HIROSHIMA MATH. J.
3 (1973), 391–406

On the Irreducibility of Induced Representations of SU (2, 1)

Minoru WAKIMOTO

(Received May 21, 1973)

§ 1. Introduction

This note is concerned with the irreducibility of representations of $SU(2, 1)$ induced from one-dimensional representations of its minimal parabolic subgroup. Let $B=MA_+N$ be the minimal parabolic subgroup of G associated with an Iwasawa decomposition KA_+N of the group $G=SU(2, 1)$. Let $\mathfrak{g}_0=\mathfrak{su}(2, 1)$ be the Lie algebra of G, and $\mathfrak{a}_+$ (resp. $\mathfrak{n}_0$) the subalgebra of $\mathfrak{g}_0$ corresponding to A_+ (resp. N), and we define a linear form ρ on $\mathfrak{a}_+$ by

$$\rho(H)=2^{-1}\mathrm{Trace}(ad_{\mathfrak{n}_0}(H))$$

for every $H\in\mathfrak{a}_+$. Then a unitary character σ of M and a complex number λ define a representation $\mu_{\sigma\lambda}$ of B by

$$\mu_{\sigma\lambda}(m(\exp H)n)=\sigma(m)\exp(\lambda\rho(H))$$

for $m\in M$, $H\in\mathfrak{a}_+$ and $n\in N$. Let $\tilde{X}^{\sigma\lambda}$ be the space of all $\boldsymbol{C}$-valued C^∞-differentiable functions f on G such that

$$f(xb)=\mu_{\sigma,\lambda+1}(b^{-1})f(x)$$

for every $x\in G$ and $b\in B$. The group G acts on $\tilde{X}^{\sigma\lambda}$ by left-translations, and there exists a canonical G-invariant non-singular pairing between $\tilde{X}^{\sigma\lambda}$ and $\tilde{X}^{\sigma,-\bar{\lambda}}$. The universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ acts on $\tilde{X}^{\sigma\lambda}$ as infinitesimal representations of left-translations, and stabilizes the subspace $X^{\sigma\lambda}$ of $\tilde{X}^{\sigma\lambda}$ consisting of all K-finite elements. The K-module $X^{\sigma\lambda}$ has the irreducible decomposition

$$X^{\sigma\lambda}=\bigoplus_{\tau\in E_K^\sigma} X_\tau^{\sigma\lambda}$$

where E_K^σ is the set of all equivalence classes of irreducible unitary representations of K which contain σ when restricted to the subgroup M, and $X_\tau^{\sigma\lambda}$ denotes the K-submodule of $X^{\sigma\lambda}$ equivalent to τ. We shall make investigations into the irreducibility of the $\mathfrak{U}$-module $X^{\sigma\lambda}$ by using its K-module structure and a canonical pairing (,) of $X^{\sigma\lambda}$ and $X^{\sigma,-\bar{\lambda}}$ The set E_K^σ contains a one-dimensional representation of K, which we shall denote by τ_0. Choose $f_0 \in X^{\sigma\lambda}_{\tau_0}$ and $f'_0 \in X^{\sigma, -\bar{\lambda}}_{\tau_0}$ such that $(f_0, f'_0)=1$. There exists a K-submodule H^* of $\mathfrak{U}$ such that

i) $\mathfrak{U}f_0 = H^* f_0, \qquad \mathfrak{U}f'_0 = H^* f'_0$

and

ii) $H^* \otimes X^{\sigma\lambda}_{\tau_0}$ is K-isomorphic to $X^{\sigma\lambda}$.

Now the set of matrix elements

$$a_{nm} = (u_{nm} f_0, u_{nm} f'_0)$$

gives us an information about the irreducibility of the $\mathfrak{U}$-module $X^{\sigma\lambda}$, where $\{u_{nm}$: n and m are non-negative integers$\}$ is a set of highest weight vectors of the K-module H^* constructed in a standard way. These matrix elements are calculated by using Casimir elements of $\mathfrak{g}$ and $\mathfrak{k}$, and our main result can be stated as follows:

THEOREM. 1) *The $\mathfrak{U}$-module $X^{1_M, \lambda}$ is irreducible if and only if $|\lambda|$ is not a positive integer, and*

2) *when $\sigma \neq 1_M$, the $\mathfrak{U}$-module $X^{\sigma\lambda}$ is irreducible if and only if $\lambda - \nu$ is not an integer, where 1_M denotes the trivial representation of M and ν is a parameter of a unitary character σ of M which will be introduced in* § 2.

§ 2. A characterization of E^{ν}_K

Throughout this paper, we put $G = SU(2, 1)$ and $\mathfrak{g}_0 = \mathfrak{su}(2, 1)$. Let θ be a Cartan involution of $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ associated to θ, where $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{h}$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{h}_0$ respectively. Let Δ be the non-zero root system of $\mathfrak{g}$ with repect to $\mathfrak{h}$. For a root α in Δ, we set

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g};\ ad(H)X = \alpha(H)X \qquad \text{for every } H \in \mathfrak{h}\}.$$

Then the set Δ is the disjoint union of $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$, where $\Delta_{\mathfrak{k}}$ (resp. $\Delta_{\mathfrak{p}}$) is the set of all compact (resp. non-compact) roots:

$$\Delta_{\mathfrak{k}} = \{\alpha \in \Delta;\ \mathfrak{g}^{\alpha} \subset \mathfrak{k}\},$$

$$\Delta_{\mathfrak{p}} = \{\alpha \in \Delta;\ \mathfrak{g}^{\alpha} \subset \mathfrak{p}\}.$$

For each $\alpha \in \Delta$, the element H_{α} in $\mathfrak{h}$ is defined by

$$B(H_{\alpha}, H) = \alpha(H)$$

for every $H \in \mathfrak{h}$, where B is the Killing form of $\mathfrak{g}$. Let $\mathfrak{h}_R$ be the real linear subspace of $\mathfrak{h}$ generated by $\{H_\alpha; \alpha \in \Delta\}$, and $\mathfrak{h}_R^*$ its dual vector space. Then a lexicographic linear order in $\mathfrak{h}_R^*$ determines a positive root system Δ^+. We set

$$\Delta_{\mathfrak{k}}^+ = \Delta^+ \cap \Delta_{\mathfrak{k}} = \text{the set of all positive compact roots,}$$

and

$$\Delta_{\mathfrak{p}}^+ = \Delta^+ \cap \Delta_{\mathfrak{p}} = \text{the set of all positive non-compact roots.}$$

