J. Sci. Hiroshima Univ. Ser. A-I
33 (1969), 41-46

Notes on Derivations of Higher Order

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(Received March 5, 1969)

Let R and S be commutative rings and assume that S is an R-algebra. Let D be a derivation of S over R. Then the power $\Delta=D^n$ is an R-linear endomorphism of S satisfying the following condition:

$$(*)\ \Delta(x_1x_2\cdots x_{n+1})=\sum_{s=1}^{n}(-1)^{s-1}\sum_{i_1<\cdots<i_s}x_{i_1}\cdots x_{i_s}\Delta(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})$$

for any $x_1, x_2, \cdots, x_{n+1}$ in S. The property (*) is used to define the notion of a derivation of order n by H. Osborn ([3]). In this note we shall prove some properties of such derivations. In the last part we shall show the following: *Let S be a field finitely generated over a subfield R. Then the set of ordinary derivations of S/R is characterized as the set of n-th order derivations D satisfying the condition that $D(x)=D(y)=0$ implies $D(xy)=0$.*

1. Let R be a commutative ring with identity 1 and let S be an R-algebra. An R-endomorphism of S is called *a derivation of order n of S/R*, if D satisfies the following identity:

$$D(x_1\cdots x_{n+1})=\sum_{s=1}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s-1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})$$

for any $x_i\in S$.

From the definition it follows easily that $D(r)=rD(1)=0$ for any $r\in R$.

First we show that the notion of n-th order derivation has a close connection with that of the higher derivations in the sense of F. K. Schmidt (cf. [1]).

Proposition 1. *Let $D=(D_0, D_1, \cdots, D_r)$ be a higher derivation of rank r (or of infinite rank) of S/R into S. Then $D_m(0<m\leqq r)$ is a derivation of order m.*

Proof. For any set of elements $x_1, \cdots, x_{m+1}$ of S, we have

$$\sum_{s=1}^{m}\sum_{i_1<\cdots<i_s}(-1)^{S+1}x_{i_1}\cdots x_{i_s}D_m(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{m+1})$$

$$=\sum_{s=1}^{m}\sum_{\substack{i_1<\cdots<i_s,\, i_j\neq i_{s+k}\\ i_{s+1}<\cdots<i_{m+1}}}{}^{S+1}x_{i_1}\cdots x_{i_s}\left(\sum D_{v_{s+1}}(x_{i_{s+1}})\cdots D_{v_{m+1}}(x_{i_{m+1}})\right).$$

The coefficient of $x_{i_1}\cdots x_{i_s}D_{v_{s+1}}(x_{i_{s+1}})\cdots D_{v_{m+1}}(x_{i_{m+1}})$ is

$$(-1)^{s+1}+(-1)^s\binom{s}{1}+\cdots+\binom{s}{s-1}=1-(1-1)^s=1,$$

while

$$D_m(x_1\cdots x_{m+1})=\sum_{m=\nu_1+\cdots+\nu_{m+1}} D_{\nu_1}(x_1)\cdots D_{\nu_{m+1}}(x_{m+1}).$$

Hence D_m is a derivation of order m.

PROPOSITION 2. *A derivation D of order $n-1$ is a derivation of order n.*

PROOF. For any set of elements $x_1, x_2, \cdots, x_{n+1}$ of S, we have

$$\begin{aligned}D(x_1\cdots x_{n+1})=&\sum_{i=1}^{n-1} x_i D(x_1\cdots\hat{x}_i\cdots x_{n-1}x_n x_{n+1})+x_n x_{n+1}D(x_1\cdots x_{n-1})\\&+\sum_{s=2}^{n-1}\sum_{i_1<\cdots<i_s\leqq n-1}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_n x_{n+1})\\&+\sum_{s=2}^{n-1}\sum_{i_1<\cdots<i_s\leqq n-1}(-1)^{s+1}x_{i_1}\cdots x_{i_{s-1}}x_n x_{n+1}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_{s-1}}\cdots x_{n-1})\\=&\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_{s+1}}(-1)^{s+1}s\,x_{i_1}\cdots x_{i_{s+1}}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_{s+1}}\cdots x_{n+1}),\end{aligned}$$

