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On a Class of Lie Algebras

Shigeaki Tôgô

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Introduction

In the previous paper [4], we have given an estimate for the dimensionality of the derivation algebra of a Lie algebra L satisfying the condition that $(\mathrm{ad}\,x)^2=0$ for $x \in L$ implies $\mathrm{ad}\ x=0$. Such a Lie algebra will be referred to as an (A_2)-algebra in this paper according to the definition given in Jôichi [2], which investigates the (A_k)-algebras, $k \geqq 2$, with intention to obtain the analogues to the (A)-algebras. He showed that the (A_2)-algebras have a different situation from the other classes of (A_k)-algebras, $k \geqq 3$. But the problem of characterizing the (A_2)-algebras remains unsolved. The purpose of this paper is to make a detailed study of this class of Lie algebras.

It is known [3] that every semisimple Lie algebra over the field of complex numbers contains no non-zero element x with $(\mathrm{ad}\ x)^2=0$. We shall show that every Lie algebra over a field Φ of characteristic $\neq 2$ whose Killing form is non-degenerate has the same property. By making use of this result we shall show that, when the basic field Φ is of characteristic 0, L is an (A_2)-algebra if and only if every element x of the nil radical N such that $(\mathrm{ad}\,x)^2=0$ belongs to the center $Z(L)$, and if and only if L is either reductive, or $L \supset N \supset Z(N)=Z(L) \supseteqq N^2 \neq (0)$ and $(\mathrm{ad}\,x)^2 \neq 0$ for any $x \in N \setminus Z(L)$. This characterization will be used in classifying certain types of solvable (A_2)-algebras. A solvable (A_2)-algebra is not generally abelian. We shall show that if Φ is an algebraically closed field of characteristic 0, then every solvable (A_2)-algebra over a field Φ is abelian. The latter half of the paper will be devoted to the study of solvable (A_2)-algebras, in particular, to the study of solvable (A_2)-algebras L such that $\dim N/Z(L)$ is 2 or 3 and of solvable (A_2)-algebras of low dimensionalities.

§1.

Throughout this paper we denote by L a finite dimensional Lie algebra over a field Φ and denote by R, N and $Z(L)$ the radical, the nil radical and the center of L respectively.

Following the terminology employed in [2], we call L to be an (A_2)-*algebra* provided that it satisfies the following condition:

(A_2) *Every element* x *of* L *such that* $(\mathrm{ad}\,x)^2=0$ *satisfies* $\mathrm{ad}\,x=0$, *that is, belongs to* $Z(L)$.

We first quote a result shown in Theorem 1 in [2] as the following

Lemma 1. *Let L be an* (A_2)*-algebra over a field of arbitrary characteristic. Then L is nilpotent if and only if L is abelian.*

By making use of the lemma, we show a necessary condition for L to be an (A_2)-algebra in the following

Proposition 1. *Let L be an* (A_2)*-algebra over a field of arbitrary characteristic. Then either $R=Z(L)$ or*

$$L\supset N\supset Z(N)=Z(L)\supseteqq N^2\neq(0).$$

Proof. Let L be a non-abelian (A_2)-algebra. Then by Lemma 1 L is not nilpotent, that is, $L\neq N$. For every $x\in Z(N)$, we have $[x,[x,L]]\subseteqq[x,N]=(0)$. From the condition (A_2) it follows that $x\in Z(L)$. Hence $Z(N)\subseteqq Z(L)$ and therefore $Z(N)=Z(L)$.

In the case where $N=Z(L)$, if $R\neq(0)$, choose an integer n such that $R^{(n)}=(0)$ but $R^{(n-1)}\neq(0)$. Suppose $n\geqq 2$. Since $R^{(n-1)}$ is an abelian ideal of L, we have $R^{(n-1)}\subseteqq N$. It follows that $(R^{(n-2)})^3=(0)$. Hence $R^{(n-2)}$ is a nilpotent ideal of L and therefore $R^{(n-2)}\subseteqq N$. It follows that $R^{(n-1)}=(0)$, which contradicts the choice of n. Thus $R^{(1)}=(0)$ and therefore $R=N=Z(L)$.

In the case where $N\neq Z(L)$, we have $N^2\neq(0)$, for if $N^2=(0)$ then $N=Z(N)=Z(L)$. The fact that $N^3=(0)$ can be shown as in the proof of Theorem 2 in [2]. It follows that $N^2\subseteqq Z(N)=Z(L)$. Thus the proof is complete.

We shall next show a sufficient condition for L to be an (A_2)-algebra. It has been observed in [3] that every semisimple Lie algebra over the field of complex numbers contains no non-zero element x with $(\operatorname{ad}x)^2=0$. We prove this assertion for a more general class of Lie algebras in the following

Lemma 2. *Let L be a Lie algebra over a field of characteristic $\neq 2$ and assume that the Killing form of L/R is non-degenerate. If $(\operatorname{ad}x)^2=0$ for $x\in L$, then $x\in R$.*

Proof. We first consider the case where L is semisimple. Suppose that $(\operatorname{ad}x)^2=0$ for $x\in L$. This means that $[x,[x,y]]=0$ for every $y\in L$. Putting $X=\operatorname{ad}x$ and $Y=\operatorname{ad}y$, we have $X^2=0$ and $[X,[X,Y]]=0$. Since

$$[X,[X,Y]]=X^2Y-2XYX+YX^2,$$

it follows that $XYX=0$. Hence $(XY)^2=0$. Denoting by B the Killing bilinear form of L, we see that $B(x,y)=0$ for every $y\in L$. Since B is non-degenerate by our hypothesis, we have $x=0$.

We now consider the general case. Suppose that $(\operatorname{ad}x)^2=0$ for $x\in L$. Put $\bar{L}=L/R$ and denote by $\bar{x}$ the element of $\bar{L}$ corresponding to x. Then $(\operatorname{ad}\bar{x})^2=0$. Since $\bar{L}$ is semisimple, we have $\bar{x}=0$ as shown in the first case. This means that $x\in R$, completing the proof.

Proposition 2. *Let L be a Lie algebra over a field of characteristic $\neq 2$.*

If L is the direct sum of an ideal which has the non-degenerate Killing form and of the center, then L is an (A_2)*-algebra.*

Proof. If L is such a direct sum, then the radical coincides with the center. Hence by Lemma 2 L is an (A_2)-algebra.

Now we restrict the basic field Φ to a field of characteristic 0. Then we can derive the following characterizations of (A_2)-algebras from the above results.

Theorem 1. *Let L be a Lie algebra over a field Φ of characteristic* 0. *Then the following statements are equivalent:*

(1) *L is an* (A_2)*-algebra.*

(2) *Every element x of N such that* $(\mathrm{ad}_L\, x)^2=0$ *belongs to $Z(L)$.*

(3) *L is either reductive, or*

$$L\supset N\supset Z(N)=Z(L)\supseteqq N^2\neq(0)$$

and $(\mathrm{ad}_L\, x)^2\neq 0$ *for every $x\in N\setminus Z(L)$.*

Proof. Since the basic field Φ is of characteristic 0, N is the set of $x\in R$ such that $\mathrm{ad}_L\, x$ is nilpotent. Hence Lemma 2 tells us that if $(\mathrm{ad}\, x)^2=0$ for $x\in L$ then $x\in N$. Therefore (1) and (2) are equivalent. From this equivalence and Proposition 2 it follows that (3) implies (1). The assertion that (1) implies (3) is a consequence of Proposition 1.

Corollary 1. *Let L be a Lie algebra over a field of characteristic* 0 *and assume that $Z(L)=Z(R)$. If R is an* (A_2)*-algebra, then L is an* (A_2)*-algebra.*

Proof. The statement is immediate from the equivalence of (1) and (2) in Theorem 1 and the fact that the nil radicals of L and R are identical.

Corollary 2. *Let L be a non-nilpotent Lie algebra over a field Φ of characteristic* 0 *such that*

$$N=(x_1,\ y_1,\ \cdots,\ x_n,\ y_n)+Z(L),$$

$$0\neq[x_i,\ y_i]\in Z(L),$$

$$[x_i,\ x_j]=[x_i,\ y_j]=[y_i,\ y_j]=0$$

$$\text{for all}\quad i\neq j.$$

Assume that for every $i=1, 2\cdots, n$, there exists an element u_i of $L\setminus N$ satisfying the following conditions:

$$[u_i,\ x_i]=y_i,\quad [u_i,\ y_i]=\lambda_i x_i,$$

$$[u_i,\ x_j],\quad [u_i,\ y_j]\in Z(L)\qquad \text{for any}\quad j\neq i,$$

where λ_i is not a square element in Φ. Then L is an (A_2)*-algebra.*

PROOF. Suppose that $x \in N$ and $(\mathrm{ad}_L x)^2=0$. Then x is expressed as

$$x=\sum_{i=1}^{n}(\alpha_i x_i+\beta_i y_i)+z, \quad z \in Z(L).$$

By using our assumption we obtain

$$[x, [x, u_j]]=(\lambda_j\beta_j^2-\alpha_j^2)[x_j, y_j]=0$$

and therefore $\lambda_j\beta_j^2=\alpha_j^2$ for $j=1, 2, \cdots, n$. Since λ_j is not a square element in Φ, we have $\beta_j=0$ and therefore $\alpha_j=0$. Hence $x \in Z(L)$. Thus L satisfies the condition (2) in Theorem 1. By Theorem 1 L is an (A_2)-algebra, completing the proof.

We note that the examples of solvable (A_2)-algebras shown in [2] and [4] are those of the (A_2)-algebras formulated in Corollary 2.

§2.

L is called split [1] provided that it has a splitting Cartan subalgebra, that is, a Cartan subalgebra H such that the characteristic roots of every $\mathrm{ad}\, x$, $x \in H$, are in the basic field Φ. It is known that every Lie algebra over an algebraically closed field is split. For split (A_2)-algebras we first show the following

LEMMA 3. *Let L be a split (A_2)-algebra over a field Φ of characteristic $\neq 2$. Then L^2 is nilpotent if and only if L is abelian.*

PROOF. Assume that L^2 is nilpotent but L is not abelian. Then L is not nilpotent by Lemma 1. Since L^2 is a nilpotent ideal of L, we have $L^2 \subseteqq N$. Let H be a splitting Cartan subalgebra and let $L=H+\sum_\alpha L_\alpha$ be the decomposition of L to the root spaces. Then it is immediate that $L_\alpha \subseteqq L^2 \subseteqq N$ for every root $\alpha \neq 0$. Choose a non-zero root β and let k be an integer such that $2^k\beta$ is a root but $2^{k+1}\beta$ is not a root. Put $\gamma=2^k\beta$ and choose a non-zero element x of L_γ. Then

$$\begin{aligned}[x, [x, L]] &= [x, [x, H]]+[x, [x, \sum_{\alpha\neq 0} L_\alpha]] \\ &\subseteqq [L_\gamma, L_\gamma]+N^3.\end{aligned}$$

We have $N^3=(0)$ by Proposition 1 and $[L_\gamma, L_\gamma]=(0)$ since 2γ is not a root. Hence $(\mathrm{ad}\, x)^2=0$. Since L is an (A_2)-algebra, it follows that $x \in Z(L)$ and therefore $x \in H$, which contradicts the choice of x. Thus we conclude that if L^2 is nilpotent then L is abelian.

