HIROSHIMA MATH. J.
2 (1972), 19–31

On the Eigenvalues of Recurrent Potential Kernels[1]

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(Received February 21, 1972)

§ 1. Introduction.

Let S be a locally compact Hausdorff space with the second countability axiom and $k(x, y)$ be a symmetric potential kernel of a recurrent Markov process with S as its state space. Let C be a compact set in S, and denote a linear operator on $L^2(C, m)$ by

$$Kf(x)=\int_C k(x, y)f(y)m(dy)\,, \tag{1, 1}$$

where m is a positive Radon measure on S. We consider the eigenvalue problem for the equation $\lambda f=Kf$ under some assumptions given in Section 2. For some region C, not all of the eigenvalues could be positive, so we are interested in the questions: When and how many negative eigenvalues has K ?

J. Troutman [8] has studied this problem for the logarithmic potential kernel by analysing the kernel itself. To the same problem, M. Kac [6] has also given an answer using the Brownian motion on the plane. Kac's result due to probabilistic idea was then formulated by T. Bojdecki [1] in analytic terms. In this paper we also use Kac's method and show the following results: For a wider class of recurrent potential kernels, there exists at most one negative eigenvalue, which is simple if it exists; this is the case if and only if the (semi-classical) equilibrium constant R_0 for the region C is negative.

§ 2. Assumptions and results.

Let $\{x_t, P_x, x \in S\}$ denote a Markov process with state space $(S, \boldsymbol{B})$ where $\boldsymbol{B}$ is the topological Borel field in S. We suppose that the Markov process $\{x_t\}$ satisfies the following assumptions:

ASSUMPTION 1. The Markov process $\{x_t\}$ has a transition density function $p_t(x, y)$ with respect to a positive Radon measure $m(\cdot)$. The function $p_t(x, y)$ is $\boldsymbol{B}\times\boldsymbol{B}$-measurable, non-negative and symmetric, that is,

$$p_t(x, y)=p_t(y, x)\geqq 0, \qquad (t>0, \quad x,\ y \in S). \tag{A$_1$, 1}$$

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(1) The most part of this work was done during the author's stay at Hiroshima University, 1970–71.

From now on we denote by $r_\alpha(x, y)$ the resolvent function, i.e., the Laplace transform of $p_t(x, y)$ and fix an arbitrary compact set C in S with positive mass $m(C)>0$.

ASSUMPTION 2. There exist two functions $\varphi(\alpha)$ and $k(x, y)$ satisfying the following conditions:

$$(\mathrm{A}_2, 1) \qquad \lim_{\alpha\downarrow 0} \varphi(\alpha)=+\infty\,.$$

$(\mathrm{A}_2, 2)$ The function $k(x, y)$ from $C\times C$ to $(-\infty, +\infty]$[(2)] is symmetric, continuous on $C\times C$ and finite on $C\times C-\varDelta$ where $\varDelta=\{(x, x)\mid x\in C\}$.

$$(\mathrm{A}_2, 3) \qquad \lim_{\alpha\downarrow 0}\ \sup_{(x,y)\in C\times C-\varDelta'} |r_\alpha(x, y)-\varphi(\alpha)-k(x, y)|=0\,,$$

where $\varDelta'=\{(x, x)\mid k(x, x)=+\infty\}$.

ASSUMPTION 3. The operator K given by (1, 1) is a compact one from $L^2(C, m)$ to $L^2(C, m)$.

Under these assumptions the functions $\varphi(x)$ and $k(x, y)$ are uniquely determined up to additive constants; $k(x, y)$ is called the *recurrent potential kernel*. We introduce useful functions which are defined in [6] as follows:

$$(2,\ 1) \qquad g(x, u)=\lim_{\alpha\downarrow 0} \alpha\varphi(\alpha)E_x\left[\int_0^\infty \exp\left(-\alpha t-u\int_0^t \chi_C(x_\tau)d\tau\right)dt\right],^{(3)}$$

$$(2,\ 2) \qquad g(x)=\lim_{u\uparrow\infty} g(x, u)$$

whose convergences will be verified in Lemma 3 and in Section 4 respectively.