while

$$\begin{aligned}&\sum_{s=1}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})\\&=\sum_{i=1}^{n+1}x_iD(x_1\cdots\hat{x}_i\cdots x_{n+1})+\sum_{s=2}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})\\&=\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_{s+1}}(-1)^{s+1}(s+1)x_{i_1}\cdots x_{i_{s+1}}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_{s+1}}\cdots x_{n+1})\\&\quad+\sum_{s=2}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})\\&=\sum_{s=2}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}s\,x_{i_1}\cdots x_{i_{s+1}}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_{s+1}}\cdots x_{n+1}).\end{aligned}$$

Hence $D(x_1\cdots x_{n+1})=\sum_{s=1}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_{n+1})$, i. e., D is a derivation of order n.

COROLLARY. *A derivation of order n of S/R is also a derivation of order n' for any $n'\geqq n$.*

Let D be a derivation of order n of S/R. For every $x\in S$, we shall introduce a new R-linear mapping D_x of S defined by

$$D_x(y)=D(xy)-xD(y)-yD(x).$$

It is easily seen that D is an ordinary derivation if and only if $D_x=0$ for every $x\in S$. More generally we have the

THEOREM 1. *If D is a derivation of order n of S/R, then D_x is a derivation of order $n-1$ for $x \in S$. Conversely if D_x is a derivation of order $n-1$ of S/R for every $x \in S$, then D is a derivation of order n.*

PROOF. Let D be a derivation of order n of S/R. Then, for any set of elements $x_1, \cdots, x_{n+1}$ of S, we have

$$\begin{aligned}
&D_{x_1}(x_2\cdots x_{n+1})\\
&\quad=D(x_1\cdots x_{n+1})-x_1D(x_2\cdots x_{n+1})-x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{i=2}^{n+1} x_iD(x_1\cdots \hat{x}_i\cdots x_{n+1})+\sum_{s=2}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+\sum_{i=2}^{n+1}(-1)^{n+1}x_1\cdots \hat{x}_i\cdots x_{n+1}D(x_i)+\{(-1)^{n+1}-1\}\,x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D_{x_1}(x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1}).
\end{aligned}$$

Therefore D_{x_1} is a derivation of order $n-1$.

Conversely, let D_x be a derivation of order $n-1$ for any $x \in S$. By definition of D_{x_1}, we have

$$\begin{aligned}
&D(x_1\cdots x_{n+1})=D_{x_1}(x_2\cdots x_{n+1})+x_1D(x_2\cdots x_{n+1})+x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D_{x_1}(x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+x_1D(x_2\cdots x_{n+1})+x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+2}x_1x_{i_1}\cdots x_{i_s}D(x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+\sum_{s=1}^{n-1}(-1)^{s+2}\tbinom{n}{s}x_2\cdots x_{n+1}D(x_1)+x_1D(x_2\cdots x_{n+1})+x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{i=1}^{n+1} x_iD(x_1\cdots \hat{x}_i\cdots x_{n+1})+\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}(-1)^{s+2}x_1x_{i_1}\cdots x_{i_s}D(x_2\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1})\\
&\qquad+(-1)^{n+1}x_2\cdots x_{n+1}D(x_1)\\
&\quad=\sum_{s=1}^{n}\sum_{i_1<\cdots<i_s}(-1)^{s+1}x_{i_1}\cdots x_{i_s}D(x_1\cdots \hat{x}_{i_1}\cdots \hat{x}_{i_s}\cdots x_{n+1}).
\end{aligned}$$

Therefore D is a derivation of order n.