In virtue of Lemma 3, we have now the following characterization of split (A_2)-algebras.

THEOREM 2. *Let L be a split Lie algebra over a field Φ of characteristic 0. Then L is an (A_2)-algebra if and only if L is either reductive, or*

$$L \supset R \supseteqq N \supset Z(N) = Z(L) \supseteqq N^2 \neq (0)$$

and $(\mathrm{ad}\, x)^2 \neq 0$ *for every* $x \in N \setminus Z(L)$.

Proof. In the case where the basic field is of characteristic 0, Lemma 3 says that a split (A_2)-algebra is solvable if and only if it is abelian. Therefore if L is a split (A_2)-algebra over a field Φ, then we have the statement (3) in Theorem 1, in the second case of which L is not solvable. Thus the theorem follows from Theorem 1.

§3.

In this and the next sections we shall study the solvable (A_2)-algebras over a field Φ of characteristic 0 as an application of Theorem 1. As seen from Theorem 2, if Φ is algebraically closed, then every solvable (A_2)-algebra is abelian. Hence, throughout these sections, we shall assume that the basic field Φ is of characteristic 0 and not algebraically closed unless otherwise specified.

This section is devoted to the study of solvable (A_2)-algebras L such that $\dim N/Z(L) = 2$ or 3. First we prove the following

Lemma 4. *Let L be a non-abelian solvable* (A_2)*-algebra. If* $[u, x] \notin Z(L)$ *for* $u \in L$ *and* $x \in N$, *then* $[u, [u, x]] \notin Z(L)$.

Proof. Assume that $[u, [u, x]] \in Z(L)$ for $u \in L$ and $x \in N$. Put $y = [u, x]$. Then by using the fact that $N^2 \subseteqq Z(L)$, for every $v \in L$ we have

$$\begin{aligned}
(\mathrm{ad}\, y)^2 v &= [y, [[u, x], v]] \\
&= [y, [u, [x, v]]] + [y, [[u, v], x]] \\
&= [[y, u], [x, v]] + [u, [y, [x, v]]] + [y, [[u, v], x]] \\
&\in [u, N^2] + N^3 \\
&= (0).
\end{aligned}$$

Thus $(\mathrm{ad}\, y)^2 = 0$ and therefore by the condition (A_2) $y \in Z(L)$. This completes the proof.

Proposition 3. *The solvable* (A_2)*-algebras L over a field Φ of characteristic* 0 *such that* $\dim N/Z(L) = 2$ *are the following Lie algebras:*

$$\begin{aligned}
&L = (u_1, u_2, \cdots, u_n) + N, \quad N = (x, y) + Z(L), \\
&[u_i, u_j] \in Z(L), \\
&[u_i, x] = y, \quad [u_i, y] = \lambda x, \\
&0 \neq [x, y] \in Z(L) \qquad \textit{for} \quad i, j = 1, 2, \cdots, n
\end{aligned}$$

where $n=\dim L/N$ *and* λ *is not a square element in* Φ.

Proof. Since $N\supset Z(L)$ by Theorem 1, we choose x in $N\setminus Z(L)$. Then $(\mathrm{ad}\,x)^2\neq 0$ and therefore there exists $u_1\in L$ such that $(\mathrm{ad}\,x)^2u_1\neq 0$. By Theorem 1 we see that $u_1\notin N$. Put $y=[u_1, x]$ and $z=[x, y]$. Then $y\in N$, $y\notin (x)+Z(L)$ and $0\neq z\in Z(L)$. Therefore $N=(x, y)+Z(L)$. By Lemma 4 we have $[u_1, y]\notin Z(L)$. Since

$$[x, [u_1, y]]=[[x, u_1], y]=[-y, y]=0,$$

it follows that

$$[u_1, y]=\lambda x+z',\ z'\in Z(L) \qquad \text{with} \quad \lambda\neq 0.$$

Replacing x by $x+\lambda^{-1}z'$, we see that

$$[u_1, x]=y,\ [u_1, y]=\lambda x,\ [x, y]=z$$

$$\text{with} \quad 0\neq z\in Z(L) \quad \text{and} \quad \lambda\neq 0.$$

If $\dim L/N\geqq 2$, choose $u_2\in L,\ \notin (u_1)+N$. And we write

$$[u_2, x] =\alpha_1x+\beta_1y+z_1,$$

$$[u_2, y] =\alpha_2x+\beta_2y+z_2,$$

$$[u_1, u_2]=\alpha_3x+\beta_3y+z_3,$$

where $z_i\in Z(L)$ for $i=1, 2, 3$. Then from $[[u_1, u_2], x]+[[u_2, x], u_1]+[[x, u_1], u_2]=0$ it follows that $z_2=\beta_3z$, $\alpha_1=\beta_2$ and $\alpha_2=\lambda\beta_1$. From the above formula with x replaced by y it follows that $z_1=-\lambda^{-1}\alpha_3z$. From $[[u_2, x], y]=[[u_2, y], x]$ it follows that $\alpha_1+\beta_2=0$ and therefore $\alpha_1=\beta_2=0$. Hence we obtain by changing the notations

$$[u_2, x] =\mu_2y+\nu_1z,$$

$$[u_2, y] =\lambda\mu_2x+\nu_2z,$$

$$[u_1, u_2]=-\lambda\nu_1x+\nu_2y+z', \qquad z'\in Z(L).$$

Replacing u_2 by $u_2-\nu_2x+\nu_1y$, we have

$$[u_2, x]=\mu_2y, \quad [u_2, y]=\lambda\mu_2x,$$

$$[u_1, u_2]\in Z(L).$$

We continue this procedure to choose $u_3, u_4, \cdots, u_n$ with $n=\dim L/N$ in such a way that

$$L=(u_1, u_2, \cdots, u_n)+N,$$

$$[u_i, x]=\mu_iy, \quad [u_i, y]=\lambda\mu_ix,$$

$$[u_1, u_i]\in Z(L) \qquad \text{for} \quad i=2, 3, \cdots, n.$$

Now we have $[x, [u_i, u_j]]=[y, [u_i, u_j]]=0$ for $i, j=1, 2, \cdots, n$, from which it follows that $[u_i, u_j] \in Z(N)=Z(L)$. Since $(\operatorname{ad} u_i)^2 \neq 0$, it follows that $\mu_i \neq 0$ for $i=2, 3, \cdots, n$. Hence we replace u_i by $\mu_i^{-1} u_i$ to obtain

$$[u_i, x]=y, \quad [u_i, y]=\lambda x \qquad \text{for} \quad i=1, 2, \cdots, n.$$

If $\lambda=\alpha^2$ in Φ, then $(\operatorname{ad} \alpha x+y)^2=0$. Therefore λ is not equal to any square element in Φ.

Conversely, let L be such a Lie algebra as indicated in the statement. Assume that $v \in N$ and $(\operatorname{ad} v)^2=0$. Then v is expressed as $v=\alpha x+\beta y+z'$, $z' \in Z(L)$. From $(\operatorname{ad} v)^2 u_1=0$, it follows that $\lambda\beta^2=\alpha^2$. Therefore $\beta=0$ and $\alpha=0$. Hence $v \in Z(L)$. Thus by Theorem 1 L is an (A_2)-algebra. The proof is complete.

In the remainder of this section we shall show that there is no solvable (A_2)-algebra L such that $\dim N/Z(L)=3$.

Lemma 5. *Let L be a solvable Lie algebra such that*

$$N=(x_1, x_2, x_3)+Z(L), \quad u \in L \setminus N,$$

$$[u, x_1]=x_2, \quad [u, x_2]=\alpha^2 x_1, \quad [u, x_3]=\alpha x_3+z,$$

$$[x_1, x_2]=z_1, \quad [x_1, x_3]=z_2, \quad [x_2, x_3]=-\alpha z_2$$

where $z, z_1, z_2 \in Z(L)$ and $\alpha \neq 0$. Then L is not an (A_2)-algebra.

Proof. Assume that L is an (A_2)-algebra. For every $v \in L, \notin (u)+N$, we write

$$[v, x_1]=\sum_{i=1}^{3} \alpha_i x_i+w_1,$$

$$[v, x_2]=\sum_{i=1}^{3} \beta_i x_i+w_2,$$

where $w_1, w_2 \in Z(L)$. From $[[u, v], x_1]+[[v, x_1], u]+[[x_1, u], v]=0$ it follows that

$$\alpha_1=\beta_2, \quad \alpha^2\alpha_2=\beta_1 \quad \text{and} \quad \alpha\alpha_3=\beta_3.$$

Then

$$\begin{aligned}(\operatorname{ad}-\alpha x_1+x_2)^2 v &= [-\alpha x_1+x_2, (\alpha\alpha_1-\beta_1)x_1+(\alpha\alpha_2-\beta_2)x_2] \\ &= -\alpha(\alpha\alpha_2-\beta_2)z_1-(\alpha\alpha_1-\beta_1)z_1 \\ &= 0.\end{aligned}$$

Since it is immediate that $(\operatorname{ad}-\alpha x_1+x_2)^2 u=0$, we have $(\operatorname{ad}-\alpha x_1+x_2)^2=0$, which contradicts the condition (A_2). Therefore L is not an (A_2)-algebra, completing the proof.

Proposition 4. *Let L be a solvable* (A_2)*-algebra over a field Φ of characteristic* 0. *Then* $\dim N/Z(L)\neq 3$. *In particular, if* $\dim Z(L)=1$, *then* $\dim N/Z(L)$ *is not odd.*

Proof. Assume that there exists a solvable (A_2)-algebra L such that $\dim N/Z(L)=3$. Take $x_1\in N\setminus Z(L)$ and choose $u\in L$ such that $(\operatorname{ad} x_1)^2 u\neq 0$. Then $u\notin N$. Put $x_2=[u, x_1]$ and $z_1=[x_1, x_2]$. Then $x_2\notin (x_1)+Z(L)$ and $0\neq z_1\in Z(L)$. By Lemma 4 we see that $[u, x_2]\notin Z(L)$.

Now suppose that $[u, x_2]\notin (x_1, x_2)+Z(L)$. Putting $x_3=[u, x_2]$, we have $N=(x_1, x_2, x_3)+Z(L)$. It follows that

$$\begin{aligned}[x_1, x_3]&=[x_1, [u, x_2]]\\ &=[[x_1, u], x_2]+[u, [x_1, x_2]]\\ &=0.\end{aligned}$$

By using this fact we obtain

$$\begin{aligned}[x_2, x_3]&=[[u, x_1], x_3]\\ &=[[u, x_3], x_1]+[u, [x_1, x_3]]\\ &\in[x_1, N]+[u, N^2]\\ &=(z_1).\end{aligned}$$

Put $[x_2, x_3]=\alpha z_1$ and $x'_3=x_3+\alpha x_1$. Then we have $[x_2, x'_3]=0$, from which it follows that $x'_3\in Z(N)=Z(L)$ and therefore $x_3\in(x_1)+Z(L)$. This contradicts our supposition.

