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On the Structure Space of a Direct Product of Rings

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§1. Introduction

It is known that, from the algebraic point of view, the ring E of entire functions has many interesting properties (see, for example, [3, §1, exerc. 12], [7] and [8]). Any residue ring $E/(f)$ by a non-zero entire function f is isomorphic to a direct product of homomorphic images of discrete valuation rings. This implies that, as far as the structure space is concerned, the study of the ring E is reduced to that of a direct product of discrete valuation rings. Thus, in this article, we shall mainly investigate the structure space of a direct product of commutative rings.

Every ring in this article will be assumed to be a commutative ring with an identity. In §2, as preliminaries, we shall give some relations between the structure space of a ring R, which will be denoted by Spec(R), and that of the Boolean algebra of idempotents in R. Next, in §3, we shall treat the case in which R is a direct product of local rings or integral domains; and in §5 the more restricted case, in which each factor of the product is a discrete valuation ring, will be treated by making use of some results on isolated subgroups of a totally ordered additive group which will be discussed in §4.

Finally, in §6, applying our theory to the ring of entire functions, we shall show how the algebraic properties of it, which was given by M. Henriksen, can be obtained (cf. [7], [8]).

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§2. Preliminaries

The set of idempotents in a ring R will be denoted by $B(R)$, or simply by B. The set $B(R)$ forms a Boolean algebra provided with the following order relation: for any x, y in B, $x \leq y$ if and only if $x = yx$. In this case the complement x' of x in B is $1-x$, $x \wedge y = xy$, $x \vee y = x + y - xy$, for any x, y in B.

The term "ideal" will be used with two meanings in this article. On the one hand, "ideal" will designate a ring ideal in a ring R. The word "ideal" will also be used to denote an ideal in a Boolean algebra $B(R)$, that is, a non-empty subset J of $B(R)$ such that $e \in J$, $f \in J$ implies $e \vee f \in J$, and $e \in J$, $f \leq e$ implies $f \in J$. Obviously, if A is an ideal in a ring R, then $A \cap B(R)$ is an ideal in $B(R)$. If A is prime, furthermore, then so is $A\cap B(R)$. By an ideal in each sense we shall mean a proper ideal throughout this article.

Let R be a ring. We shall denote by $\mathrm{Spec}(R)$ [resp. $\mathrm{Spec}(B)$] the set of prime ideals of R [resp. $B=B(R)$] with Zariski-topology and by $P(R)$ [resp. $M(R)$] the subspace of $\mathrm{Spec}(R)$ consisting of minimal [resp. maximal] prime ideals of R; any closed set of $\mathrm{Spec}(R)$ [resp. $\mathrm{Spec}(B)$] is of the form $V_R(S)$ [resp. $V_B(S)$] for some subset S of R [resp. B], where $V_R(S)$ [resp. $V_B(S)$] is the set of prime ideals of R [resp. B] containing S. Let us consider the following diagram of the natural mappings:

$$\begin{tikzcd}[row sep=small] M(R) \arrow[dr, "j"] & & \\ & \mathrm{Spec}(R) \arrow[r, "\alpha"] & \mathrm{Spec}(B), \\ P(R) \arrow[ur, "i"'] & & \end{tikzcd}$$

where $\alpha(M)=M\cap B$ for any $M\in\mathrm{Spec}(R)$, and i, j are natural injections. We shall show that α, αi and αj are continuous and surjective, and that α, αj are closed mappings. The above notations will be fixed throughout this section.

PROPOSITION 2.1 (Continuity of α). *Let J be an ideal of B. Then, $\alpha^{-1}(V_B(J))=V_R(J)$.*

PROOF. Let M be a prime ideal of R. Then we see easily that $M\in V_R(J)$ if and only if $\alpha(M)=M\cap B\in V_B(J)$.

For the closedness of α, we need a few lemmas.

LEMMA 2.2. *An ideal of a non-trivial Boolean algebra is prime if and only if it is maximal.*

PROOF. This is well known (see [1, Theorem II-7] or [10]).

COROLLARY 2.3. *Let M_1, M_2 be two prime ideals in R such that $M_1\subseteq M_2$. Then, $M_1\cap B=M_2\cap B$.*

LEMMA 2.4. *Let J be an ideal of B. Then the ideal of R generated by the set J is equal to the set-theoretic union: $\bigcup_{e\in J}Re$.*

By $I_R(J)$, or simply by $I(J)$, we shall denote the ideal of R generated by J.

PROOF OF LEMMA 2.4. Every element x of $I(J)$ is of the form

$$a_1e_1+\cdots+a_ne_n,$$

where $a_i\in R$ and $e_i\in J$ for $i=1, 2, \cdots, n$. Then we see that $x=xe$, where $e=e_1\vee\cdots\vee e_n$. This completes the proof.

COROLLARY 2.5. *Let J be an ideal of B. Then, $I(J) \cap B = J$.*

LEMMA 2.6. *Let A be an ideal of R and P a prime ideal of B containing $A \cap B$. Then, $I(P) + A \neq R$.*

PROOF. Suppose that $I(P) + A = R$. Then there exist elements $r \in R$, $a \in R$ and $e \in P$ such that $1 = re + a$, by Lemma 2.4. Hence $e' = ae' \in A \cap B \subseteq P$, where $e' = 1 - e$. This is a contradiction.

COROLLARY 2.7 (Closedness of α). *Let A be an ideal of R. Then, $\alpha(V_R(A)) = V_B(A \cap B)$.*

PROOF. Let P be a prime ideal of B containing $A \cap B$. Then, by Lemma 2.6, there exists a prime ideal M of R containing P and A. By Lemma 2.2, we obtain $\alpha(M) = P$.

It also follows from the proof of the above lemma that αj is closed, since the prime ideal M in the above proof can be chosen to be maximal. From this, moreover, it follows easily that αj is surjective, *a priori* α.

