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On a Class of Differential Operators with Polynomial Coefficients

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§1. Intoduction

In this paper, we study the existence and approximation of holomorphic solutions of a differential operator with polynomial coefficients. In general, we cannot expect the existence of holomorphic solutions even if the coefficients of an operator have no common zero ([7], [9], [10]). For example, in the complex two dimensional space $\mathbf{C}^2$, the equation

$$\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-1\right]u(x, y)=x$$

has no solution even in the space of formal power series.

An outline of this paper is as follows. In Section 2, we give some sufficient condition on a differential operator $L(\zeta, D)$ with polynomial coefficients under which $L(\zeta, D)\phi$ and ϕ have the same exponential type for every entire function ϕ (Theorem 1). This condition is then applied in Section 3 to show the existence and approximation of holomorphic solutions in some circular domain (Theorem 3).

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§2. Exponential type of entire solutions

Let $L(\zeta, D)$ be a differential operator with polynomial coefficients in $\mathbf{C}^n$. Then we can write

$$L(\zeta, D)=\sum_{\text{finite}} c_{\lambda\mu}\zeta^{\lambda}\left(\frac{\partial}{\partial\zeta}\right)^{\mu}, \tag{1}$$

where λ and μ are multi-indices, $c_{\lambda\mu}\in\mathbf{C}$, $\zeta^{\lambda}=\zeta_1^{\lambda_1}\cdots\zeta_n^{\lambda_n}$ and $\left(\frac{\partial}{\partial\zeta}\right)^{\mu}=\left(\frac{\partial}{\partial\zeta_1}\right)^{\mu_1}\cdots\left(\frac{\partial}{\partial\zeta_n}\right)^{\mu_n}$. We decompose L as follows:

$$L=L_l+L_{l+1}+\cdots+L_{l+k}, \qquad (k\geqq 0), \tag{2}$$

where $L_j=\sum_{|\lambda|-|\mu|=j} c_{\lambda\mu}\zeta^{\lambda}\left(\frac{\partial}{\partial\zeta}\right)^{\mu}$. We note that l may be a negative integer.

Definition 1. In the decomposition (2) of L, we shall call L_l the *leading part* of L. When the leading part L_l is written as

$$L_l(\zeta, D)=\sum_{\nu \text{ finite}} c_\nu \zeta^{\nu+\nu_0}\left(\frac{\partial}{\partial\zeta}\right)^\nu$$

for some multi-index $\nu_0=(\nu_1^{(0)}, \dots, \nu_n^{(0)})$, L_l is called ν_0-*simple*. In this case, l is equal to $|\nu_0|=\nu_1^{(0)}+\dots+\nu_n^{(0)}$. The ν_0-simple leading part L_l is said to be *of degree m*, if for any multi-index α such that $|\alpha|$ is sufficiently large the following inequality holds:

$$(3) \qquad |L_l[\zeta^\alpha]|=|\Sigma c_\nu \zeta^{\nu+\nu_0}\left(\frac{\partial}{\partial\zeta}\right)^\nu \zeta^\alpha| \geqq C|\alpha|^m|\zeta^{\nu_0+\alpha}|,$$

where $C>0$ is a constant independent of α and ζ.

Remark. In the one dimensional case ($n=1$), for every differential operator with polynomial coefficients $L(\zeta, D)$, $\zeta^\tau L(\zeta, D)$ has a ν_0-simple leading part for some integers $\tau\geqq 0$ and $\nu_0\geqq 0$, and in this case, the degree of its leading part is the highest order of differentiation in the leading part.