We have thus $[u, x_2]\in(x_1, x_2)+Z(L)$. Choose a basis of $Z(L)$ so that $Z(L)=(z_1, z_2, \cdots, z_m)$. Since $[x_1, [u, x_2]]=0$, it follows that

$$[u, x_2]=\lambda x_1+\sum_{i=1}^{m}\mu_i z_i \qquad \text{with} \quad \lambda\neq 0.$$

Replacing x_1 by $x_1+\lambda^{-1}\sum_{i=1}^{m}\mu_i z_i$, we have

$$[u, x_1]=x_2, \quad [u, x_2]=\lambda x_1, \quad [x_1, x_2]=z_1.$$

We now choose $x_3\in N$, $\notin(x_1, x_2)+Z(N)$. Then $N=(x_1, x_2, x_3)+Z(L)$. Put

$$[u, x_3]=\sum_{i=1}^{3}\alpha_i x_i+\sum_{i=1}^{m}\alpha'_i z_i,$$

$$[x_1, x_3]=\sum_{i=1}^{m}\beta_i z_i,$$

$$[x_2, x_3]=\sum_{i=1}^{m}\gamma_i z_i.$$

Then it follows from $[[u, x_3], x_i]=[[u, x_i], x_3]$ for $i=1, 2$ that

$$\alpha_1-\lambda\beta_1-\alpha_3\gamma_1=0, \quad \alpha_2+\alpha_3\beta_1+\gamma_1=0,$$
$$\lambda\beta_i=-\alpha_3\gamma_i, \quad -\alpha_3\beta_i=\gamma_i \qquad \text{for} \quad i\geqq 2.$$

Replacing x_3 by $x_3+\gamma_1x_1-\beta_1x_2$, we have

$$[u, x_3]=\alpha_3x_3+\sum_{i=1}^{m}\alpha'_iz_i,$$

$$[x_1, x_3]=\sum_{i=2}^{m}\beta_iz_i,$$

$$[x_2, x_3]=-\alpha_3\sum_{i=2}^{m}\beta_iz_i$$

and $\lambda\beta_i=\alpha_3^2\beta_i$ for $i\geqq 2$. If $\lambda\neq\alpha_3^2$, then $\beta_i=0$ for all $i\geqq 2$. Hence $[x_3, N]=(0)$ and therefore $x_3\in Z(N)=Z(L)$, which contradicts the choice of x_3. If $\lambda=\alpha_3^2$, then $\alpha_3\neq 0$. Hence L satisfies the hypothesis of Lemma 5 and therefore L is not an (A_2)-algebra, which contradicts our assumption. Thus the first part is proved.

We now consider the special case where $\dim Z(L)=1$. Choose $x_1\in N\setminus Z(L)$. Since $\dim N/Z(L)\geqq 2$ and $x_1\notin Z(N)$, there exists $x_2\in N\setminus Z(L)$ such that $[x_1, x_2]\neq 0$. Put $[x_1, x_2]=z$. Then $Z(L)=(z)$. Assume that we have already chosen $x_1, x_2, \cdots, x_{2k}$ in N which are linearly independent over Φ and such that

$$[x_{2h-1}, x_{2h}]=z \qquad \text{for} \quad h=1, 2, \cdots, k,$$
$$[x_i, x_j]=0 \qquad \text{for all other} \quad i<j$$

and furthermore assume that $\dim N/Z(L)>2k$. Then choose $y\in N$, $\notin (x_1, x_2, \cdots, x_{2k})+Z(L)$ and put

$$x_{2k+1}=y+(-\alpha_2x_1+\alpha_1x_2)+\cdots+(-\alpha_{2k}x_{2k-1}+\alpha_{2k-1}x_{2k})$$

where α_i is such that $[y, x_i]=\alpha_iz$. It follows that $[x_{2k+1}, x_i]=0$ for $i=1, 2, \cdots, 2k$. Since $x_{2k+1}\notin Z(N)$, we have $\dim N/Z(L)>2k+1$ and there exists $x_{2k+2}\in N$ such that $[x_{2k+1}, x_{2k+2}]=z$. Replacing x_{2k+2} by a sum of x_{2k+2} and a suitable linear combination of $x_1, x_2, \cdots, x_{2k}$ as above, we may suppose that $[x_{2k+2}, x_i]=0$ for $i=1, 2, \cdots, 2k$. Hence by using induction we can conclude that $\dim N/Z(L)$ is not odd.

Thus the proof is complete.

§4.

Throughout this section we use the following notations for a Lie algebra L:

$$n_1 = \dim L/N, \quad n_2 = \dim N/Z(L) \quad \text{and} \quad n_3 = \dim Z(L).$$

We shall then call L to be of type (n_1, n_2, n_3). Owing to Theorem 1 we see that for every non-reductive (A_2)-algebra L $n_1 \geqq 1$, $n_2 \geqq 2$ and $n_3 \geqq 1$. Hence every 1 dimensional and 2 dimensional (A_2)-algebra is abelian and every 3 dimensional (A_2)-algebra is abelian or simple. By making use of the propositions in the preceding section, we shall study the structures of the 4, 5 and 6 dimensional solvable (A_2)-algebras.

As for the 4 dimensional (A_2)-algebras we have the following

Proposition 5. *The 4 dimensional non-reductive* (A_2)*-algebras over a field* Φ *of characteristic* 0 *are the following Lie algebras*:

$L_\lambda = (x_1, x_2, x_3, x_4)$ *with the multiplication table*

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = \lambda x_2,$$

$$[x_2, x_3] = x_4, \quad [x_i, x_4] = 0$$

$$\text{for} \quad i = 1, 2, 3$$

where λ *is not a square element in* Φ.

L_{λ_1} *and* L_{λ_2} *are isomorphic if and only if* $\lambda_1 \lambda_2^{-1}$ *is a square element in* Φ.

When Φ *is the field of real numbers, every* L_λ *is isomorphic to* L_{-1}.

Proof. Let L be a 4 dimensional non-reductive (A_2)-algebra. Then L is obviously of type (1, 2, 1). By Proposition 3 we see that L is equal to L_λ with some λ.

Assume that f is an isomorphism of L_{λ_1} onto L_{λ_2}. Then f sends the nil radical and the center of L_{λ_1} onto those of L_{λ_2} respectively. Hence, denoting $L_{\lambda_2} = (y_1, y_2, y_3, y_4)$, we have

$$f(x_1) = \sum_{j=1}^{4} \alpha_{1j} y_j,$$

$$f(x_i) = \sum_{j=2}^{4} \alpha_{ij} y_j \qquad \text{for} \quad i = 2, 3,$$

$$f(x_4) = \alpha_{44} y_4.$$

Since the rank of f is 4, we have

$$\alpha_{11}\alpha_{44} \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} \neq 0.$$

From $[f(x_1), f(x_2)] = f(x_3)$ and $[f(x_1), f(x_3)] = \lambda_1 f(x_2)$, it follows that

$$\lambda_1 \alpha_{22} = \lambda_2 \alpha_{11} \alpha_{33}$$

$$\lambda_1 \alpha_{23} = \alpha_{11} \alpha_{32}$$

$$\alpha_{32} = \lambda_2 \alpha_{11} \alpha_{23}$$

$$\alpha_{33} = \alpha_{11} \alpha_{22}.$$

Hence we have

$$\alpha_{22}(\lambda_1-\lambda_2\alpha_{11}^2)=0$$

$$\alpha_{23}(\lambda_1-\lambda_2\alpha_{11}^2)=0.$$

Since we cannot have $\alpha_{22}=\alpha_{23}=0$, it follows that $\lambda_1=\lambda_2\alpha_{11}^2$.

Conversely, assume that for L_{λ_1} and L_{λ_2}, $\lambda_1\lambda_2^{-1}=\alpha^2$ with $\alpha \in \Phi$. Then $\alpha\neq 0$. Define a linear transformation f of L_{λ_1} into L_{λ_2} in such a way that

$$f(x_1)=\alpha y_1,$$

$$f(x_2)=y_2,$$

$$f(x_3)=\alpha y_3,$$

$$f(x_4)=\alpha y_4.$$

Then it is easy to see that f is an isomorphism of L_{λ_1} onto L_{λ_2}.

When Φ is the field of real numbers, if λ is not a square element then $\lambda<0$. Therefore every L_λ is isomorphic to L_{-1}, and the proof is complete.

As for the 5 dimensional (A_2)-algebras we have the following

Proposition 6. *The* 5 *dimensional non-reductive* (A_2)*-algebras over a field* Φ *of characteristic* 0 *are the following Lie algebras*:

(1) *The direct sum of a* 4 *dimensional non-reductive* (A_2)*-algebra and the* 1 *dimensional Lie algebra.*

(2) $L_{\lambda,\mu}=(x_1, x_2, x_3, x_4, x_5)$ *with the multiplication table*

$$[x_1, x_2]=\lambda x_5,$$

$$[x_1, x_3]=[x_2, x_3]=x_4,$$

$$[x_1, x_4]=[x_2, x_4]=\mu x_3,$$

$$[x_3, x_4]=x_5,$$

$$[x_i, x_5]=0 \qquad \text{for} \quad i=1, 2, 3, 4$$

where μ *is not a square element in* Φ.

L_{λ_1,μ_1} *and* L_{λ_2,μ_2} *are isomorphic if and only if both* λ_1 *and* λ_2 *are* 0 *or* $\neq 0$ *at the same time and* $\mu_1\mu_2^{-1}$ *is a square element in* Φ.

When Φ *is the field of real numbers, every* $L_{\lambda,\mu}$ *is isomorphic to one of the non-isomorphic* (A_2)*-algebras* $L_{0,-1}$ *and* $L_{1,-1}$.

Proof. Let L be a 5 dimensional (A_2)-algebra. Then by Proposition 4 L is either of type (1, 2, 2) or of type (2, 2, 1). In the first case, by Proposition 3 we see that L is a Lie algebra in (1) of the statement. In the second case, by Proposition 3 we see that L is one of $L_{\lambda,\mu}$ in (2) of the statement.

