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On Regularity of Boundary Points for Dirichlet Problems of the Equation $\Delta u=qu$ $(q\geqq 0)$

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Introduction

Regularity of boundary points for Dirichlet problems became an important subject as soon as the notion of generalized solutions was introduced by O. Perron and N. Wiener (1923). We now know various characterizations of regularity for the Laplace equation $\Delta u=0$ (see, e.g., [9; Chaps. 8 and 10]). Perron-Wiener's method has been applied also for Dirichlet problems of a more general elliptic partial differential equation $Lu=0$; and, more generally, with respect to an axiomatic harmonic structure (see M. Brelot [2], R.-M. Hervé [10], N. Boboc, C. Constantinescu and A. Cornea [1], etc.).

There are many investigations to determine under what conditions the regularity for the given equation $Lu=0$ coincides with that for $\Delta u=0$. Some of the recent results in this direction may be found in G. Stampacchia [19; §10] and R.-M. and M. Hervé [11; Théorème 3]. However, in these investigations, boundary points are assumed to be on the relative boundary of the domain which is contained in a larger domain where the equation is defined. For instance, consider the case where the domain Ω is a bounded one in the Euclidean space R^d $(d\geqq 2)$ and the equation is

$$(1)\qquad L_q u\equiv \Delta u-qu=0$$

with $q\geqq 0$. For this equation, Perron-Wiener's method can be applied whenever $q\in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ for some $p>d/2$. The results by Stampacchia and Hervés, however, only imply that if $q\in \boldsymbol{L}^p(\Omega)$ (or, $q\in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega')$ for some domain $\Omega'\supset\bar{\Omega}$), then the regularity of $\xi\in\partial\Omega$ for (1) is equivalent to that for $\Delta u=0$.

The main purpose of this paper is to investigate under what conditions on the function q, regularity of a boundary point for (1) follows from that for $\Delta u=0$, in case q does not necessarily belong to $\boldsymbol{L}^p(\Omega)$. We note that some results in this direction were obtained by M. Brelot ([3], [4] and [5]), but our results are more general and finer.

In the first chapter, we shall develop a general theory concerning regularity of *ideal* boundary points with respect to Brelot's axiomatic harmonic structures. Then, in the second chapter, we shall discuss regularity for the equation (1) on a Riemannian manifold Ω, where Δ is the Laplace-Beltrami operator, still considering an ideal boundary Γ of Ω. We shall say that $\xi \in \Gamma$ is q-regular if it is regular for the equation (1). Given two non-negative q_1 and q_2, we shall give a necessary and sufficient condition that a locally q_1-regular point on Γ is also locally q_2-regular (Corollary 1 to Theorem 2.4). The condition will be given in terms of the Green function for the equation $\Delta u = q_1 u$. This condition is then applied in Chapter 3 to the special case where Ω is a bounded domain in R^d and the boundary is the usual relative boundary. We shall obtain conditions on the growth of q near the boundary point ξ under which we can assure q-regularity of ξ (Theorems 3.2, 3.3, 3.4 and 3.5) and q-irregularity of ξ (Theorem 3.6).

CHAPTER I Regularity of ideal boundary points of a hamonic space

§1.1. Dirichlet problems on a harmonic space.

Let $(\Omega, \mathfrak{H}) = \{\mathcal{H}(\omega)\}_{\omega:\mathrm{open}\subset\Omega}$ be a harmonic space satisfying Axioms 1, 2 and 3 of M. Brelot [2]. By definition, Ω is a connected, locally connected, locally compact Hausdorff space. For an open set ω in Ω, the set of all superharmonic (resp. non-negative superharmonic) functions on ω with respect to $(\Omega, \mathfrak{H})$ is denoted by $\mathcal{S}_{\mathfrak{H}}(\omega)$ (resp. $\mathcal{S}_{\mathfrak{H}}^{+}(\omega)$). The set of all potentials with respect to $(\Omega, \mathfrak{H})$ is denoted by $\mathcal{P}_{\mathfrak{H}}$. We furthermore assume

Axiom 4. $1 \in \mathcal{S}_{\mathfrak{H}}^{+}(\Omega)$ *and* $\mathcal{P}_{\mathfrak{H}} \neq \{0\}$.

Let Ω^* be a compactification of Ω and let $\Gamma = \Omega^* - \Omega$. For an open set ω in Ω, let ω^* be the closure of ω in Ω^* and $\partial^*\omega$ be the set $\omega^* - \omega$. In particular, $\partial^*\Omega = \Gamma$. Given an extended real valued function σ on $\partial^*\omega$, we define

$$\mathcal{U}_{\sigma}^{\omega,\mathfrak{H}} = \left\{ v \in \mathcal{S}_{\mathfrak{H}}(\omega);\ \begin{matrix}\text{bounded below on } \omega, \\ \liminf_{x\to\xi,\, x\in\omega} v(x) \geqq \sigma(x) \text{ for all } \xi \in \partial^*\omega\end{matrix} \right\} \cup \{\infty\}.$$

and $\mathcal{L}_{\sigma}^{\omega,\mathfrak{H}} = -\mathcal{U}_{-\sigma}^{\omega,\mathfrak{H}}$. If $\inf \mathcal{U}_{\sigma}^{\omega,\mathfrak{H}} = \sup \mathcal{L}_{\sigma}^{\omega,\mathfrak{H}}$ and it belongs to $\mathcal{H}(\omega)$, then we say that σ is $\mathfrak{H}$-resolutive with respect to ω and denote this harmonic function by $H_{\sigma}^{\omega,\mathfrak{H}}$. In case $\omega = \Omega$, we simply say that σ is $\mathfrak{H}$-resolutive and write $H_{\sigma}^{\mathfrak{H}} = H_{\sigma}^{\Omega,\mathfrak{H}}$. The following lemma is easily obtained by standard arguments (cf. [2] and [7]):

LEMMA 1.1. (i) *If σ_1, σ_2 are $\mathfrak{H}$-resolutive with respect to ω and if λ_1, λ_2 are reals, then $\lambda_1\sigma_1 + \lambda_2\sigma_2$* (this function may take any value at a point where $+\infty - \infty$ or $-\infty + \infty$ occurs) *is $\mathfrak{H}$-resolutive with respect to ω and*

$$H_{\lambda_1\sigma_1+\lambda_2\sigma_2}^{\omega,\mathfrak{H}} = \lambda_1 H_{\sigma_1}^{\omega,\mathfrak{H}} + \lambda_2 H_{\sigma_2}^{\omega,\mathfrak{H}}.$$

(ii) *If σ_1, σ_2 are $\mathfrak{H}$-resolutive with respect to ω and $\sigma_1 \leqq \sigma_2$ on $\partial^*\omega$, then $H_{\sigma_1}^{\omega,\mathfrak{H}}$*

$\leqq H_{\sigma_2}^{\omega,\mathfrak{H}}$ *on* ω. *In particular, if* σ *is* $\mathfrak{H}$*-resolutive with respect to* ω *and* $\sigma \geqq 0$ *on* $\partial^*\omega$, *then* $H_\sigma^{\omega,\mathfrak{H}} \geqq 0$.

(iii) *Constant functions on* $\partial^*\omega$ *are* $\mathfrak{H}$*-resolutive with respect to* ω *and* $H_1^{\omega,\mathfrak{H}} \leqq 1$ *on* ω.

Also, we have (see, e.g., [2; Part IV, Theorem 10] or [13; pp. 286–287]):

Lemma 1.2. *Let* ω *and* ω' *be two non-empty open sets in* Ω *such that* $\omega \subset \omega'$. *Let* σ *be a* $\mathfrak{H}$*-resolutive function on* $\partial^*\omega'$ *and put*

$$\sigma_1 = \begin{cases} \sigma & \text{on } \partial^*\omega' \cap \partial^*\omega \\ H_\sigma^{\omega',\mathfrak{H}} & \text{on } \omega' \cap \partial^*\omega. \end{cases}$$

Then, σ_1 *is* $\mathfrak{H}$*-resolutive with respect to* ω *and*

$$H_{\sigma_1}^{\omega,\mathfrak{H}} = H_\sigma^{\omega',\mathfrak{H}}$$

on ω.

If every $\sigma \in \boldsymbol{C}(\Gamma)$ (=the set of all finite continuous functions on Γ) is $\mathfrak{H}$-resolutive, then Ω^* is called a $\mathfrak{H}$*-resolutive compactification of* Ω. By Corollary 3 and Theorem 8 of [1], we have

Lemma 1.3. *Let* Ω^* *be a* $\mathfrak{H}$*-resolutive compactification of* Ω *and let* ω *be a non-empty open subset of* Ω. *Then* ω^* *is a* $\mathfrak{H}$*-resolutive compactification of* ω *in the sense that every* $\sigma \in \boldsymbol{C}(\partial^*\omega)$ *is* $\mathfrak{H}$*-resolutive with respect to* ω.

§1.2. $\mathfrak{H}$-regular boundary points.

In this section, let Ω^* be a $\mathfrak{H}$-resolutive compactification. For a non-empty open set ω in Ω, a point $\xi \in \partial^*\omega$ is called $\mathfrak{H}$*-regular with respect to* ω (or, more precisely, with respect to (Ω^*, ω)) if

$$\lim_{x \to \xi,\, x \in \omega} H_\sigma^{\omega,\mathfrak{H}}(x) = \sigma(\xi)$$

for all $\sigma \in \boldsymbol{C}(\partial^*\omega)$. $\xi \in \Gamma$ is called simply $\mathfrak{H}$*-regular* if it is $\mathfrak{H}$-regular with respect to Ω. $\xi \in \Gamma$ is called *locally* $\mathfrak{H}$*-regular* if there is a fundamental system $\mathfrak{V}_\xi^*$ of open neighborhoods of ξ such that ξ is $\mathfrak{H}$-regular with respect to $V \cap \Omega$ for any $V \in \mathfrak{V}_\xi^*$.

Proposition 1.1. *Let* $\xi \in \Gamma$ *and let* V, V' *be two open neighborhoods of* ξ *such that* $V \subset V'$. *If* ξ *is* $\mathfrak{H}$*-regular with respect to* $V \cap \Omega$, *then it is* $\mathfrak{H}$*-regular with respect to* $V' \cap \Omega$. *Thus, if* $\xi \in \Gamma$ *is locally* $\mathfrak{H}$*-regular, then it is* $\mathfrak{H}$*-regular.*

Proof. Given $\sigma \in \boldsymbol{C}(\partial^*(V' \cap \Omega))$, let

$$\tau = \begin{cases} \sigma & \text{on } \partial^*(V' \cap \Omega) \cap \partial^*(V \cap \Omega) \\ H_\sigma^{V' \cap \Omega, \mathfrak{H}} & \text{on } V' \cap \Omega \cap \partial^*(V \cap \Omega). \end{cases}$$

Then, by Lemmas 1.2 and 1.3, τ is $\mathfrak{H}$-resolutive with respect to $V \cap \Omega$ and $H_\tau^{V \cap \Omega, \mathfrak{H}} = H_\sigma^{V' \cap \Omega, \mathfrak{H}}$ on $V \cap \Omega$. Obviously, τ is bounded on $\partial^*(V \cap \Omega)$ and continuous on $V \cap \partial^*(V \cap \Omega)$ $(\subset \partial^*(V' \cap \Omega) \cap \partial^*(V \cap \Omega))$. Then, we can find τ_1, $\tau_2 \in \boldsymbol{C}(\partial^*(V \cap \Omega))$ such that $\tau_1 \leqq \tau \leqq \tau_2$ on $\partial^*(V \cap \Omega)$ and $\tau_1(\xi) = \tau_2(\xi) = \tau(\xi)$. By assumption,

$$\lim_{x \to \xi, x \in V \cap \Omega} H_{\tau_1}^{V \cap \Omega, \mathfrak{H}}(x) = \lim_{x \to \xi, x \in V \cap \Omega} H_{\tau_2}^{V \cap \Omega, \mathfrak{H}}(x) = \tau(\xi).$$

Hence, using Lemma 1.1, (ii), we see that

$$\lim_{x \to \xi, x \in V \cap \Omega} H_\tau^{V \cap \Omega, \mathfrak{H}}(x) = \tau(\xi),$$

and hence

$$\lim_{x \to \xi, x \in V' \cap \Omega} H_\sigma^{V' \cap \Omega, \mathfrak{H}}(x) = \tau(\xi) = \sigma(\xi).$$

Thus, ξ is $\mathfrak{H}$-regular with respect to $V' \cap \Omega$.

