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On the Multiplicative Products of Distributions

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In the theory of distributions of L. Schwartz [6], multiplication for two distributions leads to difficulties. Schwartz [6] has observed that the multiplicative product is well defined if locally one is "more regular" than the other is "irregular". An approach to define multiplication for distributions has been made by Y. Hirata and H. Ogata [2]. In like manner J. Mikusiński [5] has also given a definition of multiplication. The main purpose of this paper is to show that these two definitions lead to equivalence (§1. Theorem). §2 is devoted to the discussions on the multiplicators of normal spaces of distributions. We show that, in case of functions, the ordinary product is not in general the product in the above sense even if it is a function. In §1 and §3 we make some remarks on the exchange formula for Fourier transformation.

Throughout this paper we assume that unless otherwise specified a Euclidean space on which distributions are defined is the same N-dimensional space.

1. Multiplicative products. By a *δ-sequence* or a *sequence of regularizations* we understand every sequence of non-negative functions $\rho_n \in \mathcal{D}$ with the following properties:

(1) Supp ρ_n converges to 0 when $n \to \infty$;

(2) $\int \rho_n(x)dx = 1$, the integral being extended to the whole N-dimensional space.

Given any distribution S and any δ-sequence $\{\rho_n\}$, the sequence $S_n = S*\rho_n$ will be called a *regular sequence* of S. Every regular sequence of S converges to S in $\mathcal{D}'$.

Recall the definitions of multiplication for two distributions S and T given by Y. Hirata and H. Ogata ([2], p. 150) and J. Mikusiński ([5], p. 254):

Definition 1 (Hirata and Ogata). *By $[S]T$ we understand the distributional limit of the sequence $\{S_nT\}$, if it exists for every regular sequence of S. Similarly for $S[T]$. If both $[S]T$ and $S[T]$ exist and coincide, then $[ST] = [S]T = S[T]$ is called a multiplicative product of S and T.*

DEFINITION 2 (Mikusiński). *By ST we understand the distributional limit of the sequence $\{S_nT_n\}$, if it exists for every regular sequences of S and T.*

For any $\alpha \in \mathcal{E}$ and any $S \in \mathcal{D}'$, the multiplicative product αS is usually defined by the equation ([6], I, p. 115);

$$<\alpha S, \phi> = <S, \alpha\phi>, \qquad {}^{\forall}\phi \in \mathcal{D}.$$

It is clear that Definitions 1 and 2 applied to α and S lead to the same product αS just considered.

The main purpose of this section is to show that the two definitions are equivalent. To this end, we shall first prove

PROPOSITION 1. *If ST exists, then $[ST]$ exists also and $ST=[ST]$.*

PROOF. It is sufficient to show that $\lim_{n,m\to\infty} S_nT_m$ exists. Assume the contrary, then there would exist a zero neighbourhood $\mathcal{U}$ of $\mathcal{D}'$ such that for every positive integer k we can find $n, m \geqq k$ for which $S_nT_m - ST \notin \mathcal{U}$. Therefore we can choose subsequences S_{n_p}, T_{m_p} in such a way that $n_p, m_p \uparrow \infty$ and $S_{n_p}T_{m_p} - ST \notin \mathcal{U}$. This is a contradiction since each of $\{S_{n_p}\}$ and $\{T_{m_p}\}$ is a regular sequence. The proof is complete.

The next two lemmas are needed for our further discussions.

LEMMA 1. *Let $\{\sigma_n\}$ be a sequence of functions $\in \mathcal{D}$ such that*

(1) $\operatorname{supp} \sigma_n \to 0$ *when* $n \to \infty$,

(2) $\int |\sigma_n| dx \leqq 1$ *and* $\lim_{n\to\infty} \int \sigma_n dx = c$.

If $[S]T$ exists, then $\lim_{n\to\infty} (S*\sigma_n)T = c[S]T$.

PROOF. It suffices to prove the lemma in the case where σ_n are real valued functions. Suppose $\sigma_n \geqq 0$ and $c_n = \int \sigma_n(x)dx > 0$. If we put $\rho_n(x) = \dfrac{\sigma_n(x)}{c_n}$, then $\{\rho_n\}$ is a δ-sequence. Therefore it follows that $(S*\sigma_n)T = c_n(S*\rho_n)T$ tends to $c[S]T$ as $n \to \infty$. Next we shall consider the general case. Now σ_n is written in the form $\sigma_n^+ - \sigma_n^-$, where σ_n^+, σ_n^- are the positive and negative parts of σ_n respectively. We can easily construct the sequence $\{\sigma_n'\}$, $\sigma_n' \in \mathcal{D}$, such that $\sigma_n^+ \leqq \sigma_n'$, $\int (\sigma_n' - \sigma_n^+)dx \leqq \dfrac{1}{n}$ and $\operatorname{supp} \sigma_n' \subset K_{2\varepsilon}$ if $\operatorname{supp} \sigma_n \subset K_\varepsilon$, where K_ε stands for the ball with center 0 and radius ε. If we put