PROPOSITION 3. *Let Δ, D be derivations of order r, s respectively of S/R. Then we have the identity*

$$(\Delta D)_x = \Delta D_x + \Delta_x D + \Delta_{D(x)} + \Delta(x)D + D(x)\Delta \tag{1}$$

PROOF. For any $y \in S$, we have by the definition

$$(\Delta D)_x(y) = \Delta D(xy) - x\Delta D(y) - y\Delta D(x)$$

$$\Delta_x(D(y)) = \Delta(xD(y)) - x\Delta D(y) - D(y)\Delta(x)$$

$$\Delta_{D(x)}(y) = \Delta(yD(x)) - y\Delta D(x) - \Delta(y)D(x)$$

$$\Delta(D_x(y)) = \Delta D(xy) - \Delta(xD(y)) - \Delta(yD(x)).$$

From these formula we easily arrive at the conclusion.

PROPOSITION 4. *If Δ, D are derivations of order r, s of S/R respectively, then ΔD is a derivation of order $r+s$ of S/R.*

PROOF. It is trivial that ΔD is an R-enomorphism of S. We shall prove the proposition by the induction on $r+s$. When $r=s=1$, this is immediate from proposition 3. Every member of the right hand side of (1) is a derivation of order $\leqq r+s-1$ by induction assumption. Therefore, by Theorem 1, ΔD is a derivation of order $r+s$.

COROLLARY. *If D is an ordinary derivation, D^n is a derivation of order n.*

2. Let S be an R-algebra as before and let φ be the homomorphism of the ring $S\otimes_R S$ into S defined by $\varphi(\sum x\otimes y) = \sum xy$. Let us set $J = \mathrm{Ker}\ (\varphi)$. We shall endow on $S\otimes_R S$ an S-module structure by $a(x\otimes y) = ax\otimes y$. Then the mapping $\delta^{(n)}$ of S into $\Omega_R^{(n)}(S) = J/J^{n+1}$ such that $\delta^{(n)}(x) = \{$the class of $1\otimes x - x\otimes 1$ modulo $J^{n+1}\}$ is an n-th order derivation of S into $\Omega_R^{(n)}(S)$. It is known that $\Omega_R^{(n)}(S)$ has the universal mapping property with respect to n-th order derivations of S/R (cf. [3]), and is called the module of n-th order (Kähler) differentials.

We shall denote by $\mathcal{D}_R^{(n)}(S)$ the left S-module consisting of n-th order derivations of S/R. From the universal mapping property of $\Omega_R^{(n)}(S)$, it follows that $\mathcal{D}_R^{(n)}(S)$ is isomorphic to Hom $(\Omega_R^{(n)}(S), S)$ (cf. [3]).

PROPOSITION 5. Let P, Φ be two fields such that $P \supset \Phi$ and P is finitely generated over Φ. Then P is separably algebraic over Φ if and only if $\mathcal{D}_\Phi^{(n)}(P) = 0$ for some $n > 0$.

PROOF. It is well known that P is separably algebraic over Φ if and only if $\Omega_\Phi^{(1)}(P) = J/J^2 = 0$ (cf. [2]). On the other hand, $J = J^2$ if and only if $J = J^{n+1}$ for some $n > 0$. Hence, P is separably algebraic over Φ if and only if $\Omega_\Phi^{(n)}(P) = 0$ for some $n > 0$. Since P is a field, $\Omega_\Phi^{(n)}(P) = 0$ if and only if $\mathcal{D}_\Phi^{(n)}(P) = 0$.

Let us denote by $C^{(n)}$ the set of elements D of $\mathfrak{D}_{\Phi}^{(n)}(P)$ such that $D(x)=D(y)=0$ implies $D(xy)=0$. Obviously we have $\mathfrak{D}_{\Phi}^{(1)}(P)\subset C^{(n)}$ for all $n>0$.