Since $G = SU(2, 1)$ is a simple Lie group of Hermitian type, a lexicographic linear order in $\mathfrak{h}_R^*$ can be so chosen that $\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}^+$ and $\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}^-$ are additively closed subsets of Δ. We fix a linear order in Δ as above. Let $\Pi = \{\alpha_1, \alpha_2\}$ be the fundamental root system of Δ with respect to this linear order, where we may assume that α_1 is compact and α_2 is non-compact. For a root $\alpha \in \Delta$, we define a linear form α^* on $\mathfrak{h}_R$ by

$$\alpha^* = 2 < H_\alpha, H_\alpha >^{-1} \alpha,$$

where $< \ , \ >$ is the inner product on $\mathfrak{h}$ via the Killing form B of $\mathfrak{g}$. The set $\{\alpha_1^*, \alpha_2^*\}$ is a basis of $\mathfrak{h}_R^*$, and let $\{\varepsilon_1^*, \varepsilon_2^*\}$ be its dual basis of $\mathfrak{h}_R$. The inner product $< \ , \ >$ on $\mathfrak{h}$ defines a linear isomorphism of $\mathfrak{h}_R^*$ onto $\mathfrak{h}_R$, and under this linear isomorphism, we have

$$\alpha_1 = 2\varepsilon_1^* - \varepsilon_2^*,$$

and

$$\alpha_2 = -\varepsilon_1^* + 2\varepsilon_2^*.$$

LEMMA 2.1. *For each $\alpha \in \Delta$, a vector $X_\alpha \in \mathfrak{g}^\alpha$ can be chosen so that*
1) $B(X_\alpha, X_{-\alpha}) = 1$,
2) $\sigma X_\alpha = -X_{-\alpha}$ *if* $\alpha \in \Delta_{\mathfrak{k}}$,
3) $\sigma X_\alpha = X_{-\alpha}$ *if* $\alpha \in \Delta_{\mathfrak{p}}$,

where σ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

PROOF. For each $\alpha \in \Delta$, we select $E_\alpha \in \mathfrak{g}^\alpha$ such that

$$B(E_\alpha, E_{-\alpha}) = 1 \qquad \text{for all } \alpha \in \Delta.$$

Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$, we have $\sigma(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ for every $\alpha \in \Delta$. So there exists a non-zero scalar $a_\alpha \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ such that

$$\sigma E_\alpha = a_\alpha E_{-\alpha}.$$

Since $B(\sigma E_\alpha, \sigma E_{-\alpha}) = \overline{B(E_\alpha, E_{-\alpha})} = 1$, we have

$$a_\alpha a_{-\alpha}=1.$$

Also, by $\sigma^2=1$, we have

$$a_\alpha \overline{a_{-\alpha}}=1.$$

So a_α is real, and by setting $X_\alpha=|a_\alpha|^{-\frac{1}{2}}E_\alpha$, we have

$$B(X_\alpha, X_{-\alpha})=1,$$

and

$$\begin{aligned}\sigma X_\alpha &= |a_\alpha|^{-\frac{1}{2}}\sigma E_\alpha=|a_\alpha|^{-\frac{1}{2}}a_\alpha E_{-\alpha}=(|a_\alpha|^{-1}a_\alpha)|a_\alpha|^{\frac{1}{2}}E_{-\alpha}\\ &=(\operatorname{sgn} a_\alpha)|a_{-\alpha}|^{-\frac{1}{2}}E_{-\alpha}=(\operatorname{sgn} a_\alpha)X_{-\alpha},\end{aligned}$$

where $\operatorname{sgn} a$ (a is a non-zero real number) designates the signature of a.

2) Suppose that α is a compact root. If $\sigma X_\alpha=X_{-\alpha}$, then $X_\alpha+X_{-\alpha}$ belongs to $\mathfrak{k}_0$. Since B is negative definite on $\mathfrak{k}_0$, we have

$$B(X_\alpha+X_{-\alpha}, X_\alpha+X_{-\alpha})<0.$$

This implies $B(X_\alpha, X_{-\alpha})<0$, which contradicts $B(X_\alpha, X_{-\alpha})=1$. Thus we have $\sigma X_\alpha=-X_{-\alpha}$ for every $\alpha\in\Delta_{\mathfrak{k}}$.

3) Suppose that α is a non-compact root. If $\sigma X_\alpha=-X_{-\alpha}$, then $X_\alpha+X_{-\alpha}$ belongs to $\sqrt{-1}\mathfrak{p}_0$. Since B is negative definite on $\sqrt{-1}\mathfrak{p}_0$, we have

$$B(X_\alpha+X_{-\alpha}, X_\alpha+X_{-\alpha})<0,$$

which is inconsistent with $B(X_\alpha, X_{-\alpha})=1$. Thus we have $\sigma X_\alpha=X_{-\alpha}$ for every $\alpha\in\Delta_{\mathfrak{p}}$. Q.E.D.

We define the number $N_{\alpha\beta}$ ($\alpha, \beta\in\Delta$) by

$$\begin{aligned}&[X_\alpha, X_\beta]=N_{\alpha\beta}X_{\alpha+\beta} &&\text{if } \alpha+\beta\in\Delta,\\ &N_{\alpha\beta}=0 &&\text{if } \alpha+\beta\notin\Delta.\end{aligned}$$

Then

LEMMA 2.2. *$|N_{\alpha\beta}|^2=2^{-1}q(1-p)\alpha(H_\alpha)$, where $\beta+n\alpha$ ($p\leqq n\leqq q$) is the α-series containing β.*

PROOF. Let $\tau=\sigma\theta$ be the conjugation of $\mathfrak{g}$ with respect to a compact real form $\mathfrak{g}_u=\mathfrak{k}_0+\sqrt{-1}\mathfrak{p}_0$. Then the vectors in Lemma 2.1 satisfy

$$\tau X_\alpha=-X_{-\alpha}$$

for every $\alpha\in\Delta$. Now we have

$$\tau[X_\alpha, X_\beta]=[\tau X_\alpha, \tau X_\beta]=[-X_{-\alpha}, -X_{-\beta}]$$
$$=[X_{-\alpha}, X_{-\beta}]=N_{-\alpha, -\beta}X_{-(\alpha+\beta)},$$

and

$$\tau[X_\alpha, X_\beta]=\tau(N_{\alpha\beta}X_{\alpha+\beta})=\overline{N_{\alpha\beta}}\tau X_{\alpha+\beta}=-\overline{N_{\alpha\beta}}X_{-(\alpha+\beta)}.$$

Hence

$$N_{-\alpha,-\beta}=-\overline{N_{\alpha\beta}}.$$

From Lemma 5.2 (Chap. III) of Helgason [2], we have

$$N_{\alpha\beta}N_{-\alpha,-\beta}=-2^{-1}q(1-p)\alpha(H_\alpha).$$

Thus we have

$$|N_{\alpha\beta}|^2=2^{-1}q(1-p)\alpha(H_\alpha).$$

Q.E.D.