$\sigma''_n=\sigma'_n-\sigma_n$, then $\sigma''_n\geqq 0$, $\int(\sigma''_n-\sigma^-_n)dx=\int(\sigma'_n-\sigma^+_n)dx\leqq\frac{1}{n}$ and $\mathrm{supp}\,\sigma''_n$ tends to 0. For any subsequence $\{\sigma'_{j_n}\}$ for which $\{\int\sigma'_{j_n}dx\}$ converges, it is clear that $\{\int\sigma''_{j_n}dx\}$ converges also. From the result proved above for the positive case it follows that $(S*\sigma_{j_n})T$ converges to $c[S]T$ as $n\to\infty$. Therefore it follows that $\{(S*\sigma_n)T\}$ converges to $c[S]T$. The proof is complete.

LEMMA 2. *Suppose* $[S]T$ *exists. Let* A_ε *be the set of* $\sigma\in\mathcal{D}$ *such that* $\mathrm{supp}\,\sigma\subset K_\varepsilon$ *and* $\int|\sigma(x)|dx\leqq 1$. *Then the set* $\{(S*\sigma)T\}_{\sigma\in A_\varepsilon}$ *is bounded in* $(\mathcal{D}_K)'$, K *being any compact ball in* R^N, *if* ε *is sufficiently small.*

PROOF. Putting

$$F_n=\{\phi;\ \phi\in\mathcal{D}_K,\ |<(S*\sigma)T,\ \phi>|\leqq n \text{ for } \sigma\in A_{1/n}\},$$

a closed disk, we shall first show that $\mathcal{D}_K=\cup F_n$. Assume the contrary, then there would exist an element $\phi\in\mathcal{D}_K$, but not $\in\cup F_n$. Then for every positive integer n we may choose an element $\sigma_n\in A_{1/n}$ in such a way that $|<(S*\sigma_n)T,\ \phi>|>n$. Since $\int|\sigma_n(x)|dx\leqq 1$ and $\mathrm{supp}\,\sigma_n$ tends to 0, there exists a subsequence $\{\sigma_{j_n}\}$ such that $\{\int\sigma_{j_n}(x)dx\}$ converges. On the other hand, by virtue of Lemma 1 $\{(S*\sigma_{j_n})T\}$ converges, so that $\{<(S*\sigma_{j_n})T,\ \phi>\}$ is bounded. This is a contradiction. Therefore $\mathcal{D}_K=\cup F_n$. Now since $\mathcal{D}_K$ is of type (**F**), it follows that F_n is a zero neighbourhood of $\mathcal{D}_K$ for some n. This means that $\{(S*\sigma)T\}_{\sigma\in A_{\frac{1}{n}}}$ is bounded. The proof is complete.

PROPOSITION 2. *If* $[S]T$ *exists, then we have*

(1) $[\alpha S]T$ *exists and* $[\alpha S]T=\alpha[S]T$, ${}^{\forall}\alpha\in\mathcal{E}$;
(2) $S[\alpha T]$ *exists and* $S[\alpha T]=\alpha S[T]$, ${}^{\forall}\alpha\in\mathcal{E}$;
(3) $[ST]$ *exists and* $[\alpha S]T=S[\alpha T]=\alpha[ST]$, ${}^{\forall}\alpha\in\mathcal{E}$.

PROOF. Let $\phi\in\mathcal{D}$ and $\alpha\in\mathcal{E}$. Let l be a positive integer such that $\mathrm{supp}\,\phi$ is contained in the cube $Q_l:\ \{x;\ |x_i|<l\}$. Since for a large n the value $<(\alpha S*\rho_n)T,\ \phi>$ depends only on the behaviors of α in a compact set $\subset Q_l$, so that we may assume that α is a periodic function with period $2l$ for each coordinate. Consider the Fourier expansion of α:

$$\alpha(x)=\sum c_m e^{i\frac{\pi}{l}<m,\,x>},$$

where $\{c_m\}$ is rapidly decreasing, namely $\sum|c_m|(1+|m|)^k<\infty$ for any positive

k ([6], II, p. 83). Then we can write

$$<(\alpha S*\rho_n)T, \phi> = \sum c_m <(e^{i\frac{\pi}{l}<m, x>} S*\rho_n)T, \phi>$$
$$= \sum c_m <(S*e^{-i\frac{\pi}{l}<m, x>}\rho_n)T, e^{i\frac{\pi}{l}<m, x>}\phi>$$