COROLLARY. *If $\xi \in \Gamma$ is locally $\mathfrak{H}$-regular, then it is $\mathfrak{H}$-regular with respect to $V \cap \Omega$ for any open neighborhood V of ξ.*

REMARK 1.1. The converse of the above proposition is not always true. For example, if $\Omega = \{x \in R^d;\ 0 < |x| < 1\}$ $(d \geqq 2)$ and Ω^* is the one point compactification, then the point at infinity is regular (for the classical harmonic structure) but is not locally regular. However, it is known (see, e.g., [2; Part IV, Proposition 20]) that if Ω is a relatively compact domain in a larger harmonic space Ω' and Ω^* is the closure of Ω in Ω', then local $\mathfrak{H}$-regularity of $\xi \in \Gamma$ is equivalent to $\mathfrak{H}$-regularity.

REMARK 1.2. If 1 is harmonic and there exists a barrier at $\xi \in \Gamma$, then ξ is locally regular. Here a barrier means a positive superharmonic function w defined on $V_0 \cap \Omega$ for some neighborhood V_0 of ξ such that $\inf_{x \in (V_0 - V) \cap \Omega} w(x) > 0$ for any neighborhood V of ξ and $\lim_{x \to \xi} w(x) = 0$. Proof of the above fact may be carried out in the same way as in the classical case (see, e.g., [20; Theorem I.9] or the proof of [9; Lemma 8.20]). In particular, in case Ω is a Green space or a hyperbolic Riemann surface with the classical harmonic structure, a non-minimal Kuramochi boundary point is locally regular if and only if it is regular (see [6; Satz 17.25] in case Ω is a Riemann surface).

§1.3. Comparison of $\mathfrak{H}$-regularity for comparable harmonic structures.

We now consider two harmonic spaces $(\Omega, \mathfrak{H}_1)$ and $(\Omega, \mathfrak{H}_2)$ with the same base space Ω and assume that both satisfy Axioms 1~4. Let Ω^* be a compactification of Ω. As for resolutiveness, we have the following:

LEMMA 1.4. *Let ω be an open set in Ω. If there is a compact set K in ω such that*

(1.1) $$\mathcal{S}^+_{\mathfrak{H}_1}(\omega - K) \subset \mathcal{S}^+_{\mathfrak{H}_2}(\omega - K),$$

then any bounded function on $\partial^*\omega$ which is $\mathfrak{H}_1$-resolutive with respect to ω is $\mathfrak{H}_2$-resolutive with respect to ω.

The proof of this lemma is similar to the proofs of Theorems 1 and 2 of [15]. (Note that condition (1.1) may be replaced by a weaker condition C) in [15] on each component of ω.)

LEMMA 1.5. *Let ω be an open set in Ω. If $\mathcal{S}^+_{\mathfrak{H}_1}(\omega) \subset \mathcal{S}^+_{\mathfrak{H}_2}(\omega)$ and σ is a non-negative function on $\partial^*\omega$ which is $\mathfrak{H}_1$-resolutive with respect to ω, then*

$$H^{\omega, \mathfrak{H}_1}_{\sigma} \geqq H^{\omega, \mathfrak{H}_2}_{\sigma}.$$

This lemma is easily verified by the definition. Observe that σ is $\mathfrak{H}_2$-resolutive with respect to ω by the previous lemma.

In the rest of this section, let Ω^* be a compactification which is both $\mathfrak{H}_1$- and $\mathfrak{H}_2$-resolutive.

THEOREM 1.1. *Let ω be an open set in Ω and $\xi_0 \in \partial^*\omega$. If $\mathcal{S}^+_{\mathfrak{H}_1}(\omega) \subset \mathcal{S}^+_{\mathfrak{H}_2}(\omega)$ and ξ_0 is $\mathfrak{H}_2$-regular with respect to ω, then ξ_0 is $\mathfrak{H}_1$-regular with respect to ω.*

PROOF. We shall prove

(1.2) $$\lim_{x \to \xi_0, x \in \omega} H^{\omega, \mathfrak{H}_1}_{\sigma}(x) = \sigma(\xi_0)$$

for all $\sigma \in \boldsymbol{C}(\partial^*\omega)$. If $\sigma \geqq 0$ on $\partial^*\omega$ and $\sigma(\xi_0) = \sup_{\xi \in \partial^*\omega} \sigma(\xi)$, then, by Lemmas 1.1 and 1.5, we have

$$0 \leqq H^{\omega, \mathfrak{H}_2}_{\sigma} \leqq H^{\omega, \mathfrak{H}_1}_{\sigma} \leqq \sigma(\xi_0).$$

Since $H^{\omega, \mathfrak{H}_2}_{\sigma}(x) \to \sigma(\xi_0)$ $(x \to \xi_0)$ by assumption, (1.2) holds for such σ. By virtue of Lemma 1.1, (i), if (1.2) holds for σ_1 and σ_2, then it also holds for $\sigma_1 - \sigma_2$. Therefore, by the above result, (1.2) holds for σ for which $\sigma(\xi_0) = \sup_{\xi \in \partial^*\omega} \sigma(\xi)$ or $\sigma(\xi_0) = \inf_{\xi \in \partial^*\omega} \sigma(\xi)$. Finally, an arbitrary $\sigma \in \boldsymbol{C}(\partial^*\omega)$ can be written as

$$\sigma = \max(\sigma - \sigma(\xi_0), 0) + \min(\sigma - \sigma(\xi_0), 0) + \sigma(\xi_0).$$

Since each term of the right hand side has the above property, (1.2) holds for σ.

COROLLARY. *If there is an open neighborhood V of $\xi_0 \in \Gamma$ such that*

$\mathcal{S}^+_{\mathfrak{H}_1}(V\cap\Omega)\subset\mathcal{S}^+_{\mathfrak{H}_2}(V\cap\Omega)$ and if ξ_0 is locally $\mathfrak{H}_2$-regular, then ξ_0 is locally $\mathfrak{H}_1$-regular.

The converse of Theorem 1.1 is not true in general. As for the converse direction, we have the following theorem:

THEOREM 1.2. *Let ω be an open set in Ω and suppose $\mathcal{S}^+_{\mathfrak{H}_1}(\omega)\subset\mathcal{S}^+_{\mathfrak{H}_2}(\omega)$. If $\xi_0\in\partial^*\omega$ is $\mathfrak{H}_1$-regular with respect to ω and*

$$\lim_{x\to\xi_0,\, x\in\omega} H^{\omega,\mathfrak{H}_2}_{\sigma_1}(x)=1$$

for some $\sigma_1\in \boldsymbol{C}(\partial^*\omega)$ such that $\sigma_1(\xi_0)=1$, then ξ_0 is $\mathfrak{H}_2$-regular with respect to ω.

PROOF. If $\sigma\geqq 0$ on $\partial^*\omega$ and $\sigma(\xi_0)=0$, then

$$0\leqq H^{\omega,\mathfrak{H}_2}_{\sigma}\leqq H^{\omega,\mathfrak{H}_1}_{\sigma}$$

by Lemma 1.5. Hence, $\lim_{x\to\xi_0}H^{\omega,\mathfrak{H}_1}_{\sigma}(x)=0$ implies $\lim_{x\to\xi_0}H^{\omega,\mathfrak{H}_2}_{\sigma}(x)=0$. If $\sigma\in\boldsymbol{C}(\partial^*\omega)$ and $\sigma(\xi_0)=0$, then, by considering σ^+ and σ^-, the above result implies $\lim_{x\to\xi_0} H^{\omega,\mathfrak{H}_2}_{\sigma}(x)=0$. For an arbitrary $\sigma\in\boldsymbol{C}(\partial^*\omega)$, we write it in the form

$$\sigma=\{\sigma-\sigma(\xi_0)\sigma_1\}+\sigma(\xi_0)\sigma_1.$$

Then

$$\lim_{x\to\xi_0} H^{\omega,\mathfrak{H}_2}_{\sigma}(x)=\lim_{x\to\xi_0}\{H^{\omega,\mathfrak{H}_2}_{\sigma-\sigma(\xi_0)\sigma_1}(x)+\sigma(\xi_0)H^{\omega,\mathfrak{H}_2}_{\sigma_1}(x)\}=\sigma(\xi_0)$$

by the above results and the assumption on σ_1.

Finally we prove:

PROPOSITION 1.2. *Suppose there is an open neighborhood V_0 of $\xi_0\in\Gamma$ such that $\mathcal{S}^+_{\mathfrak{H}_1}(V_0\cap\Omega)\subset\mathcal{S}^+_{\mathfrak{H}_2}(V_0\cap\Omega)$. If ξ_0 is locally $\mathfrak{H}_1$-regular and $\mathfrak{H}_2$-regular, then it is locally $\mathfrak{H}_2$-regular.*

PROOF. Let V be any open neighborhood of ξ_0 contained in V_0. By assumption, ξ_0 is $\mathfrak{H}_1$-regular with respect to $V\cap\Omega$. On the other hand, since ξ_0 is $\mathfrak{H}_2$-regular,

$$\lim_{x\to\xi_0} H^{\mathfrak{H}_2}_{1}(x)=1. \tag{1.3}$$

Since $H^{\mathfrak{H}_2}_1(x)\leqq 1$, it follows from Lemma 1.2 that $H^{\mathfrak{H}_2}_1(x)\leqq H^{V\cap\Omega,\mathfrak{H}_2}_1(x)$ for $x\in V\cap\Omega$. Since $H^{V\cap\Omega,\mathfrak{H}_2}_1\leqq 1$, (1.3) implies that

$$\lim_{x\to\xi_0} H^{V\cap\Omega,\mathfrak{H}_2}_1(x)=1.$$

Hence, by Theorem 1.2, ξ_0 is $\mathfrak{H}_2$-regular with respect to $V\cap\Omega$.

CHAPTER II q-regularity of ideal boundary points of a differentiable manifold

§2.1. Harmonic structure $\mathfrak{H}_q$ on a differentiable manifold.

In this chapter, let Ω be a connected non-compact C^1-manifold of dimension $d \geqq 2$ and let (g_{ij}) be a symmetric covariant tensor on Ω satisfying the following condition (G) (cf. [16])

(G): On each relatively compact coordinate neighborhood U in Ω, each g_{ij} is a bounded measurable function on U and there is $\lambda > 0$ (which depends on U and the coordinate) such that

$$\lambda \sum_{i=1}^{d} \xi_i^2 \leqq \sum_{i,j=1}^{d} g_{ij}(x) \xi_i \xi_j$$

for all $x \in U$ and real numbers $\xi_1, \cdots, \xi_d$.

Let $dx = \sqrt{g}\, dx_1 \cdots dx_d$ be the corresponding volume element on Ω, where $g = \det(g_{ij})$. The Laplace-Beltrami operator Δ determined by (g_{ij}) is formally given by

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{d} \sqrt{g}\, g^{ij} \frac{\partial u}{\partial x_i} \right)$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

We shall use the same notation as in [16] for the following function spaces on an open set ω in Ω: $\boldsymbol{L}^p_{\mathrm{loc}}(\omega)$, $\boldsymbol{C}^1_0(\omega)$, $\boldsymbol{D}(\omega)$, $\boldsymbol{D}_0(\omega)$ and $\boldsymbol{D}_{\mathrm{loc}}(\omega)$. In this chapter, we shall always assume

$$1 \notin \boldsymbol{D}_0(\Omega), \tag{2.1}$$

i.e., Ω is Δ-semi-adapted in the sense of [16]. We shall also use the notation $D_\omega[u, v]$ to denote the mutual Dirichlet integral of $u, v \in \boldsymbol{D}(\omega)$ over an open set ω (see [16]).

Now, let q be a non-negative measurable function on Ω belonging to $\boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ for some $p > d/2$. A continuous function u on an open set ω is called a solution of $L_q u \equiv \Delta u - qu = 0$ on ω, or *q-harmonic* on ω (with respect to (g_{ij})) if $u \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$ and

$$D_\omega[u, \phi] + \int_\omega qu\phi\, dx = 0$$

for all $\phi \in \boldsymbol{C}^1_0(\omega)$. (Note that q-harmonic functions are called L_q-harmonic in [16].) Let $\mathcal{H}_q(\omega) = \{u;\ q\text{-harmonic on } \omega\}$. Then, it is known that $\mathfrak{H}_q = \{\mathcal{H}_q(\omega)\}_{\omega:\mathrm{open}}$ defines a harmonic structure on Ω satisfying Axioms 1~3 (see [16; Theorem 2.1], [11; Théorème 1] or [8; Theorem 3.1]). Also, by virtue of the assumptions $q \geqq 0$ and (2.1), we see that Axiom 4 is also satisfied by this harmonic structure (see [16; Corollary to Proposition 2.5 and Corollary 1 to Proposition 3.2]; also [11; §3]). For simplicity, we shall use the following notation: $\mathcal{S}_q^+(\omega)$ for $\mathcal{S}_{\mathfrak{H}_q}^+(\omega)$, $\mathcal{P}_q$ for $\mathcal{P}_{\mathfrak{H}_q}$ and $\boldsymbol{D}_0^q(\omega)$ for $\boldsymbol{D}_{L_q,0}(\omega)$ (see [16] for the space $\boldsymbol{D}_{L,0}(\omega)$).

