J. Sci. Hiroshima Univ. Ser. A-I
29 (1965), 217-221

An Application of the Minimax Theorem to the Theory of Capacity

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(Received September 22, 1965)

Let K be a compact Hausdorff space and $\Phi(x, y)$ be an extended real-valued lower semicontinuous function on $K\times K$ which does not assume the value $-\infty$. We denote by $\mathcal{U}_K$ the class of non-negative unit Radon measures on K, and by S_μ the support of a measure μ. The potentials

$$\int\Phi(x, y)\,d\mu(y) \qquad \text{and} \qquad \int\Phi(y, x)\,d\mu(y)$$

will be denoted by $\Phi(x, \mu)$ and $\Phi(\mu, x)$ respectively. Our aim in this paper is to prove

Theorem. *It holds that*

$$\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)=\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(\mu, x) \tag{1}$$

and

$$\sup_{\mu\in\mathcal{U}_K}\inf_{x\in S_\mu}\Phi(x, \mu)=\sup_{\mu\in\mathcal{U}_K}\inf_{x\in S_\mu}\Phi(\mu, x). \tag{2}$$

Remark. The reciprocal of the value in (1) is taken as the definition of capacity in the Newtonian case, namely, when $\Phi(x, y)=|x-y|^{-1}$ in the Euclidean space E_3. In this case both sides of (2) are always equal to ∞ and hence (2) is trivially true. In case $\Phi(x, y)$ is (finite-valued) continuous on $K\times K$, either one of (1) and (2) follows from the other because

$$\sup_{\mu\in\mathcal{U}_K}\inf_{x\in S_\mu}\begin{matrix}\Phi(x, \mu)\\ \Phi(\mu, x)\end{matrix}=-\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\begin{matrix}(-\Phi(x, \mu))\\ (-\Phi(\mu, x)).\end{matrix}$$

Proof of the Theorem. We may assume $\Phi>0$ without loss of generality. We shall denote the left and the right hand sides of (1) by α and β respectively. First we consider the case where Φ is finite-valued and K consists of a finite number of points by induction. The case when K consists of one point is trivial. Suppose that (1) is true when K contains exactly n points, and let us consider the case where K consists of $n+1$ points. We can express $\mu\in\mathcal{U}_K$ by its masses at these points or by a vector $(\xi_1, \cdots, \xi_{n+1})$, where $\xi_1, \cdots, \xi_{n+1} \geqq 0$ and $\xi_1 + \cdots + \xi_{n+1} = 1$. We shall say that μ is non-degenerate if all ξ_i are positive. For any fixed $\mu \in \mathscr{U}_K$, $\Phi(x, \mu)$ attains its maximum at some point of K. The maximum value is then a continuous function of $(\xi_1, \cdots, \xi_n)$ on $\varDelta_n$: $\xi_1, \cdots, \xi_n \geqq 0$ and $\xi_1 + \cdots + \xi_n \leqq 1$. Hence $\inf_{\mu \in \mathscr{U}_K} \sup_{x \in K} \Phi(x, \mu)$ is attained by some $x_0 \in K$ and $\mu_0 \in \mathscr{U}_K$:

$$\inf_{\mu \in \mathscr{U}_K} \sup_{x \in K} \Phi(x, \mu) = \Phi(x_0, \mu_0).$$

We shall denote this value by α_{n+1}. It may not be equal to α because the supremum is considered on K instead on S_μ. So we denote by $\mathscr{U}_K^{(i)}$ the class of measures of $\mathscr{U}_K$ for which $\xi_i = 0$ $(1 \leqq i \leqq n+1)$, and set

$$\alpha_{n+1}^{(i)} = \inf_{\mu \in \mathscr{U}_K^{(i)}} \sup_{x \in S_\mu} \Phi(x, \mu).$$

By our assumption on induction we have

$$\alpha_{n+1}^{(i)} = \inf_{\mu \in \mathscr{U}_K^{(i)}} \sup_{x \in S_\mu} \Phi(x, \mu) = \inf_{\mu \in \mathscr{U}_K^{(i)}} \sup_{x \in S_\mu} \Phi(\mu, x) \geqq \beta.$$

Since

$$\alpha = \min(\alpha_{n+1}, \alpha_{n+1}^{(1)}, \cdots, \alpha_{n+1}^{(n+1)}),$$

it will suffice to show $\alpha_{n+1} \geqq \beta$ in case $\alpha_{n+1} = \alpha$. Let us prove $\Phi(x_0, \mu_0) \geqq \beta$.