By Lemma 2.1, the element $H_0=\sqrt{<\alpha, \alpha>/2}(X_{\alpha_1+\alpha_2}+X_{-(\alpha_1+\alpha_2)})$ is in $\mathfrak{p}_0$. Let $\mathrm{Int}(\mathfrak{g})$ denote the group of all inner automorphisms of $\mathfrak{g}$.

LEMMA 2.3 *There exists an element w in $\mathrm{Int}(\mathfrak{g})$ such that $w(H_{\alpha_1+\alpha_2})=H_0$ and $w(H_{\alpha_1}-H_{\alpha_2})=H_{\alpha_1}-H_{\alpha_2}$.*

PROOF. We shall show that

$$w=\exp\left(-\frac{\pi}{2\sqrt{2<\alpha, \alpha>}}ad\,(X_\alpha-X_{-\alpha})\right)$$

has the required properties, where $\alpha=\alpha_1+\alpha_2$. We set

$$Z=-\frac{\pi}{2\sqrt{2<\alpha, \alpha>}}(X_\alpha-X_{-\alpha}).$$

Then we have

$$ad(Z)(H_{\alpha_1}-H_{\alpha_2})=0$$
$$ad(Z)(H_\alpha)=2^{-1}\pi\sqrt{2^{-1}<\alpha, \alpha>}(X_\alpha+X_{-\alpha})$$
$$(adZ)^2(H_\alpha)=-(\pi/2)^2H_\alpha.$$

So we have

$$(\exp ad(Z))(H_{\alpha_1}-H_{\alpha_2})=H_{\alpha_1}-H_{\alpha_2},$$

and

$$(\exp ad(Z))(H_{\alpha_1+\alpha_2}) = \cos(\pi/2) + \sqrt{<\alpha, \alpha>/2}\,\sin(\pi/2)(X_\alpha + X_{-\alpha})$$
$$= \sqrt{<\alpha, \alpha>/2}\,(X_\alpha + X_{-\alpha}).$$

Q.E.D.

We set

$$\mathfrak{a}_+ = \boldsymbol{R}H_0,$$
$$\mathfrak{a}_- = \sqrt{-1}\boldsymbol{R}(H_{\alpha_1} - H_{\alpha_2}) = \sqrt{-1}\boldsymbol{R}(\varepsilon_1^* - \varepsilon_2^*),$$
$$\mathfrak{a}_0 = \mathfrak{a}_- + \mathfrak{a}_+,$$
$$\mathfrak{z}_0 = \boldsymbol{R}(\sqrt{-1}\varepsilon_2^*),$$
$$\mathfrak{k}'_0 = \boldsymbol{R}(\sqrt{-1}H_{\alpha_1}) + (\mathfrak{g}^{\alpha_1} + \mathfrak{g}^{-\alpha_1}) \cap \mathfrak{g}_0,$$

and let $\mathfrak{a}_+^c$, $\mathfrak{a}_-^c$, $\mathfrak{a}$, $\mathfrak{z}$, $\mathfrak{k}'$ be the complexifications of $\mathfrak{a}_+$, $\mathfrak{a}_-$, $\mathfrak{a}_0$, $\mathfrak{z}_0$, $\mathfrak{k}'_0$ respectively. Then $\mathfrak{a}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ with a maximal vector part. Let Λ be the non-zero root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Since $w\mathfrak{h} = \mathfrak{a}$, each element μ in $\mathfrak{h}^* = \mathrm{Hom}_c(\mathfrak{h}, \boldsymbol{C})$ is transformed to a linear form $w\mu$ on $\mathfrak{a}$:

$$(w\mu)(H) = \mu(w^{-1}H) \qquad \text{for every } H \in \mathfrak{a}.$$

Under this transformation, we have $\Lambda = w(\Delta)$. We set

$$\mathfrak{g}^{w\alpha} = w\mathfrak{g}^\alpha \qquad (\alpha \in \Delta),$$
$$H_{w\alpha} = wH_\alpha \qquad (\alpha \in \Delta),$$
$$\beta_i = w\alpha_i \qquad (i = 1, 2),$$

and

$$\Lambda^+ = w(\Delta^+).$$

Since $\mathfrak{a}_+ = \boldsymbol{R}(\beta_1 + \beta_2)$ and $<\beta_i, \beta_1 + \beta_2> > 0$ for $i = 1, 2$, this linear order in Λ is compatible relative to $(\mathfrak{a}_R, \mathfrak{a}_+)$ where $\mathfrak{a}_R = w\mathfrak{h}_R = \sqrt{-1}\mathfrak{a}_- + \mathfrak{a}_+$. We set

$$\mathfrak{n}_0 = (\sum_{\beta \in \Lambda^+} \mathfrak{g}^\beta) \cap \mathfrak{g}_0.$$

Let K, A_+ and N be the analytic subgroups of G generated by $\mathfrak{k}_0$, $\mathfrak{a}_+$ and $\mathfrak{a}_0$ respectively. The centralizer M of $\mathfrak{a}_+$ in K is connected and coincides with $A_- = A \cap K$, where A is the Cartan subgroup of G corresponding to $\mathfrak{a}_0$.

The set $\hat{M}$ of all unitary characters of M is given by $\{\sigma_\nu;\ \nu \in \frac{1}{2}\boldsymbol{Z}$ (i.e., $2\nu \in \boldsymbol{Z}$)$\}$, where σ_ν is the unitary character of M whose derivative is the restriction of