LEMMA 2.1. (cf. [8; Theorem 4.2]) *Let ω be an open set in Ω. If $q_1 \leqq q_2$ on ω, then*

$$\mathcal{S}_{q_1}^+(\omega) \subset \mathcal{S}_{q_2}^+(\omega).$$

PROOF. It is enough to show that $\mathcal{H}_{q_1}^+(\omega') \subset \mathcal{S}_{q_2}^+(\omega')$ for any open set $\omega' \subset \omega$ (cf. [12; Proposition 7.2]). If $u \in \mathcal{H}_{q_1}^+(\omega')$, then, for any $\phi \in \boldsymbol{C}_0^1(\omega')$ with $\phi \geqq 0$,

$$\begin{aligned} 0 &= D_{\omega'}[u, \phi] + \int_{\omega'} q_1 u \phi \, dx \\ &= D_{\omega'}[u, \phi] + \int_{\omega'} q_2 u \phi \, dx + \int_{\omega'} (q_1 - q_2) u \phi \, dx \\ &\leqq D_{\omega'}[u, \phi] + \int_{\omega'} q_2 u \phi \, dx. \end{aligned}$$

Hence, u is an L_{q_2}-supersolution on ω' (cf. [16; 2.1]). Since u is continuous, Proposition 2.5 of [16] implies that u is L_{q_2}-superharmonic on ω'.

§2.2. L_q-Green functions.

Since Ω is L_q-semiadapted ([16; 3.1]), there is the L_q-Green function $g_y^{L_q}(x)$ in the notation in [16; 3.4], which will be denoted by $G^q(x, y)$ in this paper. Since $L_q^* = L_q$, $G^q(x, y) = G^q(y, x)$ in our case. By a local study of L-Green function in [11; §9], we see that, given a relatively compact coordinate neighborhood U in Ω, there are constants $k_1, k_2 > 0$ such that

$$(2.2) \qquad \begin{aligned} &\frac{k_1}{|x-y|^{d-2}} \leqq G^q(x, y) \leqq \frac{k_2}{|x-y|^{d-2}} \quad \text{if } d \geqq 3; \\ &k_1 \log \frac{1}{|x-y|} \leqq G^q(x, y) \leqq k_2 \log \frac{1}{|x-y|} \quad \text{if } d = 2 \end{aligned}$$

for all $x, y \in U$. This implies that, for a fixed $x \in \Omega$, $G^q(x, \cdot) \in \boldsymbol{L}_{\mathrm{loc}}^{p'}(\Omega)$ for $p' < d/(d-2)$. Thus, by Hölder's inequality and continuity of the mapping $(x, y) \to G^q(x, y)$ $(x \neq y)$, we have

LEMMA 2.2. *For any $f \in \boldsymbol{L}_{\mathrm{loc}}^p(\Omega)$ $(p > d/2)$ and for any relatively compact open set ω in Ω,*

$$\int_\omega G^q(x,\ y)|f(y)|\,dy < \infty$$

for all $x \in \Omega$ *and* $v(x) = \int_\omega G^q(x,\ y) f(y)\,dy$ *is a continuous function on* Ω.

It also follows that, for $f \in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ $(p > d/2)$ with $f \geqq 0$, $\int_\Omega G^q(x,\ y) f(y)\,dy$ is finite everywhere on Ω if it is finite at one point.

(2.2) also implies Axiom D for $\mathfrak{H}_q$ (cf. [11; p. 338]); in particular, we have the following maximum principle of Frostman's type:

Lemma 2.3. *If* $f \in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ $(p > d/2)$ *and* $f \geqq 0$ *on* Ω, *then*

$$\sup_{x \in \Omega} \int_\Omega G^q(x,\ y) f(y)\,dy = \sup_{x \in S(f)} \int_\Omega G^q(x,\ y) f(y)\,dy,$$

where $S(f)$ *is the support of* f.

Now we prove

Lemma 2.4. *Let* $f \in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ $(p > d/2)$, $f \geqq 0$ *on* Ω *and suppose* $v(x) \equiv \int_\Omega G^q(x,\ y) f(y)\,dy$ *is finite* (at one point, and hence at every point). *Then* v *is a continuous function on* Ω *belonging to* $\mathcal{P}_q \cap \boldsymbol{D}_{\mathrm{loc}}(\Omega)$ *and*

$$D_\Omega[v,\ \phi] + \int_\Omega qv\phi\ dx = \int_\Omega f\phi\ dx \tag{2.3}$$

for all $\phi \in \boldsymbol{C}^1_0(\Omega)$.

Proof. By a general theory (cf. e.g., [10; Corollary to Proposition 17. 1]), we see that $v \in \mathcal{P}_q$.

First suppose $S(f)$ is compact. By Lemma 2.2, v is continuous on Ω. By Sobolev's lemma (see e.g., [19; Lemma 1.3]), we see that the mapping $\psi \to \int f\psi\,dx$ is continuous on $\boldsymbol{D}_0(\Omega)$, and hence on $\boldsymbol{D}^q_0(\Omega)$. Hence, there is $v_1 \in \boldsymbol{D}^q_0(\Omega)$ such that

$$D_\Omega[v_1,\ \psi] + \int_\Omega qv_1\psi\ dx = \int_\Omega f\psi\ dx \tag{2.4}$$

for all $\psi \in \boldsymbol{D}^q_0(\Omega)$. Let $\phi \in \boldsymbol{C}^1_0(\Omega)$ and consider $G^q(\phi) \equiv G^{L_q, \Omega}(\phi)$ in the notation of [16]. Then, by definition, $G^q(\phi) \in \boldsymbol{D}^q_0(\Omega)$ and

$$D_\Omega[v_1,\ G^q(\phi)] + \int_\Omega qv_1 G^q(\phi)\ dx = \int_\Omega v_1\phi\ dx.$$

Thus, taking $\psi = G^q(\phi)$ in (2.4), we have

$$\int_\Omega v_1\phi\ dx = \int_\Omega fG^q(\phi)\ dx \tag{2.5}$$

for all $\phi \in \boldsymbol{C}^1_0(\Omega)$. On the other hand, by Theorem 3.1 of [14],

$$G^q(\phi)(y)=\int_\Omega G^q(x, y)\phi(x)\,dx.$$

Hence, (2.5) implies

$$\int_\Omega v_1(x)\phi(x)dx=\iint_{\Omega\times\Omega} G^q(x, y)\phi(x)f(y)dy\,dx=\int_\Omega v(x)\phi(x)\,dx$$

for any $\phi\in \boldsymbol{C}_0^1(\Omega)$. This implies that $v=v_1$ almost everywhere on Ω. Hence, $v\in \boldsymbol{D}_0^q(\Omega)\subset \boldsymbol{D}_{\mathrm{loc}}(\Omega)$ and (2.4) shows that v satisfies (2.3).

Next let f be arbitrary ($\geqq 0$). Let ω be a relatively compact open set in Ω and let χ_ω be the characteristic function of ω. Then $v_\omega(x)=\int_\Omega G^q(x, y)f(y)\chi_\omega(y)dy$ belongs to $\boldsymbol{D}_0^q(\Omega)$ by the above result. Obviously, $v-v_\omega$ is q-harmonic on ω. Hence, $v\in \boldsymbol{D}_{\mathrm{loc}}(\omega)$. Since ω is arbitrary, we have $v\in \boldsymbol{D}_{\mathrm{loc}}(\Omega)$. Given $\phi\in \boldsymbol{C}_0^1(\Omega)$, choose ω such that $\omega\supset S(\phi)$. Since $v-v_\omega$ is q-harmonic on ω,

$$D_\Omega[v, \phi]+\int_\Omega qv\phi\,dx=D_\Omega[v_\omega, \phi]+\int_\Omega qv_\omega\phi\,dx. \tag{2.6}$$

By the above result for compact $S(f)$, we see that the right hand side of (2.6) is equal to $\int_\Omega f\chi_\omega\phi\,dx=\int_\Omega f\phi\,dx$. Hence, we obtain (2.3).

LEMMA 2.5. *If $q_1\leqq q_2$ on Ω, then $G^{q_1}(x,y)\geqq G^{q_2}(x, y)$.*

PROOF. Let $f\in \boldsymbol{C}_0^1(\Omega)$ and $f\geqq 0$. By the previous lemma, both $v_1(x)=\int_\Omega G^{q_1}(x, y)f(y)\,dy$ and $v_2(x)=\int_\Omega G^{q_2}(x, y)f(y)\,dy$ belong to $\boldsymbol{D}_{\mathrm{loc}}(\Omega)$ and

$$D_\Omega[v_i, \phi]+\int_\Omega q_iv_i\phi\,dx=\int_\Omega f\phi\,dx$$

for any $\phi\in \boldsymbol{C}_0^1(\Omega)$, $i=1, 2$. Hence, if $\phi\geqq 0$, then

$$D_\Omega[v_1-v_2, \phi]+\int_\Omega q_2(v_1-v_2)\phi\,dx=\int_\Omega(q_2-q_1)v_1\phi\,dx\geqq 0.$$

This means that v_1-v_2 is an L_{q_2}-supersolution on Ω. Since $v_2\in D_0^{q_2}(\Omega)$ (see the proof of the previous lemma), it follows from Proposition 3.1 of [16] that $v_1\geqq v_2$. Since $f(\in C_0^1(\Omega), \geqq 0)$ is arbitrary, we conclude that $G^{q_1}(x, y)\geqq G^{q_2}(x, y)$.

§2.3. Dirichlet solution H_σ^q and q-regular ideal boundary points.

Now we consider a $\mathfrak{H}_0$-resolutive compactification Ω^* of Ω. By Lemmas 1.4 and 2.1, it is also $\mathfrak{H}_q$-resolutive for any $q\geqq 0$. For a $\mathfrak{H}_q$-resolutive function σ on $\partial^*\omega$ (resp. on $\Gamma=\Omega^*-\Omega$), the Dirichlet solution $H_\sigma^{\omega,\mathfrak{H}_q}$(resp. $H_\sigma^{\mathfrak{H}_q}$) will be denoted by $H_\sigma^{\omega,q}$ (resp. H_σ^q), for simplicity.

We shall need the following result to prove our main theorem in the next section:

PROPOSITION 2.1. *Let $q_1 \leqq q_2$ on Ω and let σ be a bounded non-negative $\mathfrak{H}_{q_1}$-resolutive function on Γ. Then*

$$(2.7) \qquad H_\sigma^{q_1}(x) \geqq H_\sigma^{q_2}(x) + \int_\Omega G^{q_1}(x, y)\{q_2(y) - q_1(y)\} H_\sigma^{q_2}(y)\, dy$$

for all $x \in \Omega$. If, furthermore, $\int_\Omega G^{q_1}(x, y)\{q_2(y) - q_1(y)\} H_1^{q_1}(y)\, dy < \infty$ for some $x \in \Omega$, then

$$(2.8) \quad H_\sigma^{q_1}(x) = H_\sigma^{q_2}(x) + \int_\Omega G^{q_1}(x, y)\{q_2(y) - q_1(y)\} H_\sigma^{q_2}(y)\, dy$$

for all $x \in \Omega$.

The proof of this proposition will be given in the Appendix at the end of this chapter. Remark that, in case Ω is a locally Euclidean space and q_1 and q_2 are locally Hölder continuous, this is an easy consequence of the results in [14; §3.4 and §3.5]. Also, inequality (2.7) is given in [3; Lemma 4] for a special case.

