HIROSHIMA MATH. J.
4 (1974), 1–8

Oscillations of Differential Equations with Retardations

Yiannis G. SFICAS and Vasilios A. STAIKOS

(Received May 31, 1973)

This paper is concerned with the oscillatory and asymptotic behavior of the n-th order ($n>1$) differential equation with retarded arguments

$$(*)\qquad x^{(n)}(t)+f(t, x[g_1(t)], x[g_2(t)], \dots, x[g_m(t)])=0$$

where the functions g_i, $i=1, 2, \dots, m$ are differentiable on the half line $[t_0, \infty)$ and such that

(I) $g_i(t)\leqq t$ for every $t\geqq t_0$
(II) $g_i'(t)\geqq 0$ for every $t\geqq t_0$
(III) $\lim_{t\to\infty} g_i(t)=\infty$

Our results extend previous ones concerning retarded differential equations of the form

$$x^{(n)}(t)+f(t, x[g(t)])=0$$

(Cf. [8] and [2]). Moreover, the results given here can be used in order to obtain other ones concerning retarded differential equations of a more general form than (*), i.e., when f depends on the derivatives too. This can be done by the comparison principle introduced by the authors in [9] and [10]. Thus, recent related results given by Onose [5] and Kusano and Onose [2] could be improved.

In what follows we consider only solutions of (*) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e., a solution of (*) is called *oscillatory* if it has no last zero, otherwise it is called *nonoscillatory*.

To obtain our results we need the following three lemmas, the first of which is an adaptation of a lemma due to Kiguradze [1] and the others of lemmas in [7] and [9].

LEMMA 1. *If u is an n-times differentiable function on $[a, \infty)$ with $u^{(k)}$, $k=0, 1, \dots, n-1$, absolutely continuous on $[a, \infty)$ and if*

$$u(t)\neq 0 \text{ and } u(t)u^{(n)}(t)\leqq 0 \qquad \text{for every } t\in[a, \infty)$$

then there exists an integer l with $0\leqq l<n$, $n+l$ odd and such that

$$u(t)u^{(k)}(t)\geqq 0 \text{ for every } t\in[a, \infty) \qquad (k=0, 1, \dots, l)$$

$$(-1)^{n+k-1}u(t)u^{(k)}(t) \geqq 0 \qquad \text{for every } t \in [a, \infty) \qquad (k=l+1, l+2, \dots, n)$$

and

$$|u(t)| \geqq \frac{(t-a)^{n-1}|u^{(n-1)}(2^{n-l-1}t)|}{(n-1)(n-2)\cdots(n-l)}, \quad |u'(t)| \geqq \frac{(t-a)^{n-2}|u^{(n-1)}(2^{n-l-1}t)|}{(n-2)(n-3)\cdots(n-l)}$$

for every $t \in [a, \infty)$.

LEMMA 2. *If* u *is as in Lemma* 1 *and for some* $k=0, 1, \dots, n-2$ $\lim_{t\to\infty} u^{(k)}(t) = c$, $c \in R$, *then* $\lim_{t\to\infty} u^{(k+1)}(t) = 0$.

LEMMA 3. *If* u *is as in Lemma* 1 *and* $\lim_{t\to\infty} u(t) \neq 0$, *then there exists a constant* θ *such that for any* $i=1, 2, \dots, m$

$$\left|\frac{u^{(n-1)}(t)}{u[g_i(t)]}\right| \leqq \frac{\theta}{g_i^{n-1}(t)} \quad \text{for all large } t$$

THEOREM. *Consider the functions* p_1, p_2, φ, ρ *subject to the following conditions*:

(i) p_1, p_2 *are nonnegative and locally integrable on* $[t_0, \infty)$

(ii) φ *is defined at least on* $R^m - \{(0, 0, \dots, 0)\}$ *and such that for any* $y_1, y_2, \dots, y_m$

$$(\forall i=1, 2, \dots, m) y_i > 0 \Rightarrow \varphi(y_1, y_2, \dots, y_m) > 0$$

$$(\forall i=1, 2, \dots, m) y_i < 0 \Rightarrow \varphi(y_1, y_2, \dots, y_m) < 0$$