THEOREM 1. *The operator K can have at most one negative eigenvalue. If λ is the negative eigenvalue, then the associated eigenfunction is a constant multiple of $g(x, -\lambda^{-1})$.*

As usual we shall denote the mutual energy of two Radon measures μ and ν by

$$(2,\ 3) \qquad I(\mu, \nu)=\iint_{C\times C} k(x, y)\mu(dx)\nu(dy)\,,$$

if it is well-defined. We denote by $I(\mu)$ the energy $I(\mu, \mu)$ of μ. Let us set

$$(2,\ 4) \qquad \begin{cases} M=\{\mu\mid \text{positive Radon measure on } C,\ |I(\mu)|<\infty,\ \mu(C)=1\}, \\ M_0=\{\mu\in M\mid \mu(\{x\mid g(x)>0\})=0\},^{(4)} \end{cases}$$

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(2) $(-\infty, +\infty]=R\cup\{+\infty\}$.

(3) E_x denotes the integral with respect to P_x, and χ_C the indicator function of C.

(4) M_0 is not empty since $m\in M_0$.

and define

$$(2, 5) \qquad R=\inf_{\mu\in M} I(\mu), \qquad R_0=\inf_{\mu\in M_0} I(\mu).$$

R and R_0 are called the equilibrium constant (or Robin constant) and the semi-classical equilibrium constant respectively (see[1]).

THEOREM 2. *The negative eigenvalue λ happens if and only if R_0 is negative, in which case $R_0 \leqq \lambda^{-1}\|g(\cdot, -\lambda^{-1})\|^2$.*(5)

The inequality $R\leqq R_0$ follows from their definitions. When $M=M_0$, the function $g(x)$ is zero on C except for a set of the transfinite diameter zero(6). According to [2] we call the point x *s-regular* if $P_x(\tau_0>0)=0$, where τ_0 is the first penetration time through C:

$$\tau_0=\begin{cases} \inf\left\{t \,\middle|\, \int_0^t \chi_C(x_\tau)\,d\tau>0\right\}, & \text{if } \{\ \}\neq\phi \\ \infty, & \text{if } \{\ \}=\phi\,. \end{cases}$$

THEOREM 3. *It holds that $g(x)=0$ at any s-regular point x in C. If the set of s-irregular*(7) *points in C is of transfinite diameter zero, then $R=R_0$.*

In these theorems we find the answer to our questions: The recurrent potential kernel considered on a (s-regular) set which is large enough has only one negative eigenvalue. We now examine typical examples which satisfy the Assumptions.

EXAMPLE 1. Let $\{x_t\}$ be a one-dimensional symmetric stable process with index $\lambda(1\leqq\lambda\leqq 2)$. For the case of $1<\lambda\leqq 2$, its resolvent function is expressed as

$$r_\alpha(x, y)=\frac{1}{\pi}\int_0^\infty \frac{\cos|x-y|\xi}{\alpha+\xi^\lambda}d\xi\,.$$

The following functions

$$\varphi(\alpha)=\frac{1}{\pi}\int_0^\infty \frac{d\xi}{\alpha+\xi}, \qquad k(x, y)=\frac{1}{2\Gamma(\lambda)\cos\frac{\pi\lambda}{2}}|x-y|^{\lambda-1}\ (8)$$

satisfy (A_2, 1) and (A_2, 2). (A_2, 3) can be shown by using another expression

(5) $\|\ \|$ denotes the norm of $L^2(C, m)$.

(6) "transfinite diameter zero" means "μ-measure zero" for any $\mu\in M$.

(7) "s-irregular" means not "s-regular".

(8) See [9].

$$r_\alpha(x,\ y)=\frac{1}{\pi}\int_0^\infty \frac{\cos\xi}{b+\xi^\lambda}d\xi\cdot|x-y|^{\lambda-1}, \qquad b=\alpha|x-y|^\lambda$$

and the fact that b tends to zero uniformly in x and y in any compact set as $\alpha\downarrow 0$. For the case of $\lambda=1$, the resolvent function is the limit

$$r_\alpha(x,\ y)=\lim_{T\to\infty}\int_0^T \frac{\cos|x-y|\xi}{\alpha+\xi}d\xi\,,$$

and so the functions in Assumption 2 are given by

$$\varphi(\alpha)=\frac{1}{\pi}\log\frac{1}{\alpha}+p_1 \qquad \left(p_1=\frac{1}{\pi}\left(\int_0^1 \frac{\cos\xi-1}{\xi}\,d\xi+\int_1^\infty \frac{\cos\xi}{\xi}d\xi\right)\right),$$

$$k(x,\ y)=\frac{1}{\pi}\log\frac{1}{|x-y|}\,.$$

In fact (A_2, 1) and (A_2, 2) are satisfied. (A_2, 3) is verified as follows:

$$\begin{aligned}
&r_\alpha(x,\ y)-\varphi(\alpha)-k(x,\ y)\\
&=\frac{1}{\pi}\left[\left(\int_0^b \frac{\cos\xi-1}{b+\xi}\,d\xi+\int_0^b \frac{d\xi}{b+\xi}\right)+\left[\left\{\left(\int_{2b}^1 \frac{\cos\xi-1}{\xi}d\xi+\int_{2b}^1 \frac{d\xi}{\xi}\right)\right.\right.\right.\\
&\quad\left.\left.+\int_1^\infty \frac{\cos\xi}{\xi}d\xi\right\}\cos b+\int_{2b}^\infty \frac{\sin\xi}{\xi}d\xi\cdot\sin b\right]-\log\frac{1}{b}\\
&\quad\left.-\left(\int_0^1 \frac{\cos\xi-1}{\xi}\,d\xi+\int_1^\infty \frac{\cos\xi}{\xi}d\xi\right)\right],
\end{aligned}$$

where $b=\alpha|x-y|$. As $\alpha\downarrow 0$, b decreases to 0 uniformly in x and y in any compact set, and the above converges to 0. The Assumption 3 for both the cases is a part of the fact that

$$\iint_{C\times C}|k(x,\ y)|^2 m(dx)m(dy)<\infty\,.$$

If $\lambda\neq 1$ or if $\lambda=1$ and $C=[-r,\ r]$ $(r>2^{-1}e^{3/2})$, then the constant $R_0<0$. Therefore by Theorem 2 the operator K in these cases has only one negative eigenvalue.

EXAMPLE 2. ([1], [6], [8]). In the case that $\{x_t\}$ is a two-dimensional Brownian motion, the Assumptions have been ascertained in [6]. Let C_r be a disc with radius r. Then the smallest eigenvalue of the operator K on $L^2(C_r)$ is negative if and only if r is larger than 1.

EXAMPLE 3. Let $\{x_t\}$ be a one-dimensional symmetric birth and death process whose jumping time has an exponential distribution with the expectation c^{-1} ([4]). In this case, we have

$$r_\alpha(m, n)=\frac{1}{\sqrt{\alpha(c+2\alpha)}}\left(\frac{c+\alpha-\sqrt{\alpha(c+2\alpha)}}{C}\right)^{|m-n|} \qquad (m, n: \quad \text{integers}),$$

$$\varphi(\alpha)=\frac{1}{\sqrt{\alpha(b+2\alpha)}}, \qquad k(m, n)=-|m-n|.$$

The Assumptions for $\{x_t\}$ are verified, and the operator K on $L^2(C)$ has one negative eigenvalue for any finite set $C(\neq\phi)$.

EXAMPLE 4. Let $\{x_t\}$ be a Brownian motion with reflecting barriers on $[0, 1]$, that is,

$$r_\alpha(x, y)=\frac{1}{\sqrt{2\alpha}}\{e^{-|x+y|\sqrt{2\alpha}}+e^{-|x-y|\sqrt{2\alpha}}+(e^{2\sqrt{2\alpha}}-1)^{-1}\sum_{p,q=\pm1} e^{(px+qy)\sqrt{2\alpha}}\}.$$

We can prove by a method similar to the one in the case of $\lambda=1$ in Example 1 that

$$\varphi(\alpha)=\sqrt{\frac{2}{\alpha}}\{1+2(e^{2\sqrt{2\alpha}}-1)^{-1}\}, \qquad k(x, y)=x^2+y^2-x-y-|x-y|$$

satisfy the Assumptions. The negativity of $k(x, y)$ implies that the operator K has one negative eigenvalue.

§ 3. The proof of Theorem 1.

We begin with some properties of the potential kernel.

LEMMA 1. (1) *$m(\Delta_x')=0$ where $\Delta_x'=\{y \in C \mid k(x, y)=\infty\}$.*

(2) $\lim_{\alpha\downarrow 0} \sup_{x\in C} \int_C |r_\alpha(x, y)-\varphi(\alpha)-k(x, y)|m(dy)=0.$

(3) *The function $\int_C |k(x,y)|m(dy)$ is bounded in x on C.*

PROOF. (1) is clear from Assumption 3, and (2) follows from (1) and $(A_2, 3)$. To prove (3), we notice that for each fixed x in C

$$|k(x, y)|\leqq|r_\alpha(x, y)-\varphi(\alpha)-k(x, y)|+\varphi(\alpha)+r_\alpha(x, y)$$

for a.a. y in C, and hence

$$\int_C |k(x, y)|m(dy)\leqq\int_C |r_\alpha(x, y)-\varphi(\alpha)-k(x, y)|m(dy)+\varphi(\alpha)m(C)+\alpha^{-1}.$$

The integral on the right-hand side is bounded in x on C for a fixed $\alpha>0$. Thus we obtain (3).