Example. Let $L(\zeta, D)=\left(\frac{\partial}{\partial\zeta_1}\right)^m\zeta_1^m+\cdots+\left(\frac{\partial}{\delta\zeta_n}\right)^m\zeta_n^m$. Then L is 0-simple and its degree is equal to m. In fact,

$$L[\zeta^\alpha]=\left(\frac{\partial}{\partial\zeta_1}\right)^m\zeta_1^{m+\alpha_1}\zeta_2^{\alpha_2}\cdots\zeta_n^{\alpha_n}+\cdots+\left(\frac{\partial}{\delta\zeta_n}\right)^m\zeta_1^{\alpha_1}\cdots\zeta_{n-1}^{\alpha_{n-1}}\zeta_n^{m+\alpha_n}$$

$$=\{\sum_{j=1}^n (m+\alpha_j)(m+\alpha_j-1)\cdots(\alpha_j+1)\}\zeta^\alpha.$$

Hence

$$|L[\zeta^\alpha]|\geqq\left(\sum_{j=1}^n \alpha_j^m\right)|\zeta^\alpha|\geqq\frac{1}{n^{m-1}}(\alpha_1+\cdots+\alpha_n)^m|\zeta^\alpha|.$$

For an entirely holomorphic function $f(\zeta)$, we have the following

Proposition 1. (Fuks [2], p. 339) *Let* $f(\zeta)=\sum_\alpha c_\alpha\zeta^\alpha$ *be entirely holomorphic, and* $\sigma=\inf\{\tau>0|\ |f(\zeta)|\leqq C_\tau \exp\tau|\zeta|$ *for some* $C_\tau\}$, *where* $|\zeta|=\max_{1\leqq j\leqq n}|\zeta_j|$. *Then*

$$e\sigma=\overline{\lim_{|\alpha|\to\infty}}|\alpha|\ |c_\alpha|^{1/|\alpha|}.$$

Remark that σ is called the exponential type of $f(\zeta)$ (with respect to the norm $|\zeta|$). For the more precise relation between the Taylor coefficients of $f(\zeta)$ and the type with respect to a norm $\rho(\zeta)$ on $\mathbf{C}^n$, we refer to Fuks [2] and Martineau [5].

Now, we state the main theorem in this section.

THEOREM 1. *Let $L(\zeta, D)$ be a differential operator with polynomial coefficients and let its leading part be ν_0-simple for some multi-index ν_0 and of degree m $(m\geqq 0)$. Further we assume the following condition* (A).

$$\text{(A)} \qquad \text{in } L_{l+j}=\sum_{|\lambda|-|\mu|=l+j} c_{\lambda\mu}\zeta^{\lambda}\left(\frac{\partial}{\delta\zeta}\right)^{\mu} \quad (j=1, \dots, k),$$

$$c_{\lambda\mu}=0 \quad if \quad |\mu|>m-j-1.$$

Then, every entire function $\phi(\zeta)$ such that $L(\zeta, D)\phi$ is of exponential type ,σ is also of exponential type σ.

PROOF. It is sufficient, by Proposition 1, to examine the growth of the Taylor coefficients of $\phi(\zeta)$. We set $\phi(\zeta)=\Sigma a_{\alpha}\zeta^{\alpha}$, and $L(\zeta, D)\phi=\psi(\zeta)=\Sigma b_{\beta}\zeta^{\beta}$. Then

$$\begin{aligned} L_l\phi &= \sum_{\nu} c_{\nu}\zeta^{\nu+\nu_0}\left(\frac{\partial}{\partial\zeta}\right)^{\nu}\sum_{\nu} a_{\alpha}\zeta^{\alpha} \\ &= \sum_{\alpha} a_{\alpha}L_l[\zeta^{\alpha}] \\ &= \sum_{\alpha} a_{\alpha}c(l;\alpha)\zeta^{\alpha+\nu_0}, \end{aligned}$$

where $c(l;\alpha)$ is the coefficient of $\zeta^{\alpha+\nu_0}$ in $L_l[\zeta^{\alpha}]$. Since L_l is of degree m, we obtain by (3),

$$\text{(4)} \qquad |c(l;\alpha)|\geqq C|\alpha|^m \text{ for sufficiently large } |\alpha|.$$