It remains to show that αi is surjective, which will be obtained from the following

LEMMA 2.8. *Let P be a prime ideal of B. Then every minimal element of $V_R(I(P))$ is a minimal prime ideal in R. Furthermore, $\alpha^{-1}(P) = V_R(I(P))$, which is not empty.*

PROOF. Let M be a minimal prime ideal of R containing $I(P)$ and M' a prime ideal of R contained in M. Then, $P = M' \cap B = M \cap B$ by Lemma 2.2. Hence $I(P) \subseteq M'$. Thus $M' = M$. This completes the proof of the first assertion. The last assertion is evident from Proposition 2.1 and Corollary 2.5.

REMARK 2.9. Some basic discussions on $P(R)$ can be found in [9]. In general, αi may not be closed even if it is bijective. If $I(P)$ is a prime ideal in R for any $P \in \mathrm{Spec}(B)$, then αi is bijective. When R has no non-zero nilpotents (such a ring R is called a *reduced ring*), the converse is true. This is obtained by the next lemma.

LEMMA 2.10. *Let R be a reduced ring. Then, $R/I(J)$ is also reduced for any ideal J of $B(R)$.*

PROOF. Let x be an element of R such that $x^n \in I(J)$ for some integer n. Then there exists $e \in J$ such that $x^n = x^n e$, by Lemma 2.4. Hence $x^n e' = 0$ and so, $(xe')^n = 0$, where $e' = 1 - e$. This implies that $xe' = 0$ by our assumption. Thus, $x = xe$, which completes the proof.

COROLLARY 2.11. *Let R be a reduced ring. Then, αi is bijective if and only if $I(P)$ is a (minimal) prime ideal of R for any $P \in \mathrm{Spec}(B)$.*

Finally we shall consider the case in which α is bijective, or equivalently homeomorphic. First of all, we shall quote some results. The following conditions on a ring R are equivalent:

(a) R is an absolutely flat ring.

(b) For any element x of R, $x \in x^2R$ [2, Chap. I, §2, exerc. 17].

(c) Every principal ideal of R is generated by an idempotent of R [loc. cit].

(d) R is a reduced ring and every point of Spec(R) is closed [2, Chap. II, §4, exerc. 16].

(e) For any maximal ideal M of R, R_M is a field, or equivalently, $R_M = R/M$ [2, Chap. II, §3, exerc. 9].

It is easy to see that if R is reduced and α is homeomorphic, then R is an absolutely flat ring by the condition (d), since Spec(B) is a compact space (see [1] or [10]). Conversely, we get the following

PROPOSITION 2.12. *Let R be an absolutely flat ring and P a prime ideal of $B=B(R)$. Then $I(P)$ is a (maximal) prime ideal of R.*

PROOF. Let x, y be two elements of R such that $xy \in I(P)$. Then there exist $e_1, e_2 \in B$ such that $xR=e_1R$ and $yR=e_2R$. Since $e_1e_2 \in xyR \cap B \subseteq I(P) \cap B=P$, $e_1 \in P$ or $e_2 \in P$. Hence $x \in I(P)$ or $y \in I(P)$. This completes the proof.

COROLLARY 2.13. *Let R be an absolutely flat ring. Then*

$$P(R)=M(R)=\mathrm{Spec}(R) \overset{\alpha}{\underset{\sim}{\longrightarrow}} \mathrm{Spec}(B).$$

§ 3. A direct product of rings

Let us consider a direct product $R=\Pi_{\lambda \in X} R_\lambda$ of a family $\{R_\lambda\}$ of rings indexed by a non-empty set X. For any x in R, we shall use the following notations throughout this article:

$$x=(x_\lambda), \text{ where } x_\lambda \text{ is the } \lambda\text{-th component of } x,$$

$$Z(x)=\{\lambda \in X;\ x_\lambda=0\}.$$

Moreover, for any subset S of R, we shall denote by $Z(S)$ the collection $\{Z(x);\ x \in S\}$, and the identity element of R and those of the R_λ's will be commonly denoted by the same 1.

In the case in which Spec(R_λ) is connected for every $\lambda \in X$, i.e., $B(R_\lambda) = \{0, 1\}$ for each λ in X, every element x of $B=B(R)$ is precisely of the form (x_λ) such that $x_\lambda=0$ or 1 for each λ in X. That is to say, every element x of B is uniquely determined by $Z(x)$, which is an element of the power set

$\mathfrak{P}(X)$ of X. Such a correspondence gives the lattice-isomorphism Z of B onto $\breve{\mathfrak{P}}(X)$, where $\breve{\mathfrak{P}}(X)$ is the dual of $\mathfrak{P}(X)$; this isomorphism Z gives the natural homeomorphism Z^* of $\mathrm{Spec}(B)$ onto the ultrafilter space X'' of X, and furthermore, which is homeomorphic with the Stone-Čech compactification βX of the discrete topological space X. For the reader, we shall comment on the above briefly:

1. An ideal of $\breve{\mathfrak{P}}(X)$ is exactly a filter on X, and it is a prime ideal of $\breve{\mathfrak{P}}(X)$ if and only if it is an ultrafilter on X [1, p. 25].
2. Let X'' be the set of ultrafilters on X. For any subset A of X, we put $A^*=\{F \in X''; F \ni A\}$. Then the sets A^* form a basis of a topology on X''. This topological space X'' is called the ultrafilter space on X [5, §9, exerc. 26].
3. When we regard X as a discrete topological space, it is easy to see that the Stone-Čech compactification βX of X coincides with the ultrafilter space X'' of X, from the method of the construction of βX in [6, Theorem 6.5].

Lemma 3.1. *Let R be a direct product $\Pi_{\lambda \in X} R_\lambda$, where all the R_λ's are fields. Then,*

$$P(R)=M(R)=\mathrm{Spec}\,(R) \xrightarrow[\sim]{\alpha} \mathrm{Spec}\,(B) \xrightarrow[\sim]{Z^*} X''.$$

Proof. It is easy to see that every principal ideal of R is generated by an idempotent element of R, so that R is an absolutely flat ring. Therefore the assertion follows from Corollary 2.13 and the above discussion.

