) - V(f)]^{2m+1} = 0$. Since C_c^{3m} is dense in C_c^m, this is ture for any $f \in C_c^m$.

b) This follows from Theorem 2, b).

REMARK. In the case where $m = \infty$ and E is not a Banach space, it is possible to obtain results of the above type for certain C_c^∞-functions. For example, if $f \in C_c^\infty$ is a function such that $D^k f=0$ for some k on a compact set δ, then we have the convergence of the series in Theorem 2 on $E_{U,\delta}$ and we can see that $U(f)-V(f)$ is a quasi-nilpotent operator on $E_{U,\delta}$ such that $\langle [U(f)-V(f)]^k x, x'\rangle = 0$ for some $k=k(x, x')$ ($x \in E_{U,\delta}$, $x' \in E'$). We omit the detailed discussion of this type in this paper, since we already know that $U(f)-V(f)$ is quasi-nilpotent (Corollary to Theorem 1) and it seems, at present, to be of little value to investigate further in this direction.

§ 7. Real scalar operators.

Proposition 5. *Let T be a C_c^m-spectral operator with compact spectrum* sp(T). *Then* sp(T) *lies on the real axis if and only if there exists a C_c^m-spectral representation U corresponding to T such that $S_U = S_U^*$.*

Proof: Suppose $sp(T)$ lies on the real axis. Let φ_ε be a C^m-function on R such that $\varphi_\varepsilon = 1$ on a neighborhood of 0 and *supp* $\varphi_\varepsilon \subseteq [-\varepsilon, \varepsilon]$. For any $f \in C_c^m$, let

$$f_\varepsilon(\xi, \eta) = f(\xi, 0)\varphi_\varepsilon(\eta) \in C_c^m.$$

Given a C_c^m-spectral representation V corresponding to T, we define U by $U(f) = V(f_\varepsilon)$. Since *supp* V is contained in the real axis (in C)(Prop. 3.1 of [3]), we see that $U(f)$ does not depend on the choice of φ_ε. It is easy to see that U is a C_c^m-spectral representation commuting with T and *supp* U is contained in the real axis.

Let δ be a compact set such that $E_{U,\delta} \neq \{0\}$. For any open set σ containing δ, we can find $f \in C_c^m$ and $\varepsilon > 0$ such that $f=1$ on a neighborhood of δ and *supp* $f_\varepsilon \subset \sigma$. Then, $x \in E_{U,\delta}$ implies $x = U(f)x = V(f_\varepsilon)x \in E_{V,\sigma}$. Hence $E_{U,\delta} \subseteq E_{V,\sigma}$, which follows that $E_{U,\delta} \subseteq E_{V,\delta}$. Therefore, $E_{V,\delta} \neq \{0\}$ and $sp(T/E_{U,\delta}) \subseteq sp(T/E_{V,\delta}) \subseteq \delta$, so that U is a C_c^m-spectral representation corresponding to T. It is obvious that $S_U = S_U^*$.

Conversely, suppose that $T = S_U + Q$ and $S_U = S_U^*$. Since Q is quasi-nilpotent and T, S_U are regular elements of $L(E)$, we have $sp(T) = sp(S_U) + sp(Q) = sp(S_U)$ (see [6]). From the condition $S_U = S_U^*$, it follows that $S_U = U(\xi)$. Hence $sp(S_U) = sp(T)$ lies on the real axis by Lemma 3, b).

Proposition 6. *Suppose m is finite* (resp. $m=\infty$ and E is a Banach space with the topology τ_b in $L(E)$). *If S is a C_c^m-scalar operator whose spectrum is compact and contained in the real axis, then there exists a unique C_c^m-spectral representation U such that $S = S_U = S_U^*$.*

Proof: Let V be a C_c^m-spectral representation corresponding to S, i.e., $S = S_V$. We can construct a function $f_\varepsilon \in C_c^m$ for each ε, $0<\varepsilon<1$, such that $f_\varepsilon = 1$ on a neighborhood of $sp(S)$, $f_\varepsilon(\xi, \eta) = 0$ if $|\eta| \geq \varepsilon$ and $\|f_\varepsilon\|_l \leq M\varepsilon^{-l}$ for all ε, where $M>0$ is independent of ε. Let $g_{k,\varepsilon}(\xi, \eta) \equiv (2i\eta)^k f_\varepsilon(\xi, \eta)$. Then we have $\|g_{k,\varepsilon}\|_l \leq M_k \varepsilon^{k-l}$ for all $k=0, 1, \cdots$ and ε, where $M_k>0$ is independent of ε. Since

