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Weak Solutions for Certain Nonlinear Time-dependent Parabolic Variational Inequalities

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1. Introduction

For a (real) Banach space V, in general, we denote by V^* the dual space of V, by $\|\cdot\|_V$ and $\|\cdot\|_{V*}$ the norms in V and V^*, respectively, and by $(\cdot, \cdot)_V$ the natural pairing between V^* and V.

Let A be a (multivalued) operator from a Banach space V into V^*, that is, to each $v \in V$ a subset Av of V^* be assigned. Then we define

$$D(A) = \{v \in V; Av \neq \phi\},$$

$$R(A) = \bigcup_{v \in V} Av$$

and

$$G(A) = \{[v, v^*] \in V \times V^*; v \in D(A), v^* \in Av\},$$

which are called the domain, the range and the graph of A, respectively. An operator $A: V \to V^*$ is called monotone if

$$(v^* - w^*, v - w)_V \geq 0 \quad \text{for any } [v, v^*], [w, w^*] \in G(A).$$

If A is monotone and there is no proper monotone extension of A, then A is called maximal monotone.

As an important class of maximal monotone operators from a Banach space V into V^*, there is a class of duality mappings. Let μ be a continuous strictly increasing function from $[0, \infty)$ into itself such that $\mu(0)=0$ and $\mu(r) \uparrow \infty$ as $r \uparrow \infty$. The mapping $\mathscr{F}_\mu$: $V \to V^*$ defined by

$$\mathscr{F}_\mu(v) = \{v^* \in V^*;\ (v^*,\ v)_V = \mu(\|v\|_V)\|v\|_V \text{ and } \|v^*\|_{V*} = \mu(\|v\|_V)\}$$

is called the duality mapping of V into V^* associated with the gauge function μ. We know (cf. [6; Chapter 1]) that any duality mapping is singlevalued and demicontinuous (i.e., continuous with respect to the strong topology of V and the weak topology of V^*) provided that V is reflexive and V^* is strictly convex. Also, it is well-known (cf. [16; Proposition 1]) that a monotone operator A: $V \to V^*$ is maximal monotone if and only if the sum of A and at least one duality mapping of V into V^* is surjective, provided that V is reflexive.

By symbols "$\xrightarrow{s}$" and "$\xrightarrow{w}$" we means the convergences in the strong and the weak topology, respectively.

Throughout this paper, let H be a Hilbert space and X be a Banach space such that $X \subset H$, X is dense in H and the natural injection from X into H is continuous, and suppose that X is uniformly convex and X^* is strictly convex. Identifying H with its dual space by means of the inner product $(\cdot, \cdot)_H$ in H, we have the relations: $X \subset H \subset X^*$. Let $0<T<\infty$, $2 \leq p < \infty$ and $1/p+1/p'=1$. As V we take $L^p(0, T; X)$ which consists of p-th power summable mappings $u(t)$ of $[0, T]$ into X with norms$\left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p}$. Then $V^*=L^{p'}(0, T; X^*)$ is the dual space of V by the pairing $(\cdot, \cdot)_V = \int_0^T (\cdot, \cdot)_X dt$, and $\|\cdot\|_{V^*} = \left(\int_0^T \|\cdot\|_{X^*}^{p'} dt\right)^{1/p'}$.

We denote by $\mathscr{F}$ the duality mapping of X into X^* associated with $\mu(r)=r^{p-1}$. Then the mapping F of $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ given by $(Fu)(t)=\mathscr{F}[u(t)]$ is also the duality mapping of $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ associated with the same gauge function.

Let ψ be a function on $[0, T] \times X$ such that for each $t \in [0, T]$, $\psi(t; \cdot)$ is a lower semicontinuous convex function on X with values in $(-\infty, \infty]$ and $\psi(t; \cdot) \not\equiv \infty$ such that for each $v \in L^p(0, T; X)$, $t \to \psi(t; v(t))$ is measurable on $[0, T]$. We put

$$D_t = \{z \in X; \psi(t; z) < \infty\} \qquad \text{for each} \quad t \in [0, T]$$

and D_H = the closure of D_0 in H, and define a function Ψ on $L^p(0, T; X)$ by

$$\Psi(v) = \begin{cases} \int_0^T \psi(t; v(t)) dt & \text{if} \quad v \in D(\Psi), \\ \infty & \text{otherwise}, \end{cases}$$

where $D(\Psi) = \{v \in L^p(0, T; X); \psi(\cdot; v(\cdot)) \in L^1(0, T)\}$.