The following notations will be used in our later discussions. Let T_t be a *sojourn time* on C up to t:

$$T_t(w)=\int_0^t \chi_C(x_\tau(w))d\tau\,, \tag{3, 1}$$

and set

$$G_\alpha^u(x,\,A)=E_x\Big[\int_0^\infty \exp(-\alpha t-u\,T_t)\chi_A(x_t)dt\Big] \quad \text{for} \quad u,\,\alpha>0,\ A\in \boldsymbol{B}. \tag{3, 2}$$

Then

$$0\leqq G_\alpha^u(x,\,A)\leqq \int_A r_\alpha(x,\,y)m(dy)\leqq \alpha^{-1} \qquad (\alpha>0). \tag{3, 3}$$

Further, the so-called Kac's formula holds: For u, $\alpha>0$ and $A\in\boldsymbol{B}$

$$\begin{aligned}\int_A r_\alpha(x,\,y)m(dy)-G_\alpha^u(x,\,A) &= u\int_C G_\alpha^u(x,\,dy)\int_A r_\alpha(y,\,z)m(dz)\\ &= u\int_C r_\alpha(x,\,y)G_\alpha^u(y,\,A)m(dy).\end{aligned} \tag{3, 4}$$

(For the proof see K. Ito [5], page 2, 17, 2, Theorem 2.) Especially, in the case of $A=S$,

$$1-\alpha G_\alpha^u(x,\,S)=uG_\alpha^u(x,\,C)=u\int_C r_\alpha(x,\,y)\alpha G_\alpha^u(y,\,S)m(dy). \tag{3, 5}$$

The following Lemma 2 is immediately obtained from (3, 2) and (3, 5).

LEMMA 2. *If A in $\boldsymbol{B}$ is contained in C, then*

$$0\leqq \lim_{\alpha\downarrow 0} G_\alpha^u(x,\,A)\equiv G_0^u(x,\,A)\leqq u^{-1} \qquad (u>0). \tag{3, 6}$$

LEMMA 3. (1) *There exists a constant d such that*

$$0\leqq \alpha\varphi(\alpha)G_\alpha^u(x,\,S)\leqq d \qquad (x\in C \text{ and } \alpha>0). \tag{3, 7}$$

(2) *The function $g(x,\,u)$ of (2, 1) is well-defined and bounded in x for each fixed $u>0$.*

PROOF. The equality (3, 4) is rewritten in the form:

$$\begin{aligned}G_\alpha^u(x,\,C) &= \int_C r_\alpha(x,\,y)m(dy)-u\int_C G_\alpha^u(x,\,dy)\int_C r_\alpha(y,\,z)m(dz)\\ &= \int_C\{r_\alpha(x,\,y)-\varphi(\alpha)-k(x,\,y)\}m(dy)+\int_C k(x,\,y)m(dy)\\ &\quad -u\int_C G_\alpha^u(x,\,dy)\int_C\{r_\alpha(y,\,z)-\varphi(\alpha)-k(y,\,z)\}m(dz)\end{aligned} \tag{3, 8}$$

$$-u\int_C G_\alpha^u(x,\ dy)\int_C k(y,\ z)m(dz)+\varphi(\alpha)(1-uG_\alpha^u(x,\ C))m(C).$$

As $\alpha \downarrow 0$, the left-hand side converges to $G_0^u(x, C)$. By Lemma 1 and (3, 6) the first and the third terms in the above converge to zero, and the fourth term also converges. Therefore the convergence of the last term also does. Since $\alpha\varphi(\alpha)G_\alpha^u(x, S)=\varphi(\alpha)(1-uG_\alpha^u(x, C))$ by (3, 5), we have

$$g(x,\ u)=\lim_{\alpha\downarrow 0}\alpha\varphi(\alpha)G_\alpha^u(x,\ S)=\lim_{\alpha\downarrow 0}\varphi(\alpha)(1-uG_\alpha^u(x,\ C)). \tag{3, 9}$$

Furthermore for sufficiently small $\alpha>0$ and $\varepsilon>0$

$$|\varphi(\alpha)(1-uG_\alpha^u(x,\ C))|\leqq m(C)^{-1}\{G_\alpha^u(x,\ C)+2\varepsilon+2\sup_{x\in C}\int_C|k(x,\ y)|m(dy)\}.$$

Lemma 1 and (3, 6) provide the boundedness of the right-hand side, which proves (1).