(iii) ρ *is defined at least on* $R - \{0\}$ *and such that for any* $y \neq 0$

$$y\rho(y) > 0$$

(iv) *the function* $y\rho(y)$ *is nondecreasing for* $y>0$, *nonincreasing for* $y<0$ *and such that*

$$\int^{\infty} \frac{dy}{y\rho(y)} < \infty \quad \text{and} \quad \int_{-\infty} \frac{dy}{y\rho(y)} < \infty$$

(v) *the function* $\dfrac{\varphi(y_1, y_2, \dots, y_m)}{y_1 y_2 \cdots y_m \rho(y_1 y_2 \cdots y_m)}$ *is nonincreasing on the set* $\{(y_1, y_2, \dots, y_m) \in R^m : (\forall i) y_i > 0\}$ *and nondecreasing on the set* $\{(y_1, y_2, \dots, y_m) \in R^m : (\forall i) y_i < 0\}$ *with respect to each* y_i

(vi) *for every* μ *sufficiently large*

$$\int^{\infty} p_1(t) \frac{\sum_{i=1}^{m} g_i^{n-1}(t)}{\prod_{i=1}^{m} g_i^{n-1}(t)} \, \frac{\varphi[\mu g_1^{n-1}(t), \mu g_2^{n-1}(t), \dots, \mu g_m^{n-1}(t)]}{\rho(\mu^m \prod_{i=1}^{m} g_i^{n-1}(t))} \, dt = \infty$$

and

$$\int^{\infty} p_2(t) \frac{\sum_{i=1}^{m} g_i^{n-1}(t)}{\prod_{i=1}^{m} g_i^{n-1}(t)} \frac{\varphi[-\mu g_1^{n-1}(t), -\mu g_2^{n-1}(t), \dots, -\mu g_m^{n-1}(t)]}{(-1)^{m-1}\rho((-\mu)^m \prod_{i=1}^{m} g_i^{n-1}(t))} dt = \infty$$

If for any $t \geqq t_0$,

$$p_1(t)\varphi(y_1, y_2, \dots, y_m) \leqq f(t, y_1, y_2, \dots, y_m) \qquad \text{for } y_1 > 0, \dots, y_m > 0$$

and

$$f(t, y_1, y_2, \dots, y_m) \leqq p_2(t)\varphi(y_1, y_2, \dots, y_m) \qquad \text{for } y_1 < 0, \dots, y_m < 0$$

then for n *even all solutions of* $(*)$ *are ocsillatory, while for* n *odd all solutions of* $(*)$ *are either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with their first* $n-1$ *derivatives.*

Note. $g_i^{n-1}(t)$ stands in place of $(g_i(t))^{n-1}$.

PROOF. Let x be a nonoscillatory solution of $(*)$ with $\lim_{t\to\infty} x(t) \neq 0$. This solution can be supposed with domain $[t_0, \infty)$ and positive, since the substitution $u = -x$ transforms $(*)$ into an equation of the same form satisfying the assumptions of the theorem. Moreover, by (III), we can choose t_1, $t_1 \geqq t_0$, so that for any $i = 1, 2, \dots, m$