Owing to Lemma 2, $\{(S*e^{-i\frac{\pi}{l}<m, x>}\rho_n)T\}$ is bounded in any $(\mathcal{D}_K)'$. Therefore there exist a positive constant M and a non-negative integer k such that

$$|<(S*e^{-i\frac{\pi}{l}<m, x>}\rho_n)T, e^{i\frac{\pi}{l}<m, x>}\phi>| \leqq M(1+|m|)^k.$$

Consequently

$$|<(\alpha S*\rho_n)T, \phi>| \leqq M\sum |c_m| (1+|m|)^k < \infty.$$

Since by Lemma 1 each $(S*e^{-i\frac{\pi}{l}<m, x>}\rho_n)T$ tends to $[S]T$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \sum c_m <(S*e^{-i\frac{\pi}{l}<m, x>}\rho_n)T, e^{i\frac{\pi}{l}<m, x>}\phi>$$
$$= \sum c_m <[S]T, e^{i\frac{\pi}{l}<m, x>}\phi> = <\alpha[S]T, \phi>,$$

Hence $[\alpha S]T$ exists and coincides with $\alpha[S]T$, which completes the proof of (1).

Now we shall show that $S[\alpha T]$ exists. For every $\phi \in \mathcal{D}$ and $\alpha \in \mathcal{E}$, we have

$$<S(\alpha T)_n, \phi> = <\phi S, \alpha T*\rho_n'> = <(\phi S*\check{\rho}_n')T, \alpha>.$$

Passing to the limit as $n \to \infty$, we see that $S[\alpha T]$ exists for every $\alpha \in \mathcal{E}$, since $[\phi S]T$ exists for every $\phi \in \mathcal{D}$ by (1). By a similar reasoning as in the proof of (1), we have $S[\alpha T]=\alpha(S[T])$. The proof of (2) is complete.

Finally we shall show that (3) holds. From (1) and (2) we have

$$<[\alpha S]T, \phi> = <\alpha, S[\phi T]> = <\alpha, \phi(S[T])> = <\alpha(S[T]), \phi>.$$

Consequently, $[\alpha S]T=\alpha(S[T])$. Especially when $\alpha=1$, $[S]T=S[T]$, that is, $[ST]$ exists. Therefore we have $[\alpha S]T=S[\alpha T]=\alpha[ST]$.

Thus the proof is complete.

Owing to this proposition, we see that it is sufficient for us to show only the existence of either of $[S]T$ and $S[T]$ in order that the multiplicative product of S and T may be defined according to Definition 1.

PROPOSITION 3. *If* $[S]T$ *exists, then* $(\alpha S)(\beta T)$ *exists for every* $\alpha, \beta \in \mathcal{E}$.

PROOF. Let $\phi \in \mathcal{D}$. It is sufficient to show that $\lim_{n\to\infty} <(\alpha S)_n(\beta T)_n, \phi>$ exists. Let l be a positive integer such that $\operatorname{supp}\phi$ is contained in the cube Q_l. Since for a large n the value $<(\alpha S)_n(\beta T)_n, \phi>$ depend only on the behaviors of α and β in a compact set $\subset Q_l$, so that we may assume that $\alpha, \beta \in \mathcal{D}$. Furthermore we may also assume that ϕ is a periodic function with period $2l$ for each coordinate. Let $\phi(x)=\sum c_m e^{i\frac{\pi}{l}<m,x>}$ be the Fourier expansion of ϕ, then $\sum|c_m|(1+|m|)^k<\infty$ for any positive integer k as already remarked. Now we can write

$$\begin{aligned}<(\alpha S)_n(\beta T)_n, \phi> &= \sum c_m <(\alpha S)_n(\beta T)_n, e^{i\frac{\pi}{l}<m,x>}>\\ &= \sum c_m <(\alpha S)_n, e^{i\frac{\pi}{l}<m,x>}(\beta T)_n>\\ &= \sum c_m <(\alpha S)_n, e^{i\frac{\pi}{l}<m,x>}\beta T * e^{i\frac{\pi}{l}<m,x>}\rho_n'>\\ &= \sum c_m <(\alpha S*\rho_n*\check{\rho}_n' e^{-i\frac{\pi}{l}<m,x>})T, e^{i\frac{\pi}{l}<m,x>}\beta>.\end{aligned}$$