LEMMA 4. *The limit*

$$R(u)=\lim_{\alpha\downarrow 0}\varphi(\alpha)\{1-u\int_C\alpha\varphi(\alpha)G_\alpha^u(x,\ S)m(dx)\} \tag{3, 10}$$

exists and satisfies

$$g(x,\ u)=R(u)-u\int_C k(x,\ y)g(y,\ u)m(dy)\,. \tag{3, 11}$$

Further, we have

$$\int_C g(x,\ u)m(dx)=u^{-1}\qquad (u>0). \tag{3, 12}$$

PROOF. From (3, 5), (3, 3) and Lemma 1, we get the following

$$\alpha G_\alpha^u(x,\ S)=1-u\int_C\alpha\varphi(\alpha)G_\alpha^u(y,\ S)m(dy)-u\int_C k(x,\ y)\alpha G_\alpha^u(y,\ S)m(dy)$$
$$-u\int_C\{r_\alpha(x,\ y)-\varphi(\alpha)-k(x,\ y)\}\alpha G_\alpha^u(y,\ S)m(dy)\,.$$

Multiplying both sides by $\varphi(\alpha)$, we see

$$\alpha\varphi(\alpha)G_\alpha^u(x,\ S)=\varphi(\alpha)\{1-u\int_C\alpha\varphi(\alpha)G_\alpha^u(y,\ S)m(dy)\}$$
$$-u\int_C k(x,\ y)\alpha\varphi(\alpha)G_\alpha^u(y,\ S)m(dy)$$
$$-u\int_C\{r_\alpha(x,\ y)-\varphi(\alpha)-k(x,\ y)\}\alpha\varphi(\alpha)G_\alpha^u(y,\ S)m(dy).$$

Since each term except for the first one on the right-hand side converges as $\alpha \downarrow 0$ by Lemmas 1 and 3, the rest also converges. Hence we see that $R(u)$ is well-defined and we have (3, 11) by (3, 9). Further we have

$$\lim_{\alpha \downarrow 0} \{1-u\int_C \alpha\varphi(\alpha)G_\alpha^u(y,\ S)m(dy)\}=0$$

by $(A_2, 1)$. Hence by Lebesgue's convergence theorem, $\int_C g(x,\ u)m(dx)= u^{-1}$ $(u>0)$. Thus the proof of Lemma 4 is completed.

There exists a complete orthonormal system $\{f_j\}$ in $L^2(C,\ m)$ associated with the eigenvalues $\{\lambda_j\}$ (we admit λ_j to be zero, if necessary) of the compact operator K. By making use of $\{f_j\}$ and $\{\lambda_j\}$ the constant $R(u)$ can be expanded.

LEMMA 5. *For $u(>0)$ such that $u^{-1}+\lambda_j\neq 0$ for all j,*

$$R(u)^{-1}=\sum_{j=1}(f_j,\ 1)^2(u^{-1}+\lambda_j)^{-1}. \tag{3, 13}$$

If $\lambda_j<0$, then $(f_j,\ 1)=0$ or $R(-\lambda_j^{-1})=0$.

PROOF. By (3, 11) we get $(1+u\lambda_j)(f_j,\ g)=R(u)(f_j, 1)$, where $(\cdot,\ \cdot)$ denotes the inner product in $L^2(C,\ m)$. This implies the latter half of Lemma 5. Now we assume that $u^{-1}+\lambda_j\neq 0$ (for all j). Since

$$(f_j,\ g)(f_j,\ 1)=R(u)(1+u\lambda_j)^{-1}(f_j,\ 1)^2 \qquad (j=1,\ 2,\ \cdots),$$

we obtain by Parseval's relation

$$(g,\ 1)=R(u)\sum_{j=1}(1+\mu\lambda_j)^{-1}(f_j,\ 1)^2.$$

Combining this with (3,12), we have (3, 13).

We are now in the position to establish a kind of the so-called energy principle which is a generalized one in the logarithmic potential theory.