$$g_i(t) > \max\{t_0, 0\} \qquad \text{for every } t \geqq t_1$$

Since

$$x^{(n)}(t) = -f(t, x[g_1(t)], x[g_2(t)], \dots, x[g_m(t)])$$

it is easy to verify that

$$x^{(n)} \leqq 0 \qquad \text{on } [t_1, \infty)$$

and hence, by Lemma 1,

$$x^{(n-1)} \geqq 0 \qquad \text{on } [t_1, \infty)$$

More precisely,

$$x^{(n-1)} > 0 \qquad \text{on } [t_1, \infty)$$

since, otherwise, for some $T > t_0$,

$$x^{(n-1)} = 0 \qquad \text{on } [T, \infty)$$

and consequently

$$x^{(n)} = 0 \qquad \text{on } [T, \infty)$$

Thus, by (i) and (ii), for any $t > T$,

$$0 \leqq p_1(t)\varphi(x[g_1(t)], \ldots, x[g_m(t)]) \leqq f(t, x[g_1(t)], \ldots, x[g_m(t)]) = -x^{(n)}(t) = 0$$

and hence

$$p_1 = 0 \qquad \text{on } [T, \infty)$$

which contradicts (vi).

Now, by Taylor's formula, we obtain that for any $t \geqq t_1$ and $i = 1, 2, \ldots, m$

$$x[g_i(t)] \leqq x(t_0) + \frac{x'(t_0)}{1!}[g_i(t) - t_0] + \cdots + \frac{x^{(n-1)}(t_0)}{(n-1)!}[g_i(t) - t_0]^{n-1}$$

and consequently that there exist a sufficiently large constant μ and $t_2 \geqq t_1$ such that for any $i = 1, 2, \ldots, m$

$$x[g_i(t)] \leqq \mu g_i^{n-1}(t) \qquad \text{for every } t \geqq t_2 \tag{1}$$

As in [8] we consider the following two cases.

Case 1. $x' \geqq 0$ on $[t_1, \infty)$.

Let

$$z(t) = -x^{(n-1)}(t)\int_{t_1}^{t} \frac{\Sigma_i g_i^{n-2}(s)g_i'(s)}{(\Pi_i x[g_i(s)])\,\rho(\Pi_i x[g_i(s)])}ds \tag{2}$$

Then, for any $t \geqq t_1$, we have

$$\begin{aligned}
z'(t) &= f(t, x[g_1(t)], \ldots, x[g_m(t)])\int_{t_1}^{t} \frac{\Sigma_i g_i^{n-2}(s)g_i'(s)}{(\Pi_i x[g_i(s)])\rho(\Pi_i x[g_i(s)])}ds \\
&\quad - x^{(n-1)}(t)\frac{\Sigma_i g_i^{n-2}(t)g_i'(t)}{(\Pi_i x[g_i(t)])\,\rho(\Pi_i x[g_i(t)])} \\
&\geqq p_1(t)\frac{\varphi(x[g_1(t)], \ldots, x[g_m(t)])}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}\int_{t_1}^{t} \Sigma_i g_i^{n-2}(s)g_i'(s)ds \\
&\quad - x^{(n-1)}(t)\frac{\Sigma_i g_i^{n-2}(t)g_i'(t)}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])} \\
&= \frac{1}{n-1}p_1(t)\frac{\varphi(x[g_1(t)], \ldots, x[g_m(t)])}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}[\Sigma_i g_i^{n-1}(t) - \Sigma_i g_i^{n-1}(t_1)] \\
&\quad - x^{(n-1)}(t)\frac{\Sigma_i g_i^{n-2}(t)g_i'(t)}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}
\end{aligned}$$

Since, by (III), there exist a constant $c_1 > 0$ and $t_3 > t_2$ such that

$$\Sigma_i g_i^{n-1}(t) - \Sigma_i g_i^{n-1}(t_1) \geqq c_1 \Sigma_i g_i^{n-1}(t) \text{ for every } t \geqq t_3$$

by (1) and (v), we shall have

$$z'(t) \geqq c_2 p_1(t) \frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)} \frac{\varphi[\mu g_1^{n-1}(t),\dots,\mu g_m^{n-1}(t)]}{\rho(\mu^m \Pi_i g_i^{n-1}(t))} - x^{(n-1)}(t) \frac{\Sigma_i g_i^{n-2}(t) g_i'(t)}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])} \tag{3}$$

where
$$c_2 = \frac{c_1}{(n-1)\mu^m}.$$

Let us now consider the term

$$F(t) = x^{(n-1)}(t) \frac{\Sigma_i g_i^{n-2}(t) g_i'(t)}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