$\nu(\varepsilon_1^*-\varepsilon_2^*)$ to $\mathfrak{a}_-$. We define a linear form ρ on $\mathfrak{a}_+$ by

$$\rho(H)=2^{-1}\sum_{\beta\in\Delta_+}\beta(H)=(\beta_1+\beta_2)(H).$$

Then the set $\hat{A}$ of all characters of A is given by $\{\xi_\lambda;\ \lambda\in\boldsymbol{C}\}$, where ξ_λ is the character of A defined by $\xi_\lambda(\exp H)=\exp(\lambda\rho)(H)$ for every $H\in\mathfrak{a}_+$. For $\nu\in\frac{1}{2}\boldsymbol{Z}$ and $\lambda\in\boldsymbol{C}$, we set

$$\tilde{X}^{\nu\lambda}=\left\{\begin{array}{c} f\in C^\infty(G);\ f(xman)=\sigma_\nu(m^{-1})\xi_{\lambda+1}(a^{-1})f(x) \\ \text{for every } x\in G,\ m\in M,\ a\in A_+ \text{ and} \\ n\in N \end{array}\right\},$$

and define a G-module structure $\tilde{\pi}^{\nu\lambda}$ on $\tilde{X}^{\nu\lambda}$ by

$$(\tilde{\pi}^{\nu\lambda}(x)f)(y)=f(x^{-1}y)$$

for $x,\ y\in G$ and $f\in\tilde{X}^{\nu\lambda}$. The representation $\tilde{\pi}^{\nu\lambda}$ determines the infinitesimal representation $\tilde{\pi}_*^{\nu\lambda}$ of $\mathfrak{g}_0$ on $\tilde{X}^{\nu\lambda}$, which can be extended to the representation of the universal enveloping algebra $\mathfrak{U}=\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Let $X^{\nu\lambda}$ be the subspace of $\tilde{X}^{\nu\lambda}$ consisting of all $\tilde{\pi}^{\nu\lambda}(K)$-finite vectors in $\tilde{X}^{\nu\lambda}$. The space $X^{\nu\lambda}$ is stable under $\tilde{\pi}^{\nu\lambda}(K)$ and $\tilde{\pi}_*^{\nu\lambda}(\mathfrak{U})$. Let $\pi^{\nu\lambda}$ (resp. $\pi_*^{\nu\lambda}$) denote the representation of K (resp. $\mathfrak{U}$) on $X^{\nu\lambda}$.

Let E_K be the set of all equivalence classes of irreducible unitary representations of K. For $\nu\in\frac{1}{2}\boldsymbol{Z}$, we set

$$E_K^\nu=\{\tau\in E_K;\ [\tau|M:\sigma_\nu]\geqq 1\},$$

where $[\tau|M:\sigma_\nu]$ denotes the multiplicity of σ_ν in the representation $\tau|M$ which is the restriction of τ to the subgroup M. Let K' (resp. Z) be the semisimple part (resp. the center) of K. Then K' and Z are isomorphic to $SU(2)$ and $U(1)$, and are the analytic subgroups of K generated by $\mathfrak{k}'_0$ and $\mathfrak{z}_0$ respectively. A unitary representation of K is determined by a representation of K' and a character of Z. A representation of K' is characterized by its highest weight, while unitary characters of Z are parametrized by integers. So the set E_K is characterized by $\{a\varepsilon_1^*+b\varepsilon_2^*;\ a\in\boldsymbol{N}_0,\ b\in\boldsymbol{Z}\}$, where $\boldsymbol{N}_0$ is the set of all non-negative integers. The irreducible representation of K corresponding to $a\varepsilon_1^*+b\varepsilon_2^*$ is denoted by $\tau_{(a,b)}$.

PROPOSITION 2.4. *For a half integer $\nu\in\frac{1}{2}\boldsymbol{Z}$,*

$$E_K^\nu=\left\{\begin{array}{l}\tau_{(a,b)}:\ a\in\boldsymbol{N}_0,\ b\in\boldsymbol{Z},\ b=a-3k-2\nu \\ \qquad\text{for some integer } k \text{ such that } 0\leqq k\leqq a\end{array}\right\}.$$

PROOF. Since $\alpha_1=2\varepsilon_1^*-\varepsilon_2^*\in\mathfrak{k}'$ and $\varepsilon_2^*\in\mathfrak{z}'$, we decompose $a\varepsilon_1^*+b\varepsilon_2^*$ to the sum of $\mathfrak{k}'$-part and $\mathfrak{z}'$-part:

$$a\varepsilon_1^* + b\varepsilon_2^* = a(\varepsilon_1^* - \varepsilon_2^*/2) + (b + a/2)\varepsilon_2^*.$$

Since $\mathfrak{k}'$ is isomorphic to $\mathfrak{su}(2)$ and $\alpha_1 = 2\varepsilon_1^* - \varepsilon_2^*$, the weights of $\tau_{(a,b)}$ are given by

$$\{(a-2k)(\varepsilon_1^* - \varepsilon_2^*/2) + (b + a/2)\varepsilon_2^*;\ k \in \boldsymbol{Z},\ 0 \leqq k \leqq a\}.$$

By the condition that $\tau_{(a,b)} \in E_K^\nu$, there exists an integer $k(0 \leqq k \leqq a)$ such that $(a-2k)(\varepsilon_1^* - \varepsilon_2^*/2) + (b + a/2)\varepsilon_2^*$ is equal to $\nu(\varepsilon_1^* - \varepsilon_2^*)$ when restricted to $\mathfrak{a}_- = \sqrt{-1}\boldsymbol{R}(\alpha_1^* - \alpha_2^*)$ So we have

$$a - b - 3k = 2\nu$$

for some integer $k(0 \leqq k \leqq a)$. Q.E.D.

COROLLARY 2.5. $[\tau|M : \sigma_\nu] = 1$ *for every* $\tau \in E_K^\nu$.

From Proposition 2.4, we can see that $\tau_{(0,-2\nu)}$ belongs to E_K^ν, in other words, there exists a (unique) one-dimensional unitary representation in E_K^ν. Henceforward we fix a half integer $\nu \in \frac{1}{2}\boldsymbol{Z}$ and, for the sake of simplicity, we write τ_0 instead of $\tau_{(0,-2\nu)}$.

For $\tau \in E_K$, let $X_\tau^{\nu\lambda}$ denote the isotypic component of $X^{\nu\lambda}$ of type τ, that is, $X_\tau^{\nu\lambda}$ is the sum of all K-submodules of $X^{\nu\lambda}$ which is isomorphic to τ. Then, by the Frobenius' reciprocity theorem, $X^{\nu\lambda}$ is the direct sum of K-submodules $\{X_\tau^{\nu\lambda};\ \tau \in E_K^\nu\}$:

$$X^{\nu\lambda} = \bigoplus_{\tau \in E_K^\nu} X_\tau^{\nu\lambda}.$$

And, by Corollary 2.5, $X_\tau^{\nu\lambda}$ is the irreducible K-submodule of $X^{\nu\lambda}$ isomorphic to τ.