LEMMA 6. *$I(\mu)\geqq 0$ for each Radon measure $\mu=\mu_1-\mu_2(\mu_i\in M)$ on C with the finite mutual energy $|I(\mu_i,\ \mu_j)|<\infty(i,\ j=1,\ 2)$.*

PROOF. If $k(x,\ x)=\infty$, then $\mu(\{x\})=0$ follows from the finiteness of $|I(\mu)|$. Hence we have, as in proving Lemma 1,

$$I(\mu)=\lim_{\alpha\downarrow 0}\int_C\int_C r_\alpha(x,\ y)\mu(dx)\mu(dy).$$

Using the semi-group property and $(A_1,\ 1)$ of p_t we obtain our result as follows:

$$I(\mu)=\lim_{\alpha\downarrow 0}\int_C\int_C\int_0^\infty e^{-\alpha t}p_t(x,\ y)dt\mu(dx)\mu(dy)$$

$$=\lim_{\alpha\downarrow 0}\int_0^\infty e^{-\alpha t}dt\int_C\int_C\int_S p_{t/2}(x, z)p_{t/2}(z, y)m(dz)\mu(dx)\mu(dy)$$

$$=\lim_{\alpha\downarrow 0}\int_0^\infty e^{-\alpha t}dt\int_S\left\{\int_C p_{t/2}(x, z)\mu(dx)\right\}^2 m(dz)\geqq 0.$$

COROLLARY. *If f in $L^2(C, m)$ is a non-zero eigenfunction of K, and if $(f, 1)=0$, then the eigenvalue λ corresponding to f is non-negative.*

PROOF. Let $\nu(dx)=f(x)m(dx)$ and $\nu^{\pm}(dx)=f^{\pm}(x)m(dx)$, where $f^{\pm}(x)=\max.(\pm f(x), 0)$. Then the each mutual energy is finite. In fact,

$$I(\nu^{\pm}, \nu^{\pm})=(Kf^{\pm}, f^{\pm})\leqq\|K\|\ \|f^{\pm}\|\ \|f^{\pm}\|\leqq\|K\|\ \|f\|^2.$$

It follows from $\nu(C)=0$ and Lemma 6 that

$$0\leqq I(\nu)=(Kf, f)=\lambda(f, f).$$

Thus $\lambda\geqq 0$ is proved.

PROOF OF THEOREM 1. Let λ_j be a negative eigenvalue, f_j its associated eigenfunction ($\neq 0$). By the Corollary we have $(f_j, 1)\neq 0$. Hence $R(-\lambda_j^{-1})=0$ holds by Lemma 5. Now we suppose that there exist at least two negative eigenvalues $\lambda_1<\lambda_2<0$ such that $R(-\lambda_j^{-1})=0$ $(j=1, 2)$. Then (3, 11) and (3, 12) imply that $g(x, -\lambda_j^{-1})$ is a non-zero eigenfunction associated with $\lambda_j (j=1, 2)$. Since $-\lambda_1^{-1}<-\lambda_2^{-1}$, we have $g(x, -\lambda_1^{-1})\geqq g(x, -\lambda_2^{-1})\geqq 0$ from (2, 1). But this contradicts the orthogonality between $g(x, -\lambda_j^{-1})$ $(j=1, 2)$. Thus K has at most one negative eigenvalue. Finally, we suppose that there would be two non-zero eigenfunctions f_1 and $f_2((f_1, f_2)=0)$ corresponding to a negative eigenvalue λ_1. We can find real numbers a and b $(a^2+b^2\neq 0)$ such that $(af_1+bf_2, 1)=0$. Hence af_1+bf_2 belongs to a non-negative eigenvalue by the Corollary. This is contradictory to $\lambda_1<0$.

§ 4. The proof of Theorem 2.

We denote by λ_1 the unique negative eigenvalue of K if it exists, or the largest eigenvalue if not. The other eigenvalues of K are arranged in the order of their values: $\lambda_2\geqq\lambda_3\geqq\cdots$. By (3, 13) there exists $\lim_{u\uparrow\infty} R(u)$, which is denoted by $R(\infty)$. $R(\infty)=0$ if there exists a function f such that $Kf=0$ and $(f, 1)\neq 0$ or if $\sum_{j=1}\lambda_j^{-1}(f_j, 1)^2=\infty$. Otherwise, $R(\infty)$ satisfies the equality:

$$1=R(\infty)\sum_{j=1}\lambda_j^{-1}(f_j, 1)^2. \tag{4, 1}$$

In addition, by (2, 1) $g(x, u)$ is monotonically decreasing as u tends to infinity. We can define $g(x)=\lim_{u\uparrow\infty} g(x, u)$. Then we have

LEMMA 7. (1) $g(x)=0$ *a.e. with respect to* m *on* C.