Similarly,

$$\begin{aligned} L_{l+j}\phi &= \sum_{\alpha} a_{\alpha}\Big(\sum_{|\lambda|-|\mu|=l+j} c_{\lambda\mu}\zeta^{\lambda}\left(\frac{\partial}{\partial\zeta}\right)^{\mu}\zeta^{\alpha}\Big) \\ &= \sum_{\alpha} a_{\alpha}\sum_{\lambda,\mu} c_j(\lambda,\mu;\alpha)\zeta^{\lambda+\alpha-\mu} \end{aligned}$$

where $c_j(\lambda,\mu;\alpha)$ is the coefficient of $\zeta^{\lambda+\alpha-\mu}$ in $L_{l+j}[\zeta^{\alpha}]$. By the condition (A), there exists a constant $C'>0$ which is independent of α such that

$$\text{(5)} \qquad |c_j(\lambda,\mu;\alpha)|\leqq C'|\alpha|^{m-j-1} \qquad \text{for any } \alpha.$$

Now, we compare the coefficients of $\zeta^{\lambda_0+\nu_0}$ in the both sides of the equation $L_l\phi=\psi-(L-L_l)\phi$. Then we have

$$\text{(6)} \qquad c(l;\lambda_0)a_{\lambda_0}=b_{\lambda_0+\nu_0}-\sum_{j=1}^{k}\sum_{\substack{|\lambda|-|\mu|=l+j \\ \lambda+\alpha-\mu=\lambda_0+\nu_0}} a_{\alpha}c_j(\lambda,\mu;\alpha).$$

We set $a_p=\max\{|a_{\alpha}|;|\alpha|=p\}$ and $b_p=\max\{|b_{\beta}|;|\beta|=p\}$. Then, by (5), for some constants C'_j,

$$|c(l;\lambda_0)|\ |a_{\lambda_0}|\leqq b_{|\lambda_0+\nu_0|}+\sum_{j=1}^{k}\sum_{\substack{|\lambda|-|\mu|=l+j\\ \lambda+\alpha-\mu=\lambda_0+\nu_0}}|a_\alpha c(\lambda,\mu;\alpha)|$$

$$\leqq b_{|\lambda_0+\nu_0|}+\sum_{j=1}^{k}C'_j(|\lambda_0|-j)^{m-j-1}a_{|\lambda_0|-j}.$$

If $|\lambda_0|=p$, we have by (4),

$$Cp^m a_p\leqq b_{p+l}+\sum_{j=1}^{k}C'_j p^{m-j-1}a_{p-j}.$$

Consequently we have the following inequality for sufficiently large p

$$a_p\leqq C_0p^{-m}b_{p+l}+\sum_{j=1}^{k}C_jp^{-j-1}a_{p-j} \tag{7}$$

where $C_0, C_1, \dots, C_k$ are independent of p. Now, if $\psi(\zeta)$ is of exponential type σ, by Proposition 1 we have for every $\tau<\sigma$,

$$b_p\leqq\left(\frac{e\tau}{p}\right)^p \quad \text{for sufficiently large } p.$$

Then we can choose a constant $M_0>0$ such that

$$b_p\leqq M_0\left(\frac{e\tau}{p}\right)^p \text{ for every } p\geqq 0.$$

If we suppose that $a_q\leqq M'_q\left(\frac{e\tau}{q}\right)^q$ for $0\leqq q\leqq p-1$, then

$$a_p\leqq C_0p^{-m}M_0\left(\frac{e\tau}{p+l}\right)^{p+l}+\sum_{j=1}^{k}C_jp^{-j-1}M'_{p-j}\left(\frac{e\tau}{p-j}\right)^{p-j}$$

$$=\left(\frac{e\tau}{p}\right)^p\{C_0M_0p^{p-m}(p+l)^{-(p+l)}(e\tau)^l+$$

$$\sum_{j=1}^{k}C_jM'_{p-j}p^{p-j-1}(p-j)^{-(p-j)}(e\tau)^{-j}\}.$$

Let $\tilde{M}_p$ be the max$\{M'_{p-1}, M'_{p-2}, \dots, M'_{p-k}\}$. Then

$$a_p\leqq\left(\frac{e\tau}{p}\right)^p\{C_0M_0(e\tau)^lp^{-(m+l)}+\tilde{M}_p\sum_{j=1}^{k}C_j(e\tau)^{-j}\left(\frac{p}{p-k}\right)^p\frac{1}{p-k}\}.$$