If $S_{\mu_0} \neq K$, $\Phi(x_0, \mu_0) \geqq \sup_{x \in S_{\mu_0}} \Phi(x, \mu_0) \geqq \alpha_{n+1}^{(i)}$ for some i and hence $\Phi(x_0, \mu_0) \geqq \alpha_{n+1}^{(i)} \geqq \beta$. Therefore we assume that we can not find μ_0 with $S_{\mu_0} \neq K$. This implies that $\mu_0 = (\xi_1^{(0)}, \cdots, \xi_{n+1}^{(0)})$ is non-degenerate: $\xi_1^{(0)}, \cdots, \xi_{n+1}^{(0)} > 0$. We shall interprete the situation in geometrical terms. For a fixed $x \in K$, the graph of $\Phi(x, \mu) = \Phi(x, (\xi_1, \cdots, \xi_n, 1 - \xi_1 - \cdots - \xi_n))$ as a function of $\xi_1, \cdots, \xi_n$ on $\varDelta_n$ is the part above $\varDelta_n$ of a hyperplane in the $(n+1)$-dimensional space E_{n+1}, so that the graph of $\sup_{x \in K} \Phi(x, \mu)$ as a function of $\xi_1, \cdots, \xi_n$ on $\varDelta_n$ is the upper envelope of the $n+1$ hyperplanes above $\varDelta_n$ and hence is a convex surface. The fact that μ_0 must be non-degenerate implies that the point $P_0 = (\xi_1^{(0)}, \cdots, \xi_n^{(0)}, \alpha_{n+1})$ in E_{n+1} is the lowest extreme point of the convex surface, which lies above an interior point of $\varDelta_n$. We shall show that

$$\Phi(x_0, \mu_0) \geqq \sup_{\mu \in \mathscr{U}_K} \inf_{x \in K} \Phi(x, \mu).$$

Suppose that

$$\sup_{\mu \in \mathscr{U}_K} \inf_{x \in K} \Phi(x, \mu) = \Phi(x^*, \mu^*) > \Phi(x_0, \mu_0)$$

for $x^* \in K$ and $\mu^*=(\xi_1^*, \cdots, \xi_{n+1}^*) \in \mathscr{U}_K$. Denote the point $(\xi_1^*, \cdots, \xi_n^*, \Phi(x^*, \mu^*))$ in E_{n+1} by P^*. Consider the points where the straight line, connecting P_0 and P^*, intersects the wall of the vertical cylinder having Δ_n as its base. The height of one of them, say of $\widetilde{P}$, is less than $\Phi(x_0, \mu_0)$ on account of the inequality $\Phi(x_0, \mu_0) < \Phi(x^*, \mu^*)$. Since all the hyperplanes considered above pass through or lie below P_0 and they either pass through or lie above P^*, they either pass through or lie below $\widetilde{P}$. This means that

$$\inf_{\mu\in\mathscr{U}_K} \sup_{x\in K} \Phi(x, \mu) \leqq \text{the height of } \widetilde{P} < \Phi(x_0, \mu_0) = \inf_{\mu\in\mathscr{U}_K} \sup_{x\in K} \Phi(x, \mu),$$

which is impossible. In virtue of the minimax theorem in the theory of games we obtain

$$\beta \leqq \inf_{\nu\in\mathscr{U}_K} \sup_{y\in K} \Phi(\nu, y) = \sup_{\mu\in\mathscr{U}_K} \inf_{x\in K} \Phi(x, \mu) \leqq \alpha_{n+1} = \alpha.$$

Thus $\alpha \geqq \beta$ is true in all cases. Since the discussion is symmetric, we have $\alpha \leqq \beta$ and hence $\alpha = \beta$.

Next we shall consider the case when K is a general compact Hausdorff space and $\Phi(x, y)$ is continuous on $K \times K$. We can divide K into a finite number of mutually disjoint Borel sets $B_1^{(m)}, \cdots, B_{p_m}^{(m)}$ such that $|\Phi(x, y') - \Phi(x, y'')| < 1/m$ for all x whenever y' and y'' belong to the same one of $\{B_i^{(m)}\}$ and $|\Phi(x', y) - \Phi(x'', y)| < 1/m$ for all y whenever x' and x'' belong to the same one of $\{B_i^{(m)}\}$. We define Φ_m on $K \times K$ by

$$\Phi_m(x, y) = \inf_{B_i^{(m)} \times B_j^{(m)}} \Phi(x, y) \qquad \text{if } x \in B_i^{(m)} \text{ and } y \in B_j^{(m)}.$$

It holds that $0 \leqq \Phi(x, y) - \Phi_m(x, y) < 2/m$. We know that

$$\inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi_m(x, \mu) = \inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi_m(\mu, x) \qquad \text{for every } m.$$

Observing that

$$0 \leqq \Phi(x, \mu) - \Phi_m(x, \mu) \leqq \int \{\Phi(x, y) - \Phi_m(x, y)\}\, d\mu(y) \leqq \frac{2}{m} \qquad \text{for all } x,$$

we have

$$\alpha = \inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi(x, \mu) \leqq \frac{2}{m} + \inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi_m(x, \mu)$$

$$= \frac{2}{m} + \inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi_m(\mu, x) \leqq \frac{2}{m} + \inf_{\mu\in\mathscr{U}_K} \sup_{x\in S_\mu} \Phi(\mu, x) = \frac{2}{m} + \beta.$$

By letting $m\to\infty$ we obtain $\alpha\leqq\beta$. Similarly $\alpha\geqq\beta$ and hence $\alpha=\beta$ is concluded.