Remark 3.2. Form now on, we shall write $\mathrm{Spec}(B)=X''$ instead of $\mathrm{Spec}\,(B) \xrightarrow{\sim} X''$.

Corollary 3.3. *With the same notations as in Lemma 3.1, let M, P and F be corresponding elements of $\mathrm{Spec}\,(R)$, $\mathrm{Spec}\,(B)$ and X'', respectively. Then, $\alpha(M)=P$, $M=I(P)=\{x \in R; Z(x) \in F\}$ and $Z(P)=Z(M)=F$.*

Proof. The first two equalities have already been shown in Proposition 2.12. The fact that the isomorphism $Z: B \longrightarrow \breve{\mathfrak{P}}(X)$ induces the isomorphism $Z^*: \mathrm{Spec}\,(B) \longrightarrow X''$, amounts to the fact that $Z(P)=F$ and $Z^{-1}(P)=F$, or equivalently $P=\{x \in B; Z(x) \in F\}$. Hence $F=Z(P) \subseteq Z(M)=Z(I(P))$. Let $Z(re)$ be any element of $Z(I(P))$, where $r \in R$ and $e \in P$ (see Lemma 2.4). Then $Z(re)=Z(r) \cup Z(e) \supseteq Z(e)$, where $Z(e) \in F$; hence $Z(re) \in F$. Thus we have $Z(P)=Z(M)=F$. Finally, we see directly that $M \subseteq \{x \in R; Z(x) \in F\}$ since $Z(M)=F$. We shall show the converse. Let x be an element of R such that $Z(x) \in F$. Then there exists $e \in B$ such that $Z(e)=Z(x)$. Since $Z(e)=Z(x) \in F$, we have $e \in P$. Thus $x=xe \in I(P)=M$. This completes the proof.

Now, let us consider the two cases: (i) when all the R_λ's are integral domains, and (ii) when all the R_λ's are local rings with the maximal ideals M_λ's respectively.

The first case: Let Q_λ be the quotient field of R_λ and $Q(R)$ the total quotient ring of $R=\Pi_{\lambda\in X}R_\lambda$. Then it is easy to see that $Q(R)=\Pi_{\lambda\in X}Q_\lambda$, $B(Q(R))=B(R)$ and $P(Q(R))\xrightarrow{\sim}P(R)$. As for the fact that $P(Q(R))\xrightarrow{\sim}P(R)$, more general results can be found in [9, Theorem 5.1 p. 124]. By the next lemma and Lemma 3.1, we get the following commutative diagram:

$$\begin{array}{ccccccc} P(R) & \xrightarrow{i} & \operatorname{Spec}(R) & \xrightarrow{\alpha} & \operatorname{Spec}(B) & = & X'' \\ \wr\uparrow & & \varphi\uparrow & & \| & & \| \\ P(Q(R)) & = & \operatorname{Spec}(Q((R)) & = & \operatorname{Spec}(B(Q(R))) & = & X'', \end{array}$$

when φ is the continuous function which is induced by the natural injection: $R\longrightarrow Q(R)$.

LEMMA 3.4. *Under the same situation as above, let M be a minimal prime ideal of R and put $P=\alpha(M)$. Then, $M=MQ(R)\cap R$ and $P=M\cap B=MQ(R)\cap B$. Furthermore, let F be an ultrafilter on X, which corresponds to P. Then, $F=Z(P)=Z(M)$ and $I_R(P)=M=\{x\in R;\ Z(x)\in F\}$.*

PROOF. Obviously, $M=MQ(R)\cap R$ and $P=M\cap B\subseteq MQ(R)\cap B$. Since $MQ(R)\cap B$ is a prime ideal of B, it coincides with P by Lemma 2.2. This completes the proof of the first assertion, which amounts to the commutativity of the above diagram. By Corollary 3.3, $Z(MQ(R))=Z(P)=F$; hence $Z(M)=Z(P)=F$, since $Z(P)\subseteq Z(M)\subseteq Z(MQ(R))$. From the above diagram, we see that αi is bijective; hence $I_R(P)=M$ by virtue of Corollary 2.11. Finally, since $M\subseteq\{x\in R;\ Z(x)\in F\}=\{x\in Q(R);\ Z(x)\in F\}\cap R=MQ(R)\cap R=M$, we get the last equality in our lemma. This completes the proof.

The second case: Let K_λ be the residue field R_λ/M_λ of R_λ and $J(R)$ the Jacobson radical of $R=\Pi_{\lambda\in X}R_\lambda$. Then we see that $J(R)=\Pi_{\lambda\in X}M_\lambda$, $R/J(R)=\Pi_{\lambda\in X}K_\lambda$, and that the canonical epimorphism ψ: $R\longrightarrow R/J(R)$ induces the isomorphism $B(\psi)$: $B(R)\xrightarrow{\sim}B(Q(R))$ and the homeomorphism ψ^*: $M(R/J(R))\xrightarrow{\sim}M(R)$. By Lemma 3.1, we get the following commutative diagram:

$$\begin{array}{ccccccc} M(R/J(R)) & = & \operatorname{Spec}(R/J(R)) & = & \operatorname{Spec}(B(R/J(R))) & = & X'' \\ \psi^*\downarrow\wr & & \psi^*\downarrow & & B(\psi)^*\downarrow\wr & & \\ M(R) & \xrightarrow{j} & \operatorname{Spec}(R) & \xrightarrow{\alpha} & \operatorname{Spec}(B) & = & X'' \end{array}$$

The commutativity of this diagram can be obtained by routine calculations so the proof is omitted.