Owing to Lemma 2, $\{(\alpha S*\rho_n*\check{\rho}_n' e^{-i\frac{\pi}{l}<m,x>})T\}$ is bounded in any $(\mathcal{D}_K)'$, since, by Proposition 2, $[\alpha S]T$ exists. Therefore we have

$$|<(\alpha S*\rho_n*\check{\rho}_n' e^{-i\frac{\pi}{l}<m,x>})T, e^{i\frac{\pi}{l}<m,x>}\beta>| \leqq M(1+|m|)^k,$$

where M is a positive constant and k is a non-negative integer. Consequently

$$|<(\alpha S)_n(\beta T)_n, \phi>| \leqq M\sum|c_m|(1+|m|)^k<\infty.$$

By virtue of Lemma 1, each $(\alpha S*\rho_n*\check{\rho}_n' e^{-i\frac{\pi}{l}<m,x>})T$ tends to $[\alpha S]T$ as $n\to\infty$, so that we have

$$\begin{aligned}&\lim_{n\to\infty} <(\alpha S)_n(\beta T)_n, \phi>\\ &= \lim_{n\to\infty}\sum c_m <\alpha S*\rho_n*\check{\rho}_n' e^{-i\frac{\pi}{l}<m,x>})T, e^{i\frac{\pi}{l}<m,x>}\beta>\\ &= \sum c_m <[\alpha S]T, e^{i\frac{\pi}{l}<m,x>}\beta>\\ &= <\beta([\alpha S]T), \phi>,\end{aligned}$$

which completes the proof.

As a consequence of the preceding propositions, we have

THEOREM. *Definitions* 1 *and* 2 *are entirely equivalent. The existence of* *either of* $[S]T$ *and* $S[T]$ *assures the existence of the product of* S *and* T, *and then* $(\alpha S)T$, $S(\alpha T)$ *are also defined for every* $\alpha \in \mathcal{E}$ *and hold the relations*:

$$(\alpha S)T = S(\alpha T) = \alpha(ST).$$

If $(\alpha S)T$ *is defined for every* $\alpha \in \mathcal{D}$, *then* ST *exists.*

The multiplicative product is commutative and distributive, but not associative in general as the well known example shows ([6], I, p. 119): $\left(\frac{1}{x}x\right)\delta = \delta$, $\frac{1}{x}(x\delta) = 0$. J. Mikusiński [5] gives sufficient criteria for the existence of the product and the law of associativity by introducing the concept of order of a distribution. Now we shall introduce the definition of multiplication for three distributions.

DEFINITION 3. *Let* S, T, $W \in \mathcal{D}'$. *If the distributional limit*: $\lim_{n\to\infty} S_n T_n W_n$ *exists for every regular sequence* S_n, T_n *and* W_n, *then the limit will be defined as the multiplicative product of* S, T *and* W, *and denoted by* STW.

PROPOSITION 4. *If* ST, TW *and* STW *exist, then* $(ST)W$ *and* $S(TW)$ *exist and* $(ST)W = S(TW)$.

PROOF. Similarly as in the proof of Proposition 1, we can show that $\lim_{m,n,p\to\infty} S_m T_n W_p = STW$. Then we have

$$(ST)W = \lim_{p\to\infty}(ST)W_p = \lim_{m,n,p\to\infty} S_m T_n W_p = \lim_{m\to\infty} S_m(TW) = S(TW),$$

which completes the proof.

The value of distribution T at a point x_0 is defined [3] as the distributional limit

$$\lim_{h\to 0} T(x_0 + h\hat{x})$$

provided that such limit exists, where h stands for an N-dimensional vector $h = (h_1, h_2, \cdots, h_N)$ with $h_j \neq 0$, $j = 1, 2, \cdots, N$, and $hx = (h_1x_1, h_2x_2, \cdots, h_Nx_N)$ and $T(x_0 + h\hat{x})$ is a distribution defined by

$$< T(x_0 + h\hat{x}), \phi(\hat{x}) > = < T(\hat{x}), \frac{1}{|h_1| \cdots |h_N|}\phi\left(\frac{1}{h}(\hat{x} - x_0)\right) >,$$

where $\frac{1}{h}=\left(\frac{1}{h_1}, \cdots, \frac{1}{h_N}\right)$. If the limit exists, it is always a constant function [9]. After Mikusiński [4] we understand by value $T(x_0)$ of T at x_0 the value of this constant function. If T is a function continuous at x_0 with value c, then it is clear that the value of the distribution T at x_0 is also equal to c.

LEMMA 3. *If, for every δ-sequence $\{\rho_n\}$, $\lim_{n\to\infty} <T, \rho_n> = c$ exists, then there exists a zero neighbourhood of R^N in which T is equivalent to a bounded function continuous at 0 with value c, which is also the value of the distribution T at 0.*

PROOF. We may assume that $c=0$. Let A_ε denote the set defined in Lemma 2. Similarly as there we can show that $\sup_{\sigma\in A_\varepsilon} |<T, \sigma>| \to 0$ as $\varepsilon\to 0$. Therefore T is a bounded function $f(x)$ in a zero neighbourhood K_ε of R^N, and $\operatorname{ess.sup}_{x\in K_\varepsilon} |f(x)| \to 0$ as $\varepsilon\to 0$. The last statement is evident because of the remark preceding Lemma 3. The proof is complete.