$\mathfrak{H}_q$-regular (resp. locally $\mathfrak{H}_q$-regular) boundary points will be simply called *q-regular* (resp. *locally q-regular*). By virtue of Lemma 2.1, the results in Chapter I can be stated as follows:

THEOREM 2.1. *Let ω be an open set in Ω and $\xi \in \partial^*\omega$. If $q_1 \leqq q_2$ on ω and ξ is q_2-regular with respect to ω, then ξ is q_1-regular with respect to ω.*

COROLLARY. *If there is a neighborhood V of $\xi \in \Gamma$ such that $q_1 \leqq q_2$ on $V \cap \Omega$ and if ξ is locally q_2-regular, then ξ is locally q_1-regular.*

THEOREM 2.2. (cf. [4; n° 12]) *Let ω be an open set in Ω and suppose $q_1 \leqq q_2$ on ω. If $\xi \in \partial^*\omega$ is q_1-regular with respect to ω and*

$$\lim_{x \to \xi, x \in \omega} H_{\sigma_1}^{\omega; q_2}(x) = 1$$

for some $\sigma_1 \in \boldsymbol{C}(\partial^*\omega)$ such that $\sigma_1(\xi) = 1$, then ξ is q_2-regular with respect to ω.

PROPOSITION 2.2. *Suppose there is a neighborhood V of $\xi \in \Gamma$ such that $q_1 \leqq q_2$ on $V \cap \Omega$. If ξ is locally q_1-regular and q_2-regular, then it is locally q_2-regular.*

§2.4. Criterions for q-regularity.

Let $\xi_0 \in \Gamma = \Omega^* - \Omega$. The filter of all neighborhoods of ξ_0 will be denoted by $\mathfrak{B}_{\xi_0}$. First, we prepare

LEMMA 2.6. *Let f be a non-negative function in $\boldsymbol{L}_{\mathrm{loc}}^{p}(\Omega)$ with $p>d/2$. If there is $V_0 \in \mathfrak{V}_{\xi_0}$ such that*

$$\lim_{x\to\xi_0}\int_{V_0\cap\Omega} G^q(x, y)f(y)dy=0, \tag{2.9}$$

then

$$\lim_{V\in\mathfrak{V}_{\xi_0}} \sup_{x\in\Omega}\int_{V\cap\Omega} G^q(x, y)f(y)dy=0.$$

PROOF. By Lemma 2.3,

$$\sup_{x\in\Omega}\int_{V\cap\Omega} G^q(x, y)f(y)\,dy=\sup_{x\in\Omega\cap V^*}\int_{V\cap\Omega} G^q(x, y)f(y)dy.$$

Hence, for any $V\in\mathfrak{V}_{\xi_0}$ such that $V\subset V_0$,

$$\begin{aligned}0&\leqq\sup_{x\in\Omega}\int_{V\cap\Omega} G^q(x, y)f(y)dy\\ &=\sup_{x\in\Omega\cap V^*}\int_{V\cap\Omega} G^q(x, y)f(y)\,dy\\ &\leqq\sup_{x\in\Omega\cap V^*}\int_{V_0\cap\Omega} G^q(x, y)f(y)\,dy.\end{aligned}$$

The last term tends to 0 along $V\in\mathfrak{V}_{\xi_0}$ by virtue of the assumption (2.9).

In case $q=0$, the corresponding Green function $G^0(x, y)$ will be denoted by $G(x, y)$. We have

THEOREM 2.3. *If ξ_0 is q-regular, then*

$$\lim_{x\to\xi_0}\int_{V_0\cap\Omega} G(x, y)q(y)\,dy=0 \tag{2.10}$$

for some $V_0\in\mathfrak{V}_{\xi_0}$ and

$$\lim_{V\in\mathfrak{V}_{\xi_0}} \sup_{x\in\Omega}\int_{V\cap\Omega} G(x, y)q(y)\,dy=0.$$

PROOF. By virtue of the previous lemma, it is enough to prove (2.10). Since $\lim_{x\to\xi_0} H_1^q(x)=1$, there is $V_0\in\mathfrak{V}_{\xi_0}$ such that $H_1^q(y)\geqq 1/2$ for $y\in V_0\cap\Omega$. On the other hand, by Proposition 2.1 (applied for $q_1=0$ and $q_2=q$),

$$1-H_1^q(x)\geqq\int_{\Omega} G(x, y)q(y)H_1^q(y)\,dy.$$

Hence,

$$0\leqq\int_{V_0\cap\Omega} G(x, y)q(y)\,dy$$

$$\leqq 2\int_{V_0\cap\Omega} G(x, y)q(y)H_1^q(y)\,dy$$

$$\leqq 2\int_{\Omega} G(x, y)q(y)H_1^q(y)\,dy \leqq 2(1-H_1^q(x)).$$

The last term tends to 0 as $x\to\xi_0$. Hence, we have (2.10).

COROLLARY 1. (cf. [5; n° 4]) *If every* $\xi\in\Gamma$ *is q-regular, then* $\int_\Omega G(x, y)q(y)dy<\infty$.

COROLLARY 2. *if* ξ_0 *is q-regular, then, for any* $q'\in L_{\mathrm{loc}}^p(\Omega)(p>d/2)$ *such that* $q'\geqq 0$ *on* Ω *and* $q'\leqq q$ *on a neighborhood of* ξ_0,

$$\lim_{x\to\xi_0}\int_{V_0\cap\Omega} G^{q'}(x, y)\{q(y)-q'(y)\}\,dy=0$$

for some $V_0\in\mathfrak{V}_{\xi_0}$ *and*

$$\lim_{V\in\mathfrak{V}_{\xi_0}}\sup_{x\in\Omega}\int_{V\cap\Omega} G^{q'}(x, y)\{q(y)-q'(y)\}\,dy=0.$$

PROOF. By Lemma 2.5, $G^{q'}(x, y)\leqq G(x, y)$. Thus, this corollary immediately follows from the theorem.

As for the converse direction, we have the following result, which is our main theorem in this chapter.

THEOREM 2.4. *Suppose* $q_1\leqq q_2$ *on* $V_0\cap\Omega$ *for some* $V_0\in\mathfrak{V}_{\xi_0}$. *If* ξ_0 *is locally* q_1*-regular and if either*

$$\lim_{x\to\xi_0}\int_{V_0'\cap\Omega} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy=0$$

for some $V_0'\in\mathfrak{V}_{\xi_0}$ *such that* $V_0'\subset V_0$, *or*

$$\lim_{V\in\mathfrak{V}_{\xi_0}}\sup_{x\in\Omega}\int_{V\cap\Omega} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy=0, \tag{2.11}$$

then ξ_0 *is locally* q_2*-regular (and hence* q_2*-regular).*

PROOF. By virtue of Lemma 2.6, it is enough to prove the theorem under the assumption (2.11). First remark that, by (2.11), there is $V_1\in\mathfrak{V}_{\xi_0}$ such that $V_1\subset V_0$ and

$$\sup_{x\in\Omega}\int_{V_1\cap\Omega} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy<\infty. \tag{2.12}$$

For any $\varepsilon>0$ ($\varepsilon<1$), choose $V_\varepsilon\in\mathfrak{V}_{\xi_0}$ such that $V_\varepsilon^*\subset V_1$ and

$$\sup_{x\in\Omega}\int_{V_\varepsilon\cap\Omega} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy<\varepsilon. \tag{2.13}$$

Let $f_1=\chi_{V_1\cap\Omega}$ and $f_\varepsilon=\chi_{V_\varepsilon\cap\Omega}$(the characteristic functions of $V_1\cap\Omega$ and $V_\varepsilon\cap\Omega$). By Proposition 2.1 and (2.12), we have

$$(2.14)\qquad H_1^{q_1}(x)=H_1^{q_1+f_1(q_2-q_1)}(x)$$
$$+\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}f_1(y)H_1^{q_1+f_1(q_2-q_1)}(y)\,dy.$$

Now,

$$\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}f_1(y)H_1^{q_1+f_1(q_2-q_1)}(y)\,dy$$
$$=\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\{f_1(y)-f_\varepsilon(y)\}H_1^{q_1+f_1(q_2-q_1)}(y)\,dy$$
$$+\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}f_\varepsilon(y)H_1^{q_1+f_1(q_2-q_1)}(y)\,dy$$
$$\leqq\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\{f_1(y)-f_\varepsilon(y)\}H_1^{q_1+(f_1-f_\varepsilon)(q_2-q_1)}(y)\,dy$$
$$+\int_{\Omega\cap V_\varepsilon} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy.$$

By (2.7) and (2.13), the last expression is less than

$$H_1^{q_1}(x)-H_1^{q_1+(f_1-f_\varepsilon)(q_2-q_1)}(x)+\varepsilon.$$

Hence, (2.14) implies

$$(2.15)\qquad H_1^{q_1+f_1(q_2-q_1)}(x)\geqq H_1^{q_1+(f_1-f_\varepsilon)(q_2-q_1)}(x)-\varepsilon$$

for all $x\in\Omega$. Since $q_1+(f_1-f_\varepsilon)(q_2-q_1)=q_1$ on $V_\varepsilon\cap\Omega$ and ξ_0 is locally q_1-regualr, we see that ξ_0 is locally $q_1+(f_1-f_\varepsilon)(q_2-q_1)$-regular, and hence it is $q_1+(f_1-f_\varepsilon)(q_2-q_1)$-regular. Hence,

$$\lim_{x\to\xi_0} H_1^{q_1+(f_1-f_\varepsilon)(q_2-q_1)}(x)=1.$$

Hence, by (2.15),

$$\liminf_{x\to\xi_0} H_1^{q_1+f_1(q_2-q_1)}(x)\geqq 1-\varepsilon.$$

Since $\varepsilon>0$ is arbitrary and $H_1^{q_1+f_1(q_2-q_1)}\leqq 1$, we conclude that

$$\lim_{x\to\xi_0} H_1^{q_1+f_1(q_2-q_1)}(x)=1.$$

Thus, by Theorem 2.2, ξ_0 is $q_1+f_1(q_2-q_1)$-regular. Then, by Proposition 2.2, ξ_0 is locally $q_1+f_1(q_2-q_1)$-regular. Since $q_1+f_1(q_2-q_1)=q_2$ on $V_1\cap\Omega$, it follows that ξ_0 is locally q_2-regular.

Combining the above theorem with Corollary 2 to Theorem 2.3, we obtain

COROLLARY 1. *Suppose $q_1\leqq q_2$ on $V_0\cap\Omega$ for some $V_0\in\mathfrak{V}_{\xi_0}$ and suppose ξ_0 is locally q_1-regular. Then the following four assertions are equivalent:*

(a) *ξ_0 is locally q_2-regular;*

(b) *ξ_0 is q_2-regular;*

(c) $\lim_{x\to\xi_0}\int_{V_0'\cap\Omega} G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy=0$ *for some V_0' such that $V_0'\subset V_0$;*

(d) $\lim_{V\in\mathfrak{V}_{\xi_0}}\sup_{x\in\Omega}\int_{V\cap\Omega} G^{q_1}(x, y)\,\{q_2(y)-q_1(y)\}\,dy=0.$

COROLLARY 2. *If ξ_0 is locally q-regular and if $q'\leqq\lambda q$ on $V_0\cap\Omega$ for some $V_0\in\mathfrak{V}_{\xi_0}$ and $\lambda>0$, then ξ_0 is locally q'-regular.*

PROOF. By Theorem 2.3, $\lim_{x\to\xi_0}\int_{V_0'\cap\Omega} G(x, y)\,q(y)\,dy=0$ for some $V_0'\in\mathfrak{V}_{\xi_0}$ contained in V_0. Then, by the assumption $q'\leqq\lambda q$ on $V_0\cap\Omega$,

$$\lim_{x\to\xi_0}\int_{V_0'\cap\Omega} G(x, y)q'(y)\,dy=0.$$

Since ξ_0 is locally 0-regular (Corollary to Theorem 2.1), it is locally q'-regular by Theorem 2.4.

By a similar method, we also have

COROLLARY 3. *If ξ_0 is locally q_1-regular as well as locally q_2-regular, then it is locally $(\lambda_1 q_1+\lambda_2 q_2)$-regular for any nonnegative numbers λ_1 and λ_2.*

COROLLARY 4. *Let q_1, $q_2(\geqq 0)$ be given. If ξ_0 is locally q_1-regular and if either*

$$\lim_{x\to\xi_0}\int_{V_0\cap\Omega} G^{q_1}(x, y)\max\{q_2(y)-q_1(y), 0\}\,dy=0$$

for some $V_0\in\mathfrak{V}_{\xi_0}$, or

$$\lim_{V\in\mathfrak{V}_{\xi_0}}\sup_{x\in\Omega}\int_{V\cap\Omega} G^{q_1}(x, y)\max\{q_2(y)-q_1(y), 0\}\,dy=0,$$

then ξ_0 is locally q_2-regular.