COROLLARY 3.5. *With the same notations as above, let M be a maximal ideal of R and put $P=\alpha(M)$. If F is the ultrafilter on X, which corresponds* *to P, then $F=Z(P)=Z(M/J(R))$ and $M=J(R)+I(P)=\{x \in R;\ Z(\bar{x}) \in F\}$, where $\bar{x}$ denotes the residue class of x modulo $J(R)$. Furthermore, $R/I(P)$ is a local ring with the maximal ideal $M/I(P)$.*

Proof. For any subset S of R, we shall denote by $\bar{S}$ the image of S in $R/J(R)$. By Corollary 3.3, $F=Z(\bar{P})=Z(\bar{M})$, and $\bar{M}=I_{\bar{R}}(\bar{P})=\{x \in \bar{R};\ Z(x) \in F\}$. From this, it is easy to see that $F=Z(P)=Z(\bar{P})$, and $M=\{x \in R;\ Z(\bar{x}) \in F\}$. It remains to show that $M=J(R)+I(P)$. Since $J(R)+I(P)\subseteq M$, we shall show that $M\subseteq J(R)+I(P)$. Let x be an element of M. Then, there exists $e \in P$ such that $\bar{x}=\bar{x}\bar{e}$. Hence $x\equiv xe$ (mod $J(R)$). Therefore, $x \in J(R)+I(P)$. Thus, $M=J(R)+I(P)$. Finally, from the above diagram, we see that αj is bijective, so that M is the unique maximal ideal of R containing P. Thus, we see that $R/I(P)$ is a local ring. This completes the proof.

Theorem 3.6. *Let $\{R_\lambda\}$ be a family of local integral domains indexed by a non-empty set X and put $R=\Pi_{\lambda\in X}R_\lambda$. Then, αj: $M(R)\longrightarrow \mathrm{Spec}(B)$ and αi: $P(R)\longrightarrow \mathrm{Spec}(B)$ are homeomorphisms. Moreover, let M be a maximal ideal of R and M' a minimal prime ideal of R such that $M\cap B=M'\cap B(=P)$. Then,*

(1) $Z(M')=Z(P)=Z(M/J(R))\,(=F)$,

(2) $I(P)=M'=\{x \in R;\ Z(x) \in F\}$,

(3) $M=J(R)+I(P)=\{x \in R;\ Z(\bar{x}) \in F\}$, *and*

(4) $R_M=R/I(P)$,

where we denote by $J(R)$ the Jacobson radical of R and by $\bar{x}$ the residue class of x modulo $J(R)$.

Proof. The assertions (1), (2) and (3) have already been shown. We shall show that $R_M=R/I(P)$. By Corollary 3.5, $R/I(P)$ is a local ring with the maximal ideal $M/I(P)$. To complete the proof, it suffices to show that the kernel of the canonical homomorphism: $R\longrightarrow R_M$, coincides with $I(P)$. This will be done in the following

Lemma 3.7. *Let M be a prime ideal of a ring R and A the kernel of the canonical homomorphism: $R\longrightarrow R_M$. Then, $I(M\cap B)\subseteq A\subseteq M$.* If *$I(M\cap B)$ is a prime ideal, furthermore, then $I(M\cap R)=A$ and R_M is a local domain.*

Proof. Let x be an element of $I(M\cap B)$. Then there exists $e \in M\cap B$ such that $x=xe$ by Lemma 2.4. Hence $x(1-e)=0$, which implies that $x \in A$ since $1-e \notin M$. Thus $I(M\cap B)\subseteq A$, which proves the first assertion. Next, suppose that $I(M\cap B)$ is prime. To complete the proof, we have only to show that $A\subseteq I(M\cap B)$. Let a be an element of A. Then there exists $s \in R-M$ such that $as=0$. Since $as=0 \in I(M\cap B)$, $a \in I(M\cap B)$. Thus $A\subseteq I(M\cap B)$.

Corollary 3.8 to Theorem 3.6. *In Theorem* 3.6, *if each R_λ is a valuation* *ring, then $R/I(P)$ is a valuation ring. Therefore, in this case, the set $V(P)$ of prime ideals in R containing $I(P)$ is linearly ordered under set-inclusion and* Spec(R) *as a set is the disjoint union:* $\cup_{P\in\mathrm{Spec}(B)}V(P)$.

Proof. Obviously, if each R_λ is a valuation ring, then $R=\Pi_{\lambda\in X}R_\lambda$ is a Bezout ring, i.e., every finitely generated ideal of R is principal [4, §1, exerc. 20]. Therefore, $R/I(P)$ is a local Bezout domain, whence it is a valuation ring (cf. [4, §2, exerc. 12]). The other assertion in our corollary is trivial and we omit the proof.

Theorem 3.9. *With the same notations as in Theorem* 3.6, *suppose that each R_λ is a discrete valuation ring, and set $Q=\cap_{n=1}^{\infty}M^n$. Then we have*

(1) *R/Q is a discrete valuation ring.*

(2) *$Q=I(P)$ if and only if F has the countable intersection property.*

Remark: In general, a filter on a set is said to be fixed if the intersection of all members of it is not empty. It is said to be free otherwise. If F is fixed, in Theorem 3.9, then F has the countable intersection property, i.e., any intersection of countable members of F is also a member of F; some discussions on this context are found in [6, Chap. 12].

Proof of Theorem 3.9. Let π_λ be a prime element of R_λ for each $\lambda\in X$ and set $\pi=(\pi_\lambda)$. Then the maximal ideal $M/I(P)$ of $R/I(P)$ is a principal ideal generated by the residue class of π modulo $I(P)$. Therefore, $\cap_{n=1}^{\infty}(M/I(P))^n=\cap_{n=1}^{\infty}\pi^nR_M$ is the largest non-maximal prime ideal (such a prime ideal is called a *submaximal* prime ideal) of $R/I(P)=R_M$. Since $I(P)^n=I(P)\subseteq M$, $Q/I(P)=\cap_{i=1}^{\infty}M^n/I(P)=\cap_{n=1}^{\infty}(M/I(P))^n$. Thus we see that R/Q is a Noetherian local domain with the maximal ideal which is principal, so that R/Q is a discrete valuation ring. Next, we shall prove the second assertion. Suppose that $Q=I(P)$. We shall prove dually that F has the countable intersection property. Let $\{N_i\}$ be an ascending sequence of subsets of X such that $N_i\notin F$ for $i=1, 2, \cdots$. It suffices to show that $\cup_{i=1}^{\infty}N_i\notin F$. Now we take the element $x=(x_\lambda)$ of R as follows:

$$\begin{aligned} x_\lambda &=0 \quad \text{for } \lambda\in X-\cup_{i=1}^{\infty}N_i,\\ &=\pi_\lambda \quad \text{for } \lambda\in N_1,\\ &=\pi_\lambda^2 \quad \text{for } \lambda\in N_2-N_1, \text{ and so on.}\end{aligned}$$

Let e_i be an element of B such that $Z(e_i)=N_i$ for $i=1, 2, \cdots$. By (3) in Theorem 3.6, $N_i\notin F$ amounts to $e_i\notin M$. On the other hand, we see that $xe_i\in\pi^{i+1}R$, so that $x\in\pi^{i+1}R_M$ for $i=1, 2, \cdots$. Hence $x\in Q=I(P)$, which amounts to $Z(x)\in F$ by (2) in Theorem 3.6. Thus $\cup_{i=1}^{\infty}N_i=X-Z(x)\notin F$.