For sufficiently large p, we have

$$\sum_{j=1}^{k}C_j(e\tau)^{-j}\left(\frac{p}{p-k}\right)^p\frac{1}{p-k}\leqq 1$$

because $\left(\frac{p}{p-k}\right)^p$ converges to e^k as $p\to\infty$. Therefore, if we set $M_p=\tilde{M}_p+C_0M_0(e\tau)^lp^{-(m+l)}$, then

$$a_p \leqq M_p \left(\frac{e\tau}{p}\right)^p.$$

By induction, for some integer N and a constant M_N,

$$a_p \leqq \{M_N + C_0 M_0 (e\tau)^l \sum_{q=N}^{p} q^{-(m+l)}\} \left(\frac{e\tau}{p}\right)^p \quad \text{for any } p \geqq N.$$

Then if $m \geqq 2$, $\sum_{q=N}^{p} q^{-(m+l)}$ is bounded, so that, $\overline{\lim}_{p\to\infty} p a_p^{1/p} \leqq e\tau$. For the case $m=1$ or 0, the condition (A) shows that $L_{l+j}=0$ for $1 \leqq j \leqq k$. Hence $a_p \leqq \frac{C_0}{p} b_{p+l} \leqq \frac{C_0 M_0}{p}\left(\frac{e\tau}{p+l}\right)^{p+l}$, so we also have $\overline{\lim}_{p\to\infty} p a_p^{1/p} \leqq e\tau$. Since τ is an arbitrary number larger than σ, the exponential type of $\phi(\zeta)$ is, in any case, less than or equal to σ. Since the exponential type of $\psi(\zeta)$ is σ, it follows that $\phi(\zeta)$ must be of exponential type σ. This completes the proof.

We give an example which shows that the conclusion of Theorem 1 fails to hold if the condition (A) is not satisfied.

EXAMPLE. Let $L\left(\zeta, \frac{d}{d\zeta}\right) = \frac{d}{d\zeta}\zeta - \zeta$, $\zeta \in \mathbf{C}^1$. Then L has a 0-simple leading part of degree 1, but the condition (A) is not satisfied. We set, as before, $\phi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$, $\psi(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n$ and $L\left(\zeta, \frac{d}{d\zeta}\right)\phi(\zeta) = \psi(\zeta)$. Then comparing the coefficients of the both sides, we have

$$a_n = \frac{b_n}{n+1} + \frac{a_{n-1}}{n+1} \quad \text{for} \quad n \geqq 0, \quad (a_{-1}=0).$$

If $b_n = n^{-n} (n \geqq 1)$ and $b_0 = 1$, by induction

$$a_n = \frac{1}{(n+1)!}\left\{\frac{n!}{n^n} + \frac{(n-1)!}{(n-1)^{n-1}} + \cdots + \frac{2!}{2^2} + \frac{1!}{1^1} + 1\right\}.$$

From Stirling's formula $n! = n^n e^{-n}\sqrt{2\pi n}\, e^{\delta_n/12n} (0 < {}^{\exists}\delta_n < 1)$,

$$a_n \geqq \frac{1}{(n+1)!}\left\{\frac{1}{e^n} + \frac{1}{e^{n-1}} + \cdots + \frac{1}{e} + 1\right\} = \frac{1}{(n+1)!}\left\{\frac{1-\left(\frac{1}{e}\right)^{n+1}}{1-\frac{1}{e}}\right\}.$$

Then,

$$\overline{\lim}_{n\to\infty} n a_n^{1/n} \geqq \lim_{n\to\infty}\left(\frac{n^n}{(n+1)!}\right)^{1/n} \left\{\frac{1-\left(\frac{1}{e}\right)^{n+1}}{1-\frac{1}{e}}\right\}^{1/n} = e.$$

On the other hand, $\lim_{n\to\infty} nb_n^{1/n}=1$. That is, ψ is of exponential type e^{-1}, while the exponential type of ϕ is larger than or equal to 1.