PROOF. Since $\max\{q_2(y)-q_1(y), 0\}=\max\{q_2(y), q_1(y)\}-q_1(y)$, Theorem 2.4 implies that ξ_0 is locally $\max(q_1, q_2)$-regular. Hence, by the corollary to Theorem 2.1, ξ_0 is locally q_2-regular.

§2.5. An application to Dirichlet problems of non-homogeneous equations.

Let Ω^* be a $\mathfrak{H}_0$-resolutive compactification again and we consider a boundary value problem

$$\text{(2.16)} \qquad \begin{cases} L_q u \equiv \Delta u - qu = f \text{ on } \Omega \\ u = 0 \text{ on } \Gamma. \end{cases}$$

By a solution of (2.16), we mean a continuous function u on Ω^* such that $u \mid \Omega \in \boldsymbol{D}_{\text{loc}}(\Omega)$,

$$D_\Omega[u, \phi] + \int_\Omega qu\phi \, dx = -\int_\Omega f\phi \, dx$$

for all $\phi \in \boldsymbol{C}_0^1(\Omega)$ and $u=0$ on Γ.

As an application of the results in the previous section, we have the following theorem (cf. [4]):

THEOREM 2.5. *Suppose* $f \in \boldsymbol{L}_{\text{loc}}^p(\Omega)$ *with* $p > d/2$.

(a) *If every* $\xi \in \Gamma$ *is locally* q*-regular and* $(q+|f|)$*-regular (or, equivalently, if every* $\xi \in \Gamma$ *is locally* $(q+|f|)$*-regular), then the boundary value problem* (2.16) *has a (unique) solution.*

(b) *If every* $\xi \in \Gamma$ *is locally* q*-regular and if* (2.16) *has a solution for a given* $f \geqq 0$, *then every* $\xi \in \Gamma$ *is* $(q+f)$*-regular.*

PROOF. (a) By Corollary 1 to Theorem 2.3, $\int_\Omega G^q(x, y)|f(y)|\, dy < \infty$ for any $x \in \Omega$. By assumption, $H_1^q(x) \to 1$ and $H_1^{q+|f|}(x) \to 1$ as $x \to \xi$ for any $\xi \in \Gamma$. It also follows from Harnack's principle that $\alpha \equiv \inf_{x \in \Omega} H_1^{q+|f|}(x) > 0$. Then, using Proposition 2.1, we have

$$\begin{aligned} 0 &\leqq \int_\Omega G^q(x, y)|f(y)|\, dy \\ &\leqq \frac{1}{\alpha}\int_\Omega G^q(x, y)|f(y)| H_1^{q+|f|}(y)\, dy \\ &= \frac{1}{\alpha}\{H_1^q(x) - H_1^{q+|f|}(x)\} \to 0 \quad (x \to \xi), \end{aligned}$$

i.e.,

$$\lim_{x \to \xi} \int_\Omega G^q(x, y)|f(y)|\, dy = 0$$

for any $\xi \in \Gamma$. Hence, the function

$$u(x) = \begin{cases} -\int_\Omega G^q(x, y) f(y)\, dy & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Gamma \end{cases}$$

is continuous on Ω^* and, by virtue of Lemma 2.4 (considering f^+ and f^-), it is a solution of (2.16).

(b) Let u be a solution of (2.16) for a given $f \geqq 0$. Then

$$D_{\Omega}[u, \phi] + \int_{\Omega} qu\phi \, dx \leqq 0$$

for any $\phi \in \boldsymbol{C}_0^1(\Omega)$ with $\phi \geqq 0$. Therefore, $-u$ is L_q-superharmonic on Ω (Proposition 2.5 of [16]). Since $u=0$ on Γ, $-u \in \mathcal{P}_q$. Then we can show that

$$-u(x) = \int_{\Omega} G^q(x, y) f(y) \, dy$$

for $x \in \Omega$ (cf. [11] and [16]). Therefore,

$$\lim_{x \to \xi} \int_{\Omega} G^q(x, y) f(y) \, dy = 0$$

for all $\xi \in \Gamma$. Then, by Theorem 2.4, every $\xi \in \Gamma$ is $(q+f)$-regular.

Corollary. *Let $f \in \boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$ $(p > d/2)$ and $\sigma \in C(\Gamma)$. If every $\xi \in \Gamma$ is locally q-regular and $(q+|f|)$-regular (or, equivalently, if every $\xi \in \Gamma$ is locally $(q+|f|)$-regular), then the boundary value problem*

$$\begin{cases} L_q u \equiv \Delta u - qu = f & \text{on } \Omega \\ u = \sigma & \text{on } \Gamma \end{cases}$$

has a solution which is continuous on Ω^*.

Proof. Let u_1 be the solution of (2.16). Then $u = u_1 + H^q_\sigma$ is the required solution.

Appendix: Proof of Proposition 2.1.

We prepare five propositions, which are known in case Ω is a locally Euclidean space and q's are locally Hölder continuous (see [14]).

Proposition A-1. *For any $x \in \Omega$,*

$$\int_{\Omega} G^q(x, y) q(y) \, dy \leqq 1.$$

Proof. First, suppose $S(q)$ is compact. Then, by Lemma 2.4, $v(x) \equiv \int_{\Omega} G^q(x, y) q(y) \, dy$ belongs to $\mathcal{P}_q \cap \boldsymbol{D}_{\mathrm{loc}}(\Omega)$ and

$$D_{\Omega}[v, \phi] + \int_{\Omega} qv\phi \, dx = \int_{\Omega} q\phi \, dx$$

for all $\phi \in C_0^1(\Omega)$. Hence, $D_\Omega[1-v, \phi]+\int_\Omega q(1-v)\phi dx=0$ for all $\phi \in C_0^1(\Omega)$, i.e., $1-v$ is q-harmonic on Ω. Since $v \in \mathcal{P}_q$, we have $1-v\geqq 0$.

Next, let q be arbitrary. Since Ω is countable at infinity (cf. [16]), there is a sequence $\{\chi_n\}$ of non-negative measurable functions on Ω such that each $S(\chi_n)$ is compact and $\chi_n(x) \uparrow 1$ $(n\to\infty)$ for each $x \in \Omega$. By the above result,

$$\int_\Omega G^{q\chi_n}(x, y)q(y)\chi_n(y)\, dy \leqq 1.$$

By Lemma 2.5, $G^{q\chi_n}(x, y)\geqq G^q(x, y)$. Hence,

$$\int_\Omega G^q(x, y)q(y)\chi_n(y)\, dy \leqq 1.$$

Letting $n\to\infty$, we obtain the proposition.

PROPOSITION A-2. *If $q_1\leqq q_2$ on Ω and $\sigma\geqq 0$ is $\mathfrak{H}_{q_1}$-resolutive, then the greatest q_2-harmonic minorant of $H_\sigma^{q_1}$ is equal to $H_\sigma^{q_2}$.*

Proof of this proposition is similar to [14; Lemma 3.8].

PROPOSITION A-3. *Let $q_1\leqq q_2$ on Ω and σ be a bounded $\mathfrak{H}_{q_1}$-resolutive function on Γ. Then*

$$H_\sigma^{q_1}(x)=H_\sigma^{q_2}(x)+\int_\Omega G^{q_2}(x, y)\,\{q_2(y)-q_1(y)\}H_\sigma^{q_1}(y)\, dy.$$

PROOF. It is enough to prove this for $\sigma\geqq 0$. By Proposition A-1,

$$v(x)\equiv\int_\Omega G^{q_2}(x, y)\,\{q_2(y)-q_1(y)\}\, H_\sigma^{q_1}(y)\, dy <\infty.$$

Hence, by Lemma 2.4, v is continuous and

$$D_\Omega[v, \phi]+\int_\Omega q_2 v\phi\, dx=\int_\Omega (q_2-q_1)\, H_\sigma^{q_1}\phi\, dx$$

for all $\phi \in C_0^1(\Omega)$. On the other hand, since $H_\sigma^{q_1}$ is q_1-harmonic,

$$D_\Omega[H_\sigma^{q_1}, \phi]+\int_\Omega q_2 H_\sigma^{q_1}\phi\, dx=\int_\Omega (q_2-q_1)\, H_\sigma^{q_1}\phi\, dx.$$

Hence,

$$D_\Omega[H_\sigma^{q_1}-v, \phi]+\int_\Omega q_2(H_\sigma^{q_1}-v)\phi\, dx =0$$

for any $\phi \in C_0^1(\Omega)$, i.e., $H_\sigma^{q_1}-v$ is q_2-harmonic. Since $v \in \mathcal{P}_{q_2}$, it follows that $H_\sigma^{q_1}-v$ is the greatest q_2-harmonic minorant of $H_\sigma^{q_1}$. Hence, by the above proposition, $H_\sigma^{q_1}-v=H_\sigma^{q_2}$.

PROPOSITION A-4. *Let $\{q_n\}$ be a monotone increasing sequence converging to q.* (We are assuming that q_n, q are non-negative and belong to $\boldsymbol{L}^p_{\mathrm{loc}}(\Omega)$, $p>d/2$.) *Then, for any $\mathfrak{H}_{q_1}$-resolutive non-negative function σ on Γ,*

$$H^{q_n}_\sigma(x) \downarrow H^q_\sigma(x) \quad (n\to\infty)$$

at every point $x \in \Omega$.

Proof is similar to [14; Theorem 3.4], by the aid of Proposition A-3.

PROPOSITION A-5. *If $q_1\leqq q_2$ on Ω and $q_1=q_2$ outside a compact set in Ω and if $\sigma\geqq 0$ is bounded $\mathfrak{H}_{q_1}$-resolutive, then the least q_1-harmonic majorant of $H^{q_2}_\sigma$ is equal to $H^{q_1}_\sigma$.*

PROOF. Let u be the least q_1-harmonic majorant of $H^{q_2}_\sigma$. Obviously, $H^{q_2}_\sigma\leqq u\leqq H^{q_1}_\sigma$. Let $v \in \mathcal{U}^{\Omega,q_2}_\sigma(\mathcal{U}^{\Omega,q}_\sigma\equiv\mathcal{U}^{\Omega,\mathfrak{H}_q}_\sigma)$. Then $v-H^{q_2}_\sigma \in \mathcal{S}^+_{q_2}(\Omega)$. Since $q_1=q_2$ on $\Omega-K$ for some compact set K in Ω, $v-H^{q_2}_\sigma \in \mathcal{S}^+_{q_1}(\Omega-K)$. Let $M=\sup\sigma$ and choose $w \in \mathcal{P}_{q_1}$ such that $w>M$ on a neighborhood of K. Then $v_1\equiv\min(M, v-H^{q_2}_\sigma+w) \in \mathcal{S}^+_{q_1}(\Omega)$ and $v_1+u\geqq v_1+H^{q_2}_\sigma\geqq\min(M, v)$. Hence, $v_1+u \in \mathcal{U}^{\Omega,q_1}_\sigma$, i.e., $v_1+u\geqq H^{q_1}_\sigma$. Taking infimum of v, we have

$$\min(M, w)+u\geqq H^{q_1}_\sigma.$$

Since $\min(M, w) \in \mathcal{P}_{q_1}$, it follows that $u\geqq H^{q_1}_\sigma$. Hence $u=H^{q_1}_\sigma$.

PROOF of PROPOSITION 2.1. First suppose $q_1=q_2$ outside a compact set in Ω. Then, by Lemma 2.2, $\int_\Omega G^{q_1}(x, y)\{q_2(y)-q_1(y)\}\,dy<\infty$ and an argument similar to the proof of Proposition A-3 gives (2.8), by the aid of Proposition A-5.

Next, we prove the general case. Let χ_n be as in the proof of Proposition A-1. By the above result, we have

$$(*)\qquad H^{q_1}_\sigma(x)=H^{q_1+\chi_n(q_2-q_1)}_\sigma(x)$$
$$+\int_\Omega G^{q_1}(x, y)\chi_n(y)\{q_2(y)-q_1(y)\}H^{q_1+\chi_n(q_2-q_1)}_\sigma(y)\,dy.$$

Since $H^{q_1+\chi_n(q_2-q_1)}_\sigma\geqq H^{q_2}_\sigma$, (*) implies

$$H^{q_1}_\sigma(x)\geqq H^{q_2}_\sigma(x)+\int_\Omega G^{q_1}(x, y)\chi_n(y)\{q_2(y)-q_1(y)\}\,H^{q_2}_\sigma(y)\,dy.$$

Letting $n\to\infty$, we obtain (2.7).