Conversely, suppose that $Q\neq I(P)$. Let x be an element of $Q-I(P)$.

For each λ in X, let us denote by v_λ the normalized valuation of R_λ, and set $N_i=\{\lambda \in X;\ v_\lambda(x_\lambda)=i\}$ for $i=0, 1, 2, \cdots$. Then, $Z(x)=X-\bigcup_{i=0}^\infty N_i \notin F$, since $x \notin I(P)$. Hence $\bigcup_{i=0}^\infty N_i \in F$. To complete the proof, it is sufficient to show that $N_i \notin F$ for $i=1, 2, \cdots$. Suppose that $N_i \in F$ for some i. Then there exists $e \in P$ such that $Z(e)=N_i$; the element xe', where $e'=1-e$, can be written in the form of $\pi^i ue'$, where u is a unit in R. This implies that $\pi^i R_M=xR_M \subseteq QR_M=\bigcap_{n=1}^\infty \pi^n R_M$, since $e' \notin M$. This is impossible. Thus the proof is completed.

THEOREM 3.10. *With the same notations and assumptions as in Theorem 3.9, if $Q \neq I(P)$, then R/Q is a complete discrete valuation ring.*

PROOF. By our assumption, there exists a descending sequence $\{N_i\}_{i=0,1,2,\ldots}$ of subsets of X such that $N_i \in F$ for each i and $\bigcap_{i=0}^\infty N_i \notin F$. We may assume that $\bigcap_{i=0}^\infty N_i=\phi$, by replacing N_i by N_i-N if necessary, where $N=\bigcap_{i=0}^\infty N_i \notin F$. Now we take the element e_i in $B=B(R)$ such that $Z(e_i)=N_i$ for each i. Let $\{x_i\}_{i=0,1,2,\ldots}$ be a countable family of elements of R. Then we can consider in R an infinite sum $\sum_{i=0}^\infty e_i' x_i$, where $e_i'=1-e_i$ for each i, since the λ-th components of each term are almost all zero for each λ in X. Now let $\{a_n\}_{n=1,2,\ldots}$ be a Cauchy sequence in R/Q with Zariski topology. We shall prove that it converges in R/Q. We may assume that it is a regular sequence, i.e., $a_{n+1}-a_n \in \pi^n R/Q$ for each n. For any $r \in R$, we shall denote by $\bar{r}$ the equivalence class of r modulo Q. Note that $\bar{r}=\overline{er}$ for any idempotent $e \in R-Q$ and for any $r \in R$. Now, choose representatives $\{x_n\}_{n=0,1,2\ldots}$ in R such that $a_1=\bar{x}_0$ and $a_{n+1}-a_n=\overline{\pi^n x_n}$ for $n=1, 2, \cdots$; and set $y_n=\sum_{i=0}^\infty \pi^i x_{n+i} e_{n+i}'$ for $n=0, 1, 2, \cdots$, where $e_0'=1$. Then we obtain

$$y_0=x_0+\pi x_1 e_1'+\pi^2 x_2 e_2'+\cdots+\pi^{n-1} x_{n-1} e_{n-1}'+\pi^n y_n,$$
$$\text{for } n=1, 2, \cdots.$$

This shows that $\bar{y}_0=a_n+\overline{\pi^n y_n}$. Thus the proof is completed.

§ 4. Some results on isolated subgroups

Let G be a totally ordered additive group with the order relation $\leq$. For any subsets A, B of G, when $a \leq b$ for any $a \in A$ and $b \in B$, we shall denote it by $A \leq B$. We shall write $a \leq B$ instead of $\{a\} \leq B$. Similarly, we shall use the notations like $A<B$, $a<B$, etc. Let H be an isolated subgroup of G and set $T=\{a \in G;\ H<a\}$. Throughout this section, the above notation will be fixed and furthermore, we shall assume that T satisfies the following two conditions:

(I) If A and B are non-empty countable subsets of T such that $A<B$, then there exists an element c in T such that $A \leq c \leq B$.

(II) For each element a in T, there exists a strictly descending sequence $\{a_{(n)}\}$ of elements in T such that

$$na_{(n)} \leq a \leq 2na_{(n)} \qquad \text{for } n=1, 2, \cdots$$

(such a sequence will be called an *r-descending sequence* of a).

Let a be an element of T. By H^a we shall denote the maximal isolated subgroup of G not containing a, and by H_a the minimal isolated subgroup of G containing a. Then the following lemma is immediate.

LEMMA 4.1. *Let a be an element of T. Then,*

(1) $H_a = \cup_{n=1}^{\infty} \{x \in G;\ |x| \leq na\}$ *and*

(2) $H^a = \{x \in G;\ |nx| < a$ *for any integer* $n\}$.

LEMMA 4.2. *Let $\{a_{(n)}\}$ be an r-descending sequence of $a \in T$. Then,* $H^a = \cap_{n=1}^{\infty} \{x \in G;\ |x| < a_{(n)}\}$.

PROOF. It is clear that $H^a \subseteq \{x \in G;\ |x| < a_{(n)}\}$ for any integer n. Therefore $H^a \subseteq \cap_{n=1}^{\infty} \{x \in G;\ |x| < a_{(n)}\}$. Conversely, let x be an element of $T - H^a$. Then there exists an integer n such that $a \leq nx$, by Lemma 4.1; hence $a_{(n)} \leq x$. Thus the proof is completed.