Assume that f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}, where we write $L_{\lambda_2,\mu_2}=(y_1, y_2, y_3, y_4, y_5)$. Since f sends the nil radical and the center of L_{λ_1,μ_1} onto those of L_{λ_2,μ_2} respectively, we can express f in the following form:

$$f(x_i)=\begin{cases} \sum_{j=1}^{5}\alpha_{ij}y_j & \text{for } i=1, 2 \\ \sum_{j=3}^{5}\alpha_{ij}y_j & \text{for } i=3, 4 \\ \alpha_{55}y_5 & \text{for } i=5, \end{cases}$$

where

$$\alpha_{55}\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix}\begin{vmatrix}\alpha_{33} & \alpha_{34}\\ \alpha_{43} & \alpha_{44}\end{vmatrix}\neq 0.$$

From $[f(x_1), f(x_3)]=f(x_4)$ it follows that

$$\alpha_{33}(\alpha_{11}+\alpha_{12})=\alpha_{44}$$

$$\mu_2\alpha_{34}(\alpha_{11}+\alpha_{12})=\alpha_{43}.$$

From $[f(x_1), f(x_4)]=\mu_1 f(x_3)$ it follows that

$$\mu_1\alpha_{33}=\mu_2\alpha_{44}(\alpha_{11}+\alpha_{12})$$

$$\mu_1\alpha_{34}=\alpha_{43}(\alpha_{11}+\alpha_{12}).$$

Therefore we have

$$\alpha_{33}\{\mu_1-\mu_2(\alpha_{11}+\alpha_{12})^2\}=0$$

$$\alpha_{34}\{\mu_1-\mu_2(\alpha_{11}+\alpha_{12})^2\}=0.$$

Since α_{33} and α_{34} cannot be equal to 0 at the same time, it follows that

$$\mu_1=\mu_2(\alpha_{11}+\alpha_{12})^2.$$

By this equality together with $\mu_1\neq 0$, we see that $\alpha_{11}+\alpha_{12}\neq 0$. On the other hand, from $[f(x_1), f(x_2)]=\lambda_1 f(x_5)$, it follows that

$$\alpha_{23}(\alpha_{11}+\alpha_{12})-\alpha_{13}(\alpha_{21}+\alpha_{22})=0$$

$$\alpha_{24}(\alpha_{11}+\alpha_{12})-\alpha_{14}(\alpha_{21}+\alpha_{22})=0$$

$$\lambda_2(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})+(\alpha_{13}\alpha_{24}-\alpha_{14}\alpha_{23})=\lambda_1\alpha_{55}.$$

Therefore from the first two equations above we obtain

$$\alpha_{13}\alpha_{24}-\alpha_{14}\alpha_{23}=0.$$

Then the last equation above becomes

$$\lambda_1\alpha_{55}=\lambda_2(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}).$$

This shows that $\lambda_1=0$ if and only if $\lambda_2=0$.

Conversely, assume that for L_{λ_1,μ_1} and L_{λ_2,μ_2} both λ_1 and λ_2 are 0 or $\neq 0$ at the same time and $\mu_1\mu_2^{-1}=\alpha^2$ with $\alpha \in \Phi$. In the case where $\lambda_1=\lambda_2=0$, we define a linear transformation of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$\begin{aligned}
f(x_1) &= \alpha y_1\\
f(x_2) &= \alpha y_2\\
f(x_3) &= y_3+y_4\\
f(x_4) &= \alpha(\mu_2 y_3+y_4)\\
f(x_5) &= \alpha(1-\mu_2)y_5.
\end{aligned}$$

Then the rank of f is 5, since the coefficient matrix of f has the determinant $\alpha^4(1-\mu_2)^2\neq 0$. In the case where $\lambda_1\neq 0$ and $\lambda_2\neq 0$, we define a linear transformation of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$\begin{aligned}
f(x_1) &= \alpha y_1\\
f(x_2) &= \{\alpha-\lambda_1\lambda_2^{-1}(1-\mu_2)\}\, y_1+\lambda_1\lambda_2^{-1}(1-\mu_2)y_2\\
f(x_3) &= y_3+y_4\\
f(x_4) &= \alpha(\mu_2 y_3+y_4)\\
f(x_5) &= \alpha(1-\mu_2)y_5.
\end{aligned}$$

Then the rank of f is 5, since the coefficient matrix of f has the determinant $\alpha^3\lambda_1\lambda_2^{-1}(1-\mu_2)^3\neq 0$. In each case, it is easy to see that f preserves the multiplication and therefore f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}.

If Φ is the field of real numbers, then it is immediate that every $L_{\lambda,\mu}$ is isomorphic to one of $L_{0,-1}$ and $L_{1,-1}$ which are not isomorphic.

Thus the proof is complete.

Finally we shall clarify the structure of the 6 dimensional (A_2)-algebras by restricting the basic field Φ to the field of real numbers. We first show the following

LEMMA 6. *The 6 dimensional* (A_2)*-algebras of type* (2, 2, 2) *over a field* Φ *of characteristic* 0 *are the following Lie algebras:*

(1) *The direct sum of a* 5 *dimensional* (A_2)*-algebra of type* (2, 2, 1) *and the* 1 *dimensional Lie algebra.*

(2) $L_\lambda=(x_1, x_2, \cdots, x_6)$ *with the multiplication table*

$$\begin{aligned}
&[x_1, x_2]=x_6,\\
&[x_1, x_3]=[x_2, x_3]=x_4,\\
&[x_1, x_4]=[x_2, x_4]=\lambda x_3,\\
&[x_3, x_4]=x_5,\\
&[x_i, x_j]=0 \qquad \textit{for all other} \quad i<j,
\end{aligned}$$

where λ *is not a square element in* Φ.

L_{λ_1} and L_{λ_2} are isomorphic if and only if $\lambda_1\lambda_2^{-1}$ is a square element in Φ. When Φ is the field of real numbers, every L_λ is isomorphic to L_{-1}.

PROOF. Let L be a 6 dimensional (A_2)-algebra of type (2, 2, 2). Then L is solvable. Hence by Proposition 3 L is described by a basis $x_1, x_2, \cdots, x_6$ in such a way that

$$\begin{aligned}
&[x_1, x_2] = \alpha x_5 + \beta x_6,\\
&[x_1, x_3] = [x_2, x_3] = x_4,\\
&[x_1, x_4] = [x_2, x_4] = \lambda x_3,\\
&[x_3, x_4] = x_5,\\
&[x_i, x_j] = 0 \qquad \text{for all other} \quad i<j
\end{aligned}$$

where λ is not a square element in Φ. If $\beta=0$, then L is the Lie algebra of the type indicated in (1) of the statement. If $\beta\neq 0$, then we can take $\alpha x_5+\beta x_6$ as new x_6 and L becomes the Lie algebra indicated in (2) of the statement.

Assume that f is an isomorphism of L_{λ_1} onto L_{λ_2}. Writing $L_{\lambda_2}=(y_1, y_2, \cdots, y_6)$, f can be expressed in the following form:

$$f(x_i) = \begin{cases} \sum_{j=1}^{6} \alpha_{ij} y_j & \text{for} \quad i=1, 2 \\ \sum_{j=3}^{6} \alpha_{ij} y_j & \text{for} \quad i=3, 4 \\ \sum_{j=5}^{6} \alpha_{ij} y_j & \text{for} \quad i=5, 6. \end{cases}$$

Since the rank of f is 6, we have

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} \cdot \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{vmatrix} \cdot \begin{vmatrix} \alpha_{55} & \alpha_{56} \\ \alpha_{65} & \alpha_{66} \end{vmatrix} \neq 0.$$

From $[f(x_1), f(x_3)] = f(x_4)$ it follows that

$$\alpha_{33}(\alpha_{11}+\alpha_{12}) = \alpha_{44}$$
$$\lambda_2\alpha_{34}(\alpha_{11}+\alpha_{12}) = \alpha_{43}.$$

From $[f(x_1), f(x_4)] = \lambda_1 f(x_3)$ it follows that

$$\lambda_1\alpha_{33} = \lambda_2\alpha_{44}(\alpha_{11}+\alpha_{12})$$
$$\lambda_1\alpha_{34} = \alpha_{43}(\alpha_{11}+\alpha_{12}).$$

Hence we obtain

$$\alpha_{33}\{\lambda_1 - \lambda_2(\alpha_{11}+\alpha_{12})^2\} = 0,$$
$$\alpha_{34}\{\lambda_1 - \lambda_2(\alpha_{11}+\alpha_{12})^2\} = 0.$$

Since $\alpha_{33}\neq 0$ or $\alpha_{34}\neq 0$, it follows that

$$\lambda_1=\lambda_2(\alpha_{11}+\alpha_{12})^2.$$

Conversely, assume that for L_{λ_1} and L_{λ_2}, $\lambda_1\lambda_2^{-1}=\alpha^2$ with $\alpha\in\Phi$. Then we define a linear transformation f of L_{λ_1} into L_{λ_2} in such a way that

$$\begin{aligned}
f(x_1)&=\alpha y_1\\
f(x_2)&=\alpha y_2\\
f(x_3)&=y_3+y_4\\
f(x_4)&=\alpha(\lambda_2 y_3+y_4)\\
f(x_5)&=\alpha(1-\lambda_2)y_5\\
f(x_6)&=\alpha^2 y_6.
\end{aligned}$$

The coefficient matrix of f has the determinant $\alpha^6(1-\lambda_2)^2\neq 0$ and it is immediate that f preserves the multiplication. Hence f is an isomorphism of L_{λ_1} onto L_{λ_2}.

When Φ is in particular the field of real numbers, every L_λ is obviously isomorphic to L_{-1}. Thus the proof is complete.

LEMMA 7. *The solvable* (A_2)*-algebras of type* (3, 2, 1) *over a field* Φ *of characteristic* 0 *are the following Lie algebras*:

$L_{\lambda,\mu}=(x_1, x_2, \cdots, x_6)$ *with the multiplication table*

$$\begin{aligned}
&[x_1, x_2]=\lambda x_6,\quad [x_1, x_3]=[x_2, x_3]=0,\\
&[x_1, x_4]=[x_2, x_4]=[x_3, x_4]=x_5,\\
&[x_1, x_5]=[x_2, x_5]=[x_3, x_5]=\mu x_4,\\
&[x_4, x_5]=x_6,\\
&[x_i, x_6]=0 \qquad \text{for} \quad i=1, 2, \cdots, 5
\end{aligned}$$

where μ *is not a square element in* Φ.

L_{λ_1,μ_1} *and* L_{λ_2,μ_2} *are isomorphic if and only if both* λ_1 *and* λ_2 *are* 0 *or* $\neq 0$ *at the same time and* $\mu_1\mu_2^{-1}$ *is a square element in* Φ.

When Φ *is the field of real numbers, every* $L_{\lambda,\mu}$ *is isomorphic to one of the non-isomorphic* (A_2)*-algebras* $L_{0,-1}$ *and* $L_{1,-1}$.

PROOF. Let L be a solvable (A_2)-algebra of type (3, 2, 1). Then by Proposition 3 L can be described by a basis $x_1, x_2, \cdots, x_6$ in such a way that

$$\begin{aligned}
&[x_1, x_2]=\alpha x_6,\quad [x_1, x_3]=\beta x_6,\quad [x_2, x_3]=\gamma x_6,\\
&[x_1, x_4]=[x_2, x_4]=[x_3, x_4]=x_5,\\
&[x_1, x_5]=[x_2, x_5]=[x_3, x_5]=\mu x_4,\\
&[x_4, x_5]=x_6,\quad [x_i, x_6]=0 \qquad \text{for} \quad i=1, 2, \cdots, 5
\end{aligned}$$

where μ is not a square element in Φ.