PROPOSITION 5. *The product ST exists if and only if, for every $\alpha \in \mathcal{D}$, there exists a zero neighbourhood in which $\alpha S*\check{T}$ is a bounded function continuous at 0. In this case $<ST, \alpha> = (\alpha S*\check{T})(0)$, the value at 0.*

Further, if $S(\tau_t T)$ exists for every $t \in K_\varepsilon$, then $\alpha S*\check{T}$ is a bounded function on a neighbourhood of K_ε and continuous at every point of K_ε.

PROOF. The first statement is evident from the relation $<S(T*\rho_n'), \alpha> = <\alpha S*\check{T}, \rho_n'>$, together with Lemma 3. As for the last statement, owing to the relations

$$<S((\tau_t T)*\rho_n'), \alpha> = <\alpha S*\tau_{-t}\check{T}, \rho_n'> = <\tau_{-t}(\alpha S*\check{T}), \rho_n'>,$$

we see that there corresponds to each point t of K_ε a neighbourhood of t in which $\alpha S*\check{T}$ is a bounded function continuous at t. It follows that the last part of the proposition is also true.

REMARK 1. If $S\left(\frac{\partial T}{\partial x_j}\right)$, $j=1, 2, \cdots, N$, exist, then ST and $\frac{\partial S}{\partial x_j}T$, $j=1, 2, \cdots, N$, exist, and the following relations hold:

$$\frac{\partial(ST)}{\partial x_j} = \frac{\partial S}{\partial x_j}T + S\frac{\partial T}{\partial x_j}, \qquad j=1, 2, \cdots, N.$$

In fact, let α be any element of $\mathcal{D}$. It follows from Proposition 5 that

$$\alpha S*\left(\frac{\partial T}{\partial x_j}\right)^{\vee} = -\frac{\partial}{\partial x_j}(\alpha S*\check{T}), \qquad j=1, 2, \cdots, N,$$

are bounded near the origin. Therefore, owing to a Theorem of Kryloff ([6], II, p. 37), $\alpha S*\check{T}$ is continuous near the origin, which, together with the same proposition, shows that ST exists. Similarly from the relation:

$$\frac{\partial S}{\partial x_j}*(\alpha T)^{\vee} = -S*\left(\frac{\partial \alpha}{\partial x_j}T\right)^{\vee} - S*\left(\alpha\frac{\partial T}{\partial x_j}\right)^{\vee},$$

we see also that $\left(\frac{\partial S}{\partial x_j}\right)T$, $j=1, 2, \cdots, N$, exist. Then it follows from the following relations:

$$<\frac{\partial(ST)}{\partial x_j}, \alpha> = -<ST, \frac{\partial \alpha}{\partial x_j}> = -\left(\left(\frac{\partial \alpha}{\partial x_j}S\right)*\check{T}\right)(0),$$

$$<\frac{\partial S}{\partial x_j}T, \alpha> = \left(\alpha\frac{\partial S}{\partial x_j}*\check{T}\right)(0),$$

$$<S\frac{\partial T}{\partial x_j}, \alpha> = \left(\alpha S*\left(\frac{\partial T}{\partial x_j}\right)^{\vee}\right)(0) = -\left(\frac{\partial}{\partial x_j}(\alpha S*\check{T})\right)(0),$$

that

$$\frac{\partial(ST)}{\partial x_j} = \frac{\partial S}{\partial x_j}T + S\frac{\partial T}{\partial x_j}, \qquad j=1, 2, \cdots, N.$$

REMARK 2. Using Proposition 5 we can give a simple proof of the exchange formula for Fourier transformation obtained by Y. Hirata and H. Ogata [2]. Let S and T be $\mathscr{S}'$-composable tempered distributions. Put $U=\mathfrak{F}(S)$ and $V=\mathfrak{F}(T)$. Then for any $\alpha \in \mathcal{D}$, we have because of $\alpha U \in \mathcal{E}' \subset \mathcal{O}_C'$

$$\mathfrak{F}^{-1}(\alpha U*\check{V}) = (\mathfrak{F}^{-1}(\alpha)*S)\check{T} \in \mathcal{D}_{L^1}'.$$

Therefore $\alpha U*\check{V}$ is a continuous function as a Fourier transform of an element of $\mathcal{D}_{L^1}'$. It follows from Proposition 5 that the multiplicative product UV exists, and we have

$$<UV, \alpha> = \int(\mathfrak{F}^{-1}(\alpha)*S)\check{T}dx = <S*T, \mathfrak{F}(\alpha)> = <\mathfrak{F}(S*T), \alpha>,$$

which implies that $UV=\mathfrak{F}(S*T)$.