LEMMA 4.3. $H^b \neq H_a$ *for any* $a, b \in T$.

PROOF. Suppose that $H^b = H_a$. And set $A = \{na\}_{n=1,2,\dots}$ and $B = \{b_{(n)}\}_{n=1,2,\dots}$, where $\{b_{(n)}\}$ is an r-descending sequence of b. Then, $A < B$ by Lemma 4.1 and Lemma 4.2. It follows from our assumption that there exists $c \in T$ such that $A \leq c \leq B$. The fact $c \leq B$ implies $c \in H^b = H_a$; by Lemma 4.1, there exists an integer n such that $c \leq na$. On the other hand, $A \leq c$ implies $(n+1)a \leq c$. This is a contradiction and we have the assertion.

COROLLARY 4.4. *Suppose that $H^a \subsetneqq H^b$ for some $a, b \in T$. Then, there exists an element c in T such that $H_a \subsetneqq H^c \subsetneqq H_c \subsetneqq H^b$.*

PROOF. Obviouly $H^a \subsetneqq H^b$ implies $H_a \subseteq H^b$, hence $H_a \subsetneqq H^b$ by Lemma 4.3. Then, by Lemma 4.3 again, we see that $H_a \subsetneqq H^c \subsetneqq H_c \subsetneqq H^b$, for every element c in $H^b - H_a$. This completes the proof.

PROPOSITION 4.5. *Let $\{a_i\}_{i=1,2,\dots}$ and $\{b_i\}_{i=1,2,\dots}$ be two sequences of elements in T such that $H^{a_i} \subsetneqq H^{a_j} \subsetneqq H^{b_j} \subsetneqq H^{b_i}$ for each i, j such that $i<j$. Then there exists an element c in T such that $H^{a_i} \subsetneqq H^c \subsetneqq H^{b_i}$ for $i=1, 2, \cdots$.*

PROOF. Set $A = \{na\}_{n=1,2,\dots}$ and $B = \{(b_n)_{(n)}\}_{n=1,2,\dots}$, where $\{(b_n)_{(m)}\}_m$ is an r-descending sequence of b_n for each n. Then there exists $c \in T$ such that $A \leq c \leq B$ by our assumption. This element c is a required one.

COROLLARY 4.6. *Suppose that $H^a \subsetneqq H^b$ for some $a, b \in T$. Then the totally* *ordered set* $S=\{H^c;\ H^a \subsetneqq H^c \subsetneqq H^b\}$ *is an* η_1*-set**), *and therefore, S has power at least* $2^{\aleph_1}$.

§ 5. A direct product of discrete valuation rings

Let $\{R_\lambda\}$ be a family of discrete valuation rings indexed by a non-empty set X and put $\pi=(\pi_\lambda)$ in $R=\Pi_{\lambda\in X}R_\lambda$, where each π_λ is a prime element of R_λ. And let M be a maximal ideal of R and set $P=M\cap B(R)$. In §3, we saw that $R_M=R/I(P)$ is a valuation ring with the maximal ideal πR_M and with the submaximal prime ideal $Q/I(P)$, where $Q=\cap_{n=1}^{\infty}M^n$, and that $Q=I(P)$ if and only if $F=Z(P)$ has the countable intersection property. In this section, we shall show that if $Q\neq I(P)$, then the set of prime ideals of R contained in M has power at least $2^{\aleph_1}$. The above notations and the assumption that $I(P)\neq Q$, will be fixed throughout this section. Further, we shall denote by v a valuation of $R/I(P)$ and by G its value group and by T the image of $Q/I(P)$ under v. And, for any element x of R, we shall denote by $\bar{x}$ the residue class of x modulo $I(P)$. To begin with, we shall prove that T satisfies the condition (I) and (II) in §4.

Definition 5.1. Let $x=(x_\lambda)$, $y=(y_\lambda)$ be two elements in R and N a subset of X. Then we denote $x\leq y$ on N, when $v_\lambda(x_\lambda)\leq v_\lambda(y_\lambda)$ for every $\lambda\in N$, where each v_λ is the normalized valuation of R_λ, and similarly, $x<y$ on N, when $v_\lambda(x_\lambda)<v_\lambda(y_\lambda)$ for every $\lambda\in N$. Then we have the following lemma immediately.

Lemma 5.2. *Let x, y be any two elements in R. Then, $v(\bar{x})\leq v(\bar{y})$ if and only if $x\leq y$ on some member in F. Therefore, $v(\bar{x})=v(\bar{y})$ if and only if $x\leq y$ and $y\leq x$ on some member in F at the same time. Moreover, $v(\bar{x})<v(\bar{y})$ if and only if $x<y$ on some member in F.*

Proposition 5.3. *For any element a in T, there exists an r-descending sequence of a.*

Proof. Choose an element $x=(x_\lambda)$ in $Q-I(P)$ such that $v(\bar{x})=a$ and set $N_i=\{\lambda\in X;\ v_\lambda(x_\lambda)=i\}$ for $i=0, 1, 2, \dots$. Then, we have already seen in the proof of Theorem 3.9 that $N_i\notin F$ for every i and $X-\cup_{i=0}^{\infty}N_i=Z(x)\notin F$. Now, for every positive integer n, we take the element x_n of R as follows:

$$\begin{aligned}(x_n)_\lambda &=0 \quad \text{for } \lambda\in X-\cup_{i=0}^{\infty}N_i,\\ &=1 \quad \text{for } \lambda\in N_0\cup\cdots\cup N_{n-1},\\ &=\pi_\lambda \quad \text{for } \lambda\in N_n\cup\cdots\cup N_{2n-1},\end{aligned}$$

*) A totally ordered set S is called an η_1-*set* when S satisfies the condition: for any countable subsets A, B of S such that $A<B$, there exists $c\in S$ such that $A<c<B$ (cf. [6, Chap 13]).

$$=\pi_\lambda^2 \quad \text{for } \lambda \in N_{2n}\cup\cdots\cup N_{3n-1}, \text{ and so on.}$$

Now set $a_{(n)}=v(\bar{x}_n)$ for each n. Then the fact that $(x_n)^{2n}\geq x$ on $\cup_{m\geq n}N_m$ and $x\geq(x_n)^n$ on X, implies $2na_{(n)}\geq a\geq na_{(n)}$, by Lemma 5.2, since $\cup_{m\geq n}N_m \in F$. On the other hand, we see that $a_{(n)}>a_{(n+1)}$, since $x_n>x_{n+1}$ on $\cup_{m>2n(n+1)}N_m$. Thus $\{a_{(n)}\}$ is an r-descending sequence of a. This completes the proof.

