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Some Examples Related to Duality Theorem in Linear Programming

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The duality problems in linear programming may read as follows. Suppose an $m\times n$ matrix $A=(a_{ij})$, a column vector $\boldsymbol{b}=(b_1, \cdots, b_m)$ and a row vector $\boldsymbol{c}=(c_1, \cdots, c_n)$ are given.

The primal problem: *Find a column vector* $\boldsymbol{u}=(u_1, \cdots, u_n)$ *which maximizes the linear form* $\boldsymbol{cu}$ *subject to the conditions* $A\boldsymbol{u}\leqq\boldsymbol{b}$ *and* $\boldsymbol{u}\geqq 0$.

The dual problem: *Find a row vector* $\boldsymbol{v}=(v_1, \cdots, v_n)$ *which minimizes the linear form* $\boldsymbol{vb}$ *subject to the conditions* $\boldsymbol{v}A\geqq\boldsymbol{c}$ *and* $\boldsymbol{v}\geqq 0$.
In each problem a vector satisfying the required conditions is called feasible, and if it attains the maximum or minimum it is called optimal.

These problems can be represented by the following tableau:

$(\geqq 0)$	u_1	$\cdots$	u_j	$\cdots$	u_n	$\leqq$
v_1	a_{11}	$\cdots$	a_{1j}	$\cdots$	a_{in}	b_1
$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
v_i	a_{i1}	$\cdots$	a_{ij}	$\cdots$	a_{in}	b_i
$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
v_m	a_{m1}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	b_m
$\geqq$	c_1	$\cdots$	c_j	$\cdots$	c_n	min / max

By taking inner products of the row of u's with the rows of A and the row of c's, we obtain the constraints $A\boldsymbol{u}\leqq\boldsymbol{b}$ and the linear form $\boldsymbol{cu}$ of the primal; the inner products of the column of v's with the columns of A and the column of b's yield the dual constraints $A\boldsymbol{v}\geqq\boldsymbol{c}$ and the linear form $\boldsymbol{vb}$.

Associated with these problems is the following well-known theorem:

The Duality Theorem. If the primal is feasible and if $\sup \boldsymbol{cu}<\infty$, *then there exist optimal solutions in the dual as well as in the primal, and moreover the extremal values of the linear forms coincide, i.e.,* $\max \boldsymbol{cu}=\min \boldsymbol{vb}$.

In the foregoing paper [1], M. Ohtsuka investigated the problems in a very general situation, and obtained extensions of the duality theorem. We refer necessary notions and notations to [1]. We shall show in the present paper that the conditions imposed in Ohtsuka's Theorems 2 and 3 are in a way necessary. Actually, even if $\mathscr{M}\neq\emptyset$, $-\infty<M<\infty$ and Φ, f and g are all continuous and non-negative, we have examples where respectively

1. There exist no optimal measures in the primal.
2. Though an optimal measure exits in the primal, $\mathcal{M}' = \emptyset$.
3. There exists an optimal measure in the primal and $\mathcal{M}' \neq \emptyset$, $M = M'$, nevertheless there exist no optimal in the dual.
4. Though there exist optimal measures both in the primal and the dual, nevertheless $M < M'$.

Examples shall be given by the tableaux which are so explanatory that further explanations will be superfluous.

1. $X = \{1\}$, $Y = \{N, \omega\}$: the Alexandroff one point compactification of the discrete space N of all natural numbers.

	u_1	u_2	$\cdots$	u_n	$\cdots$	u_ω	$\leqq$
v	1	$\frac{1}{2}$	$\cdots$	$\frac{1}{n}$	$\cdots$	0	1
$\geqq$	$\frac{1}{1}\left(1-\frac{1}{1}\right)$	$\frac{1}{2}\left(1-\frac{1}{2}\right)$	$\cdots$	$\frac{1}{n}\left(1-\frac{1}{n}\right)$	$\cdots$	0	min / max

2. $X = Y = \{N, \omega\}$

	u_1	u_2	$\cdots$	u_n	$\cdots$	u_ω	$\leqq$
v_1	1						1
v_2		$\frac{1}{2^2}$		0			$\frac{1}{2^3}$
$\vdots$			$\ddots$				$\vdots$
v_n				$\frac{1}{n^2}$			$\frac{1}{n^3}$
$\vdots$		0			$\ddots$		$\vdots$
v_ω						0	0
$\geqq$	1	$\frac{1}{2}$	$\cdots$	$\frac{1}{n}$	$\cdots$	0	min / max

3. $X=\{N, \omega\}$, $Y=\{1\}$

	u	$\leqq$
v_1	1	1
v_2	$\frac{1}{2}$	$\frac{1}{2^2}$
$\vdots$	$\vdots$	$\vdots$
v_n	$\frac{1}{n}$	$\frac{1}{n^2}$
$\vdots$	$\vdots$	$\vdots$
v_ω	0	0
$\geqq$	1	min / max

4. $X=Y=\{N, \omega\}$

	$u_1\ u_2 \cdots u_n \cdots\cdots\cdots$	u_ω	$\leqq$
v_1	$\cdots\cdots \frac{1}{n} \cdots\cdots\cdots$	0	1
v_2	$\cdots\cdots \frac{1}{n}\ \frac{1}{2^2} \cdots\cdots$	0	$\frac{1}{2^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	$\cdots\cdots \frac{1}{n}\ \frac{1}{n^n} \cdots\cdots$	0	$\frac{1}{n^n}$
v_{n+1}	$\cdots\cdots \frac{1}{n}\ \frac{1}{(n+1)^n} \cdots$	0	$\frac{1}{(n+1)^{(n+1)}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_k	$\cdots\cdots \frac{1}{n}\ \frac{1}{k^n} \cdots\cdots$	0	$\frac{1}{k^k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_ω	$\cdots\cdots 0 \cdots\cdots\cdots$	0	0
$\geqq$	$1\ \frac{1}{2} \cdots \frac{1}{n} \cdots\cdots\cdots$	0	min / max

Reference

[1] M. Ohtsuka: *A generalization of duality theorem in the theory of linear programming*, this journal.

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