In the case $\lim_{t\to\infty} x'(t) \neq 0$, by Lemma 3, we have that for some $t_4 \geqq t_3$ and any $t \geqq t_4$,

$$F(t) = \frac{x^{(n-1)}(t)}{(\Pi_i x[g_i(t)])'} (\Sigma_i g_i^{n-2}(t) g_i'(t)) \frac{(\Pi_i x[g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

$$= \frac{1}{\frac{x'[g_1(t)]}{x^{(n-1)}(t)} x[g_2(t)]\cdots x[g_m(t)] g_1'(t) + \cdots + x[g_1(t)]\cdots x[g_{m-1}(t)] \cdot \frac{x'[g_m(t)]}{x^{(n-1)}(t)} g_m'(t)} \cdot$$

$$\cdot (\Sigma_i g_i^{n-2}(t) g_i'(t)) \frac{(\Pi_i x[g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

$$\leqq \frac{\theta}{c_3^{m-1} \Sigma_i g_i^{n-2}(t) g_i'(t)} (\Sigma_i g_i^{n-2}(t) g_i'(t)) \frac{(\Pi_i x[g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

$$= \frac{\theta}{c_3^{m-1}} \frac{(\Pi_i x[g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

where $c_3 = x(t_4)$. Hence (3) gives

$$z'(t) \geqq c_2 p_1(t) \frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)} \frac{\varphi[\mu g_1^{n-1}(t),\dots,\mu g_m^{n-1}(t)]}{\rho(\mu^m \Pi_i g_i^{n-1}(t))} - \frac{\theta}{c_3^{m-1}} \frac{(\Pi_i x[g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

for every $t \geqq t_4$. Integrating the last inequality from t_4 to t and taking into account (iv) and (vi), we derive that z is eventually positive, which contradicts (2). It remains to derive a contradiction in the case $\lim_{t\to\infty} x'(t) = 0$. To do this, we observe that, by Lemma 1, x' is nonincreasing on $[t_1, \infty)$ and there exist

$c_4>0$ and $t_5 \geqq t_3$ such that for any $i=1, 2, \dots, m$

$$\lambda g_i(t) \geqq t_3 \text{ and } x'[\lambda g_i(t)] \geqq c_4 t^{n-2} x^{(n-1)}(t) \qquad \text{for every } t \geqq t_5$$

where $\lambda = 2^{-n+l+1}$, i.e., $0<\lambda \leqq 1$. Thus, by (I) and (iv), for any $t \geqq t_5$,

$$F(t)=\frac{x^{(n-1)}(t)\Sigma_i g_i^{n-2}(t)g_i'(t)}{(\Pi_i x[\lambda g_i(t)])'}\;\frac{(\Pi_i x[\lambda g_i(t)])'}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}$$

$$\leqq \frac{\Sigma_i g_i^{n-2}(t)g_i'(t)}{\lambda\left[\frac{x'[\lambda g_1(t)]}{x^{(n-1)}(t)}x[\lambda g_2(t)]\cdots x[\lambda g_m(t)]g_i'(t)+\cdots+x[\lambda g_1(t)]\cdots x[\lambda g_{m-1}(t)]\cdot\frac{x'[\lambda g_m(t)]}{x^{(n-1)}(t)}g_m'(t)\right]}\cdot$$

$$\cdot\frac{(\Pi_i x[\lambda g_i(t)])'}{(\Pi_i x[\lambda g_i(t)])\rho(\Pi_i x[\lambda g_i(t)])}$$

$$\leqq \frac{1}{\lambda c_4 c_5^{m-1}}\;\frac{\Sigma_i g_i^{n-2}(t)g_i'(t)}{\Sigma_i t^{n-2}g_i'(t)}\;\frac{(\Pi_i x[\lambda g_i(t)])'}{(\Pi_i x[\lambda g_i(t)])\rho(\Pi_i x[\lambda g_i(t)])}$$