There exists a K-invariant non-singular pairing (,) between $X^{\nu\lambda}$ and $X^{\nu,-\bar{\lambda}}$, which is given by

$$(f, g) = \int_K f(k)\overline{g(k)}dk$$

for $f \in X^{\nu\lambda}$ and $g \in X^{\nu,-\bar{\lambda}}$, where dk is the Haar measure on K normalized by $\int_K dk = 1$. This pairing (,) is $\mathfrak{U}$-invariant in the sense that the following equality holds:

$$(\pi_*^{\nu\lambda}(u)f, g) = (f, \pi_*^{\nu,-\bar{\lambda}}(u^s)g)$$

for every $u \in \mathfrak{U}, f \in X^{\nu\lambda}$ and $g \in X^{\nu,-\bar{\lambda}}$, where $u \to u^s$ is the $\boldsymbol{R}$-linear automorphism of the linear space $\mathfrak{U}$ such that i) $X^s = -X$ for $X \in \mathfrak{g}_0$, ii) $(\alpha u)^s = \bar{\alpha}u^s$ for $\alpha \in \boldsymbol{C}$

and $u \in \mathfrak{U}$, and iii) $(uv)^s = v^s u^s$ for every $u, v \in \mathfrak{U}$. Since the K-module $X^{\nu, -\bar{\lambda}}$ is isomorphic to $X^{\nu\lambda}$, it decomposes into the direct sum of irreducible K-submodules:

$$X^{\nu,-\bar{\lambda}} = \bigoplus_{\tau \in E_K^\nu} X_\tau^{\nu,-\bar{\lambda}}.$$

Choose $f_0 \in X_{\tau_0}^{\nu\lambda}$ and $f_0' \in X_{\tau_0}^{\nu,-\bar{\lambda}}$ such that $(f_0, f_0') = 1$.

The space $\mathfrak{p}$ admits the canonical K-module structure. Let $\mathfrak{p}'$ be the K-module dual to $\mathfrak{p}$, and $S' = S(\mathfrak{p}')$ (resp. $S = S(\mathfrak{p})$) the symmetric algebra over $\mathfrak{p}'$ (resp. $\mathfrak{p}$). The algebra S' may be regarded as the polynomial ring on $\mathfrak{p}$, while S as the ring of differential operators on S' with constant coefficients, and each algebra carries the canonical K-module structure extended from that on $\mathfrak{p}$ or $\mathfrak{p}'$. We set

$$J = \{x \in S;\ kx = x \qquad \text{for every } k \in K\}$$

and

$$J_+ = \{x \in J;\ \text{the constant part of } x \text{ is zero}\}$$

$$= J \cap \sum_{i=1}^{\infty} S^i,$$

where S^i is the subspace of S consisting of all homogeneous elements of the degree i. And we define the space H' of all harmonic polynomials on $\mathfrak{p}$ by

$$H' = \{f \in S';\ xf = 0 \qquad \text{for every } x \in J_+\}.$$

The K-modules $\mathfrak{p}$ and $\mathfrak{p}'$ are isomorphic via the Killing form B of $\mathfrak{g}$, and this isomorphism can be extended to the K-isomorphism of S' onto S. The image of H' under this isomorphism is denoted by H.

It is well known that there exists a linear isomorphism β of the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$ onto the universal enveloping algebra $\mathfrak{U}$ such that (i) $\beta(X^k) = (\beta(X))^k$ for every $X \in \mathfrak{g}$ and $k \in \boldsymbol{N}_0$ and (ii) (with the obvious identification) β is the identity map on $\mathfrak{g}$. This mapping is called the symmetrization and has the following property:

$$\beta(X_1 \dots X_k) = (k!)^{-1} \sum_{\sigma \in \mathfrak{S}_k} X_{\sigma(1)} \cdots X_{\sigma(k)}$$

for $X_1, \dots, X_k \in \mathfrak{g}$, where $\mathfrak{S}_k$ denotes the permutation group of k-numbers $\{1, \dots, k\}$.

We set $H^* = \beta(H)$. Note that the restriction $\beta | H$ of β on H is a K-isomorphism of H onto H^*.

LEMMA 2.6. ([5], Proposition 10)

$$\pi_*^{\nu\lambda}(\mathfrak{U})f_0=\pi_*^{\nu\lambda}(H^*)f_0.$$

Let φ_λ (resp. $\varphi_{-\bar{\lambda}}$) be a linear mapping of H^* to $X^{\nu\lambda}$ (resp. $X^{\nu,-\bar{\lambda}}$) defined by

$$\varphi_\lambda(u)=\pi_*^{\nu\lambda}(u)f_0,$$

and

$$\varphi_{-\bar{\lambda}}(u)=\pi_*^{\nu,-\bar{\lambda}}(u)f_0'.$$

Lemma 2.7. *f_0 is $\mathfrak{U}$-cyclic in $X^{\nu\lambda}$ if and only if* Ker φ_λ *is zero.*

Proof. By Kostant-Rallis [4] and Corollary 2.5, the K-module H^* decomposes into the direct sum of irreducible K-submodules $\{H_\tau^*;\ \pi\in E_K^0\}$;

$$H^*=\bigoplus_{\tau\in E_K^0} H_\tau^*$$

Since τ_0 is one-dimensional, the mapping of E_K^0 to E_K^ν defined by $\tau\to\tau\otimes\tau_0$ is bijective. This proves the lemma.

Q.E.D.

Proposition 2.8. *$X^{\nu\lambda}$ is $\mathfrak{U}$-irreducible if and only if* $\mathrm{Ker}\,\varphi_\lambda=\{0\}$ *and* $\mathrm{Ker}\,\varphi_{-\bar{\lambda}}=\{0\}$.

Proof. By the existence of a $\mathfrak{U}$-invariant non-singular pairing of $X^{\nu\lambda}$ and $X^{\nu,-\bar{\lambda}}$, $X^{\nu\lambda}$ is $\mathfrak{U}$-irreducible if and only if $X^{\nu,-\bar{\lambda}}$ is $\mathfrak{U}$-irreducible. If $X^{\nu\lambda}$ is $\mathfrak{U}$-irreducible, f_0' and f_0 are $\mathfrak{U}$-cyclic, and so by Lemma 2.7, we have $\mathrm{Ker}\,\varphi_\lambda=\{0\}$ and $\mathrm{Ker}\,\varphi_{-\bar{\lambda}}=\{0\}$. Conversely, assume that $\mathrm{Ker}\,\varphi_\lambda=\{0\}$ and $\mathrm{Ker}\,\varphi_{-\bar{\lambda}}=\{0\}$. Let V be a $\mathfrak{U}$-invariant subspace of $X^{\nu\lambda}$. Since each element in $X^{\nu\lambda}$ is K-finite, V is a K-invariant subspace of $X^{\nu\lambda}$. Let $V^\perp$ be the orthogonal complement of V in $X^{\nu,-\bar{\lambda}}$ with respect to (,). Then it occurs that i) $X_{\tau_0}^{\nu\lambda}\subset V$ or ii) $X_{\tau_0}^{\nu,-\bar{\lambda}}\subset V^\perp$. Since, by our assumption, f_0 and f_0' are $\mathfrak{U}$-cyclic in $X^{\nu\lambda}$ and $X^{\nu,-\bar{\lambda}}$ respectively, i) implies $V=X^{\nu\lambda}$, while ii) implies $V^\perp=X^{\nu,-\bar{\lambda}}$ or equivalently $V=\{0\}$. Therefore $X^{\nu\lambda}$ is $\mathfrak{U}$-irreducible.