Given $u_0 \in D_H$ and $f \in L^{p'}(0, T; X^*)$, we formulate the problem $V[\psi, f, u_0]$ as follows: Find $u \in D(\Psi) \cap C([0, T]; H)$ such that

(i) $u(0) = u_0$;

(ii) $u' \,(= (d/dt)u) \in L^{p'}(0, T; X^*)$;

(iii) $\int_0^T (u'-f, u-v)_X dt \leq \Psi(v) - \Psi(u) \qquad$ for every $\quad v \in D(\Psi)$.

Such a function u is called a strong solution of $V[\psi, f, u_0]$, while a function $u \in D(\Psi)$ is called a weak solution of $V[\psi, f, u_0]$ if the following (iv) is satisfied:

(iv) $\begin{cases} \int_0^T (v'-f, u-v)_X dt - \frac{1}{2}\|u_0 - v(0)\|_H^2 \leq \Psi(v) - \Psi(u) \\ \text{for every } v \in D(\Psi) \cap C([0, T]; H) \text{ such that } v' \in L^{p'}(0, T; X^*). \end{cases}$

Now, for each $u_0 \in D_H$ we consider the following operator M_{u_0} (resp. S_{u_0}) from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$: $[u, f] \in G(M_{u_0})$ (resp. $G(S_{u_0})$) if and only if u is a weak (resp. strong) solution of $V[\psi, f, u_0]$.

Roughly speaking, the relation $f \in S_{u_0}(u)$ implies that u is a strong solution of the initial value problem

$$\begin{cases} u'(t) + \partial\psi(t; u(t)) \ni f(t) & \text{on} \quad [0, T], \\ u(0) = u_0, \end{cases}$$

where $\partial\psi(t; \cdot)$ is the subdifferential of $\psi(t; \cdot)$. Such a problem has been studied by many authors (e.g., [1, 2, 4, 5, 8, 10, 14, 15, 17]).

The aim of the present paper is to investigate the operators S_{u_0} and M_{u_0}. In fact, we shall show that M_{u_0} is a kind of closure of S_{u_0} and is a maximal monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$. Our main result extends Theorem II.2 in Brézis [5; Chapter 2] to the time-dependent case and has many applications to initial-boundary value problems for nonlinear parabolic partial differential equations (e.g., [5, 7, 11, 12]).

2. Main theorem

Our main theorem is stated as follows:

THEOREM. *Suppose that*

(a) *Ψ is lower semicontinuous, $\Psi \not\equiv \infty$ and $\Psi > -\infty$ on $L^p(0, T; X)$;*

(b) *there are subsets D of D_H and $\mathscr{D}$ of $L^{p'}(0, T; X^*)$ with the following properties: D is dense in D_H, $\mathscr{D}$ is dense in $L^{p'}(0, T; X^*)$ and for each $x \in D$ and $g \in \mathscr{D}$ there exists $u \in L^p(0, T; X)$ such that $g \in Fu + S_x(u)$.*

Then we have:

(I) *If $u_0 \in D_H$ and $u \in D(M_{u_0})$, then $u \in C([0, T]; H)$ and $u(0) = u_0$.*

(II) *Let u_0 be any element of D_H. Then $[u, f] \in G(M_{u_0})$ if and only if there are sequences $\{u_{0,n}\} \subset D_H$, $\{[u_n, f_n]\} \subset L^p(0, T; X) \times L^{p'}(0, T; X^*)$ such that $[u_n, f_n] \in G(S_{u_{0,n}})$ for each n, $u_{0,n} \xrightarrow{s} u_0$ in H, $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and $f_n \xrightarrow{w} f$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$.*

(III) *For each $u_0 \in D_H$, M_{u_0} is a maximal monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$.*

(IV) *Let $u_{0,i} \in D_H$ and $[u_i, f_i] \in G(M_{u_{0,i}})$ $(i = 1, 2)$. Then for any $s, t \in [0, T]$ with $s \leq t$,*

$$\|u_1(t) - u_2(t)\|_H^2 \leq \|u_1(s) - u_2(s)\|_H^2 + 2\int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))_X d\tau.$$

REMARK 1. If X^* is uniformly convex, then "$f_n \xrightarrow{w} f$" in the above (II) may be replaced by "$f_n \xrightarrow{s} f$". This is easily checked in the proof of the theorem.