2. Multiplicators. A space of distributions $\mathcal{H}$ is, by definition, a locally convex vector space contained in $\mathcal{D}'$ as a linear subspace with a finer topology. A space of distributions $\mathcal{H}$ is referred to as normal if $\mathcal{D}$ is contained in $\mathcal{H}$ with a finer topology and is dense in $\mathcal{H}$.

Let $\mathcal{H}$ be a normal space of distributions and $\mathcal{L}$ be a space of distributions. According to L. Schwartz ([7], p. 69), $S \in \mathcal{D}'$ is a *multiplicator* of $\mathcal{H}$ into $\mathcal{L}$, if there exists a continuous linear mapping $<S>$ of $\mathcal{H}$ into $\mathcal{L}$ which coincides with the multiplicative product by S on $\mathcal{D} \subset \mathcal{H}$. When $\mathcal{H} = \mathcal{L}$, we shall say that S is a multiplicator of $\mathcal{H}$.

PROPOSITION 6. *Let $\mathcal{H}$ be a barrelled normal space of distributions. If S is a distribution such that for every $T \in \mathcal{H}$ the multiplicative product ST exists, then S is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'$, $<S>T = ST$ for every $T \in \mathcal{H}$, and $\phi S \in \mathcal{H}'$ for every $\phi \in \mathcal{D}$.*

In addition, assume that $S\mathcal{H} \subset \mathcal{L}$ with $\mathcal{D}$ strictly dense in $\mathcal{L}'_\sigma$, $\mathcal{L}$ being a normal space of distributions, then S is a multiplicator of $\mathcal{H}$ into $\mathcal{L}$.

PROOF. By definition, $ST = \lim_{n\to\infty} S(T*\rho'_n)$. Since the mapping $T \to S(T*\rho_n)$ of $\mathcal{H}$ into $\mathcal{D}'$ is continuous and $\mathcal{H}$ is barrelled, it follows that the mapping $<S> : T \to ST$ of $\mathcal{H}$ into $\mathcal{D}'$ is continuous. Since for every $\phi \in \mathcal{D}$ the relation $<S>\phi = S\phi$ holds, S is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'$. Therefore, for every $\phi \in \mathcal{D}$, the mapping $T \to <ST, \phi>$ is obviously a continuous linear form on $\mathcal{H}$, so that there exists an element $W_\phi \in \mathcal{H}'$ such that $<ST, \phi> = <T, W_\phi>$. If $T = \psi \in \mathcal{D}$, $<S\psi, \phi> = <\psi, S\phi> = <\psi, W_\phi>$. Then it follows that $S\phi = W_\phi \in \mathcal{H}'$.

As for the last statement of the proposition, that the linear mapping $<S> : T \to ST$ of $\mathcal{H}$ into $\mathcal{L}$ is continuous is an immediate consequence of a theorem of R. Shiraishi ([8], p. 176). Therefore S is a multiplicator of $\mathcal{H}$ into $\mathcal{L}$. The proof is complete.

EXAMPLE 1. $S \in \mathcal{E}$ if and only if ST is defined for every $T \in \mathcal{D}'$. In fact, $\mathcal{D}'$ is a space of distributions $\mathcal{H}$ satisfying all the conditions of Proposition 6. Therefore if ST exists for every $T \in \mathcal{D}'$, then $\phi S \in \mathcal{D}$ for every $\phi \in \mathcal{D}$, so that $S \in \mathcal{E}$. The converse is trivial.

EXAMPLE 2. $S \in \mathcal{O}_M$ if and only if ST is defined and $ST \in \mathcal{S}'$ for every $T \in \mathcal{S}'$. In fact, $\mathcal{S}'$ is a space of distributions $\mathcal{H}$ satisfying all the conditions of Proposition 6. Therefore if ST exists for every $T \in \mathcal{S}'$, then $\phi S \in \mathcal{S}$ for every $\phi \in \mathcal{D}$, so that $S \in \mathcal{E}$. The mapping $T \to ST$ of $\mathcal{S}'$ into $\mathcal{S}'$ is continuous with its dual mapping: $\mathcal{S} \to \mathcal{S}$. Therefore $S \in \mathcal{E}$ becomes a multiplicator of $\mathcal{S}$, that is, $S \in \mathcal{O}_M$. The converse is trivial.