REMARK. When we regard $L(\zeta, D)$ in Theorem 1 as an operator from the space of the entire functions of exponential type into itself, the codimension of the image of L is, in general, infinite. Since only the terms $\{b_{\alpha+\nu_0}\}$ determine a_α, if ψ is contained in the image of L, then the coefficients $\{b_\mu\}$ where μ cannot be represented by the form $\alpha+\nu_0$ for some α, must satisfy some conditions. In particular, if $\nu_0=(0, \dots, 0)$, that is, L has a 0-simple leading part, for every ψ of exponential type with $b_\mu=0$ for $|\mu|<N$ (N is chosen so that (4) holds for $|\alpha|\geqq N$), we can construct a solution ϕ of exponential type. Therefore the codimension of the image of L is finite and the basis of the complementary space consists of polynomials. Moreover if $c(l; \alpha)=c(0; \alpha)\neq 0$ for every α, L becomes a (topological) isomorphism.

Next we consider the case where ψ is holomorphic in a polydisc. The following proposition, a generalization of Cauchy-Hadamard's formula, is well known.

PROPOSITION 2. (Biermann-Lemaire) *The formal power series* $f(\zeta)=\Sigma c_\alpha\zeta^\alpha$ *is holomorphic in a polydisc* $\Delta(r)=\{\zeta|\ |\zeta_j|<r, j=1, \dots, n\}$ *if and only if*

$$\overline{\lim_{|\alpha|\to\infty}}\ |c_\alpha|^{1/|\alpha|}\leqq\frac{1}{r}.$$

We shall prove the following

THEOREM 2. *Suppose that the operator* $L(\zeta, D)$ *has a* ν_0*-simple leading part of degree* $m(\geqq 0)$. *Further we assume that*

$$\text{(A}'\text{)}\qquad \text{in } L_{l+j}=\sum_{|\lambda|-|\mu|=l+j} c_{\lambda\mu}\zeta^\lambda\left(\frac{\partial}{\partial\zeta}\right)^\mu \quad (j=1, \dots, k),$$

$$c_{\lambda\mu}=0 \quad \text{if} \quad |\mu|>m-j.$$

Then every formal power series $\phi(\zeta)$ *such that* $L(\zeta, D)\phi$ *is holomorphic in a polydisc* $\Delta(r)$ $(0<r\leqq+\infty)$, *is also holomorphic in* $\Delta(r)$.

PROOF. We use the same method as in the proof of Theorem 1. Then we have, instead of (7),

$$a_p\leqq C_0p^{-m}b_{p+l}+\sum_{j=1}^k C_jp^{-j}a_{p-j}$$

(notations are the same as those of Theorem 1). If $b_p\leqq M_0\rho_p$ for $p\geqq 0$ and $a_q\leqq M'_q\rho^q$ for $1\leqq q\leqq p-1$, then

$$\begin{aligned} a_p &\leqq \rho^p\{C_0M_0p^{-m}\rho^l+\sum_{j=1}^k C_jp^{-j}M'_{p-j}\rho^{-j}\} \\ &\leqq \{\tilde{M}_p+C_0M_0p^{-m}\rho^l\}\rho^p \end{aligned}$$

for sufficiently large p such that $\sum_{j=1}^{k}\frac{C_j}{(\rho p)^j}\leqq 1$, where $\tilde{M}_p=\max\{M'_{p-1},\ \ldots\ldots,\ M'_{p-k}\}$. Therefore if $m\geqq 1$, we have

$$a_p\leqq\left\{M+C_0M_0\rho^l\cdot\sum_{q=1}^{p}\frac{1}{q^m}\right\}\rho^p$$
$$\leqq\{M+C_0M_0\rho^l(1+\log p)\}\rho^p$$

for some constant $M>0$. Then $\overline{\lim}_{p\to\infty}a_p^{1/p}\leqq\rho$. For the case $m=0$, $L_{l+j}=0$ because of the condition (A'). Hence $a_p\leqq C_0M_0{}^{p+l}$, so we also have $\overline{\lim}_{p\to\infty}a_p^{1/p}\leqq\rho$. This means that ϕ is holomorphic in $\Delta(r)$.

In the following example, L has a 0-simple leading part of oreder 1, but does not satisfy the condition (A'). Then we can construct a function ϕ not entire for which $L\left(\zeta,\frac{d}{d\zeta}\right)\phi$ is entire.