REMARK 2. Since Ψ is convex on $L^p(0, T; X)$, the assumption (a) implies that Ψ is weakly sequentially lower semicontinuous on $L^p(0, T; X)$ and that there are $f^* \in L^{p'}(0, T; X^*)$ and a number c such that

$$\Psi(v) \geq \int_0^T (f^*, v)_X dt + c \qquad \text{for all} \quad v \in L^p(0, T; X).$$

COROLLARY 1. *Suppose that there is a positive number C with the following property: For each s, $t \in [0, T]$ with $s \leq t$ and for each $z \in D_s$ there is $\tilde{z} \in D_t$ such that*

$$\|\tilde{z} - z\|_X \leq C|t-s|$$

and

$$\psi(t; \tilde{z}) \leq \psi(s; z) + C|t-s|(1 + \|z\|_X^p + |\psi(s; z)|).$$

Then (I), (II), (III) *and* (IV) *in the theorem hold.*

In case $X = H$, the hypothesis in Corollary 1 can be replaced by a weaker one:

COROLLARY 2. *Suppose that $X = H$ and that there is a positive non-decreasing function $r \to C(r)$ with the following property: For each $r > 0$, each s, $t \in [0, T]$ with $s \leq t$ and for each $z \in D_s$ with $\|z\|_H \leq r$ there is $\tilde{z} \in D_t$ such that*

$$\|\tilde{z} - z\|_H \leq C(r)|t-s|$$

and

$$\psi(t; \tilde{z}) \leq \psi(s; z) + C(r)|t-s|(1 + |\psi(s; z)|).$$

Then (I), (II), (III) *and* (IV) *are valid.*

In fact, these corollaries are consequences of the above theorem and results in [8] and [10].

3. Proof of the theorem

In order to prove the theorem we prepare some lemmas.

LEMMA 1 ([10; Theorem 7.1]). *If $u_{0,i} \in D_H$ and $[u_i, f_i] \in G(S_{u_{0,i}})$ $(i = 1, 2)$, then for any s, $t \in [0, T]$ with $s \leq t$,*

$$\|u_1(t) - u_2(t)\|_H^2 \leq \|u_1(s) - u_2(s)\|_H^2 + 2\int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))_X d\tau.$$

This lemma suggests us that for each $u_0 \in D_H$ the following operator $\tilde{S}_{u_0}$ from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ is important: $[u, f] \in G(\tilde{S}_{u_0})$ if and only if there are sequences $\{u_{0,n}\} \subset D_H$ and $\{[u_n, f_n]\} \subset L^p(0, T; X) \times L^{p'}(0, T; X^*)$ such that $[u_n, f_n] \in G(S_{u_{0,n}})$ for each n, $u_{0,n} \xrightarrow{s} u_0$ in H, $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and $f_n \xrightarrow{w} f$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$.

As for $\tilde{S}_{u_0}$ we have

LEMMA 2. *Suppose that (a) and (b) in the theorem are satisfied and let u_0 be any element of D_H. Then:*

(1) *If $u \in D(\tilde{S}_{u_0})$, then $u \in D(\Psi) \cap C([0, T]; H)$ and $u(0) = u_0$.*

(2) *If $[u, f] \in G(\tilde{S}_{u_0})$, then the inequality*

$$\int_0^T (v' - f, u - v)_X dt + \frac{1}{2} \| u(T) - v(T) \|_H^2 - \frac{1}{2} \| u_0 - v(0) \|_H^2 \le \Psi(v) - \Psi(u) \tag{3.1}$$

holds for every $v \in D(\Psi) \cap C([0, T]; H)$ with $v' \in L^{p'}(0, T; X^*)$.