First assume that $\alpha-\beta+\gamma=0$. Then we assert that $\alpha=\beta=\gamma=0$. In fact, we put

$$y=\begin{cases} x_3+\alpha^{-1}\gamma x_1-\alpha^{-1}\beta x_2 & \text{if} \quad \alpha\neq 0 \\ x_2-\beta^{-1}\gamma x_1-\alpha\beta^{-1}x_3 & \text{if} \quad \beta\neq 0 \\ x_1-\beta\gamma^{-1}x_2+\alpha\gamma^{-1}x_3 & \text{if} \quad \gamma\neq 0. \end{cases}$$

Then $[y, x_1]=[y, x_2]=[y, x_3]=0$,

$$[y, x_4]=\begin{cases} \alpha^{-1}(\alpha-\beta+\gamma)x_5=0 & \text{if} \quad \alpha\neq 0 \\ -\beta^{-1}(\alpha-\beta+\gamma)x_5=0 & \text{if} \quad \beta\neq 0 \\ \gamma^{-1}(\alpha-\beta+\gamma)x_5=0 & \text{if} \quad \gamma\neq 0, \end{cases}$$

and similarly $[y, x_5]=0$. Hence $y\in Z(L)$ and therefore $\dim Z(L)\geqq 2$, which contradicts the hypothesis that L is of type (3, 2, 1). Therefore $\alpha=\beta=\gamma=0$, as was asserted.

Next assume that $\alpha-\beta+\gamma\neq 0$. If $\alpha\neq 0$, put

$$x_3'=\alpha(\alpha-\beta+\gamma)^{-1}(x_3+\alpha^{-1}\gamma x_1-\alpha^{-1}\beta x_2).$$

Then $[x_1, x_3']=[x_2, x_3']=0$, $[x_3', x_4]=x_5$ and $[x_3', x_5]=\mu x_4$. If $\beta\neq 0$, put

$$x_2'=-\beta(\alpha-\beta+\gamma)^{-1}(x_2-\beta^{-1}\gamma x_1-\alpha\beta^{-1}x_3).$$

Then $[x_1, x_2']=[x_2', x_3]=0$, $[x_2', x_4]=x_5$ and $[x_2', x_5]=\mu x_4$. If $\gamma\neq 0$, put

$$x_1'=\gamma(\alpha-\beta+\gamma)^{-1}(x_1-\beta\gamma^{-1}x_2+\alpha\gamma^{-1}x_3).$$

Then $[x_1', x_2]=[x_1', x_3]=0$, $[x_1', x_4]=x_5$ and $[x_1', x_5]=\mu x_4$. Thus, in any case, we can change a basis so that two of α, β and γ are 0. Finally by rearranging x_1, x_2, x_3 if necessary, we obtain $\beta=\gamma=0$. Therefore L is one of the $L_{\lambda,\mu}$. Thus the first statement is proved.

Assume that f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}. Writing $L_{\lambda_2,\mu_2}=(y_1, y_2, \cdots, y_6)$, we can express f in the following form:

$$f(x_i)=\begin{cases} \sum_{j=1}^{6}\alpha_{ij}y_j & \text{for} \quad i=1, 2, 3 \\ \sum_{j=4}^{6}\alpha_{ij}y_j & \text{for} \quad i=4, 5 \\ \alpha_{66}y_6 & \text{for} \quad i=6. \end{cases}$$

Since the rank of f is 6, we have

$$\alpha_{66}\begin{vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{vmatrix}\begin{vmatrix}\alpha_{44} & \alpha_{45}\\ \alpha_{54} & \alpha_{55}\end{vmatrix}\neq 0.$$

From $[f(x_i), f(x_4)]=f(x_5)$ for $i=1, 2, 3$, it follows that

$$\alpha_{44}(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})=\alpha_{55}$$

$$\mu_2\alpha_{45}(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})=\alpha_{54} \qquad \text{for} \quad i=1, 2, 3.$$

From $[f(x_i), f(x_5)]=\mu_1 f(x_4)$ for $i=1, 2, 3$, it follows that

$$\mu_1\alpha_{44}=\mu_2\alpha_{55}(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})$$

$$\mu_1\alpha_{45}=\quad \alpha_{54}(\alpha_{i1}+\alpha_{i2}+\alpha_{i3}) \qquad \text{for} \quad i=1, 2, 3.$$

Hence we have

$$\alpha_{44}\{\mu_1-\mu_2(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})^2\}=0$$

$$\alpha_{45}\{\mu_1-\mu_2(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})^2\}=0$$

$$\text{for} \quad i=1, 2, 3.$$

Since $\alpha_{44}\neq 0$ or $\alpha_{45}\neq 0$, it follows that

$$\mu_1=\mu_2(\alpha_{i1}+\alpha_{i2}+\alpha_{i3})^2, \qquad i=1, 2, 3.$$

We have also

$$\alpha_{44}\{(\alpha_{11}+\alpha_{12}+\alpha_{13})-(\alpha_{21}+\alpha_{22}+\alpha_{23})\}=0$$

$$\alpha_{44}\{(\alpha_{11}+\alpha_{12}+\alpha_{13})-(\alpha_{31}+\alpha_{32}+\alpha_{33})\}=0$$

$$\alpha_{45}\{(\alpha_{11}+\alpha_{12}+\alpha_{13})-(\alpha_{21}+\alpha_{22}+\alpha_{23})\}=0$$

$$\alpha_{45}\{(\alpha_{11}+\alpha_{12}+\alpha_{13})-(\alpha_{31}+\alpha_{32}+\alpha_{33})\}=0.$$

From these and the fact that $\alpha_{44}\,\alpha_{55}-\alpha_{45}\,\alpha_{54}\neq 0$, it follows that

$$\alpha_{11}+\alpha_{12}+\alpha_{13}=\alpha_{21}+\alpha_{22}+\alpha_{23}=\alpha_{31}+\alpha_{32}+\alpha_{33}.$$

From $[f(x_1), f(x_2)]=\lambda_1 f(x_6)$ and $[f(x_1), f(x_3)]=[f(x_2), f(x_3)]=0$, it follows that

$$\alpha_{24}(\alpha_{11}+\alpha_{12}+\alpha_{13})-\alpha_{14}(\alpha_{21}+\alpha_{22}+\alpha_{23})=0$$

$$\alpha_{25}(\alpha_{11}+\alpha_{12}+\alpha_{13})-\alpha_{15}(\alpha_{21}+\alpha_{22}+\alpha_{23})=0$$

$$\alpha_{34}(\alpha_{11}+\alpha_{12}+\alpha_{13})-\alpha_{14}(\alpha_{31}+\alpha_{32}+\alpha_{33})=0$$

$$\alpha_{35}(\alpha_{11}+\alpha_{12}+\alpha_{13})-\alpha_{15}(\alpha_{31}+\alpha_{32}+\alpha_{33})=0$$

$$\alpha_{34}(\alpha_{21}+\alpha_{22}+\alpha_{23})-\alpha_{24}(\alpha_{31}+\alpha_{32}+\alpha_{33})=0$$

$$\alpha_{35}(\alpha_{21}+\alpha_{22}+\alpha_{23})-\alpha_{25}(\alpha_{31}+\alpha_{32}+\alpha_{33})=0$$

$$\lambda_2(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})+(\alpha_{14}\alpha_{25}-\alpha_{15}\alpha_{24})=\lambda_1\alpha_{66}$$

$$\lambda_2(\alpha_{11}\alpha_{32}-\alpha_{12}\alpha_{31})+(\alpha_{14}\alpha_{35}-\alpha_{15}\alpha_{34})=0$$

$$\lambda_2(\alpha_{21}\alpha_{32}-\alpha_{22}\alpha_{31})+(\alpha_{24}\alpha_{35}-\alpha_{25}\alpha_{34})=0.$$

Since $\alpha_{i1}+\alpha_{i2}+\alpha_{i3}\neq 0$ for $i=1, 2, 3$, it follows that

$$\alpha_{14}\alpha_{25}-\alpha_{15}\alpha_{24}=0$$

$$\alpha_{14}\alpha_{35}-\alpha_{15}\alpha_{34}=0$$

$$\alpha_{24}\alpha_{35}-\alpha_{25}\alpha_{34}=0$$

and therefore

$$\lambda_2(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})=\lambda_1\alpha_{66}$$

$$\lambda_2(\alpha_{11}\alpha_{32}-\alpha_{12}\alpha_{31})=0$$

$$\lambda_2(\alpha_{21}\alpha_{32}-\alpha_{22}\alpha_{31})=0.$$

If we denote

$$D=\begin{vmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{vmatrix},$$

then

$$D=(\alpha_{11}+\alpha_{12}+\alpha_{13})\{(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})-(\alpha_{11}\alpha_{32}-\alpha_{12}\alpha_{31})$$
$$+(\alpha_{21}\alpha_{32}-\alpha_{22}\alpha_{31})\}.$$

It follows that

$$\lambda_1\alpha_{66}=\lambda_2(\alpha_{11}+\alpha_{12}+\alpha_{13})^{-1}D.$$

Since $\alpha_{66}D\neq 0$, we conclude that $\lambda_1=0$ if and only if $\lambda_2=0$.

Conversely, assume that for L_{λ_1,μ_1} and L_{λ_2,μ_2} both λ_1 and λ_2 are 0 or $\neq 0$ at the same time and $\mu_1\mu_2^{-1}=\alpha^2$ with $\alpha\in\Phi$. In the case where $\lambda_1=\lambda_2=0$, we define a linear transformation f of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$f(x_1)=\alpha y_1$$

$$f(x_2)=\alpha y_2$$

$$f(x_3)=\alpha y_3$$

$$f(x_4)=y_4+y_5$$

$$f(x_5)=\alpha(\mu_2 y_4+y_5)$$

$$f(x_6)=\alpha(1-\mu_2)y_6.$$

The rank of f is 6, since the coefficient matrix of f has the determinant $\alpha^5(1-\mu_2)^2 \neq 0$.

In the case where $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, we define a linear transformation f of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$f(x_1)=\lambda_1\lambda_2^{-1}(1-\mu_2)y_1+\{\alpha-\lambda_1\lambda_2^{-1}(1-\mu_2)\}\, y_2$$

$$f(x_2)=\alpha\, y_2$$

$$f(x_3)=\alpha\, y_3$$

$$f(x_4)= y_4+ y_5$$

$$f(x_5)=\alpha(\mu_2 y_4+ y_5)$$

$$f(x_6)=\alpha(1-\mu_2)y_6.$$

The rank of f is 6, since the coefficient matrix of f has the determinant $\alpha^4\lambda_1\lambda_2^{-1}(1-\mu_2)^3 \neq 0$. It is easy to see that in any case f preserves the multiplication. Therefore f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}.