EXAMPLE. $L\left(\zeta,\frac{d}{d\zeta}\right)=\frac{d}{d\zeta}\zeta-\frac{d}{d\zeta}\zeta^2$, and $\phi(\zeta)=\frac{1}{1-\zeta}$.

We consider the topological structure of the space of entire functions of exponential type. Let B be any nonnegative number. We denote by $\widetilde{\mathrm{Exp}}(B)$ the space of all entire functions f which satisfy

$$|f(\zeta)|\leqq C\exp B|\zeta| \tag{8}$$

for some constant C and every $\zeta\in\mathbf{C}^n$, where $|\zeta|=\max_{1\leqq j\leqq n}|\zeta_j|$. For $f\in\widetilde{\mathrm{Exp}}(B)$, we define $\|f\|_B$ as the infimum of the constant C in (8). Then $\widetilde{\mathrm{Exp}}(B)$ becomes a Banach space. The space of inductive limit of these $\widetilde{\mathrm{Exp}}(B)$ as $B\to r$ and $B<r$ is denoted by $\mathrm{Exp}(r)$. It is the space of all entire functions of exponential type less than r $(0<r\leqq+\infty)$.

PROPOSITION 3. *Let $L(\zeta, D)$ be the same operator as in Theorem* 1. *Suppose that $L_l(\zeta, D)\zeta^\alpha\neq 0$ for any $\zeta^\alpha\neq 0$. Then the map $L(\zeta, D)$*: $\mathrm{Exp}(r)\to\mathrm{Exp}(r)$ *is injective and has a closed range.*

PROOF. Let the filter $\{\psi_k\}$ converge to ψ_0 in $\mathrm{Exp}(r)$, and $\psi_k=L(\zeta, D)\phi_k$ for some $\phi_k\in\mathrm{Exp}(r)$ $(k\in\Lambda$, some ordered set). Since $\{\psi_k\}$ converge to ψ_0 uniformly on every compact set in $\mathbf{C}^n$, the Taylor coefficients $b_\mu^{(k)}$ of $\psi_k=\Sigma b_\mu^{(k)}\zeta^\mu$ converge to those $b_\mu^{(0)}$ of $\psi_0=\Sigma b_\mu^{(0)}\zeta^\mu$. By the assumption, each of the Taylor coefficients $a_\lambda^{(k)}$ of $\phi_k=\Sigma a_\lambda^{(k)}\zeta^\lambda$, is expressed as a finite linear combination of $\{b_\mu^{(k)}\}$, so that L is injective and $\{a_\lambda^{(k)}\}$ ($\forall\lambda$ fixed) becomes a converging filter, that is, there exists a formal power series $\phi_0(\zeta)=\Sigma a_\lambda^{(0)}\zeta^\lambda$ which satisfies $L(\zeta, D)\phi_0$

$(\zeta)=\psi_0(\zeta)$. Since $\psi_0 \in \mathrm{Exp}(r)$, by Theorem 1, ϕ_0 must be of exponential type less than r. This completes the proof.

§ 3. Existence and Approximation

Let Ω be a domain in $\mathbf{C}^n$. We denote by $H(\Omega)$ the space of all holomorphic functions in Ω with compact convergence topology. An elements S of the dual space $H'(\Omega)$ is called an anlytic functional in Ω, for which we define the Fourier transform $\hat{S}$ as follows:

$$\hat{S}(\zeta)=S_z(e^{<z,\zeta>}),$$

where $<z, \zeta> = \sum_{j=1}^{n} z_j\zeta_j$. A compact set K in Ω is called a carrier of S if there exists a constant C_ω for every neighborhood ω of K such that

$$|S(f)| \leqq C_\omega \sup_{z\in\omega}|f(z)|, \qquad f \in H(\Omega).$$

The next proposition is well known.