PROOF. Let $[u, f]$ be any element of $G(\tilde{S}_{u_0})$. Then, by definition we find sequences $\{u_{0,n}\} \subset D_H$ and $\{[u_n, f_n]\}$ such that $[u_n, f_n] \in G(S_{u_{0,n}})$, $u_{0,n} \xrightarrow{s} u_0$ in H, $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and $f_n \xrightarrow{w} f$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$. It follows from Lemma 1 that $\{u_n\}$ converges to u in H uniformly on $[0, T]$, so that $u \in C([0, T]; H)$ and $u(0) = u_0$. For each n we have

$$\int_0^T (u_n' - f_n, u_n - v)_X dt \le \Psi(v) - \Psi(u_n) \qquad \text{whenever} \quad v \in D(\Psi),$$

because $f_n \in S_{u_{0,n}}(u_n)$. If $v \in D(\Psi) \cap C([0, T]; H)$ and $v' \in L^{p'}(0, T; X^*)$, then we have by integration by parts

$$\int_0^T (v' - f_n, u_n - v)_X dt + \frac{1}{2} \| u_n(T) - v(T) \|_H^2 - \frac{1}{2} \| u_{0,n} - v(0) \|_H^2 \le \Psi(v) - \Psi(u_n). \tag{3.2}$$

Now, note that by assumption there is at least one function $h \in D(\Psi) \cap C([0, T]; H)$ with $h' \in L^{p'}(0, T; X^*)$; in fact, for each $x_0 \in D$, $D(S_{x_0}) \neq \phi$ by assumption (b) and any function in $D(S_{x_0})$ has such properties. Substituting this h for v in (3.2), we see that $\{\Psi(u_n)\}$ is bounded above and on account of (a) and Remark 2 we have

$$-\infty < \Psi(u) \le \liminf_{n \to \infty} \Psi(u_n) < \infty,$$

so that $u \in D(\Psi)$. Letting $n \to \infty$ in (3.2), we obtain (3.1).

COROLLARY. *Suppose that (a) and (b) are satisfied and let u_0 be any element of D_H. Then M_{u_0} is an extension of $\tilde{S}_{u_0}$, i.e., $G(\tilde{S}_{u_0}) \subset G(M_{u_0})$.*

LEMMA 3. *Let u_0 be any element of D_H. If $[u_1, f_1] \in G(M_{u_0})$ and $[u_2, f_2] \in G(\tilde{S}_{u_0})$, then we have*

$$\int_0^T (f_1 - f_2, u_1 - u_2)_X dt \geq 0. \tag{3.3}$$

PROOF. By the definition of $\tilde{S}_{u_0}$ there are sequences $\{u_{0,n}\} \subset D_H$, $\{[u_{2,n}, f_{2,n}]\}$ such that $[u_{2,n}, f_{2,n}] \in G(S_{u_{0,n}})$, $u_{0,n} \xrightarrow{s} u_0$ in H, $u_{2,n} \xrightarrow{s} u_2$ in $L^p(0, T; X)$ and $f_{2,n} \xrightarrow{w} f_2$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$. For each n we see that

$$\int_0^T (u'_{2,n} - f_1, u_1 - u_{2,n})_X dt - \frac{1}{2} \|u_0 - u_{0,n}\|_H^2 \leq \Psi(u_{2,n}) - \Psi(u_1)$$

and

$$\int_0^T (u'_{2,n} - f_{2,n}, u_{2,n} - u_1)_X dt \leq \Psi(u_1) - \Psi(u_{2,n}).$$

By adding these two inequalities we get

$$\int_0^T (f_1 - f_{2,n}, u_1 - u_{2,n})_X dt \geq -\frac{1}{2} \|u_0 - u_{0,n}\|_H^2,$$

so we have (3.3) by letting $n \to \infty$.

COROLLARY. *For each $u_0 \in D_H$, $\tilde{S}_{u_0}$ is a monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$.*

This corollary is a direct consequence of Lemma 3 and the corollary of Lemma 2.

PROOF OF THE THEOREM: To prove the theorem it is enough to show that $\tilde{S}_{u_0}$ is a maximal monotone operator from $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ for each $u_0 \in D_H$. Indeed, assume the maximal monotonicity of $\tilde{S}_{u_0}$ for each $u_0 \in D_H$. Then, by Lemma 3 we have $M_{u_0} = \tilde{S}_{u_0}$ for each $u_0 \in D_H$, which implies (III), simultaneously (II) by the definition of $\tilde{S}_{u_0}$ and (I) by (1) of Lemma 2. Moreover, (IV) also easily follows from Lemma 1.