(2) *The constant* $R(\infty)$ *is equal to*

$$R_0=\inf\{I(\mu)\,|\,\mu\in M,\ \mu(\{x\,|\,g(x)>0\})=0\}.$$

PROOF. We have noticed in Lemma 4 that $\int_C g(x,u)m(dx)=u^{-1}$, and $g(x,u)$ decreases to $g(x)$ as $u\uparrow\infty$. Thus we have (1). Put $\mu_u(dx)=ug(x,u)m(dx)$. Then by (3, 11),

$$R(u)=\int_C k(x,y)\mu_u(dy)+g(x,u)\geqq\int_C k(x,y)\mu_u(dy).$$

Integrating both sides in x with μ_u, we have $R(u)\geqq I(\mu_u)$. Hence $a\equiv\underline{\lim}_{u\uparrow\infty} I(\mu_u)\leqq R(\infty)$. On the other hand, since the measure μ_u is absolutely continuous with respect to m, μ_u belongs to M_0 by (1) of Lemma 7. Hence $a\geqq R_0$. We choose $u_n\to\infty$ such that $I(\mu_n')$ tends to a as $n\to\infty$, where $\mu_n'=\mu_{u_n}$. Then, it follows from the monotonicity of $g(x,u)$ in u that, for each $\nu\in M_0$,

$$\begin{aligned}0&=\int_C g(x)\nu(dx)=\lim_{n\to\infty}\int_C g(x,u_n)\nu(dx)\\&=\lim_{n\to\infty}\int_C\left\{R(u_n)-\int_C k(x,y)u_ng(y,u_n)m(dy)\right\}\nu(dx)\\&=R(\infty)-\lim_{n\to\infty} I(\mu_n',\nu).\end{aligned}$$

Applying Lemma 6 to $I(\mu_n'-\nu)$, we have $2I(\mu_n',\nu)\leqq I(\mu_n')+I(\nu)$. Letting n tend to ∞, we have $2R(\infty)\leqq a+I(\nu)$ for any $\nu\in M_0$. Hence $2R(\infty)\leqq a+R_0$. Using this inequality together with the relations $R_0\leqq a$ and $a\leqq R(\infty)$ repeatedly, it follows that $R(\infty)=a=R_0$.

PROOF OF THEOREM 2. At first suppose that λ_1 is negative. By Theorem 1 $g(x,-\lambda_1^{-1})$ is the eigenfunction corresponding to λ_1. The measure

$$\nu(dx)=(g(\cdot,-\lambda_1^{-1}),1)^{-1}g(x,-\lambda_1^{-1})m(dx)$$

is a positive Radon measure in M_0. Hence

$$R_0\leqq I(\nu)=\lambda_1(g(\cdot,-\lambda_1^{-1}),1)^{-2}\|g(\cdot,-\lambda_1^{-1})\|^2.$$

From (3, 12) it follows that $R_0\leqq\lambda_1^{-1}\|g(\cdot,-\lambda_1^{-1})\|^2$, which implies $R_0<0$. Conversely, suppose $R_0<0$. We have by Lemma 7 $R(\infty)<0$, and then we have from (4, 1) that there is at least one negative eigenvalue. Thus we complete the proof.

§ 5. The proof of Theorem 3.

We devide the proof of Theorem 3 into three lemmas.

Lemma 8. *The following conditions for x are equivalent*:

(5, 1) $$P_x(\tau_0>0)=0,$$

(5, 2) $$\lim_{u\uparrow\infty} G_\alpha^u(x, S)=0 \qquad (\textit{for some } \alpha>0),$$

(5, 3) $$\lim_{u\uparrow\infty} uG_\alpha^u(x, C)=1 \qquad (\textit{for some } \alpha>0).$$

Proof. Since if $\tau_0<\infty$, the sojourn time $T_t>0$ for $t>\tau_0$, we have

(5, 4)
$$\lim_{u\uparrow\infty} G_\alpha^u(x, S)=\lim_{u\uparrow\infty}\left\{E_x\left[\int_0^{\tau_0}\exp(-\alpha t-u\,T_t)dt\right]+E_x\left[\int_{\tau_0}^{\infty}\exp(-\alpha t-u\,T_t)dt\right]\right\}$$
$$=E_x\left[\int_0^{\tau_0}e^{-\alpha t}dt\right].$$

Hence (5, 1) holds if and only if (5, 2) holds. By (3, 5) we obtain that (5, 2) and (5, 3) are equivalent. Thus we have finished the proof.