$$\leqq \frac{1}{\lambda c_4 c_5^{m-1}}\;\frac{(\Pi_i x[\lambda g_i(t)])'}{(\Pi_i x[\lambda g_i(t)])\rho(\Pi_i x[\lambda g_i(t)])}$$

where $c_5 = x(t_3)$. Hence, (3) gives

$$z'(t) \geqq c_2 p_1(t)\frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)}\;\frac{\varphi[\mu g_1^{n-1}(t),\dots,\mu g_m^{n-1}(t)]}{\rho(\mu^m \Pi_i g_i^{n-1}(t))}$$

$$-\frac{1}{\lambda c_4 c_5^{m-1}}\;\frac{(\Pi_i x[\lambda g_i(t)])'}{(\Pi_i x[\lambda g_i(t)])\rho(\Pi_i x[\lambda g_i(t)])}$$

for every $t \geqq t_5$. Integrating this inequality from t_5 to t and taking into account (iv) and (vi), we derive again that z is eventually positive, which also contradicts (2).

Case 2. $x' \leqq 0$ on $[t_1, \infty)$.

We consider an odd integer $\alpha>1$ and the ordinary differential equation

$$(**) \qquad y^{(n)} + p(t)y^{\alpha} = 0$$

where

$$p(t)=\frac{f(t, x[g_1(t)], \dots, x[g_m(t)])}{x^{\alpha}(t)}$$

In this case x is nonincreasing on $[t_1, \infty)$ and $c = \lim_{t\to\infty} x(t)$ exists and is positive. Thus, by (1), for any $t \geqq t_2$ we have

$$t^{n-1}p(t)=t^{n-1}\frac{f(t,\ x[g_1(t)],\ \dots,\ x[g_m(t)])}{x^\alpha(t)}$$

$$\geqq\frac{t^{n-1}}{x^\alpha(t_2)}p_1(t)\varphi(x[g_1(t)],\ \dots,\ x[g_m(t)])$$

$$=\frac{t^{n-1}}{x^\alpha(t_2)}p_1(t)\frac{\varphi(x[g_1(t)],\ \dots,\ x[g_m(t)])}{(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])}(\Pi_i x[g_i(t)])\rho(\Pi_i x[g_i(t)])$$

$$\geqq\frac{t^{n-1}}{x^\alpha(t_2)}p_1(t)\frac{\varphi[\mu g_1^{n-1}(t),\ \dots,\ \mu g_m^{n-1}(t)]}{\mu^m(\Pi_i g_i^{n-1}(t))\rho(\mu^m\Pi_i g_i^{n-1}(t))}c^m\rho(c^m)$$

$$\geqq\frac{c^m\rho(c^m)}{x^\alpha(t_2)\mu^m m}\ \frac{mt^{n-1}}{\Pi_i g_i^{n-1}(t)}\ \frac{\varphi[\mu g_1^{n-1}(t),\ \dots,\ \mu g_m^{n-1}(t)]}{\rho(\mu^m\Pi_i g_i^{n-1}(t))}$$

and consequently, for some appropriate constant $K>0$ and any $t\geqq t_2$

$$t^{n-1}p(t)\geqq Kp_1(t)\frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)}\ \frac{\varphi[\mu g_1^{n-1}(t),\ \dots,\ \mu g_m^{n-1}(t)]}{\rho(\mu^m\Pi_i g_i^{n-1}(t))}$$

which, by (vi), gives

$$\int^\infty t^{n-1}p(t)dt=\infty$$

It is well-known (Cf. [4], [6] and [8]) that under the last condition, all solutions y of (**) with $\lim_{t\to\infty} y(t)\neq 0$ are oscillatory and this is a contradiction, since x is a such solution of the equation (**).