Q.E.D.

§ 3. *K*-highest weight vectors in $X^{\nu\lambda}$

The space H decomposes into the direct sum of irreducible K-submodules:

$$H=\bigoplus_{\tau\in E_K^0} H_\tau,$$

and E_K^0 is given by

$$E_K^0=\{\tau_{(a,a-3k)};\ a,\ k\in \boldsymbol{N}_0 \quad \text{and} \quad k\leqq a\}.$$

In this section, we shall describe highest weight vectors in H_τ and $\varphi_\lambda(H_\tau)$.

We set $X_+=X_{\alpha_1+\alpha_2}$ and $X_-=X_{-\alpha_2}$. The vector X_+ (resp. X_-) is a highest weight vector of the K-module $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$), where $\mathfrak{p}_+=\sum_{\alpha\in\Delta_\mathfrak{p}^+}\mathfrak{g}^\alpha$ and $\mathfrak{p}_-=\sum_{\alpha\in\Delta_\mathfrak{p}^+}\mathfrak{g}^{-\alpha}$. As one can see easily, $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is the irreducible K-module characterized by $\tau_{(1,1)}$ (resp. $\tau_{(1,-2)}$).

LEMMA 3.1. *For n, $k\in \boldsymbol{N}_0$ $(0\leqq k\leqq n)$, $X_-^k X_+^{n-k}$ is a highest weight vector in $H_{\tau(n,n-3k)}$.*

PROOF. It is enough to prove that $X_-^k X_+^{n-k}$ is in H. By Kostant-Rallis [4], H is the linear subspace of $S(\mathfrak{p})$ generated by $\{X^m;\ X$ is a nilpotent element in $\mathfrak{p}$, $m\in \boldsymbol{N}_0\}$. And aX_-+bX_+ is a nilpotent element in $\mathfrak{p}$ for any a, $b\in \boldsymbol{C}$. So we have

$$(aX_-+bX_+)^n\in H$$

for every a, $b\in \boldsymbol{C}$. Thus we have $X_-^kX_+^{n-k}\in H$.

Q.E.D.

LEMMA 3.2. *For n, $m\in \boldsymbol{N}_0$,*

$$\pi_*^{\nu\lambda}(\beta(X_-^nX_+^m))f_0=\pi_*^{\nu\lambda}(X_-^n X_+^m)f_0.$$

PROOF. It suffices to show that

$$\pi_*^{\nu\lambda}([X_+,\ X_-])\pi_*^{\nu\lambda}(X_-^k X_+^l)f_0=0$$

for every k, $l\in \boldsymbol{N}_0$. And this equality holds, since $[X_+,\ X_-]$ is a scalar multiple of X_{α_1} and $\pi_*^{\nu\lambda}(X_-^k X_+^l)f_0$ is a highest weight vector in $X_{\tau(k+l,l-2k)}^{\nu\lambda}$.

Q.E.D.

Summing up Lemma 2.7, Lemma 3.1 and Lemma 3.2, we have the following:

LEMMA 3.3. *f_0 is $\mathfrak{U}$-cyclic in $X^{\nu\lambda}$ if and only if*

$$\pi_*^{\nu\lambda}(X_-^n X_+^m)f_0=0$$

for every n, $m\in \boldsymbol{N}_0$.

We set

$$f_{nm}=\pi_*^{\nu\lambda}(X_-^n X_+^m)f_0,$$

$$f'_{nm}=\pi_*^{\nu,-\bar{\lambda}}(X_-^nX_+^m)f'_0,$$

and

$$a_{nm}=(f_{nm}, f'_{nm})$$

for n, $m \in \boldsymbol{N}_0$. Then we have

PROPOSITION 3.4. *$X^{\nu\lambda}$ is $\mathfrak{U}$-irreducible if and only if $a_{nm} \neq 0$ for every n, $m \in \boldsymbol{N}_0$.*

PROOF. This is an easy consequence of Proposition 2.8, Lemma 3.3 and the fact that $(\ , \ )$ is a K-invariant non-singular pairing of $X^{\nu\lambda}$ and $X^{\nu,-\bar{\lambda}}$.

Q.E.D.

§ 4. The calculation of a_{nm}

Let Ω be the Casimir element in $\mathfrak{U}$, and we set

$$\omega=\sum_{\alpha\in\Delta_{\mathfrak{p}}^+} X_{-\alpha}X_{\alpha}.$$

Then, by a simple calculation, we have

$$\omega=2^{-1}\{\Omega-(H_1^2+H_2^2)-2H_{\rho'}\}-\sum_{\alpha\in\Delta_{\mathfrak{k}}^+} X_{-\alpha}X_{\alpha},$$

where $\{H_1, H_2\}$ is an orthonormal basis of $\sqrt{-1}\,\mathfrak{h}_0$ with respect to the Killing form B, and ρ' is a linear form on $\mathfrak{h}$ defined by

$$\rho'=2^{-1}\sum_{\alpha\in\Delta^+}\alpha=\alpha_1+\alpha_2.$$

LEMMA 4.1. *$\pi_*^{\nu\lambda}(\Omega)$ is a scalar operator given by*

$$\nu^2/9+(\lambda^2-1)/3.$$

PROOF. Let H'_1 (resp. H'_2) be an element in $\sqrt{-1}\,\mathfrak{a}_-$ (resp. $\mathfrak{a}_+$) normalized by $B(H'_i, H'_i)=1$ $(i=1, 2)$. Then

$$\begin{aligned}\Omega&=H_1'^2+H_2'^2+\sum_{\beta\in A^+}(X_\beta X_{-\beta}+X_{-\beta}X_\beta)\\&=H_1'^2+H_2'^2-2\rho+2\sum_{\beta\in A^+}X_\beta X_{-\beta},\end{aligned}$$

where $X_\beta\in\mathfrak{g}^\beta\,(\beta\in A)$ is chosen so that $B(X_\beta, X_{-\beta})=1$. It is known that $\pi_*^{\nu\lambda}(\Omega)$ is a scalar operator. In order to obtain this scalar, we calculate $[\pi_*^{\nu\lambda}(\Omega)f_0](e)$. Since each element in $X^{\nu\lambda}$ is invariant under the right translation by N, we have