PROPOSITION 4. (Ehrenpreis-Martineau [5]-Hörmander [4]) *If $S \in H'(\Omega)$ is carried by a compact set K in Ω, then $\hat{S}(\zeta)$ is an entire function and for every $\delta > 0$, there is a constant C_δ such that*

$$|\hat{S}(\zeta)| \leqq C_\delta \exp(H_K(\zeta)+\delta|\zeta|), \qquad \zeta \in \mathbf{C}^n,$$

where $H_K(\zeta)=\sup_{z\in K} \mathrm{Re}<z, \zeta>$. Conversely, if K is a compact convex set and $M(\zeta)$ an entire function satisfying the above inequality for every $\delta > 0$, there exists an analytic functional S carried by K such that $\hat{S}(\zeta)=M(\zeta)$.

We then study the topological structure of the space of analytic functionals. Let Ω be a convex domain in $\mathbf{C}^n$, and let $\{K_j\}$ be a sequence of compact convex sets in Ω such that

$$K_j \subset K_{j+1}, \quad \text{and} \quad \bigcup_{j=1}^{\infty} K_j = \Omega.$$

The space of all entire function $f(\zeta)$ in $\mathbf{C}^n$ such that $|f(\zeta)| \leqq C \exp(H_{K_j}(\zeta))$, $\zeta \in \mathbf{C}^n$ is denoted by $\widetilde{\mathrm{Exp}}(K_j)$. As before, $\widetilde{\mathrm{Exp}}(K_j)$ becomes a Banach space, and $\widetilde{\mathrm{Exp}}(K_j) \subset \widetilde{\mathrm{Exp}}(K_{j+1})$. $\mathrm{Exp}(\Omega)$ is defined as the inductive limit of these spaces. Since Fourier transformation is injective on the space of analytic functionals in a Runge domain, it follows by Proposition 4 that $H'(\Omega)$ is algebraically isomorphic to $\mathrm{Exp}(\Omega)$.

LEMMA. *Fourier transformation from $H'(\Omega)$ to* $\mathrm{Exp}(\Omega)$ *is continuous.*

PROOF. Since $H'(\Omega)$ is a (DFS) space, it is sufficient to show that Fourier transformation is sequentially continuous. Let $S_j \in H'(\Omega)$ be any sequence converging to 0 in $H'(\Omega)$. $H(\Omega)$ is an (FS) space, so that it becomes quasi-normable (Grothendieck [3] p. 325, Prop. 1). Then, there exists a neighborhood V of 0 in $H(\Omega)$ such that S_j converges to 0 uniformly on V. We may take V as the set $\{f \in H(\Omega) \mid \sup_{z \in K_N} |f(z)| \leqq M\}$ for some constant M and a compact convex set K_N in Ω. In this case, $\hat{S}_j(\zeta)$ converges to 0 in $\widetilde{\mathrm{Exp}}(K_N)$, hence in $\mathrm{Exp}(\Omega)$.

From this lemma and the open mapping theorem (due to Ptak), we have

PROPOSITION 5. *If Ω is a convex domain in $\mathbf{C}^n$, then $H'(\Omega)$ is topologically isomorphic to* $\mathrm{Exp}(\Omega)$. (See also Ehrenpreis [1], Martineau [6].)

Let $P(z, D_z) = \Sigma c_{\lambda\mu} z^\mu \left(\frac{\partial}{\partial z}\right)^\lambda$ be a differential operator with polynomial coefficients. Then, the Fourier transform of the adjoint operator P' of $P(z, D_z)$ is $L(\zeta, D_\zeta) = \Sigma c_{\lambda\mu} \zeta^\lambda \left(\frac{\partial}{\partial \zeta}\right)^\mu$. In fact, for any $S \in H'(\Omega)$,

$$\begin{aligned}
(\widehat{P'(z, D_z)S})(\zeta) &= \langle P'S, e^{\langle z,\zeta \rangle} \rangle \\
&= \langle S, P(z, D_z) e^{\langle z,\zeta \rangle} \rangle \\
&= \Sigma c_{\lambda\mu} \langle S, \zeta^\lambda \left(\frac{\partial}{\partial \zeta}\right)^\mu e^{\langle z,\zeta \rangle} \rangle \\
&= \Sigma c_{\lambda\mu} \zeta^\lambda \left(\frac{\partial}{\partial \zeta}\right)^\mu \hat{S}(\zeta).
\end{aligned}$$

In order to prove the existence and approximation of the holomorphic solution, we use the next proposition due to F. Treves.