Remark. The property (5, 2) or (5, 3) for some $\alpha>0$ implies by the monotonicity of G_α^u in α that it holds for any sufficiently small $\alpha>0$.

Lemma 9. *Let f be a bounded measurable function on C. Then we have for any point x where $P_x(\tau_0>0)=0$,*

(5, 5) $$\varlimsup_{u\uparrow\infty}\lim_{\alpha\downarrow 0}\int_C uG_\alpha^u(x, dy)f(y)=\lim_{\alpha\downarrow 0}\varlimsup_{u\uparrow\infty}\int_C uG_\alpha^u(x, dy)f(y).$$

Proof. By Lemma 8, we have $\lim_{u\uparrow\infty} uG_\alpha^u(x, C)=1$. From $(A_2, 1)$ and (3, 9) it follows that $\lim_{\alpha\downarrow 0} uG_\alpha^u(x, C)=1$. Hence (5, 5) is valid for $f=$constant. Next, suppose that $0\leqq f\leqq 1$. Since G_α^u increases as $\alpha\downarrow 0$, we have

$$\varlimsup_{u\uparrow\infty}\lim_{\alpha\downarrow 0}\int_C uG_\alpha^u(x, dy)f(y)\geqq\lim_{\alpha\downarrow 0}\varlimsup_{u\uparrow\infty}\int_C uG_\alpha^u(x, dy)f(y),$$

and

$$\varlimsup_{u\uparrow\infty}\lim_{\alpha\downarrow 0}\int_C uG_\alpha^u(x, dy)(1-f)(y)\geqq\lim_{\alpha\downarrow 0}\varlimsup_{u\uparrow\infty}\int_C uG_\alpha^u(x, dy)(1-f)(y).$$

Noting that $\lim_{u\uparrow\infty} uG_\alpha^u(x, C)=\lim_{\alpha\downarrow 0} uG_\alpha^u(x, C)=1$, we can rewrite the above in the form:

$$\overline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}\int_C uG_\alpha^u(x, dy)f(y)\leqq\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}\int_C uG_\alpha^u(x, dy)f(y).$$

Thus we have (5, 5) for $0\leqq f\leqq 1$, and hence for a bounded measurable function f, since such a function is expressed as $a\cdot\tilde{f}+b$ where $0\leqq\tilde{f}\leqq 1$ and $a\geqq 0$, b are constants.

LEMMA 10. *It holds that $g(x)=0$ at a point x where $P_x(\tau_0>0)=0$.*

PROOF. We have seen in the proof of Lemma 3 that

$$F_0(\alpha, u)=F_1(\alpha, u)+F_2(\alpha, u)+F_3(\alpha, u), \tag{5, 6}$$

where

$$F_0(\alpha, u)=G_\alpha^u(x, C)-\int_C\{r_\alpha(x, y)-\varphi(\alpha)-k(x, y)\}m(dy)-\int_C k(x, y)m(dy),$$

$$F_1(\alpha, u)=-\int_C uG_\alpha^u(x, dy)\int_C\{r_\alpha(y, z)-\varphi(\alpha)-k(y, z)\}m(dz),$$

$$F_2(\alpha, u)=-\int_C uG_\alpha^u(x, dy)\int_C k(y, z)m(dz),$$

$$F_3(\alpha, u)=\alpha\varphi(\alpha)G_\alpha^u(x, S)m(C).$$

Since $\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_0(\alpha, u)=\overline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_0(\alpha, u)$, we obtain that

$$\begin{aligned}&\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_1+\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_2+\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_3\\ &\geqq\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_1+\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_2+\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_3.\end{aligned}$$

The above two limits of F_1 are 0 by Lemma 1, and $\lim_{u\uparrow\infty}\lim_{\alpha\downarrow 0}F_2$ exists, since the limits of the other terms at (5, 6) exist. So by Lemma 9 $\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_2=\overline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_2=\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_2$, and hence $\lim_{\alpha\downarrow 0}\overline{\lim_{u\uparrow\infty}}F_3\geqq\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_3$. Therefore by Lemma 8 we have

$$0\geqq\underline{\lim_{u\uparrow\infty}}\lim_{\alpha\downarrow 0}F_3=g(x)m(C)\geqq 0,$$

which completes the proof.

PROOF OF THEOREM 3. Using Lemma 10 and the definitions of R and R_0, we can easily establish the proof.

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