$$\begin{aligned}[\pi_*^{\nu\lambda}(\Omega)f_0](e) &= [\pi_*^{\nu\lambda}(H_1^2)f_0](e)+[\pi_*^{\nu\lambda}(H_2^2-2H_\rho)f_0](e)\\ &=\{\nu^2\|\varepsilon_1^*-\varepsilon_2^*\|^2+(\lambda+1)^2\|\rho\|^2-2(\lambda+1)\|\rho\|^2\}f_0(e)\\ &=\{\nu^2\|\varepsilon_1^*-\varepsilon_2^*\|^2+(\lambda^2-1)\|\rho\|^2\}f_0(e),\end{aligned}$$

where $\| \quad \|$ denotes the norm on $\mathfrak{a}_R$ defined by the Killing form B. For $\mathfrak{g}_0=\mathfrak{su}(2, 1)$, the norm of each root is $1/\sqrt{3}$. So we have

$$\|\rho\|^2=\|\beta_1+\beta_2\|^2=1/3$$

and

$$\|\varepsilon_1^*-\varepsilon_2^*\|^2=\|(\alpha_1-\alpha_2)/3\|^2=1/9.$$

Thus we have

$$\pi_*^{\nu\lambda}(\Omega)=\nu^2/9+(\lambda^2-1)/3.$$

Q.E.D.

In the following, for the sake of simplicity, we write uf_{nm} or uf'_{nm} instead of $\pi_*^{\nu\lambda}(u)f_{nm}$ or $\pi_*^{\nu,-\bar{\lambda}}(u)f'_{nm}$. We set

$$\mu_{nm}=(n+m)\varepsilon_1^*+(-2\nu+m-2n)\varepsilon_2^*.$$

Then, by a simple calculation, we have

$$\begin{aligned}&<\mu_{nm}, \alpha_1> =(n+m)/6\\ &<\mu_{nm}, \alpha_2> =(-2\nu+m-2n)/6\\ &<\mu_{nm}, \rho'> =(-2\nu+2m-n)/6\\ &<\mu_{nm}, \mu_{nm}> =9^{-1}\{4\nu^2-6\nu(m-n)+3(m^2-mn+n^2)\}.\end{aligned}$$

LEMMA 4.2. *For $n, m\in \boldsymbol{N}_0$,*

$$\omega f_{nm}=(1/6)\{\lambda^2-(\nu+n-m-1)^2-n(m+1)\}f_{nm}.$$

PROOF. Since f_{nm} is a highest weight vector in $\tau_{(n+m,-2\nu+m-2n)}$, we have

$$\begin{aligned}\omega f_{nm}&=2^{-1}\Omega f_{nm}-2^{-1}(H_1^2+H_2^2)f_{nm}-H_{\rho'}f_{nm}\\ &=\{2^{-1}(\nu^2/9+(\lambda^2-1)/3)-2^{-1}\|\mu_{nm}\|^2-<\mu_{nm}, H_{\rho'}>\}f_{nm}\\ &=\{(18)^{-1}(\nu^2+3(\lambda^2-1))-(18)^{-1}[4\nu^2-6\nu(m-n)+3(m^2-mn+n^2)]\\ &\quad -6^{-1}(-2\nu+2m-n)\}f_{nm}=6^{-1}\{\lambda^2-(\nu+n-m-1)^2-n(m+1)\}f_{nm}.\end{aligned}$$

Q.E.D.

PROPOSITION 4.3. *For $n, m \in \boldsymbol{N}_0$,*

$$a_{n,m+1}+a_{n+1,m}=(-1/6)a_{nm}[\lambda^2-(\nu+n-m)^2-(n+1)(m+1)].$$

PROOF. By the definition of a_{nm}, we have

$$\begin{aligned} a_{n,m+1} &= (X_+f_{nm}, X_+f'_{nm}) = (X_+^s X_+ f_{nm}, f'_{nm}) \\ &= -(X_{-(\alpha_1+\alpha_2)}X_{\alpha_1+\alpha_2}f_{nm}, f'_{nm}), \\ a_{n+1,m} &= (X_-f_{nm}, X_-f'_{nm}) = (X_-^s X_- f_{nm}, f'_{nm}) \\ &= -(X_{\alpha_2}X_{-\alpha_2}f_{nm}, f'_{nm}) \\ &= -\{(X_{-\alpha_2}X_{\alpha_2}f_{nm}, f'_{nm}) + (H_{\alpha_2}f_{nm}, f'_{nm})\} \\ &= -(X_{-\alpha_2}X_{\alpha_2}f_{nm}, f'_{nm}) - <\alpha_2, \mu_{nm}> a_{nm} \\ &= -(X_{-\alpha_2}X_{\alpha_2}f_{nm}, f'_{nm}) - (1/6)(-2\nu+m-2n)a_{nm}, \end{aligned}$$

where we have used Lemma 2.1. So we have

$$\begin{aligned} &a_{n,m+1}+a_{n+1,m} \\ &= -(\omega f_{nm}, f'_{nm}) - (1/6)(-2\nu+m-2n)(f_{nm}, f'_{nm}) \\ &= -(1/6)a_{nm}[\lambda^2-(\nu+n-m)^2-(n+1)(m+1)]. \end{aligned}$$

Q.E.D.

PROPOSITION 4.4. *For $m \in \boldsymbol{N}$,*

$$6(m+1)a_{0m} = -m\{\lambda^2-(\nu-m)^2\}a_{0,m-1}.$$

PROOF. We calculate $(X_{-\alpha_1}f_{0m}, X_{-\alpha_1}f'_{0m})$. By Lemma 2.1, we have

$$\begin{aligned} (X_{-\alpha_1}f_{0m}, X_{-\alpha_1}f'_{0m}) &= (X_{-\alpha_1}^s X_{-\alpha_1}f_{0m}, f'_{0m}) \\ &= (X_{\alpha_1}X_{-\alpha_1}f_{0m}, f'_{0m}) \\ &= ((X_{-\alpha_1}X_{\alpha_1}+H_{\alpha_1})f_{0m}, f'_{0m}) \\ &= (H_{\alpha_1}f_{0m}, f'_{0m}) \\ &= <\mu_{0m}, \alpha_1> a_{0m} \\ &= (m/6)a_{0m}. \end{aligned}$$