Now, since

$$G^{q_1}(x, y)\chi_n(y)\{q_2(y)-q_1(y)\}\,H^{q_1+\chi_n(q_2-q_1)}_\sigma(y)$$

$$\leqq (\sup \sigma) G^{q_1}(x, y) \{q_2(y) - q_1(y)\} H_1^{q_1}(y),$$

the condition $\int_\Omega G^{q_1}(x, y) \{q_2(y) - q_1(y)\} H_1^{q_1}(y) dy < \infty$ guarantees the application of Lebesgue's convergence theorem on letting $n \to \infty$ in (*). Thus, since $H_\sigma^{q_1 + \chi_n(q_2 - q_1)} \downarrow H_\sigma^{q_2}$ by Proposition A-4, we obtain (2.8).

CHAPTER III Conditions for q-regularity of relative boundary points

In this chapter, we consider a bounded domain Ω in the Euclidean space R^d and the usual closure $\bar{\Omega}$ in R^d as a compactification of Ω. For simplicity, we shall consider only the case $(g_{ij}) = (\delta_{ij})$, so that Δ is the ordinary Laplacian. We shall always assume that q is a non-negative function on Ω belonging to $\boldsymbol{L}_{\mathrm{loc}}^p(\Omega)$, $p > d/2$.

It is well-known that $\bar{\Omega}$ is a resolutive compactification of Ω (see, e.g., [9; Theorem 8.11]). Also, as remarked in Chapter I (Remark 1.1), if $\xi \in \partial\Omega = \bar{\Omega} - \Omega$ is regular (=0-regular), then it is also locally regular. Thus, by Theorem 2.1 and Corollary 1 to Theorem 2.4, we have

THEOREM 3.1. (a) *$\xi_0 \in \partial\Omega$ is locally q-regular if and only if it is q-regular.*

(b) *Suppose $\xi_0 \in \partial\Omega$ is regular. Then it is q-regular if and only if either*

$$\lim_{r \to 0} \sup_{x \in \Omega} \int_{\Omega \cap B(\xi_0; r)} G(x, y) q(y)\, dy = 0,$$

or, for some $r_0 > 0$

$$\lim_{x \to \xi_0, x \in \Omega} \int_{\Omega \cap B(\xi_0; r_0)} G(x, y) q(y)\, dy = 0.$$

Here, $B(\xi_0; r) = \{x \in R^d;\ |x - \xi_0| < r\}$.

§3.1. The case where there is no condition on the boundary.

First, we state a theorem, which is a consequence of a general theory (see [19; Théorème 10.2] and [11; Théorème 3]; also cf. [12; Corollary 7.7]):

THEOREM 3.2. *If $\xi_0 \in \partial\Omega$ is regular and if $q \in \boldsymbol{L}^p(\Omega \cap V)$, $p > d/2$, for some neighorhood V of ξ_0, then ξ_0 is q-regular.*

We can give an elementary proof to this theorem using Theorem 3.1 and Hölder's inequality. Note that a result in [5; n° 6] is an immediate consequence of this theorem.

THEOREM 3.3. *If $q(x) \leqq \psi(|x - \xi_0|)$ on $B(\xi_0; r_0) \cap \Omega$ $(r_0 > 0)$ for a non-*

negative locally summable function ψ *on* $(0, r_0]$ *such that*

$$(3.1) \qquad \begin{aligned} &\int_0^{r_0} t\psi(t)\,dt < \infty, \ \textit{if } d \geqq 3 \\ &\int_0^{r_0} t \log\frac{1}{t}\psi(t)\,dt < \infty, \ \textit{if } d=2, \end{aligned}$$

then ξ_0 *is* q*-regular whenever it is regular.*

Proof. Let $F(x, y) = |x-y|^{2-d}$ if $d \geqq 3$, $F(x, y) = \log(k/|x-y|)$ if $d=2$, where $k>0$ is so chosen that $F(x, y) \geqq 2\pi G(x, y)$ for all $x, y \in \Omega$. For $0<t<r$, let

$$U_t(x) = \int_{S(0;1)} F(x, \xi_0 + t\theta)\,dS(\theta),$$

where $S(0; 1)$ is the unit sphere and dS is the surface element on $S(0; 1)$. By a classical theory, it is known that U_t is constant on $B(\xi_0; t)$ and

$$\begin{aligned} \sup_{x \in \Omega} U_t(x) &= U_t(\xi_0) \\ &= \int_{S(0;1)} F(0; t\theta)\,dS(\theta) = \begin{cases} \sigma_d t^{2-d} & (d \geqq 3) \\ 2\pi \log(k/t) & (d=2), \end{cases} \end{aligned}$$

where $\sigma_d = \int_{S(0;1)} dS(\theta)$. Let $c_d = (d-2)\sigma_d$ if $d \geqq 3$, $c_d = 2\pi$ if $d=2$. Then $F(x,y) \geqq c_d G(x, y)$ for all $x, y \in \Omega$. Hence, for $0<r<r_0$,

$$\begin{aligned} c_d \int_{B(\xi_0;r)\cap\Omega} G(x, y)q(y)\,dy &\leqq \int_{B(\xi_0;r)} F(x, y)\psi(|y-\xi_0|)\,dy \\ &= \int_0^r \psi(t)t^{d-1}U_t(x)\,dt \\ &\leqq \begin{cases} \sigma_d \displaystyle\int_0^r \psi(t)t\,dt & (d \geqq 3) \\ 2\pi \displaystyle\int_0^r \psi(t)\left(\log\frac{k}{t}\right)t\,dt & (d=2) \end{cases} \end{aligned}$$

for any $x \in \Omega$. By (3.1), the last term tends to 0 as $r \to 0$. Hence, Theorem 3.1 implies that ξ_0 is q-regular if it is regular.

Remark 3.1. Either Theorem 3.2 or Theorem 3.3 implies that if ξ_0 is regular and $q(x) \leqq \lambda |x-\xi_0|^{-2+\varepsilon}$ on $B(\xi_0; r_0) \cap \Omega$ for some $\lambda \geqq 0$ and $\varepsilon>0$, then ξ_0 is q-regular. Applying this result to the corollary to Theorem 2.5, we obtain a theorem given by K. Miller [17; Theorem 4]. We can improve it by using Theorem 3.5 below (§3.3).

§3.2. The case where the boundary is a Liapunov-Dini surface.

A non-negative continuous function $\varepsilon(t)$ defined on $[0, t_0]$ $(t_0>0)$ is called a Dini function ([22]) if it is monotone increasing on $[0, t_0]$, $\varepsilon(t)/t$ is monotone decreasing on $(0, t_0]$ and $\int_0^{t_0}[\varepsilon(t)/t]\,dt<\infty$. Given a bounded domain Ω and $\xi_0 \in \partial\Omega$, we shall say (following K.-O. Widman [22]) that the part of the boundary $\partial\Omega\cap B(\xi_0; r_0)$ $(r_0>0)$ is a Liapunov-Dini surface if there are a C^1-function F on $B(\xi_0; r_0)$ and a Dini function $\varepsilon(t)$ on $[0, r_0]$ satisfying the following two conditions:

(a) $S=\partial\Omega\cap B(\xi_0; r_0)$ is a C^1-surface represented by F, i.e., $S=\{\xi \in B(\xi_0; r_0); F(\xi)=0\}$ and grad $F\equiv\left(\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_d}\right)\neq 0$ on S;

(b) For any $\xi_1, \xi_2 \in S$,

$$\left|\frac{\partial F}{\partial x_i}(\xi_1)-\frac{\partial F}{\partial x_i}(\xi_2)\right|\leqq\varepsilon(|\xi_1-\xi_2|), \quad i=1, \cdots, d.$$

THEOREM 3.4. *Suppose $S=\partial\Omega\cap B(\xi_0;r_0)$ is a Liapunov-Dini surface for some $r_0>0$. Let ψ be a non-negative locally summable function on $(0, r_0]$ such that*

$$\int_0^{r_0} t\psi(t)\,dt<\infty \tag{3.2}$$

and f be a C^1-function on $B(\xi_0; r_0)$ such that $f(x)=0$ on S, $\Omega\cap B(\xi_0; r_0)=\{x \in B(\xi_0; r_0); f(x)>0\}$ and grad $f(\xi_0)\neq 0$. *If, for such ψ and f, we have*

$$q(x) \leqq \psi(f(x)) \tag{3.3}$$

for all $x \in \Omega\cap B(\xi_0;r_0)$, then ξ_0 is q-regular.

PROOF. It is easy to see that the boundary $\partial\Omega$ satisfies the Poincaré exterior cone condition at ξ_0, and hence ξ_0 is regular.

Without loss of generality, we may assume that $\xi_0=0$ and grad $f(0)=(\delta, 0, \cdots, 0)$ with $\delta>0$. For $x=(x_1, \cdots, x_d) \in R^d$, let $x'=(x_2, \cdots, x_d)$. We write $Q_r=\{x;\ |x_1|<r,\ |x'|<r\}$ and $Q_r^+=\{x \in Q_r;\ x_1>0\}$. Since grad $f(0)\neq 0$, there is $\rho_0>0$ $(\rho_0\leqq r_0/2)$ such that by the mapping

$$\Phi: x=(x_1, x')\rightarrow(f(x), x'),$$

$D_{\rho_0}\equiv\Phi^{-1}(Q_{\rho_0})\subset B(0; r_0/2)$, $\partial f/\partial x_1\geqq\delta/2$ on D_{ρ_0} and Φ is one-to-one on D_{ρ_0}. Since D_{ρ_0} is a neighborhood of ξ_0, it is enough to prove that ξ_0 is q-regular with respect to $D_{\rho_0}^+=D_{\rho_0}\cap\Omega$ (Theorem 3.1, (a)), i.e., we may assume that $\Omega=D_{\rho_0}^+$. Thus, let $G(x, y)$ be the Green function of $D_{\rho_0}^+$. For $0<r<\rho_0$, let $D_r=\Phi^{-1}(Q_r)$ and $D_r^+=\Phi^{-1}(Q_r^+)$. Obviously, $D_r^+=D_r\cap\Omega$ and $\{D_r\}_{0<r<\rho_0}$ is a fundamental system of neighborhoods of $\xi_0 = 0$. By virtue of Theorem 3.1, (b), it suffices to show

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{D_r^+} G(x, y) q(y)\, dy = 0. \tag{3.4}$$

Let $\Phi^{-1}(v) = (h(v), v')$ for $v \in Q_{\rho_0}$. Then h is a C^1-function on Q_{ρ_0},

$$0 < \frac{\partial h}{\partial v_1}(v) = \left(\frac{\partial f}{\partial x_1}(\Phi^{-1}(v))\right)^{-1} \leqq \frac{2}{\delta} \quad \text{on } Q_{\rho_0} \tag{3.5}$$

and $S \cap D_{\rho_0} = \{(h(0, v'), v');\ |v'| < \rho_0\}$. By condition (3.3) and by the change of variables $v = \Phi(y)$, we have

$$\begin{aligned}\int_{D_r^+} G(x, y) q(y)\, dy &\leqq \int_{D_r^+} G(x, y) \psi(f(y))\, dy \\ &= \int_{Q_r^+} G(x, \Phi^{-1}(v)) \psi(v_1) \frac{\partial h}{\partial v_1}(v)\, dv \\ &\leqq \frac{2}{\delta} \int_0^r \psi(t)\, dt \int_{v_1 = t, |v'| < r} G(x, \Phi^{-1}(v))\, dv',\end{aligned}$$

where, in the last inequality, we used (3.5). Now we put

$$U_t(x) = \int_{v_1 = t, |v'| < \rho_0/2} G(x, \Phi^{-1}(v))\, dv'$$

for $0 < t < \rho_0/2$. We shall show that there is $k > 0$, independent of x, t, such that

$$U_t(x) \leqq kt \tag{3.6}$$

for all $x \in \Omega$ and $0 < t < \rho_0/2$. Then, for $0 < r < \rho_0/2$,

$$\sup_{x\in\Omega} \int_{D_r^+} G(x, y) q(y)\, dy \leqq \frac{2k}{\delta} \int_0^r t\psi(t)\, dt.$$

Condition (3.2) implies that the right hand side tends to 0 as $r \to 0$. Therefore, we obtain (3.4) and the theorem is proved.

To prove (3.6), we consider two cases.