To complete the proof, we observe that in the case of a nonoscillatory solution x of (*), Lemma 1 ensures that $\lim_{t\to\infty} x(t)=0$ occurs only when n is odd. Hence, the theorem is an immediate consequence of Lemma 2.

NOTE. After this paper was written the authors received a preprint [3] which Professors Kusano and Onose had kindly sent. In [3] the case $m=1$ and $f=\varphi$ is studied, where the function φ is supposed with nonnegative derivative and in place of (vi) a closely related condition appears, which does not contain the parameter μ (This is the case $\rho(y)=\frac{\varphi(y)}{y}\phi(|y|^{1/(n-1)})$ sgn y, where ϕ is a positive function with nonnegative derivative).

Under the additional assumption that the function φ is nondecreasing on the set $\{(y_1,\ \dots,\ y_m)\in R^m\colon (\forall i)y_1y_i>0\}$ with respect to each y_i $(i=1,\ 2,\ \dots,\ m)$, condition (vi) can also be stated independently of μ, as follows:

$$\text{(vi)}'\qquad \int^\infty p_1(t)\frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)}\ \frac{\varphi[g_1^{n-1}(t),\ g_2^{n-1}(t),\ \dots,\ g_m^{n-1}(t)]}{\rho(\Pi_i g_i^{n-1}(t))}dt=\infty$$

and

$$\int^{\infty} p_2(t)\frac{\Sigma_i g_i^{n-1}(t)}{\Pi_i g_i^{n-1}(t)} \frac{\varphi[-g_1^{n-1}(t), -g_2^{n-1}(t), \dots, -g_m^{n-1}(t)]}{(-1)^{m-1}\rho((-1)^m \Pi_i g_i^{n-1}(t))} dt = \infty$$

The proof remains essentially the same by using in place of z the function w,

$$w(t) = -x^{(n-1)}(t)\int_{t_1}^{t} \frac{\Sigma_i g_i^{n-2}(s) g_i'(s)}{\Pi_i x[g_i(s)]\rho\left(\frac{\Pi_i x[g_i(s)]}{\mu^m}\right)} ds$$

where $\mu \geqq 1$ and such that (1) is satisfied.

References

[1] I. Kiguradze, The problem of oscillation of solutions of nonlinear differential equations, Differential 'nye Uravnenija **1** (1965), 995–1006 (Russian).

[2] T. Kusano and H. Onose, Oscillation of solutions of nonlinear differential delay equations of arbitrary order, Hiroshima Math. J. **2** (1972), 1–13.

[3] T. Kusano and H. Onose, Oscillations of functional differential equations with retarded argument, J. Differential Equations (to appear).

[4] I. Ličko and M. Švec, Le caractère oscillatoire des solutions de l'équation $y^{(n)} + f(x)y^{\alpha} = 0$, $n > 1$, Czech. Math. J. **13** (1963), 481–491.

[5] H. Onose, Some oscillation criteria for n-th order nonlinear delay-differential equations, Hiroshima Math. J. **1** (1971), 171–176.

[6] G. Ryder and D. Wend, Oscillation of solutions of certain ordinary differential equations of n-th order, Proc. Amer. Math. Soc. **25** (1970), 463–469.

[7] Y. Sficas, On oscillation and asymptotic behavior of a certain class of differential equations with retarded argument, Utilitas Mathematica **3** (1973), 239–249.

[8] Y. Sficas and V. Staikos, Oscillations of retarded differential equations, Proc. Camb. Phil. Soc. **75** (1974), 95–101.

[9] V. Staikos and Y. Sficas, Oscillatory and asymptotic behavior of functional differential equations, J. Differential Equations, **12** (1972), 426–437.

[10] V. Staikos and Y. Sficas, Oscillatory and asymptotic properties of differential equations with retarded arguments, Applicable Anal. (to appear).

Department of Mathematics,
University of Ioannina,
Ioannina, Greece