PROPOSITION 7. *Let $\mathscr{H}$, $\mathscr{L}$ be normal spaces of distributions with the approximation properties by regularization and truncation* ([7], *p.* 7). *Further we suppose that $\mathscr{L}$ has γ-topology. Let S be a multiplicator of $\mathscr{H}$ into $\mathscr{L}$, then ST exists for every $T \in \mathscr{H}$ and $<S>T=ST$, and S is also a multiplicator of $\mathscr{L}'_c$ into $\mathscr{H}'_c$, so that SW exists for every $W \in \mathscr{L}'$ and $<S>W=SW$.*

PROOF. Let $\{\alpha_n\}$ be any sequence of multiplicators, that is, $\alpha_n \in \mathscr{D}$, α_n tends to 1 in $\mathscr{E}$ as $n\to\infty$ and $\{\alpha_n\}$ is bounded in $\mathscr{B}$. Let $\{\rho_n\}$ be any δ-sequence and T be any element of $\mathscr{H}$. Since $\alpha_m(T*\rho_n) \in \mathscr{D}$, it follows that

$$<S>\big(\alpha_m(T*\rho_n)\big) = S\big(\alpha_m(T*\rho_n)\big).$$

Passing to the limit as $m\to\infty$, since $\mathscr{H}$ has the approximation property by truncation, we see that

$$<S>(T*\rho_n) = S(T*\rho_n).$$

Further, since $\mathscr{H}$ has the approximation property by regularization, it follows that $<S>(T*\rho_n)$ tends to $<S>T$ as $n\to\infty$, so that $S(T*\rho_n)$ converges to $<S>T$, which implies that ST exists and $ST=<S>T$. Since $<S>$ is continuous, the dual mapping denoted by the same symbol $<S>$ is also a continuous linear mapping of $\mathscr{L}'_c$ into $\mathscr{H}'_c$, and therefore S is a multiplicator of $\mathscr{L}'_c$ into $\mathscr{H}'_c$. We know that $\mathscr{L}'_c$ has the approximation properties by regularization and truncation ([7], p. 10). Therefore by a similar reasoning as above we see that the last statement of the proposition is true. The proof is complete.

REMARK 3. Let f, g be functions, that is, locally summable functions. Even if the ordinary product fg is a function, it may occur that fg is not the multiplicative product. For example, let $\mathscr{H}$, $\mathscr{K}$ be the spaces of functions defined as follows (we assume $N\geqq 2$):

$$\mathscr{H} = \left\{f\,;\, \|f\|^2 = \int \frac{|f(x)|^2}{|x|}\,dx < \infty\right\};$$

$$\mathscr{K} = \{g\,;\, \|g\|^2 = \int |g(x)|^2\,|x|\,dx < \infty\}.$$

We note that $\mathscr{H}$ is the dual Banach space of $\mathscr{K}$. Suppose multiplication for every f and every g is possible. Let $\mathscr{H}_1$ denote the subspace of $\mathscr{H}$ consisting of functions with support in the unit ball. It follows from the closed graph theorem that the mapping $(f, g)\to f*\check{g}$ of $\mathscr{H}_1\times\mathscr{K}$ into L^1_{loc} is continuous. By

Proposition 5, $f*\check{g}$ is bounded in a zero neighbourhood of R^N. If we put

$$H_n=\{f\in\mathscr{H}_1;\ \operatorname*{ess.\,sup}_{x\in K_{\frac{1}{n}}}|f*\check{g}|\leqq n\},$$

then H_n is a closed disk of $\mathscr{H}_1$ and $\mathscr{H}_1=\cup H_n$. Therefore $f*\check{g}$ is uniformly bounded in a zero neighbourhood of R^N for a fixed $g\in\mathscr{K}$ and all $f\in\mathscr{H}_1$ with $\|f\|\leqq 1$. By a similar reasoning we see that in a zero neighbourhood K_ε $(0<\varepsilon<1)$ each $f*\check{g}$ is a bounded function. For any $\phi\in\mathscr{D}$ we have

$$<(\tau_t f)(g*\rho_n),\ \phi>=\int\big(((\tau_t f)\phi)*\check{g}\big)\rho_n dx$$

$$=\int\big(f(\tau_{-t}\phi)*\check{g}\big)\tau_{-t}\rho_n dx.$$

If we take $t\in K_{\varepsilon/2}$, the sequence $\{<(\tau_t f)(g*\rho_n),\ \phi>\}$ is bounded. Moreover if g is taken from $\mathscr{D}$, the sequence converges to $<(\tau_t f)g,\ \phi>$. Therefore, by a Theorem of Banach-Steinhaus, the sequence $\{<(\tau_t f)(g*\rho_n),\ \phi>\}$ converges. This means that the multiplicative product of $\tau_t f$ and every $g\in\mathscr{K}$ exists. Then it follows from Proposition 6 that $\tau_t f\in\mathscr{H}$, that is, $\int\frac{|f(x-t)|^2}{|x|}dx<\infty$ for every $f\in\mathscr{H}_1$. Therefore $\frac{|x|}{|x+t|}$ is bounded in $K_{\varepsilon/2}$, a contradiction. Thus we see that there are functions $f\in\mathscr{H}$ and $g\in\mathscr{K}$ such that the multiplicative product of f and g does not exist.