The case $d \geqq 3$: In this case, first we remark that

$$G(x, y) \leqq k_1 |x - y|^{2-d} \tag{3.7}$$

for all $x, y \in \Omega$, where $k_1 = c_d^{-1}$ is independent of x, y. On the other hand, since S is a Liapunov-Dini surface, the arguments in the proof of Theorem 2.3 of [22] can be repeated and we obtain

$$G(x, y) \leqq k_2 \operatorname{dist}(x, S) \cdot \operatorname{dist}(y, S) |x-y|^{-d} \tag{3.8}$$

for all $x, y \in D^+_{\rho_0/2}$, where $k_2 > 0$ is independent of x, y. Now

$$\operatorname{dist}(x, S) \leqq x_1 - h(0, x') \leqq \frac{2}{\delta} f(x).$$

Hence, (3.8) implies

$$G(x, y) \leqq k_3 f(x) f(y) |x-y|^{-d} \tag{3.9}$$

for all $x, y \in D^+_{\rho_0/2}$, where $k_3 > 0$ is independent of x, y. Let $S_t = \{x; f(x) = t, |x'| < \rho_0/2\}$. For $0 < t < \rho_0/2$, $S_t \subset D^+_{\rho_0/2}$. If $x \in S_t$ $(0 < t < \rho_0/2)$, then, by (3.7) and (3.9), we have

$$\begin{aligned} U_t(x) &\leqq \int_{v_1=t, |x'-v'| \leqq t, |v'| < \rho_0/2} G(x, \Phi^{-1}(v))\, dv' \\ &\quad + \int_{v_1=t, |x'-v'| > t, |v'| < \rho_0/2} G(x, \Phi^{-1}(v))\, dv' \\ &\leqq k_1 \int_{v_1=t, |x'-v'| \leqq t, |v'| < \rho_0/2} |x - \Phi^{-1}(v)|^{2-d} dv' \\ &\quad + k_3 t^2 \int_{v_1=t, |x'-v'| > t, |v'| < \rho_0/2} |x - \Phi^{-1}(v)|^{-d} dv' \\ &\leqq k_1 \int_{|x'-v'| \leqq t} |x'-v'|^{2-d} dv' \\ &\quad + k_3 t^2 \int_{|x'-v'| > t} |x'-v'|^{-d} dv' \\ &= k_1 \sigma_{d-1} \int_0^t d\rho + k_3 \sigma_{d-1} t^2 \int_t^\infty \rho^{-2} d\rho = kt, \end{aligned}$$

where $k = (k_1 + k_3)\sigma_{d-1}$ is independent of x, t. Hence

$$\sup_{x \in S_t} U_t(x) \leqq kt$$

for all $t \in (0, \rho_0/2)$. Now (3.6) follows from Lemma 2.3.

The case $d=2$: In this case, we identify R^2 with a complex plane, so that $x = (x_1, x_2) \in R^2$ is identified with $z = x_1 + i x_2$. Let $w = \zeta(z)$ be a conformal mapping of the simply connected domain $D^+_{\rho_0}$ onto the right half plane $\{\mathrm{Re}\, w > 0\}$ such that ζ has a continuous extension to $\overline{D^+_{\rho_0}}$ for which $\zeta(0) = 0$. Then, by a theorem of S. Warschawski ([21; Zusatz 1 zum Satze 10]; also cf. [20; Theorem IX. 9, (ii)]),

$$0 < \mu_1 \leqq |\zeta'(z)| \leqq \mu_2 < \infty \tag{3.10}$$

for all $z \in D^+_{\rho_0/2}$. On the other hand, since $G(\zeta^{-1}(w), \zeta^{-1}(u))$ is the Green function of $\{\mathrm{Re}\, w > 0\}$,

$$G(\zeta^{-1}(w), \zeta^{-1}(u)) = \log\left|\frac{w+\bar{u}}{w-u}\right| = \frac{1}{2}\log\left[1+4(\mathrm{Re}\, w)(\mathrm{Re}\, u)|u-w|^{-2}\right].$$

Hence,

$$G(x, y) = \frac{1}{2}\log\left[1+4(\mathrm{Re}\,\zeta(x))(\mathrm{Re}\,\zeta(y))|\zeta(x)-\zeta(y)|^{-2}\right].$$

If $x, y \in D^+_{\rho_0/2}$, then, by (3.10),

$$\mathrm{Re}\,\zeta(x) \leqq \mu_2 \cdot \{x_1 - h(0, x_2)\} \leqq \frac{2\mu_2}{\delta} f(x)$$

and

$$|\zeta(x)-\zeta(y)| \geqq \mu_1 |x-y| \geqq \mu_1 |x_2 - y_2|.$$

Hence, if $x, y \in S_t$ $(0 < t < \rho_0/2)$, then

$$G(x, y) \leqq \frac{1}{2}\log\left[1+M^2 t^2 |x_2 - y_2|^{-2}\right],$$

where $M = (4\mu_2)/(\delta\mu_1)$ is independent of x, y. Hence, for $x \in S_t$ $(0 < t < \rho_0/2)$,

$$\begin{aligned} U_t(x) &= \int_{v_1 = t, |v_2| \leqq \rho_0/2} G(x, \Phi^{-1}(v))\, dv_2 \\ &\leqq \frac{1}{2}\int_{|v_2| \leqq \rho_0/2} \log\left[1+M^2 t^2 |x_2 - v_2|^{-2}\right] dv_2 \\ &\leqq Mt\int_0^\infty \log(1+s^{-2})\, ds = M\pi t. \end{aligned}$$

Therefore, we obtain (3.6) also for the case $d=2$.

Corollary 1. *Suppose $S = \partial\Omega \cap B(\xi_0; r_0)$ is a Liapunov-Dini surface for some $r_0 > 0$. If*

(3.11) $\quad q(x) \leqq \psi(\mathrm{dist}(x, S)) \quad$ *for all $x \in \Omega \cap B(\xi_0; r_0)$*

for a non-negative monotone decreasing function ψ on $(0, r_0]$ satisfying (3.2), then ξ_0 is q-regular.

Proof. Let S be represented by a C^1-function F satisfying condition (a) for a Liapunov-Dini surface. We may assume that $F(x)>0$ on $\Omega\cap B(\xi_0; r_0)$. It is easy to see that there is $\delta_1>0$ such that $\mathrm{dist}(x, S)\geqq\delta_1 F(x)$ for all $x\in\Omega\cap B(\xi_0; r_0/2)$. Since ψ is monotone decreasing,

$$q(x)\leqq\psi(\mathrm{dist}(x, S))\leqq\psi(\delta_1 F(x))$$

for $x\in\Omega\cap B(\xi_0; r_0/2)$. Hence, taking $f(x)=\delta_1 F(x)$ in the theorem, we obtain this corollary.

COROLLARY 2. *Suppose $S=\partial\Omega\cap B(\xi_0; r_0)$ is a C^2-surface for some $r_0>0$. If* (3.11) *holds for a non-negative locally summable function ψ on* $(0, r_0]$ *satisfying* (3.2), *then ξ_0 is q-regular.*

PROOF. Obviously, a C^2-surface is a Liapunov-Dini surface with a Dini function $\varepsilon(t)=at$ (a: const.>0). Furthermore, if S is a C^2-surface, then

$$f(x)=\begin{cases}\mathrm{dist}(x, S) & \text{for } x\in\bar{\Omega}\cap B(\xi_0; r_0)\\ -\mathrm{dist}(x, S) & \text{for } x\in B(\xi_0; r_0)-\bar{\Omega}\end{cases}$$

is a C^1-function on $B(\xi; r_1)$ for a sufficiently small $r_1>0$ ($r_1\leqq r_0$) and grad $f(\xi_0)\neq 0$. Hence this corollary follows from the theorem.

REMARK 3.2. As an immediate consequence of Corollary 1 above, we see that if Ω is bounded by a closed Liapunov-Dini surface S(i.e., $\partial\Omega=S$) and if

$$q(x)\leqq k\,\{\mathrm{dist}\,(x, S)\}^{-2+\varepsilon}$$

for all $x\in\Omega$ for some $k>0$ and $\varepsilon>0$, then every point of $\partial\Omega$ is q-regular. Thus, applying Theorem 2.5, (a), we have the following result:

If $\partial\Omega$ is a Liapunov-Dini surface and if

$$|f(x)|\leqq k\,\{\mathrm{dist}\,(x, S)\}^{-2+\varepsilon} \tag{3.12}$$

for all $x\in\Omega$ with $k>0$ and $\varepsilon>0$, then the boundary value problem

$$\Delta u=f \text{ on } \Omega \quad\text{and}\quad u=0 \text{ on } \partial\Omega$$

has a continuous solution.

In this connection, we remark that G. Prodi [18] gave an existence theorem for a similar problem under the assumption (3.12).

§3.3. The case where the boundary satisfies cone conditions.

We shall say that the boundary $\partial\Omega$ satisfies the *exterior* (resp. *interior*) *cone condition at* $\xi_0\in\partial\Omega$ if there exists a truncated circular open cone C with vertex at ξ_0 such that $\Omega\cap C=\emptyset$ (resp. $C\subset\Omega$). In this section, we give conditions for q-regularity of ξ_0 when $\partial\Omega$ satisfies such cone conditions at ξ_0.

LEMMA 3.1. *Let C_0 be the cone*

$$C_0 \equiv \{x=(x_1, x') \in R^d;\ x_1 > a|x'|\} \quad (a>0)$$

and let $C_0^* = R^d - \overline{C_0}$. Let $G_0(x, y)$ and $G_0^*(x, y)$ be the Green functions of C_0 and C_0^*, respectively. Also, let $e=(1, 0, \cdots, 0)$, $S=\{x \in C_0;\ |x|=1,\ x_1 \geqq ka|x'|\}$ with $k>1$ and $S^* = \{x \in R^d - C_0;\ |x|=1\}$. Then, for some constants $\alpha_0>0$, $\beta_0>0$, $c_1>0$ and $c_2>0$, we have

$$(3.13) \qquad \int_{S^*} G_0^*(x, \theta)\, dS(\theta) \leqq \begin{cases} c_1|x|^{\alpha_0} & \text{for } x \in C_0^* \cap B(0; 1) \\ c_1|x|^{-\alpha_0-d+2} & \text{for } x \in C_0^* - B(0; 1), \end{cases}$$

and

$$(3.14) \qquad G_0(te, \theta) \geqq c_2 t^{-\beta_0-d+2} \quad \text{for } t \geqq 1$$

for all $\theta \in S$.

PROOF. Let u be the harmonic measure of S^* with respect to the domain $C_0^* \cap B(0; 1)$. For $0<t \leqq 1$, let $\omega(t) = \sup_{\theta \in S^*} u(t\theta)$. Obviously, $\omega(1)=1$ and $0<\omega(t)<1$ for $0<t<1$. By the maximum principle, we have

$$u(x) \leqq \omega(t) \cdot u(x/t)$$

for $|x|<t$. Hence, if $t'<t$, then

$$\omega(t') \leqq \omega(t) \cdot \omega(t'/t) \leqq \omega(t).$$

It follows that $\omega(t) \leqq 2^{\alpha_0} t^{\alpha_0}$ for $0<t \leqq 1$, where $\alpha_0>0$ is so chosen that $2^{\alpha_0} = [\omega(1/2)]^{-1}$. Therefore, $u(x) \leqq 2^{\alpha_0}|x|^{\alpha_0}$ for $0<|x| \leqq 1$. Now, $w(x) = \int_{S^*} G_0^*(x, \theta)\, dS(\theta)$ is non-negative harmonic on $C_0^* \cap B(0; 1)$, $w=0$ on $\partial C_0^* \cap B(0; 1)$ and w is bounded on S^*. Hence, $w \leqq c'u$ on $C_0^* \cap B(0; 1)$ for some $c'>0$. Thus we obtain (3.13) for $x \in C_0^* \cap B(0; 1)$. By considering the Kelvin transformation of u, we similarly obtain (3.13) for $x \in C_0^* - B(0; 1)$.

Next, let $\beta_0 = \max\{2, (d-1)a^2+1\}$ and consider the function

$$v(x) = (x_1^2 - a^2|x'|^2)^{\beta_0/2} |x-e|^{-2\beta_0-d+2}$$

on $\overline{C_0} - \{e\}$. v is a C^∞-function on $C_0 - \{e\}$ and, by a direct computation, we see that $\Delta v \geqq 0$ on $C_0 - \{e\}$, so that v is subharmonic there. Let U be a neighborhood of S such that $\bar{U} \subset C_0$ and let $\mu = \inf_{x \in \partial U, \theta \in S} G_0(x, \theta)$ and $\lambda = \sup_{x \in \partial U} v(x)$. Then, $0<\mu$, $\lambda<\infty$ and

$$(3.15) \qquad G_0(x, \theta) \geqq \frac{\mu}{\lambda} v(x)$$

for all $x \in \partial U$ and $\theta \in S$. Since v is subharmonic on $C_0 - \bar{U}$, $v=0$ on ∂C_0 and $v(x) \to 0$ as $|x| \to \infty$, (3.15) holds for all $x \in C_0 - U$ and $\theta \in S$. Thus we obtain (3.14), since

$$v(te) = t^{\beta_0} |t-1|^{-2\beta_0 - d + 2} \geqq t^{-\beta_0 - d + 2}$$

for $t \geqq 1$.