When Φ is in particular the field of real numbers, it is immediate that every $L_{\lambda,\mu}$ is isomorphic to one of the (A_2)-algebras $L_{0,-1}$ and $L_{1,-1}$, which are not isomorphic.

Thus the proof is complete.

LEMMA 8. *The* (A_2)-*algebras of type* $(1, 4, 1)$ *over the field* Φ *of real numbers are the following Lie algebras*:

(1) $L_{\lambda,\mu}=(x_0, x_1,\cdots, x_5)$ *with the multiplication table*

$$\begin{aligned}
&[x_0, x_1]=x_2,\\
&[x_0, x_2]=\lambda x_1,\\
&[x_0, x_3]=x_4,\\
&[x_0, x_4]=\mu x_3,\\
&[x_1, x_2]=[x_3, x_4]=x_5,\\
&[x_i, x_j]=0 \qquad \textit{for all other} \quad i<j
\end{aligned}$$

where $\lambda<0$ *and* $\mu<0$.

(2) $L_{\lambda,\mu,\nu}=(x_0, x_1, \cdots, x_5)$ *with the multiplication table*

$$\begin{aligned}
&[x_0, x_1]=x_2,\\
&[x_0, x_2]=\lambda x_1+x_3,\\
&[x_0, x_3]=x_4,\\
&[x_0, x_4]=\nu x_1+\mu x_3,\\
&[x_1, x_2]=x_5, \quad [x_3, x_4]=\nu x_5,\\
&[x_i, x_j]=0 \qquad \textit{for all other} \quad i<j
\end{aligned}$$

where $\lambda<0$, $\mu<0$ *and* $0<\nu<\lambda\mu$.

PROOF. Let L be an (A_2)-algebra of type (1, 4, 1). Take $x_1 \in N\setminus Z(L)$. Then there exists $x_0 \in L\setminus N$ such that $[x_1, [x_1, x_0]]\neq 0$. Put $x_2=[x_0, x_1]$ and $x_5=[x_1, x_2]$. Then $x_2 \in N\setminus Z(L)$ and $Z(L)=(x_5)$.

Case I. $[x_0, x_2] \in (x_1, x_2, x_5)$: We write

$$[x_0, x_2]=\lambda x_1+\mu x_2+\nu x_5.$$

Then from $[[x_0, x_2], x_1]=[[x_0, x_1], x_2]$ it follows that $\mu=0$. If $\lambda=0$, then $(\operatorname{ad} x_2)^2=0$. Therefore $\lambda\neq 0$. Replacing x_1 by $x_1+\lambda^{-1}\nu x_5$, we may suppose that

$$[x_0, x_2]=\lambda x_1.$$

Take $y \in N\setminus(x_1, x_2, x_5)$. When $[y, x_1]=\alpha_1 x_5$ and $[y, x_2]=\alpha_2 x_5$, we put

$$x_3=y-\alpha_2 x_1+\alpha_1 x_2.$$

Then $[x_3, x_1]=[x_3, x_2]=0$. Put $x_4=[x_0, x_3]$. Since $[x_3, [x_3, x_0]]\neq 0$, we have $x_4 \in N$ and $[x_3, x_4]=\alpha x_5$ with $\alpha\neq 0$. It follows that $N=(x_1, x_2, \cdots, x_5)$. We infer

$$\begin{aligned}[x_4, x_1]&=[[x_0, x_3], x_1]\\&=[[x_0, x_1], x_3]+[x_0, [x_3, x_1]]\\&=[x_2, x_3]\\&=0\end{aligned}$$

and similarly $[x_4, x_2]=0$. We write

$$[x_0, x_4]=\sum_{i=1}^{5}\alpha_i x_i.$$

Then from $[[x_0, x_4], x_i]=[[x_0, x_i], x_4]$ for $i=1, 2, 3$, it follows that $\alpha_1=\alpha_2=\alpha_4=0$. Since $[x_4, [x_4, x_0]]\neq 0$, we see that $\alpha_3\neq 0$. Replacing x_3 by $x_3+\alpha_3^{-1}\alpha_5 x_5$ and α_3 by μ, we have

$$[x_0, x_4]=\mu x_3 \qquad \text{with} \quad \mu\neq 0.$$

Thus the structure of L is described by a basis $x_0, x_1, \cdots, x_5$ as follows:

$$\begin{aligned}&[x_0, x_1]=x_2, \quad [x_0, x_2]=\lambda x_1,\\&[x_0, x_3]=x_4, \quad [x_0, x_4]=\mu x_3,\\&[x_1, x_2]=x_5, \quad [x_3, x_4]=\alpha x_5,\\&[x_i, x_j]=0 \qquad \text{for all other} \quad i<j\end{aligned}$$

where $\alpha \neq 0$, $\lambda \neq 0$ and $\mu \neq 0$.

Furthermore we have $\lambda<0$ and $\mu<0$, for if $\lambda>0$ then $(\mathrm{ad}\,\lambda^{\frac{1}{2}} x_1 + x_2)^2 = 0$ and if $\mu>0$ then $(\mathrm{ad}\,\mu^{\frac{1}{2}} x_3 + x_4)^2 = 0$. If $\alpha<0$, then

$$(\mathrm{ad}\,(-\alpha)^{\frac{1}{2}} x_1 + (\alpha\mu)^{\frac{1}{2}} x_2 + x_3 + (-\lambda)^{\frac{1}{2}} x_4)^2 = 0.$$

Hence α must be >0. Now we may take $\alpha=1$, since by replacing x_3 and x_4 by $\alpha^{-\frac{1}{2}} x_3$ and $\alpha^{-\frac{1}{2}} x_4$ respectively we have the same multiplication table with the exception that $[x_3, x_4]=x_5$. Thus L has the form indicated in (1) of the statement.

Case II. $[x_0, x_2] \notin (x_1, x_2, x_5)$: Put $y=[x_0, x_2]$. Then it is immediate that $[y, x_1]=0$. We put $[y, x_2]=\lambda x_5$ and $x_3=y-\lambda x_1$. Then we obtain

$$[x_0, x_2] = \lambda x_1 + x_3, \qquad \lambda \neq 0,$$

$$[x_3, x_1] = [x_3, x_2] = 0.$$

We next put $x_4=[x_0, x_3]$. Then $[x_3, x_4]=\nu x_5$ with $\nu \neq 0$. It is immediate that

$$[x_4, x_1] = [x_4, x_2] = 0.$$

Now let us write

$$[x_0, x_4] = \sum_{i=1}^{5} \beta_i x_i.$$

Then from $[[x_0, x_4], x_i]=[[x_0, x_i], x_4]$ for $i=1, 2, 3$ it follows that $\beta_1=\nu$ and $\beta_2=\beta_4=0$. Since $[x_4, [x_4, x_0]] \neq 0$, it follows that $\beta_3 \neq 0$. After replacing x_0 by $x_0-\beta_5\nu^{-1}x_3$, we change the notations to see that L is described by a basis $x_0, x_1, \cdots, x_5$ as follows:

$$\begin{aligned}
&[x_0, x_1] = x_2, && [x_0, x_2] = \lambda x_1 + x_3,\\
&[x_0, x_3] = x_4, && [x_0, x_4] = \nu x_1 + \mu x_3,\\
&[x_1, x_2] = x_5, && [x_3, x_4] = \nu x_5,\\
&[x_i, x_j] = 0 && \text{for all other} \quad i<j
\end{aligned}$$

where $\lambda \neq 0$, $\mu \neq 0$ and $\nu \neq 0$.

Furthermore we have $\lambda<0$ and $\mu<0$, for if $\lambda>0$ then $(\mathrm{ad}\,\lambda^{\frac{1}{2}} x_1 + x_2)^2 = 0$ and if $\mu>0$ then $(\mathrm{ad}\,\mu^{\frac{1}{2}} x_3 + x_4)^2 = 0$. Since L is an (A_2)-algebra, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq 0$ implies $(\mathrm{ad}\sum_{i=1}^{4} \alpha_i x_i)^2 \neq 0$, that is,

$$\alpha_1^2 + \nu\alpha_3^2 + (-\lambda\alpha_2^2 - 2\nu\alpha_2\alpha_4 - \mu\nu\alpha_4^2) \neq 0.$$

Put $f(\alpha_2, \alpha_4) = -\lambda\alpha_2^2 - 2\nu\alpha_2\alpha_4 - \mu\nu\alpha_4^2$. If $f(\alpha_2, \alpha_4) \leqq 0$ for some $(\alpha_2, \alpha_4) \neq 0$, we take α_1 and α_3 so that

$$\alpha_1 = (-f(\alpha_2, \alpha_4))^{\frac{1}{2}} \quad \text{and} \quad \alpha_3 = 0.$$

Then we obtain $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq 0$ and $\alpha_1^2 + \nu\alpha_3^2 + f(\alpha_2, \alpha_4) = 0$. Hence we have $f(\alpha_2, \alpha_4) > 0$ for every $(\alpha_2, \alpha_4) \neq 0$, that is, f is positive definite. It follows that

$$-\mu\nu > 0 \quad \text{and} \quad \lambda\mu\nu - \nu^2 > 0,$$

and therefore $0 < \nu < \lambda\mu$. Thus L has the structure indicated in (2) of the statement.

Conversely, if L is $L_{\lambda,\mu}$ or $L_{\lambda,\mu,\nu}$, then it is easy to see that L is an (A_2)-algebra.

Thus the proof is complete.

LEMMA 9. *Under the notations in Lemma* 8, *every* (A_2)*-algebra* $L_{\lambda,\mu}$ *is isomorphic to one of the non-isomorphic* (A_2)*-algebras* $L_{-1,\theta}$ *where* $-1 \leqq \theta < 0$.