PROPOSITION 5.4. *Let A, B be two non-empty countable subsets of T such that $A<B$. Then there exists $c \in T$ such that $A\leq c\leq B$.*

PROOF. We may assume that $A=\{a_n\}_{n=1,2,\dots}$ and $B=\{b_n\}_{n=1,2,\dots}$ such that $a_n<a_m<b_m<b_n$ for all $n<m$. Now, choose elements $\{x_n\}_{n=1,2,\dots}$ and $\{y_n\}_{n=1,2,\dots}$ in $Q-I(P)$ such that $v(\overline{x_n})=a_n$, and $v(\overline{y_n})=b_n$ for each n. Then there exists a descending sequence $\{N_i\}_{i=1,2,\dots}$ of members of F, such that $x_1\leq x_2\leq\cdots\leq x_n\leq y_n\leq\cdots\leq y_2\leq y_1$ on N_n for each n. First, suppose that $\cap_{i=1}^\infty N_i=N \in F$. Then, we take two elements $x=(x_\lambda)$ and $y=(y_\lambda)$ of R as follows:

$$x_\lambda=0 \quad \text{for } \lambda \in X-N,$$
$$=\pi_\lambda^{\alpha_\lambda} \quad \text{for } \lambda \in N, \text{ where } \alpha_\lambda=\sup_n v_\lambda((x_n)_\lambda).$$
$$y_\lambda=0 \quad \text{for } \lambda \in X-N,$$
$$=\pi_\lambda^{\beta_\lambda} \quad \text{for } \lambda \in N, \text{ where } \beta_\lambda=\inf_n v_\lambda((y_n)_\lambda).$$

In the above, we put $\pi_\lambda^{\alpha_\lambda}=0$ when $\alpha_\lambda=\infty$. Then, $x_n\leq x\leq y\leq y_n$ on N for every n. This implies that $a_n\leq v(\bar{x})\leq v(\bar{y})\leq b_n$ for every n. Thus the assertion settles in this case. Next, suppose that $\cap_{i=1}^\infty N_i=N \notin F$. Then we may assume that N is an empty set, by replacing N_i by N_i-N if necessary. Now, we take two elements $x=(x_\lambda)$ and $y=(y_\lambda)$ of R as follows:

$$x_\lambda=y_\lambda=0 \quad \text{for } \lambda \in X-\cup_{i=1}^\infty N_i,$$
$$x_\lambda=(x_1)_\lambda,\ y_\lambda=(y_1)_\lambda \quad \text{for } \lambda \in N_1-N_2,$$
$$x_\lambda=(x_2)_\lambda,\ y_\lambda=(y_2)_\lambda \quad \text{for } \lambda \in N_2-N_3, \text{ and so on.}$$

Then we see that $x\leq y$ on X and $x_n\leq x\leq y\leq y_n$ on N_n for every n. This implies that $a_n\leq v(\bar{x})\leq v(\bar{y})\leq b_n$ for every n. In either case, $v(\bar{x})$ or $v(\bar{y})$ is a required element in T. Thus the proof is completed.

PROPOSITION 5.5. *For any element a in T, there exist b, $c \in T$ such that $H^b\subsetneqq H^a\subsetneqq H^c$.*

PROOF. Let $x=(x_\lambda)$ be an element in $Q-I(P)$ such that $v(\bar{x})=a$, set $N_i=\{\lambda \in X;\ v_\lambda(x_\lambda)=i\}$ for $i=1, 2, \dots$, and we take the element $y=(y_\lambda)$ of R as follows:

$$y_\lambda=0 \qquad \text{for } \lambda \in X-\cup_{i=1}^{\infty}N_i,$$

$$=\pi_\lambda^{\alpha_\lambda} \quad \text{for } \lambda \in N_i, \text{ where } \alpha_\lambda=[\sqrt{i}], \text{ for } i=1, 2, \cdots.$$

Then we see that $x>y^n$ on $\cup_{m\geq n^2+1}N_m$ for each n. This implies that $nb<a$ for every n, where $b=v(y)$. By Lemma 4.1, we see that $b \in H^a$, so that $H^b \subsetneqq H^a$.

Similarly, we take the element $z=(z_\lambda)$ of R as follows:

$$z_\lambda=0 \qquad \text{for } \lambda \in X-\cup_{i=1}^{\infty}N_i,$$

$$=\pi_\lambda^{i^2} \quad \text{for } \lambda \in N_i \ (i=1, 2, \cdots).$$

Then by routine calculations, we get that $H^a \subsetneqq H^c$, where $c=v(z)$.

COROLLARY 5.6. *Let P_1 be a prime ideal of R such that $I(P)\subsetneqq P_1 \subsetneqq Q$. Then there exist prime ideal P_2, P_3 of R such that $I(P)\subsetneqq P_2 \subsetneqq P_1 \subsetneqq P_3 \subsetneqq Q$. Moreover, the set of prime ideals of R between $I(P)$ and Q has power at least $2^{\aleph_1}$.*

PROOF. The first assertion is obtained from Proposition 5.5 directly; the last by Corollary 4.6.