V is continuous from C_c^m into $L(E)$,

$$|\langle V(g_{k,\varepsilon})x, x'\rangle| \leq A(x, x')\varepsilon^{k-m}$$

for $x \in E$ and $x' \in E'$, where $A(x, x')>0$ is independent of ε and k. (resp. $\|V(g_{k,\varepsilon})\| \leq A\varepsilon^{k-m_0}$ for some m_0, where $A>0$ is independent of ε and k.)

Since $sp(S)$ lies on the real axis, *supp* V is contained in the real axis. Therefore, we have $V(g_{k,\varepsilon})=(S_V-S_V^*)^k$ for all ε and $k=0, 1, \cdots$ Hence, if we let $Q=S_V-S_V^*$, then

$$|\langle Q^k x, x'\rangle| \leq A(x, x')\varepsilon^{k-m} \quad (\text{resp. } \|Q^k\| \leq A\varepsilon^{k-m_0}).$$

Since ε is arbitrary $(0<\varepsilon<1)$, it follows that $Q^k=0$ for all $k>m$(resp. $k>m_0$). Now it is easy to see that

$$U(f)=\sum_{k=0}^{m}\frac{1}{k!}V(D^k f)Q^k \quad (\text{resp.} =\sum_{k=0}^{m_0}\frac{1}{k!}V(D^k f)Q^k)$$

satisfies the proposition, where $D=\frac{1}{2}\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right)$. The uniqueness follows from Proposition 2.

DEFINITION. A linear transformation S is called a *real C_c^m-scalar operator* if there exists a C_c^m-spectral representation U such that $S/E_U=S_U=S_U^*$. The C_c^m-spectral representation U satisfying this relation is uniquely determined by Proposition 2 and is called *the canonical representation of S*.

The above two propositions imply that, in the case where m is finite or $m=\infty$ and E is a Banach space with the topology τ_b in $L(E)$, a C_c^m-scalar operator S with compact spectrum is real if and only if $sp(S)$ lies on the real axis.

REMARK. If E is a Hilbert space, then any Hermitian operator on E is a real C_c^0-scalar operator and vice versa. Therefore, the notion of real scalar operators is a generalization of that of Hermitian operators.

PROPOSITION 7. i) *If S is a real C_c^m-scalar operator, then any C_c^m-spectral representation corresponding to S is commuting with the canonical representation of S.*

ii) *If S_1 and S_2 are commuting real C_c^m-scalar operators, then their canonical representations are commuting.*

iii) *If U is a C_c^m-spectral representation and $\varphi \in C_c^m$ is real valued, then $U(\varphi)$ is a real C_c^m-scalar operator and its canonical representation is given by U_φ in* Lemma 3.

iv) *Let S_1 and S_2 be commuting real C_c^∞-scalar operators and suppose* $\mathrm{sp}(S_1)$ *and* $\mathrm{sp}(S_2)$ *are both compact. Then $P(S_1, S_2)$ is a real C_c^∞-scalar operator for any polynomial P in two variables with real coefficients. In particular, S_1+S_2 and S_1S_2 are real C_c^∞-scalar operators.*

PROOF: i) and ii) are immediate consequences of Proposition 3 and its corollary. iii) follows from Lemma 3. iv) is a consequence of the corollary to Proposition 3.1 in [4] and ii) above.

EXAMPLE. Let $E=\mathscr{S}(R^n)=$ the space of rapidly decreasing C^∞-functions on R^n. (Or, we may let $E=(\mathscr{S}(R^n))'$.) Then any differential operator of the form

$$D=P\left(i\frac{\partial}{\partial x_1},\ \cdots,\ i\frac{\partial}{\partial x_n}\right),$$

where P is a polynomial in n variables with *real* coefficients, is a real C_c^∞-scalar operator on E. (Cf. Example 2.5 of [3])

An indication of further development: It may be possible to consider similar canonical representations for other type of generalized scalar operators, e.g., for C_c^m-scalar operators whose spectra lie in a C^m-curve in C.

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