Since, for each $u_0 \in D_H$, $\tilde{S}_{u_0}$ is monotone by the corollary of Lemma 3, in order to show the maximal monotonicity of $\tilde{S}_{u_0}$ it is sufficient to prove that $\tilde{S}_{u_0} + F$ is surjective.

Let u_0 and f be any elements of D_H and $L^{p'}(0, T; X^*)$, respectively. Now, choose sequences $\{u_{0,n}\} \subset D$ and $\{f_n\} \subset \mathscr{D}$ so that $u_{0,n} \xrightarrow{s} u_0$ in H and $f_n \xrightarrow{s} f$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$. In view of assumption (b), for each n there exists

$u_n \in D(S_{u_{0,n}})$ such that $f_n - Fu_n \in S_{u_{0,n}}(u_n)$, or equivalently,

$$\int_0^T (u_n' - f_n + Fu_n, u_n - v)_X dt \leq \Psi(v) - \Psi(u_n) \qquad \text{for all} \quad v \in D(\Psi). \tag{3.4}$$

Taking h (the same function as in the proof of Lemma 2) as v in (3.4), we have by integration by parts

$$\int_0^T (h' - f_n + Fu_n, u_n - h)_X dt + \frac{1}{2} \|u_n(T) - h(T)\|_H^2$$
$$- \frac{1}{2} \|u_{0,n} - h(0)\|_H^2 \leq \Psi(h) - \Psi(u_n).$$

Since

$$\|\mathscr{F}[u_n(t)]\|_{X*} = \|u_n(t)\|_X^{p-1}$$

and

$$(\mathscr{F}[u_n(t)], u_n(t))_X = \|u_n(t)\|_X^p,$$

the above inequality yields that

$$\Psi(u_n) + \int_0^T \|u_n\|_X^p dt$$
$$\leq \frac{1}{2} \|u_{0,n} - h(0)\|_H^2 + \Psi(h) + \int_0^T (\|h'\|_{X*} + \|f_n\|_{X*})(\|u_n\|_X + \|h\|_X)\, dt$$
$$+ \int_0^T \|u_n\|_X^{p-1} \|h\|_X dt.$$

Hence, by the assumption (a) and Remark 2 we see that $\{u_n\}$ is bounded in $L^p(0, T; X)$ and $\{\Psi(u_n)\}$ is bounded. We apply Lemma 1 to $[u_n, f_n - Fu_n] \in G(S_{u_{0,n}})$ and $[u_m, f_m - Fu_m] \in G(S_{u_{0,m}})$. Using the monotonicity of F we have

$$\|u_n(t) - u_m(t)\|_H^2 \leqq 2\int_0^t (f_n - Fu_n - f_m + Fu_m, u_n - u_m)_X d\tau + \|u_{0,n} - u_{0,m}\|_H^2$$
$$\leqq 2\int_0^T \|f_n - f_m\|_{X*} \|u_n - u_m\|_X d\tau + \|u_{0,n} - u_{0,m}\|_H^2 \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty.$$

Hence $\{u_n\}$ converges in H uniformly on $[0, T]$ to a function $u \in C([0, T]; H)$ with $u(0) = u_0$. Then $u \in L^p(0, T; X)$, $u_n \xrightarrow{w} u$ in $L^p(0, T; X)$ as $n \to \infty$ and

$$-\infty < \Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n) < \infty$$

because of (a) and Remark 2 again. We may assume, taking a subsequence if necessary, that $Fu_n \xrightarrow{w} g$ in $L^{p'}(0, T; X^*)$ as $n \to \infty$ for some $g \in L^{p'}(0, T; X^*)$.

Since $u \in D(\Psi)$ as was seen above, we infer from (3.4) and the monotonicity of F that

$$\limsup_{n\to\infty}\left\{\int_0^T (u_n', u_n-u)_X dt + \Psi(u_n) - \Psi(u)\right\} \leq 0. \tag{3.5}$$