On the other hand, by the ordinary multiplication, fg is a function $\in L^1$ because of the equality: $|f(x)g(x)|=\frac{|f(x)|}{|x|^{1/2}}|x|^{1/2}|g(x)|$. And it is easy to see that the mapping $g\to fg$ (ordinary product) of $\mathscr{K}$ into L^1 is continuous, that is, f is a multiplicator of $\mathscr{K}$ into L^1. $\mathscr{K}$ has obviously a barrelled normal space of distributions with the approximation property by truncation. This together with Proposition 7 shows that $\mathscr{K}$ has not the approximation property by regularization.

3. Digressions. Let S, T be tempered distributions. If S, T are $\mathscr{S}'$-composable, $\mathscr{F}(S)\mathscr{F}(T)$ is defined and $\mathscr{F}(S)\mathscr{F}(T)=\mathscr{F}(S*T)$ ([2], p. 151). Here we shall show that the sequence $\{(\mathscr{F}(S))_n(\mathscr{F}(T))_n\}$ converges in $\mathscr{S}'$.

First we note that if $\{\rho_n\}$ is a δ-sequence, then $\mathscr{F}(\rho_n)$ converges to 1 in $\mathscr{B}_c$. This is a consequence of direct calculation. Let ϕ be any element of $\mathscr{S}$. Then we have

$$<(\mathcal{F}(S)*\rho_n)\,(\mathcal{F}(T)*\rho'_n),\ \phi> = <\mathcal{F}(S)*\rho_n,\ (\mathcal{F}(T)*\rho'_n)\phi>$$

$$= \int (\mathcal{F}(S)*\rho_n)\,(\mathcal{F}(T)*\rho'_n)\phi dx.$$

According to Parseval's formula, it follows that

$$<(\mathcal{F}(S)*\rho_n)\,(\mathcal{F}(T)*\rho'_n),\ \phi> = \int (S\mathcal{F}(\check{\rho}_n))\,((\check{T}\mathcal{F}(\rho'_n)*\mathcal{F}(\phi))dx$$

$$= \iint S(x)\mathcal{F}(\check{\rho}_n)\,(x)\check{T}(x-y)\mathcal{F}(\rho'_n)\,(x-y)\mathcal{F}(\phi)\,(y)dxdy$$

$$= \iint S(x)\,T(y)\mathcal{F}(\phi)\,(x+y)\mathcal{F}(\rho_n)^{\vee}(x)\mathcal{F}(\rho'_n)^{\vee}(y)dxdy.$$

By our assumption, $(S_x \otimes T_y)\mathcal{F}(\phi)\,(\hat{x}+\hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}$ and by the preceding remark $\mathcal{F}(\rho_n)\,(x)\mathcal{F}(\rho'_n)\,(y)$ tends to 1 in $\mathcal{B}_c$ as $n\to\infty$. Hence, it follows that

$$\lim_{n\to\infty} <(\mathcal{F}(S)*\rho_n)\,(\mathcal{F}(T)*\rho'_n),\ \phi> = <S*T,\ \mathcal{F}(\phi)>$$

$$= <\mathcal{F}(S*T),\ \phi>.$$

Therefore, $(\mathcal{F}(S)*\rho_n)\,(\mathcal{F}(T)*\rho'_n)$ converges to $\mathcal{F}(S*T)$ in $\mathcal{S}'$.

Next we suppose S, T are composable and $S*T \in \mathcal{S}'$. Then in the above proof $<\mathcal{F}(S)_n\mathcal{F}(T)_n,\ \phi>$ converges to $<\mathcal{F}(S*T),\ \phi>$ if we take $\phi \in \mathcal{F}(\mathcal{D})$. On the other hand L. Ehrenpreis [1] introduced the space $\boldsymbol{D}$, the Fourier transform of $\mathcal{D}$, with the topology which makes the mapping $\phi \to \mathcal{F}(\phi)$ topological. Therefore it follows that the above consideration shows that $\mathcal{F}(S)_n\mathcal{F}(T)_n$ converges to $\mathcal{F}(S*T)$ in $\boldsymbol{D}'$, the strong dual of $\boldsymbol{D}$.

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