PROPOSITION 6. (Treves [8]) *Let E_0, F_0, E, F be locally convex topological linear spaces and E, F be Fréchet spaces. In the following commutative diagram (all maps are continuous and linear), we assume that the ranges of u_0,*

$$\begin{array}{ccc}
E_0 & \xrightarrow{\ i\ } & E \\
{\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u} \\
F_0 & \xrightarrow[\ j\ ]{} & F
\end{array}$$

u and i are dense in the corresponding spaces and that in E'_0, the dual space of E_0, the range of u'_0, the adjoint operator of u_0, is equal to the polar of the null space of u_0. Then the following two properties are equivalent.

1) *u is surjective and* $i\ (N(u_0))$ *is dense in* $N(u)$,

2) $y'_0 \in F'_0$ *such that* $u'_0(y'_0) \in R(i') \Rightarrow y'_0 \in R(j')$,

For every r $(0 < r \leqq +\infty)$, we define the domain $\Omega(r)$ in $\mathbf{C}^n$ as

$$\Omega(r) = \{z \mid \|z\| < r\},$$

where $\|z\| = |z_1| + \ldots + |z_n|$. Then, by Proposition 5, $H'(\Omega(r))$ is isomorphic to $\mathrm{Exp}(r) = \{f(\zeta) \in H(\mathbf{C}^n) \mid |f(\zeta)| \leqq C \exp\tau|\zeta| \text{ for some } \tau < r\}$ where $|\zeta| = \max_{1 \leqq j \leqq n} |\zeta_j|$, for if $K = \{z \mid \|z\| \leqq \tau\}$, then $H_K(\zeta) = \tau|\zeta|$.

THEOREM 3. *Let* $P(z, D_z) = \Sigma c_{\lambda\mu} z^\mu \left(\frac{\partial}{\partial z}\right)^\lambda$ *be a differential operator with polynomial coefficients. We assume that* $L(\zeta, D_\zeta) = \Sigma c_{\lambda\mu} \zeta^\lambda \left(\frac{\partial}{\partial \zeta}\right)^\mu$ *satisfies all the conditions in Proposition* 3, *that is, L has a* v_0*-simple leading part* L_l *(for some multi-index* v_0*) of degree* $m(\geqq 0)$, *and* $L_l(\zeta^\alpha) \neq 0$ *for any* $\zeta^\alpha \neq 0$, *and the condition* (A) *in Theorem* 1 *is fulfilled. Then for every r* $(0 < r \leqq +\infty)$, *we have*

1) $P(z, D_z)$: $H(\Omega(r)) \to H(\Omega(r))$ *is surj ective*

and

2) *for* $u \in H(\Omega(r))$ *such that* $P(z, D_z)u = 0$, *there exists a sequence* $\{u_j\}$ *in* $H(\mathbf{C}^n)$ *such that* $P(z, D_z)u_j = 0$ *and* $\{u_j\}$ *convergers to u in* $H(\Omega(r))$.

PROOF. We first prove the case $r = +\infty$. In this case, 2) is trivial. To show 1), it is sufficient to prove that the adjoint operator P' of P is injective and has a weakly closed range. Since $H(\mathbf{C}^n)$ is reflexive, a subspace in H' is weakly closed if and only if it is strongly closed. By Proposition 5, P' is injective and has a closed range if and only if L: $\mathrm{Exp}(r) \to \mathrm{Exp}(r)$ is injective and has a closed range, which follows from Proposition 3. In the general case, we apply Proposition 6 with $E_0 = F_0 = H(\mathbf{C}^n)$, $E = F = H(\Omega(r))$, i and j being natural injections, and $u_0 = u = P(z, D_z)$. Since $\Omega(r)$ is convex, the ranges of i and j are dense. By the first step of this proof, u_0 is surjective, so that all the assumptions of Proposition 6 are fulfilled. Therefore it is surfficient to show that every $S \in H'(\mathbf{C}^n)$ such that $P'(z, D_z)S \in H'(\Omega(r))$ is also an analytic functional in $\Omega(r)$. But this follows from Theorem 1. The proof is complete.

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