PROOF. Assume that f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}. We here write $L_{\lambda_2,\mu_2} = (y_0, y_1, \cdots, y_5)$. Since f sends the nil radical and the center of L_{λ_1,μ_1} onto those of L_{λ_2,μ_2} respectively, we can express f in the following form:

$$f(x_0) = \sum_{j=0}^{5} \alpha_{0j} y_j$$

$$f(x_i) = \sum_{j=1}^{5} \alpha_{ij} y_j \qquad \text{for} \quad i = 1, 2, 3, 4$$

$$f(x_5) = \alpha_{55} y_5.$$

From $[f(x_0), f(x_1)] = f(x_2)$ it follows that

$$\alpha_{21} = \lambda_2\alpha_{00}\alpha_{12}$$

$$\alpha_{22} = \alpha_{00}\alpha_{11}$$

$$\alpha_{23} = \mu_2\alpha_{00}\alpha_{14}$$

$$\alpha_{24} = \alpha_{00}\alpha_{13}.$$

From $[f(x_0), f(x_2)] = \lambda_1 f(x_1)$ it follows that

$$\lambda_1\alpha_{11} = \lambda_2\alpha_{00}\alpha_{22}$$

$$\lambda_1\alpha_{12} = \alpha_{00}\alpha_{21}$$

$$\lambda_1\alpha_{13} = \mu_2\alpha_{00}\alpha_{24}$$

$$\lambda_1\alpha_{14} = \alpha_{00}\alpha_{23}.$$

Hence we have

$$\alpha_{11}(\lambda_1-\lambda_2\alpha_{00}^2)=0$$

$$\alpha_{12}(\lambda_1-\lambda_2\alpha_{00}^2)=0$$

$$\alpha_{13}(\lambda_1-\mu_2\alpha_{00}^2)=0$$

$$\alpha_{14}(\lambda_1-\mu_2\alpha_{00}^2)=0.$$

In a similar way, from $[f(x_0), f(x_3)]=f(x_4)$ and $[f(x_0), f(x_4)]=\mu_1 f(x_3)$, it follows that

$$\alpha_{41}=\lambda_2\alpha_{00}\alpha_{32}$$

$$\alpha_{42}=\alpha_{00}\alpha_{31}$$

$$\alpha_{43}=\mu_2\alpha_{00}\alpha_{34}$$

$$\alpha_{44}=\alpha_{00}\alpha_{33}$$

and also

$$\alpha_{31}(\mu_1-\lambda_2\alpha_{00}^2)=0$$

$$\alpha_{32}(\mu_1-\lambda_2\alpha_{00}^2)=0$$

$$\alpha_{33}(\mu_1-\mu_2\alpha_{00}^2)=0$$

$$\alpha_{34}(\mu_1-\mu_2\alpha_{00}^2)=0.$$

If $\lambda_1-\lambda_2\alpha_{00}^2\neq 0$ and $\lambda_1-\mu_2\alpha_{00}^2\neq 0$, it follows that $\alpha_{11}=\alpha_{12}=\alpha_{13}=\alpha_{14}=0$. If $\lambda_1-\lambda_2\alpha_{00}^2\neq 0$ and $\mu_1-\lambda_2\alpha_{00}^2\neq 0$, it follows that $\alpha_{11}=\alpha_{12}=\alpha_{31}=\alpha_{32}=0$ and therefore $\alpha_{21}=\alpha_{22}=\alpha_{41}=\alpha_{42}=0$. If $\lambda_1-\mu_2\alpha_{00}^2\neq 0$ and $\mu_1-\mu_2\alpha_{00}^2\neq 0$, it follows that $\alpha_{13}=\alpha_{14}=\alpha_{33}=\alpha_{34}=0$ and therefore $\alpha_{23}=\alpha_{24}=\alpha_{43}=\alpha_{44}=0$. If $\mu_1-\lambda_2\alpha_{00}^2\neq 0$ and $\mu_1-\mu_2\alpha_{00}^2\neq 0$, it follows that $\alpha_{31}=\alpha_{32}=\alpha_{33}=\alpha_{34}=0$. Thus in any case we see that

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix}=0$$

and therefore the determinant of the coefficient matrix of f equals 0, which is impossible since f is an isomorphism. Hence, if $\lambda_1-\lambda_2\alpha_{00}^2\neq 0$ or $\mu_1-\mu_2\alpha_{00}^2\neq 0$, we have necessarily $\lambda_1-\mu_2\alpha_{00}^2=0$ and $\mu_1-\lambda_2\alpha_{00}^2=0$. Thus we obtain

$$\lambda_1-\lambda_2\alpha_{00}^2=\mu_1-\mu_2\alpha_{00}^2=0, \quad \text{or}$$

$$\lambda_1-\mu_2\alpha_{00}^2=\mu_1-\lambda_2\alpha_{00}^2=0.$$

Since $\alpha_{00}\neq 0$, it follows that $\lambda_1\lambda_2^{-1}=\mu_1\mu_2^{-1}$ or $\lambda_1\mu_2^{-1}=\lambda_2^{-1}\mu_1$.

Conversely, assume that for L_{λ_1,μ_1} and L_{λ_2,μ_2} we have $\lambda_1\lambda_2^{-1}=\mu_1\mu_2^{-1}$ or

$\lambda_1\mu_2^{-1}=\lambda_2^{-1}\mu_1$. In the case where $\lambda_1\lambda_2^{-1}=\mu_1\mu_2^{-1}$, we put $\alpha=(\lambda_1\lambda_2^{-1})^{\frac{1}{2}}$. Then $\lambda_1=\alpha^2\lambda_2$ and $\mu_1=\alpha^2\mu_2$. Define a linear transformation f of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$
\begin{aligned}
f(x_0)&=\alpha y_0\\
f(x_1)&= y_1+(-\mu_1)^{\frac{1}{2}}y_2\\
f(x_2)&=\alpha\{\lambda_2(-\mu_1)^{\frac{1}{2}}y_1+y_2\}\\
f(x_3)&= y_3+(-\lambda_1)^{\frac{1}{2}}y_4\\
f(x_4)&=\alpha\{(-\lambda_1)^{\frac{1}{2}}\mu_2 y_3+y_4\}\\
f(x_5)&=\alpha(1+\lambda_1\mu_2)y_5.
\end{aligned}
$$

Then the coefficient matrix of f has the determinant $\alpha^4(1+\lambda_1\mu_2)^3>0$, and it is easy to see that f preserves the multiplication. Therefore f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}.

In the case where $\lambda_1\mu_2^{-1}=\lambda_2^{-1}\mu_1$, we put $\alpha=(\lambda_1\mu_2^{-1})^{\frac{1}{2}}$. Then $\lambda_1=\alpha^2\mu_2$ and $\mu_1=\alpha^2\lambda_2$. Define a linear transformation f of L_{λ_1,μ_1} into L_{λ_2,μ_2} in such a way that

$$
\begin{aligned}
f(x_0)&=\alpha y_0\\
f(x_1)&= y_3+(-\mu_1)^{\frac{1}{2}}y_4\\
f(x_2)&=\alpha\{(-\mu_1)^{\frac{1}{2}}\mu_2 y_3+y_4\}\\
f(x_3)&= y_1+(-\lambda_1)^{\frac{1}{2}}y_2\\
f(x_4)&=\alpha\{(-\lambda_1)^{\frac{1}{2}}\lambda_2 y_1+y_2\}\\
f(x_5)&=\alpha(1+\lambda_1\lambda_2)y_5.
\end{aligned}
$$

Then it is easy to see that f is an isomorphism of L_{λ_1,μ_1} onto L_{λ_2,μ_2}.

Thus in order that L_{λ_1,μ_1} and L_{λ_2,μ_2} are isomorphic it is necessary and sufficient that $\lambda_1\lambda_2=\mu_1\mu_2$ or $\lambda_1\mu_2=\lambda_2\mu_1$.

We now see that $L_{\lambda,\mu}$ is isomorphic to $L_{-1,-\lambda\mu^{-1}}$ if $\lambda\geqq\mu$ and to $L_{-1,-\lambda^{-1}\mu}$ if $\lambda<\mu$. Hence every $L_{\lambda,\mu}$ is isomorphic to one of $L_{-1,\theta}$ with $-1\leqq\theta<0$. It is immediate that $L_{-1,\theta}$ with $-1\leqq\theta<0$ are not isomorphic for different θ.

Thus the proof is complete.

Lemma 10. *Under the notations in Lemma* 8, *every* (A_2)-*algebra* $L_{\lambda,\mu,\nu}$ *is isomorphic to one of the* (A_2)-*algebras* $L_{\lambda,\mu}$.

Proof. Let $L_{\lambda,\mu,\nu}$ be an (A_2)-algebra in the statement (2) of Lemma 8. We consider the following equation:

$$(\lambda\mu-\nu)x^2+(\lambda+\mu)x+1=0.$$

Since $\nu>0$, it is immediate that the equation has two different roots in Φ. Since $\lambda<0$, $\mu<0$ and $\lambda\mu-\nu>0$, their sum and product are both positive. Let us denote by α^2 the larger one of them. Then by making use of the fact that $\lambda<0$ and $0<\nu<\lambda\mu$, we can show that $\alpha^2\lambda+1<0$. Put $\lambda_1=-1$ and $\mu_1=\alpha^2(\lambda+\mu)+1$. Then $\mu_1<0$. We have also $\lambda_1\neq\mu_1$. In fact, if $\lambda_1=\mu_1$, then $\alpha^2(\lambda+\mu)=-2$ and therefore $\alpha^2\mu+1=-\alpha^2\lambda-1$. Hence

$$\begin{aligned}\alpha^4\nu&=(\alpha^2\lambda+1)(\alpha^2\mu+1)\\&=-(\alpha^2\lambda+1)^2\\&<0,\end{aligned}$$

which contradicts the fact that $\nu>0$. By the definitions of λ_1 and μ_1, we have obviously

$$\begin{aligned}\alpha^4\nu&=(\alpha^2\lambda-\lambda_1)(\alpha^2\mu-\lambda_1)\\&=-(\alpha^2\lambda-\lambda_1)(\alpha^2\lambda-\mu_1)\\&=(\alpha^2\lambda-\mu_1)(\alpha^2\mu-\mu_1).\end{aligned}$$

Thus we can choose two non-zero real numbers β and γ satisfying the condition

$$\beta^2(\alpha^2\lambda-\lambda_1)=-\gamma^2(\alpha^2\lambda-\mu_1).$$

Writing $L_{\lambda,\mu,\nu}=(y_0,\ y_1,\ \cdots,\ y_5)$, we define a linear transformation f of L_{λ_1,μ_1} into $L_{\lambda,\mu,\nu}$ in such a way that

$$\begin{aligned}f(x_0)&=\alpha y_0\\f(x_1)&=\alpha^{-2}\beta\nu^{-1}\{\alpha^2\nu y_1-(\alpha^2\lambda-\lambda_1)y_3\}\\f(x_2)&=\alpha^{-1}\beta\nu^{-1}\{\alpha^2\nu y_2-(\alpha^2\lambda-\lambda_1)y_4\}\\f(x_3)&=\alpha^{-2}\gamma\nu^{-1}\{\alpha^2\nu y_1-(\alpha^2\lambda-\mu_1)y_3\}\\f(x_4)&=\alpha^{-1}\gamma\nu^{-1}\{\alpha^2\nu y_2-(\alpha^2\lambda-\mu_1)y_4\}\\f(x_5)&=-\alpha^{-3}\beta^2\nu^{-1}(\lambda_1-\mu_1)(\alpha^2\lambda-\lambda_1)y_5.\end{aligned}$$

Then the coefficient matrix of this transformation has the determinant

$$-\alpha^{-4}\beta^4\gamma^2\nu^{-3}(\lambda_1-\mu_1)^3(\alpha^2\lambda-\lambda_1)\neq0.$$