From (3.4) again we obtain by integration by parts

$$\int_0^T (Fu_n, u_n - v)_X dt \leq \int_0^T (v', v-u_n)_X dt + \frac{1}{2}\|u_{0,n} - v(0)\|_H^2$$

$$+ \Psi(v) - \Psi(u_n) + \int_0^T (f_n, u_n - v)_X dt$$

for every $v \in D(\Psi) \cap C([0, T]; H)$ with $v' \in L^{p'}(0, T; X^*)$. Hence,

$$\limsup_{n\to\infty}\int_0^T (Fu_n, u_n-u)_X dt$$

$$\leq \limsup_{n\to\infty}\int_0^T (Fu_n, u_n-v)_X dt + \int_0^T (g, v-u)_X dt$$

$$\leq \int_0^T (v', v-u)_X dt + \Psi(v) - \Psi(u) + \int_0^T (f-g, u-v)_X dt$$

$$+ \frac{1}{2}\|u_0 - v(0)\|_H^2$$

for every $v \in D(\Psi) \cap C([0, T]; H)$ with $v' \in L^{p'}(0, T; X^*)$. In the last expression of these inequalities, take $v = u_n$ and let $n \to \infty$. Then by (3.5) we have

$$\limsup_{n\to\infty}\int_0^T (Fu_n, u_n-u)_X dt \leq 0.$$

This implies (cf. [6; Chapter 1]) that $u_n \xrightarrow{s} u$ in $L^p(0, T; X)$ and $Fu_n \xrightarrow{w} Fu$ in $L^{p'}(0. T; X^*)$, since $L^p(0, T; X)$ is uniformly convex. Thus by the definition of $\tilde{S}_{u_0}$, $f - Fu \in \tilde{S}_{u_0}(u)$. As f was an arbitrary function in $L^{p'}(0, T; X^*)$, we conclude that $\tilde{S}_{u_0} + F$ is surjective.

4. Application

In this section we give an application.

Let A be a singlevalued bounded pseudomonotone operator (see [3]) from the closure of $D(\Psi)$ in $L^p(0, T; X)$ into $L^{p'}(0, T; X^*)$ and suppose that there exists $w \in D(\Psi) \cap C([0, T]; H)$ with w' in $L^{p'}(0, T; X^*)$ such that

$$\frac{\int_0^T (Av,\ v-w)_X dt + \Psi(v)}{\|v\|_{L^p(0,T;X)}} \longrightarrow \infty \ as \ \|v\|_{L^p(0,T;X)} \longrightarrow \infty,\ v \in D(\Psi)\,.$$

Then we have

PROPOSITION. *Under the same assumption as in Corollary* 1, *for each* $u_0 \in D_H$ *and each* $f \in L^{p'}(0, T; X^*)$ *there exists* $u \in D(\Psi) \cap C([0, T]; H)$ *such that* $u(0)=u_0$ *and*

$$\int_0^T (v'-f+Au,\ u-v)_X dt - \frac{1}{2}\|u_0 - v(0)\|_H^2 \leq \Psi(v) - \Psi(u)$$

for every $v \in D(\Psi) \cap C([0, T]; H)$ *with* $v' \in L^{p'}(0, T; X^*)$.

PROOF. Since M_{u_0} is maximal monotone by Corollary 1, it follows from a result in Brèzis [3] that $M_{u_0}+A$ is surjective for any $u_0 \in D_H$. Given any $f \in L^{p'}(0, T; X^*)$, there exists $u \in D(\Psi) \cap C([0, T]; H)$ such that $f - Au \in M_{u_0}(u)$. This implies the above inequality. The fact that $u(0)=u_0$ follows from (I) in the Theorem.

EXAMPLE. Let Ω be a bounded domain in R^m $(m \geq 2)$ with smooth boundary Γ and Γ_0 be a closed subset of Γ. We set $Q=(0, T)\times\Omega$, $\Sigma=(0, T)\times\Gamma$ and $\Sigma_0=(0, T)\times\Gamma_0$. Given functions u_0 on Ω, f on Q, l on Σ and g on Σ, we consider the initial-boundary value problem of mixed type

$$\text{(P)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{k=1}^m \frac{\partial}{\partial x_k}\left(\alpha_k \left|\frac{\partial u}{\partial x_k}\right|^{p-2} \frac{\partial u}{\partial x_k}\right) + \alpha_0 |u|^{p-2} u = f & \text{in } Q. \\ u(0, \cdot) = u_0 & \text{on } \Omega, \\ u = l & \text{on } \Sigma_0, \\ -\displaystyle\sum_{k=1}^m \alpha_k \left|\frac{\partial u}{\partial x_k}\right|^{p-2} \frac{\partial u}{\partial x_k} \nu_k + g = e^u & \text{on } \Sigma - \Sigma_0, \end{cases}$$

where $\nu(x)=(\nu_1(x), \nu_2(x),\dots, \nu_m(x))$ is the unit vector which is normal to Γ at $x \in \Gamma$ and oriented toward the exterior of Ω.