§6. Application

Let E be the ring of entire functions, and let z be the identity mapping of the complex number field C, which is regarded as an element of E. Then, for any complex number c, we obtain a discrete valuation ring $R_c=E_{(z-c)}$ with a prime element $\pi_c=z-c$. Now, let us fix a non-zero and non-unit element f in E for a little while; and let A be the set of zeros of f and, for each $c \in A$, let $0(c)$ be the order of f at c. Then, from the theorem of Mittag-Leffler, we obtain the following natural isomorphisms by purely algebraic calculations:

$$E/fE \xrightarrow{\Pi\varphi_c} \Pi_{c\in A}E/\pi_c^{0(c)}E=\Pi_{c\in A}R_c/\pi_c^{0(c)}R_c, \tag{6.1}$$

where $\varphi_c\colon E/fE \longrightarrow E/\pi_c^{0(c)}E$ is the natural surjection for each c.

PROPOSITION 6.1 [7, Lemma 1, p. 183]. *If M is a maximal ideal of E, then the field E/M is algebraically closed.*

PROOF. Let f be a non-zero element of M and A the set of zeros of f; and set $R=\Pi_{c\in A}R_c$. Then E/M is a residue field of E/fE; and by virtue of (6.1), it is a residue field of a residue ring of R; hence, it is a residue field of R. Let $J(R)$ be the Jacobson radical of R. Then, since $R/J(R)=\Pi_{c\in A}R_c/\pi_cR_c$

$=C^A$, our proposition follows from the next lemma.

LEMMA 6.2. *Let $\{K_\lambda\}$ be a family of algebraically closed fields indexed by non-empty set X, and set $R=\Pi_{\lambda\in X}K_\lambda$. Then, every residue field of R is algebraically closed.*

The proof is routine and omitted.

PROPOSITION 6.3. *If M is a maximal ideal of E, then ME_M is a principal ideal.*

PROOF. Observing the proof of Proposition 6.1, we can take an element $f(\neq 0)\in M$ such that the order of f at each zero of f is one. Then, since E/fE is an absolutely flat ring by virtue of (6.1), we obtain

$$(E/fE)_{M/fE}=E_M/fE_M=E/M,$$

so that $fE_M=ME_M$. This completes the proof.

COROLLARY 6.4 [8, Theorem 3, p. 714]. *Let M be a maximal ideal of E. Then, $Q=\cap_{n=1}^{\infty}M^n$ is the largest nonmaximal prime ideal contained in M and E/Q is a discrete valuation ring.*

PROOF. Let f be an element of M such that $ME_M=fE_M$. Then our corollary follows from the fact

$$Q=\cap_{n=1}^{\infty}M^n=\cap_{n=1}^{\infty}M^nE_M\cap E=\cap_{n=1}^{\infty}f^nE_M\cap E.$$

PROPOSITION 6.5 [8, Corollary p. 716]. *Let M be a maximal ideal of E. Then the set of prime ideals of E contained in M is linearly ordered under set-inclusion.*

PROOF. Let P_1, P_2 be two prime ideals of E contained in M and suppose that $P_1\not\subseteq P_2$ and $P_2\not\subseteq P_1$. Then there exist $f_1\in P_1-P_2$ and $f_2\in P_2-P_1$. Considering (6.1) with $f=f_1f_2$, we obtain our proposition from Crollary 3.8.

PROPOSITION 6.6 [8, Theorem 2, p. 713]. *Every non-zero prime ideal P of E is contained in a unique maximal ideal.*

PROOF. Considering (6.1) with $f(\neq 0)\in P$, our proposition follows from Corollary 3.8.

REMARK 6.7. As the converse of (6.1), for any (countable) discrete subset $A(\neq\phi)$ of C and for any sequence $\{0(c)\}_{c\in A}$ of non-negative integers indexed by A, there exists an element f in E such that (6.1) holds.

PROPOSITION 6.8. *Let R be a N-copy of the discrete valuation ring $E_{(z)}$, where N is the set of positive integers, and let P_1 be a prime ideal of R which* *is neither maximal nor minimal. Then,* $R/P_1 = E/P'$ *for some prime ideal* P' *of* E.

PROOF. Let M be a maximal ideal of R containing P_1 and P_0 a minimal prime ideal of R contained in P_1. Then observing the proof of Theorem 3.9 with $X=N$, we can take an element x in $P_1 - P_0$ such that $Z(x)=\phi$. Thus $R/xR = E/fE$ for some element f of E by Remark 6.7, which proves our proposition.

COROLLARY 6.9 [8, p. 719]. *There exists a maximal ideal* M *of* E *such that* $Q=\cap_{n=1}^{\infty} M^n \neq 0$, *and with this prime ideal* Q, E/Q *is a complete discrete valuation ring.*

PROOF. This follows from Proposition 6.8 and Theorem 3.10.

COROLLARY 6.10 [8, Theorem 5, p. 717]. *With the same* M *as in Corollary* 6.9, *the set of prime ideals of* E *contained in* M *has power at least* $2^{\aleph_1}$.

PROOF. This follows from Proposition 6.8 and Corollary 5.6.

References

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Collog. Pabl. XXV, New York, 1967.

[2] N. Bourbaki, Eléments de Mathématique, *Algèbre Commutative*, Chap. I and Chap. II, Hermann, Paris, 1961.

[3] N. Bourbaki, Eléments de Mathématique, *Algèbre Commutative*, Chap. V, Hermann, Paris, 1964.

[4] N. Bourbaki, Eléments de Mathématique, *Algèbre Commutative*, Chap. VII, Hermann, Paris. 1965.

[5] N. Bourbaki, Eléments de Mathématique, *Topologie Générale*, Chap. I, Hermann, Paris. 1961.

[6] L. Gillmann and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960.

[7] M. Henriksen, *On the ideal structure of the ring of entire functions*, Pacific J. Math. **2** (1952), 179-184.

[8] M. Henriksen, *On the prime ideals of the ring of entire functions*, Pacific J. Math. **3** (1953), 711-720.

[9] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative rings*, Trans. Amer. Math. Soc. **115** (1965), 110-130.

[10] M. H. Stone, *The theory of representations for Boolean algebras*, Trans. AMS **40** (1936), 37-111.

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