Hence the rank of f is 6. We have to show that f preserves the multiplications. By making use of the product expressions of $\alpha^4\nu$ and the equality defining β and γ, we have the following equalities:

$$\begin{aligned}
[f(x_0), f(x_1)] &= \alpha^{-1}\beta\nu^{-1}[y_0, \alpha^2\nu y_1-(\alpha^2\lambda-\lambda_1)y_3]\\
&= \alpha^{-1}\beta\nu^{-1}\{\alpha^2\nu y_2-(\alpha^2\lambda-\lambda_1)y_4\}\\
&= f(x_2),\\
[f(x_0), f(x_2)] &= \beta\nu^{-1}[y_0, \alpha^2\nu y_2-(\alpha^2\lambda-\lambda_1)y_4]\\
&= \beta\nu^{-1}\{\alpha^2\nu(\lambda y_1+y_3)-(\alpha^2\lambda-\lambda_1)(\nu y_1+\mu y_3)\}\\
&= \alpha^{-2}\beta\nu^{-1}\{\alpha^2\lambda_1\nu y_1+\alpha^4\nu y_3-\alpha^2\mu(\alpha^2\lambda-\lambda_1)y_3\}\\
&= \alpha^{-2}\beta\nu^{-1}\{\alpha^2\lambda_1\nu y_1-\lambda_1(\alpha^2\lambda-\lambda_1)y_3\}\\
&= \lambda_1 f(x_1),\\
[f(x_0), f(x_3)] &= \alpha^{-1}\gamma\nu^{-1}[y_0, \alpha^2\nu y_1-(\alpha^2\lambda-\mu_1)y_3]\\
&= \alpha^{-1}\gamma\nu^{-1}\{\alpha^2\nu y_2-(\alpha^2\lambda-\mu_1)y_4\}\\
&= f(x_4),\\
[f(x_0), f(x_4)] &= \gamma\nu^{-1}[y_0, \alpha^2\nu y_2-(\alpha^2\lambda-\mu_1)y_4]\\
&= \gamma\nu^{-1}\{\alpha^2\nu(\lambda y_1+y_3)-(\alpha^2\lambda-\mu_1)(\nu y_1+\mu y_3)\}\\
&= \alpha^{-2}\gamma\nu^{-1}\{\alpha^2\mu_1\nu y_1+\alpha^4\nu y_3-\alpha^2\mu(\alpha^2\lambda-\mu_1)y_3\}\\
&= \alpha^{-2}\gamma\nu^{-1}\{\alpha^2\mu_1\nu y_1-\mu_1(\alpha^2\lambda-\mu_1)y_3\}\\
&= \mu_1 f(x_3),\\
[f(x_1), f(x_2)] &= \alpha^{-3}\beta^2\nu^{-2}[\alpha^2\nu y_1-(\alpha^2\lambda-\lambda_1)y_3, \alpha^2\nu y_2-(\alpha^2\lambda-\lambda_1)y_4]\\
&= \alpha^{-3}\beta^2\nu^{-1}\{\alpha^4\nu y_5+(\alpha^2\lambda-\lambda_1)^2 y_5\}\\
&= -\alpha^{-3}\beta^2\nu^{-1}(\lambda_1-\mu_1)(\alpha^2\lambda-\lambda_1)y_5\\
&= f(x_5),\\
[f(x_1), f(x_3)] &= 0,\\
[f(x_1), f(x_4)] &= \alpha^{-3}\beta\gamma\nu^{-2}[\alpha^2\nu y_1-(\alpha^2\lambda-\lambda_1)y_3, \alpha^2\nu y_2-(\alpha^2\lambda-\mu_1)y_4]\\
&= \alpha^{-3}\beta\gamma\nu^{-1}\{\alpha^4\nu y_5+(\alpha^2\lambda-\lambda_1)(\alpha^2\lambda-\mu_1)y_5\}\\
&= 0,\\
[f(x_2), f(x_3)] &= \alpha^{-3}\beta\gamma\nu^{-2}[\alpha^2\nu y_2-(\alpha^2\lambda-\lambda_1)y_4, \alpha^2\nu y_1-(\alpha^2\lambda-\mu_1)y_3]\\
&= -\alpha^{-3}\beta\gamma\nu^{-1}\{\alpha^4\nu y_5+(\alpha^2\lambda-\lambda_1)(\alpha^2\lambda-\mu_1)y_5\}\\
&= 0,
\end{aligned}$$

$$[f(x_2), f(x_4)] = 0,$$

$$\begin{aligned}[f(x_3), f(x_4)] &= \alpha^{-3}\gamma^2\nu^{-2}[\alpha^2\nu y_1-(\alpha^2\lambda-\mu_1)y_3, \alpha^2\nu y_2-(\alpha^2\lambda-\mu_1)y_4]\\ &=\alpha^{-3}\gamma^2\nu^{-1}\{\alpha^4\nu y_5+(\alpha^2\lambda-\mu_1)^2 y_5\}\\ &= \alpha^{-3}\gamma^2\nu^{-1}(\lambda_1-\mu_1)(\alpha^2\lambda-\mu_1)y_5\\ &= -\alpha^{-3}\beta^2\nu^{-1}(\lambda_1-\mu_1)(\alpha^2\lambda-\lambda_1)y_5\\ &= f(x_5),\end{aligned}$$

$$[f(x_i), f(x_5)] = 0 \qquad \text{for} \quad i=0, 1, \cdots, 4.$$

Thus we conclude that f is an isomorphism of L_{λ_1,μ_1} onto $L_{\lambda,\mu,\nu}$.

The proof of the lemma is complete.

REMARK. The multiplication tables of the (A_2)-algebras $L_{\lambda,\mu}$ and $L_{\lambda,\mu,\nu}$ in Lemma 8 define (A_2)-algebras over the field of rational numbers when λ, μ and ν are especially rational numbers. Let us denote them by $L^*_{\lambda,\mu}$ and $L^*_{\lambda,\mu,\nu}$ respectively. We here note that, contrary to the assertion in Lemma 10, $L^*_{\lambda,\mu,\nu}$ is not necessarily isomorphic to any one of the $L^*_{\lambda,\mu}$.

Assume that there exists an isomorphism f of $L^*_{\lambda,\mu}$ onto $L^*_{-1,-2,1}$. Writing $L^*_{-1,-2,1}=(y_0, y_1, \cdots, y_5)$, we can express f in the following form:

$$f(x_0) = \sum_{j=0}^{5}\alpha_{0j}y_j$$

$$f(x_i) = \sum_{j=1}^{5}\alpha_{ij}y_j \qquad \text{for} \quad i=1, 2, 3, 4$$

$$f(x_5) = \alpha_{55}y_5.$$

From $[f(x_0), f(x_1)]=f(x_2)$ it follows that

$$\alpha_{21} = \alpha_{00}(-\alpha_{12}+\alpha_{14})$$

$$\alpha_{22} = \alpha_{00}\alpha_{11}$$

$$\alpha_{23} = \alpha_{00}(\alpha_{12}-2\alpha_{14})$$

$$\alpha_{24} = \alpha_{00}\alpha_{13}.$$

From $[f(x_0), f(x_2)]=\lambda f(x_1)$ it follows that

$$\lambda\alpha_{11} = \alpha_{00}(-\alpha_{22}+\alpha_{24})$$

$$\lambda\alpha_{12} = \alpha_{00}\alpha_{21}$$

$$\lambda\alpha_{13} = \alpha_{00}(\alpha_{22}-2\alpha_{24})$$

$$\lambda\alpha_{14} = \alpha_{00}\alpha_{23}.$$

Hence we obtain

$$\alpha_{11}(\lambda+\alpha_{00}^2)-\alpha_{00}^2\alpha_{13}=0$$

$$\alpha_{12}(\lambda+\alpha_{00}^2)-\alpha_{00}^2\alpha_{14}=0$$

$$\alpha_{00}^2\alpha_{11}-\alpha_{13}(\lambda+2\alpha_{00}^2)=0$$

$$\alpha_{00}^2\alpha_{12}-\alpha_{14}(\lambda+2\alpha_{00}^2)=0.$$

If $\alpha_{11}=\alpha_{12}=\alpha_{13}=\alpha_{14}=0$, then the rank of f is not 6. Hence some of α_{11}, α_{12}, α_{13} and α_{14} are not equal to 0. It follows that

$$(\lambda+\alpha_{00}^2)(\lambda+2\alpha_{00}^2)-\alpha_{00}^4=0$$

and therefore

$$\lambda^2+3\lambda\alpha_{00}^2+\alpha_{00}^4=0.$$

It is however immediate that there are no rational numbers λ and α_{00} satisfying the equality above, which is a contradiction. Thus $L^*_{-1,-2,1}$ is not isomorphic to any $L^*_{\lambda,\mu}$.

Finally, by making use of the preceding propositions and lemmas, we shall determine the structure of the 6 dimensional solvable (A_2)-algebras over the field of real numbers in the following

PROPOSITION 7. *The 6 dimensional non-abelian solvable* (A_2)*-algebras over the field of real numbers are, up to isomorphism, the Lie algebras described by a basis* $x_1, x_2, \cdots, x_6$ *with the following multiplication tables.*

(1): $[x_1, x_2]=x_3, \quad [x_1, x_3]=-x_2,$
$[x_2, x_3]=x_4.$

(2): $[x_1, x_2]=\lambda x_5,$
$[x_1, x_3]=[x_2, x_3]=x_4,$
$[x_1, x_4]=[x_2, x_4]=-x_3,$
$[x_3, x_4]=x_5$ *with* $\lambda=0, 1.$

(3): $[x_1, x_2]=x_6,$
$[x_1, x_3]=[x_2, x_3]=x_4,$
$[x_1, x_4]=[x_2, x_4]=-x_3,$
$[x_3, x_4]=x_5.$

(4): $[x_1, x_2]=\mu x_6,$
$[x_1, x_4]=[x_2, x_4]=[x_3, x_4]=x_5,$
$[x_1, x_5]=[x_2, x_5]=[x_3, x_5]=-x_4,$
$[x_4, x_5]=x_6$ *with* $\mu=0, 1.$

(5): $[x_1, x_2]=x_3$, $[x_1, x_3]=-x_2$,
$[x_1, x_4]=x_5$, $[x_1, x_5]=\nu x_4$,
$[x_2, x_3]=[x_4, x_5]=x_6$ *with* $-1\leqq\nu<0$.

Here in each of the tables $[x_i, x_j]=0$ *for all* $i<j$ *if it is not in the table.*

PROOF. Let L be a 6 dimensional non-abelian solvable (A_2)-algebra over the field of real numbers. Then $n_1\geqq 1$, $n_2\geqq 2$ and $n_3\geqq 1$. By Proposition 4 we have the following cases:

$$n_1=1,\quad n_2=2,\quad n_3=3;$$
$$n_1=2,\quad n_2=2,\quad n_3=2;$$
$$n_1=3,\quad n_2=2,\quad n_3=1;$$
$$n_1=1,\quad n_2=4,\quad n_3=1.$$

In the first case we have the table (1) by Propositions 3 and 5. In the second case we have the tables (2) and (3) by Proposition 6 and Lemma 6. In the third case we have the table (4) by Lemma 7. In the fourth case we have the table (5) by Lemmas 8, 9 and 10. The proof is complete.

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Department of Mathematics,
Faculty of Science,
Hiroshima University