We remark that there are $\mu_0\in\mathcal{U}_K$ and $x_0\in S_{\mu_0}$ such that $\Phi(x_0, \mu_0)=\alpha$. In fact, we choose a directed set $\{\mu_\iota; \iota\in I\}$ in $\mathcal{U}_K$ such that $\sup_{x\in S_{\mu_\iota}}\Phi(x, \mu_\iota)$ tends to α along I. We may assume that $\{\mu_\iota\}$ converges vaguely to a measure $\mu_0\in\mathcal{U}_K$. Take any $x\in S_{\mu_0}$. We find $\{x_\iota\}$ such that $x_\iota\in S_{\mu_\iota}$ and $x_\iota\to x$. It follows that $\Phi(x, \mu_0)=\lim\Phi(x_\iota, \mu_\iota)\leqq\lim\sup_{x\in S_{\mu_\iota}}\Phi(x, \mu_\iota)=\alpha$. Hence $\alpha\leqq\sup_{x\in S_{\mu_0}}\Phi(x, \mu_0)\leqq\alpha$. This shows $\sup_{x\in S_{\mu_0}}\Phi(x, \mu_0)=\alpha$. It is easy to find $x_0\in S_{\mu_0}$ satisfying $\Phi(x_0, \mu_0)=\alpha$.

Now we are concerned with the final case that $\Phi(x, y)$ is lower semi-continuous on $K\times K$. We denote by H the family, directed by the relation $\leqq$, of non-negative continuous functions $h\leqq\Phi$ on $K\times K$. For each $h\in H$, let $\mu_h\in\mathcal{U}_K$ give

$$\sup_{x\in S_{\mu_h}} h(x, \mu_h)=\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu).$$

Let us prove

$$\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)=\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu). \tag{3}$$

Take any $h'\in H$. For any x and $h\in H$ such that $h\geqq h'$ we have $h'(x, \mu_h)\leqq h(x, \mu_h)$. Since $\mathcal{U}_K$ is compact with respect to the vague topology, there is a value μ_0 of accumulation of the mapping $h\to\mu_h$ along H. We may suppose that μ_h converges to μ_0 vaguely. Let x belong to S_{μ_0}. There are points $\{x_h\}$ such that $x_h\in S_{\mu_h}$ and $x_h\to x$ along H. It holds that

$$h'(x, \mu_0)=\lim_{h\in H} h'(x_h, \mu_h)\leqq\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu),$$

and that

$$\Phi(x, \mu_0)\leqq\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu).$$

Since $x\in S_{\mu_0}$ is arbitrary,

$$\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)\leqq\sup_{x\in S_{\mu_0}}\Phi(x, \mu_0)\leqq\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu).$$

Now (3) follows because $\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu)\leqq\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)$ for all $h\in H$.

The equality $\alpha=\beta$ is an immediate consequence of (3). Actually

$$\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(x, \mu)=\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(x, \mu)=\sup_{h\in H}\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu} h(\mu, x)=\inf_{\mu\in\mathcal{U}_K}\sup_{x\in S_\mu}\Phi(\mu, x).$$

To establish (2) it will be sufficient to show

$$\sup_{\mu\in\mathcal{U}_K}\inf_{x\in S_\mu}\Phi(x,\mu)=\sup_{h\in H}\sup_{\mu\in\mathcal{U}_K}\inf_{x\in S_\mu}h(x,\mu). \tag{4}$$

Take any $\mu\in\mathcal{U}_K$. Since $\Phi(x,\mu)$ is lower semicontinuous on S_μ, given any $m<\inf_{x\in S_\mu}\Phi(x,\mu)$, we can find $h\in H$ such that $h(x,\mu)>m$ for all $x\in S_\mu$. Hence, for this h,

$$m\leqq\inf_{x\in S_\mu}h(x,\mu)\leqq\sup_{\nu\in\mathcal{U}_K}\inf_{x\in S_\nu}h(x,\nu)\leqq\sup_{h\in H}\sup_{\nu\in\mathcal{U}_K}\inf_{x\in S_\nu}h(x,\nu).$$

We obtain (4) because m may be arbitrarily close to $\inf_{x\in S_\mu}\Phi(x,\mu)$ and $\mu\in\mathcal{U}_K$ is arbitrary. Our theorem is now completely proved.

At the end we give bibliographical remarks. In [4] the author denoted α and β by $V(K)$ and $\check{V}(K)$ respectively. He proved there $V(K)-m\leqq 2(\check{V}(K)-m)$ with $m=\inf_{K\times K}\Phi(x,y)$ and raised the question whether or not it is possible to improve the coefficient 2 (2.2 of open questions at p. 284 of [4]). This question is now settled by the result in the present paper. In the above discussion we applied the minimax theorem. The first application of this theorem to the theory of capacity was made in [1].*) Usefulnesses in potential theory of some other techniques in the theory of games or in the theory of linear programming are found also in [2] and [3].

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*) This was first published as "Une application du théorème du minimax à la théorie du potentiel, Colloque Internat. C.N.R.S. Théorie du Potentiel, Exposé n° 8, Orsay (S.-et-O.), 1964".