Now, we give a weak formulation of the variational inequality associated with (P). Place the following restrictions on α_k, u_0, f, l and g:

(a) $\alpha_0, \alpha_1,\dots, \alpha_m$ are bounded measurable functions on $[0, T]\times\Omega$ such that $\alpha_k \geq C_1$ a.e. on $[0, T]\times\Omega$, $k=0, 1,\dots, m$, for some positive constant C_1.

(b) $u_0 \in L^2(\Omega)$, $f \in L^{p'}(Q)$ and $g \in L^{p'}(0, T; W^{-1/p', p'}(\Gamma))$.

(c) l is a bounded measurable function on $[0, T]\times\Gamma$ such that

$$\|l(t, \cdot) - l(s, \cdot)\|_{L^\infty(\Gamma)} + \|l(t, \cdot) - l(s, \cdot)\|_{W^{1/p', p}(\Gamma)} \leq C_2 |t-s|$$

for all $s, t \in [0, T]$, where C_2 is a positive constant.

For each $t \in [0, T]$, put $K(t) = \{z \in W^{1,p}(\Omega); \gamma z = l(t, \cdot)$ on Γ_0 in the sense of $W^{1/p',p}(\Gamma)$ [1]$\}$ (γ is the trace operator from $W^{1,p}(\Omega)$ into $W^{1/p',p}(\Gamma)$) and define

$$\psi(t; z) = \begin{cases} \int_\Gamma e^{\gamma z} d\Gamma & \text{if } z \in K(t) \text{ and } e^{\gamma z} \in L^1(\Gamma), \\ \infty & \text{otherwise.} \end{cases}$$

Then we can verify the hypothesis in Corollary 1 (see [11; § 3]). The weak variational formulation for (P) is of the following form: Find $u \in D(\Psi) \cap C([0, T]; L^2(\Omega))$ such that $u(0) = u_0$ and

$$\begin{aligned} &\int_0^T < v', u - v > dt + \sum_{k=1}^m \int_Q \alpha_k \left| \frac{\partial u}{\partial x_k} \right|^{p-2} \frac{\partial u}{\partial x_k} \left(\frac{\partial u}{\partial x_k} - \frac{\partial v}{\partial x_k} \right) dx dt \\ &\quad + \int_Q \alpha_0 |u|^{p-2} u(u - v)\, dx dt \\ &\quad - \int_Q f(u - v)\, dx dt - \int_0^T (g, \gamma u - \gamma v)_\Gamma dt - \frac{1}{2} \|u_0 - v(0)\|^2_{L^2(\Omega)} \\ &\le \int_\Gamma e^{\gamma v}\, d\Gamma dt - \int_\Gamma e^{\gamma u}\, d\Gamma dt \end{aligned}$$

for every $v \in D(\Psi) \cap C([0, T]; L^2(\Omega))$ with $v' \in L^{p'}(0, T; (W^{1,p}(\Omega))^*)$, where $<\cdot, \cdot>$ and $(\cdot, \cdot)_\Gamma$ stand for the natural pairings between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$ and between $W^{-1/p',p'}(\Gamma)$ and $W^{1/p',p}(\Gamma)$, respectively. Applying the Proposition for the above ψ and the operator A from $L^p(0, T; W^{1,p}(\Omega))$ into $L^{p'}(0, T; (W^{1,p}(\Omega))^*)$ defined by

$$< Av, w > = \sum_{k=1}^m \int_Q \alpha_k \left| \frac{\partial v}{\partial x_k} \right|^{p-2} \frac{\partial v}{\partial x_k} \frac{\partial w}{\partial x_k}\, dx dt + \int_Q \alpha_0 |v|^{p-2}\, v w\, dx dt,$$

we see that this problem has a solution u. Moreover, if u has the property that $u' \in L^{p'}(0, T; L^{p'}(\Omega))$, then we can show that u is a solution of (P) in a generalized sense (see [9; § 1] and [11; § 3]).

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1) For the definition, see [13; § 1] or [11; § 1].

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