THEOREM 2. *Let $P=\Phi(\xi_1, \cdots \xi_m)$ be a field finitely generated over Φ. Then $C^{(n)}=\mathfrak{D}_{\Phi}^{(1)}(P)$ for all $n>0$. Namely, $\mathfrak{D}_{\Phi}^{(1)}(P)$ is characterized as the set of elements D of $\mathfrak{D}_{\Phi}^{(n)}(P)$ such that $D(x)=D(y)=0$ implies $D(xy)=0$.*

PROOF. We consider a homomorphism

$$f : C^{(n)} \longrightarrow P^m = \underbrace{P\oplus\cdots\oplus P}_{m}$$

defined by $f(D)=(D(\xi_1), \cdots, D(\xi_m))$. Then f is injective. In fact, let $D \in \operatorname{Ker}(f)$. If $D(x)=D(y)=0$, we have $D(x+y)=0$ and $D(xy)=0$ by the hypothesis on $C^{(n)}$. Hence to show that D is a zero map it suffices to prove that $D(x)=D(y)=0$ implies also $D\left(\frac{x}{y}\right)=0$ $(y\neq 0)$. Let us set $\alpha=\frac{x}{y}$. Then we see immediately that $0=D(y^n\alpha)=(-1)^{n-1}y^nD(\alpha)$. Hence $D(\alpha)=0$. Thus $C^{(n)}$ is isomorphic to a subspace of P^m, and $s=\dim_p C^{(n)}\leqq m$. Let $D_1, \cdots, D_s$ be a base of $C^{(n)}$ over P. And we set $\alpha_i=f(D_i)(1\leqq i\leqq s)$. The set $\{\alpha_i\}(1\leqq i\leqq s)$ generates $\operatorname{Im}(f)$. Hence $\{\alpha_i\}(1\leqq i\leqq s)$ is a base of $\operatorname{Im}(f)$. We set

$$A=\begin{pmatrix} D_1(\xi_1)\cdots\cdots D_1(\xi_m) \\ \vdots \qquad\qquad \vdots \\ D_s(\xi_1)\cdots\cdots D_s(\xi_m)\end{pmatrix}.$$

The rank of A is s. Therefore we may assume

$$\begin{vmatrix} D_1(\xi_1)\cdots\cdots D_1(\xi_s) \\ \vdots \qquad\qquad \vdots \\ D_s(\xi_1)\cdots\cdots D_s(\xi_s)\end{vmatrix}\neq 0. \tag{**}$$

Let E be $\Phi(\xi_1, \cdots, \xi_s)$ and let Δ be an element of $C^{(n)}$ satisfying $\Delta(E)=0$. Δ can be written as a linear combination of D_i over P, i. e., $\Delta=\sum_{i=1}^{s} a_iD_i(a_i \in P)$.

$$\sum_{i=1}^{s} a_iD_i(\xi_j)=\Delta(\xi_j)=0 \qquad \text{for} \quad j=1, 2, \cdots, s.$$

By (**), $a_i=0$ for $i=1, 2, \cdots, s$, i. e. $\Delta=0$. Hence derivations of order n of P/E contained in $C^{(n)}$ is only 0, and $\mathfrak{D}_{E}^{(1)}(P)=0$. Therefore P is separably algebraic over E. Conversely, let F be a field $\Phi(\xi_{i_1}\cdots\xi_{i_t})(1\leqq i_1<\cdots<i_t\leqq m)$ such that P is separably algebraic over F. We shall consider a map $g:\mathfrak{D}_{\Phi}^{(n)}(P)\longrightarrow P^t$ defined by $g(D)=(D(\xi_{i_1}), \cdots, D(\xi_{i_t}))$. g is a P-linear mapping. As above, if $D \in C^{(n)}$ and $D(\xi_{i_j})=0$ for $j=1, 2, \cdots, t$, then $D(F)=0$. Hence D is a derivation of order n of P over F. Since P is separably algebraic over F, $D=0$ on P. Therefore g is an isomorphism of $C^{(n)}$ into P^t, and $s=\dim_p C^{(n)}\leqq t$. Thus the

dimension of $C^{(n)}$ over P is equal to the smallest number t such that P is separably algebraic over $\Phi(\xi_{i_1}\cdots\xi_{i_t})$. On the other hand, it is well known that the dimension of $\mathcal{D}_{\Phi}^{(1)}(P)$ has the same property ([1], Chap. IV, Th. 16). Therefore $\mathcal{D}_{\Phi}^{(1)}(P)=C^{(n)}$ for all $n>0$.

Acknowledgement. The author wishes to express his hearty thanks to Professor Y. Nakai for his valuable suggestions.

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