Since

$$X_{-\alpha_1}f_{0m} = X_{-\alpha_1}X_+^m f_0 = (X_{-\alpha_1}X_+^m - X_+^m X_{-\alpha_1})f_0$$

$$=\sum_{k=1}^{m} X_{+}^{k-1}[X_{-\alpha_1}, X_{+}]\, X_{+}^{m-k} f_0$$

$$=N_{-\alpha_1,\alpha_1+\alpha_2}\sum_{k=1}^{m} X_{+}^{k-1} X_{\alpha_2} X_{+}^{m-k} f_0$$

$$=N_{-\alpha_1,\alpha_1+\alpha_2}\sum_{k=1}^{m} X_{\alpha_2} X_{+}^{k-1} X_{+}^{m-k} f_0$$

$$=mN_{-\alpha_1,\alpha_1+\alpha_2} X_{\alpha_2} f_{0,m-1},$$

we have

$$\begin{aligned}(X_{-\alpha_1} f_{om}, X_{-\alpha_1} f'_{om}) &= m^2 |N_{-\alpha_1,\alpha_1+\alpha_2}|^2 (X_{\alpha_2} f_{0,m-1}, X_{\alpha_2} f'_{0,m-1}) \\ &= m^2 |N_{-\alpha_1,\alpha_1+\alpha_2}|^2 (X_{\alpha_2}^s X_{\alpha_2} f_{0,m-1}, f'_{0,m-1}) \\ &= -m^2 |N_{-\alpha_1,\alpha_1+\alpha_2}|^2 (X_{-\alpha_2} X_{\alpha_2} f_{0,m-1}, f'_{0,m-1}).\end{aligned}$$

Applying Lemma 2.2 to $\mathfrak{g}_0=\mathfrak{su}\,(2, 1)$, we have

$$|N_{-\alpha_1,\ \alpha_1+\alpha_2}|^2=1/6.$$

So we have

$$a_{0m}=-m(X_{-\alpha_2} X_{\alpha_2} f_{0,m-1}, f'_{0,m-1}) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1).$$

On the other hand, we have

$$\begin{aligned}a_{0m} &= (f_{0m}, f'_{0m}) = (X_{+} f_{0,m-1}, X_{+} f'_{0,m-1}) \\ &= (X_{+}^s X_{+} f_{0,m-1}, f'_{0,m-1}) \\ &= -(X_{-(\alpha_1+\alpha_2)} X_{\alpha_1+\alpha_2} f_{0,m-1}, f'_{0,m-1}) \quad \cdots\cdots\cdots\cdots\cdots\cdots (2).\end{aligned}$$

From (1), (2) and Lemma 4.2, we have

$$\begin{aligned}(1+m)a_{0m} &= -m(\omega f_{0,m-1}, f'_{0,m-1}) \\ &= (-m/6)\{\lambda^2-(\nu-m)^2\} a_{0,m-1}.\end{aligned}$$

Q.E.D.

COROLLARY 4.5. *For* $m\in \boldsymbol{N}_0$,

$$a_{0m}=(-1/6)^m(1/(m+1))\prod_{k=1}^{m}\{\lambda^2-(\nu-k)^2\}.$$

THEOREM 4.6. *For* $n,\ m\in \boldsymbol{N}_0$,

$$a_{nm}=(-1/6)^{n+m}B(m+1,\ n+1)\Big[\prod_{k=1}^{m}\{\lambda^2-(\nu-k)^2\}\Big]\Big[\prod_{k=1}^{n}\{\lambda^2-(\nu+k)^2\}\Big],$$

where $B(x, y)$ *is the betha function*: $B(m+1,\ n+1)=m!n!/(m+n+1)!$.

PROOF. We shall prove the theorem by induction on n. For $n=0$, the above formula coincides with Corollary 4.5. Now we assume that the theorem holds for a fixed $n\in \boldsymbol{N}_0$ and for any $m\in \boldsymbol{N}_0$. Then, by Proposition 4.3, we have

$$a_{n+1,m}=-a_{n,m+1}-(1/6)a_{nm}[\lambda^2-(\nu+n-m)^2-(m+1)(n+1)]$$

$$=-(-1/6)^{m+n+1}B(m+2,\ n+1)\,[\prod_{k=1}^{m+1}\{\lambda^2-(\nu-k)^2\}]\,[\prod_{k=1}^{n}\{\lambda^2-(\nu+k)^2\}]$$

$$-(1/6)\,(-1/6)^{m+n}B(m+1,\ n+1)\,[\prod_{k=1}^{m}\{\lambda^2-(\nu-k)^2\}]\,[\prod_{k=1}^{n}\{\lambda^2-(\nu+k)^2\}]$$

$$\times[\lambda^2-(\nu+n-m)^2-(m+1)\,(n+1)]$$

$$=-(-1/6)^{m+n+1}(m!n!/(m+n+2)!)\,[\prod_{k=1}^{m}\{\lambda^2-(\nu-k)^2\}]\,[\prod_{k=1}^{n}\{\lambda^2-(\nu+k)^2\}]$$

$$\times[(m+1)\,\{\lambda^2-(\nu-m-1)^2\}-(m+n+2)\,\{\lambda^2-(\nu+n-m)^2-(m+1)\,(n+1)\}]$$

$$=(-1/6)^{m+n+1}(m!(n+1)!/(m+n+2)!)\,[\prod_{k=1}^{m}\{\lambda^2-(\nu-k)^2\}]\,[\prod_{k=1}^{n+1}\{\lambda^2$$

$$-(\nu+k)^2\}],$$

and this completes the proof. Q.E.D.

From Theorem 4.6 and Proposition 3.4, we have

COROLLARY 4.7. 1) *The* $\mathfrak{U}$*-module* $X^{0\lambda}$ *is reducible if and only if* λ *is a non-zero integer, and*

2) *when* $\nu\neq 0$, *the* $\mathfrak{U}$*-module* $X^{\nu\lambda}$ *is reducible if and only if* $\lambda-\nu$ *is an integer.*

Added in Proof.

Recently the author is announced from Prof. K. Okamoto that Prof. N.R. Wallach has proved the same results in a quite different way and that he has also obtained the decomposition of the elementary series representations of $SU(2, 1)$.

References

[1] Harish-Chandra: Representations of a semisimple Lie group on a Banach space. I, Trans. Amer. Math. Soc., **75** (1953), 185–243.

[2] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[3] B. Kostant: On the existence and irreducibility of certain series of representations, Bull. AMS, **75** (1969), 627–642.

[4] B. Kostant and S. Rallis: Orbits and representations associated with symmetric spaces, Amer. J. Math., **93** (1971), 753–809.
[5] M. Wakimoto: On the irreducibility of some series of representations, Hiroshima Math. J., **2** (1972), 71–98.

Department of Mathematics,
Faculty of Science,
Hiroshima University