Next, we prepare an elementary lemma:

LEMMA 3.2. *Let f be a non-negative locally summable function on $(0, t_0]$ $(t_0 > 0)$ and let*

$$F_\alpha(t) = t^\alpha \int_t^{t_0} s^{-\alpha} f(s)\, ds \qquad (\alpha > 0)$$

and

$$G_\beta(t) = t^{-\beta} \int_0^t s^\beta f(s)\, ds \qquad (\beta > 0)$$

for $0 < t \leqq t_0$. Then, given α, $\beta > 0$, $\lim_{t\to 0} F_\alpha(t) = 0$ if and only if $\lim_{t\to 0} G_\beta(t) = 0$. Hence, if $\lim_{t\to 0} F_\alpha(t) = 0$ for some $\alpha > 0$, then it holds for all $\alpha > 0$ and $\lim_{t\to 0} G_\beta(t) = 0$ for all $\beta > 0$; if $\lim_{t\to 0} G_\beta(t) = 0$ for some $\beta > 0$, then it holds for all $\beta > 0$ and $\lim_{t\to 0} F_\alpha(t) = 0$ for all $\alpha > 0$.

PROOF. First, we assume that $\limsup_{t\to 0} G_\beta(t) < \infty$ (, and hence $G_\beta(t) < \infty$ for each $t \in (0, t_0]$). We have

$$t^{-\alpha} F_\alpha(t) - (2t)^{-\alpha} F_\alpha(2t) = \int_t^{2t} s^{-\alpha} f(s)\, ds$$

and

$$(2t)^\beta G_\beta(2t) - t^\beta G_\beta(t) = \int_t^{2t} s^\beta f(s)\, ds$$

for $t \in (0, t_0/2]$. Since

$$\int_t^{2t} s^{-\alpha} f(s) ds \leqq t^{-(\alpha+\beta)} \int_t^{2t} s^\beta f(s)\, ds$$

and

$$\int_t^{2t} s^\beta f(s) ds \leqq (2t)^{\alpha+\beta} \int_t^{2t} s^{-\alpha} f(s)\, ds,$$

we have

$$F_\alpha(t) - 2^{-\alpha} F_\alpha(2t) \leqq 2^\beta G_\beta(2t)$$

and

$$G_\beta(2t) - 2^{-\beta} G_\beta(t) \leqq 2^\alpha F_\alpha(t)$$

for $t \in (0, t_0/2]$. It then follows that $\limsup_{t\to 0} F_\alpha(t) < \infty$,

$$(1-2^{-\alpha}) \limsup_{t \to 0} F_\alpha(t) \leqq 2^\beta \limsup_{t \to 0} G_\beta(t)$$

and

$$(1-2^{-\beta}) \limsup_{t \to 0} G_\beta(t) \leqq 2^\alpha \limsup_{t \to 0} F_\alpha(t)$$

under the assumption $\limsup_{t\to 0} G_\beta(t) < \infty$. Hence, we obtain the lemma under this assumption.

We shall next show that $\limsup_{t\to 0} G_\beta(t) = \infty$ implies $\limsup_{t\to 0} F_\alpha(t) = \infty$. If $\limsup_{t\to 0} G_\beta(t) = \infty$, then, given $M > 0$, we can find a sequence $\{t_n\} \subset (0, t_0/2]$ such that $t_n \to 0$ and

$$\int_{t_n}^{2t_n} s^\beta f(s)\, ds \geqq 2^{\alpha+\beta} M t_n^\beta.$$

For, otherwise, there is $t' \in (0, t_0]$ such that $\int_t^{2t} s^\beta f(s)\, ds \leqq 2^{\alpha+\beta} M t^\beta$ for all $t \in (0, t']$, so that

$$G_\beta(t) = t^{-\beta} \sum_{n=0}^{\infty} \int_{2^{-(n+1)}t}^{2^{-n}t} s^\beta f(s)\, ds$$

$$\leqq t^{-\beta} 2^{\alpha+\beta} M \sum_{n=0}^{\infty} 2^{-(n+1)\beta}\, t^\beta = 2^{\alpha+\beta} (2^\beta - 1)^{-1} M$$

for all $t \in (0, t']$, which is a contradiction. Now,

$$F_\alpha(t_n) \geqq t_n^\alpha \int_{t_n}^{2t_n} s^{-\alpha} f(s)\, ds$$

$$\geqq t_n^\alpha (2t_n)^{-(\alpha+\beta)} \int_{t_n}^{2t_n} s^\beta f(s)\, ds$$

$$\geqq t_n^\alpha (2t_n)^{-(\alpha+\beta)} 2^{\alpha+\beta} M t_n^\beta = M.$$

Hence, $\limsup_{t\to 0} F_\alpha(t) \geqq M$. Since M is arbitrary, $\limsup_{t\to 0} F_\alpha(t) = \infty$.

Remark 3.3. By the above proof, we see that $\limsup_{t\to 0} F_\alpha(t) = \infty$ if and only if $\limsup_{t\to 0} G_\beta(t) = \infty$. Also, it is easy to see that if $\int_0^{t_0} f(s)\, ds$ is finite, then $\lim_{t\to 0} F_\alpha(t) = 0$ and $\lim_{t\to 0} G_\beta(t) = 0$ for all α, $\beta > 0$. The converse is not always true; for example, if $f(t) = [t \log(1/t)]^{-1}$, then $\lim_{t\to 0} F_\alpha(t) = 0$ but $\int_0 f(s) ds = \infty$.

Theorem 3.5. *If $\partial\Omega$ satisfies the exterior cone condition at $\xi_0 \in \partial\Omega$ and if*

(3.16) $$q(x) \leqq \psi(|x-\xi_0|)$$

for all $x \in \Omega \cap B(\xi_0; r_0)$ $(r_0>0)$ *for a non-negative locally summable function* ψ *on* $(0, r_0]$ *such that*

(3.17) $$\lim_{t \to 0} t^{-\alpha} \int_0^t r^{1+\alpha} \psi(r)\, dr = 0$$

for some $\alpha>0$, *then* ξ_0 *is q-regular.*

PROOF. We may assume that $\xi_0=0$ and $C_0^* \cap B(0; r_0) \supset \Omega \cap B(0; r_0)$, where C_0^* is the set defined in Lemma 3.1. Obviously, $\xi_0=0$ is regular. By virtue of Theorem 3.1 and our assumption (3.16), it is enough to prove

(3.18) $$\lim_{x \to 0} \int_{C_0^* \cap B(0; r_0)} G_0^*(x, y) \psi(|y|)\, dy = 0.$$

Using the relation $G_0^*(x, r\theta) = r^{2-d} G_0^*(x/r, \theta)$ for $r>0$ and (3.13), we have

$$\begin{aligned}
&\int_{C_0^* \cap B(0; r_0)} G_0^*(x, y) \psi(|y|)\, dy \\
&= \int_0^{r_0} r^{d-1} \psi(r)\, dr \int_{S^*} G_0^*(x, r\theta)\, dS(\theta) \\
&= \int_0^{r_0} r\psi(r)\, dr \int_{S^*} G_0^*(x/r, \theta)\, dS(\theta) \\
&\leqq c_1 \left\{ \int_0^{|x|} r\psi(r) \left(\frac{|x|}{r} \right)^{-\alpha_0-d+2} dr + \int_{|x|}^{r_0} r\psi(r) \left(\frac{|x|}{r} \right)^{\alpha_0} dr \right\} \\
&= c_1 \left\{ |x|^{-\alpha_0-d+2} \int_0^{|x|} r^{1+\alpha_0+d-2} \psi(r)\, dr + |x|^{\alpha_0} \int_{|x|}^{r_0} r^{1-\alpha_0} \psi(r)\, dr \right\}.
\end{aligned}$$

Applying Lemma 3.2 with $f(t) = t\psi(t)$, we see that the last expression tends to zero as $x \to 0$, by virtue of the assumption (3.17). Hence we have (3.18).

COROLLARY. *If* $\partial\Omega$ *satisfies the exterior cone condition at* ξ_0 *and if*

$$\lim_{x \to \xi_0, x \in \Omega} |x-\xi_0|^2 q(x) = 0,$$

then ξ_0 *is q-regular.*

Finally, we give a sufficient condition for q-irregularity of $\xi_0 \in \partial\Omega$, which shows that the results in this chapter (Theorems 3.2, 3.3, 3.4 and 3.5) are fairly sharp (cf. Corollaries 1 and 2 below).

THEOREM 3.6. *Suppose* $\partial\Omega$ *satisfies the interior cone condition at* $\xi_0 \in \partial\Omega$ *and let C be a truncated circular open cone with vertex at* ξ_0 *such that* $C \subset \Omega$. *If*

$$(3.19) \qquad \limsup_{t \to 0} t^{-\beta} \int_{C' \cap B(\xi_0; t)} |x - \xi_0|^{\beta - d + 2} q(x)\, dx > 0$$

for some $\beta > 0$ *and a truncated circular closed cone* C' *with vertex at* ξ_0 *such that* $C' - \{0\} \subset C$, *then* ξ_0 *is* q*-irregular.*

PROOF. We may assume that $\xi_0 = 0$, $C = C_0 \cap B(0; r_1)$ and $C' = \{x;\ |x| \leqq r_0,\ x_1 \geqq ka|x'|\}$ with $0 < r_0 < r_1$ and $k > 1$. By Theorem 3.1, it is enough to prove that, for any ρ with $0 < \rho \leqq r_0$,

$$\limsup_{x \to 0} \int_{C' \cap B(0; \rho)} G_0(x, y) q(y)\, dy > 0,$$

or,

$$(3.20) \qquad \limsup_{t \to 0} \int_0^\rho r^{d-1}\, dr \int_S G_0(te, r\theta) q(r\theta) dS(\theta) > 0,$$

in the notation of Lemma 3.1. Since $G_0(te, r\theta) = r^{2-d} G_0((t/r)e, \theta)$, (3.14) of Lemma 3.1 implies that, for $0 < t < \rho$,

$$\int_0^\rho r^{d-1}\, dr \int_S G_0(te, r\theta) q(r\theta) dS(\theta)$$

$$\geqq c_2 \int_0^t r \left(\frac{t}{r}\right)^{-\beta_0 - d + 2} \left\{\int_S q(r\theta) dS(\theta)\right\} dr$$

$$= c_2\, t^{-\beta_0 - d + 2} \int_0^t r^{1 + \beta_0 + d - 2} \left\{\int_S q(r\theta) dS(\theta)\right\} dr.$$

Let $f(r) = r \int_S q(r\theta) dS(\theta)$, which is defined almost everywhere and is locally summable on $(0, r_0]$ by Fubini's theorem. By (3.19),

$$\limsup_{t \to 0} t^{-\beta} \int_0^t r^\beta f(r)\, dr > 0.$$

Hence, by Lemma 3.2,

$$\limsup_{t \to 0} t^{-\beta_0 - d + 2} \int_0^t r^{1 + \beta_0 + d - 2} \left\{\int_S q(r\theta) dS(\theta)\right\} dr > 0.$$

Thus we have (3.20).

COROLLARY 1. *If* $\partial\Omega$ *satisfies the interior cone condition at* $\xi_0 \in \partial\Omega$ *and if*

$$q(x) \geqq \psi(|x - \xi_0|)$$

for all $x \in C$ *for a truncated circular cone* C *with vertex at* ξ_0 *such that* $C \subset \Omega$ *and for a non-negative locally summable function* ψ *on* $(0, r_0]$ *such that*

$$\limsup_{t \to 0} t^{-\beta} \int_0^t r^{1+\beta} \psi(r)\, dr > 0$$

for some $\beta > 0$, *then* ξ_0 *is* q*-irregular.*

COROLLARY 2. *If $\partial\Omega$ satisfies the interior cone condition at $\xi_0 \in \partial\Omega$ and if* $\liminf_{x\to\xi_0, x\in C} |x-\xi_0|^2 q(x) > 0$ *for a truncated circular cone C with vertex at ξ_0 such that $C \subset \Omega$, then ξ_0 is q-irregular.*

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