HIROSHIMA MATH. J.
6 (1976), 95–111

Generalized Extremal Length of an Infinite Network

Tadashi NAKAMURA and Maretsugu YAMASAKI

(Received September 2, 1975)

Introduction

The extremal length of a network, which is the reciprocal of the value of a quadratic programming problem, was first investigated by R. J. Duffin [4] on a finite graph and next by the second author [7] on an infinite graph. In this paper we shall be concerned with a generalized form of the extremal length as in [5] along the same lines as in [4] and [7]. The generalized extremal length of an infinite network may be regarded as the reciprocal of the value of a convex programming problem. One of our main purposes is to establish a reciprocal relation between the generalized extremal distance and the generalized extremal width of an infinite network which was established by M. Ohtsuka [5] for the continuous case. We shall also study the generalized extremal length of an infinite network relative to a finite set and the ideal boundary of the network. A concept of non-linear flows which was studied in [1] and [3] will appear in § 3 and § 4 in connection with the extremal width of a network.

§ 1. Preliminaries

Let X be a set of nodes and let Y be a set of directed arcs. Since we always consider the case where X and Y consist of a countably infinite number of elements, we put

$$X = \{0, 1, 2,\dots, n,\dots\},$$

$$Y = \{1, 2,\dots, n,\dots\}.$$

Let $K=(K_{vj})$ be the node-arc incidence matrix. Namely $K_{vj}=1$ if arc j is directed toward node v, $K_{vj}=-1$ if arc j is directed away from node v and $K_{vj}=0$ if arc j and node v do not meet.

We assume that X, Y and K satisfy the following conditions:

(1.1) $\{j \in Y;\ K_{vj} \neq 0\}$ is a nonempty finite set for each $v \in X$.

(1.2) $e(j)=\{v \in X;\ K_{vj} \neq 0\}$ consists of exactly two nodes for each $j \in Y$.

(1.3) For any $\alpha, \beta \in X$, there are $v_1,\dots, v_n \in X$ and $j_1,\dots, j_{n+1} \in Y$ such that $e(j_i)=\{v_{i-1}, v_i\}$, $i=1,\dots, n+1$ with $v_0=\alpha$ and $v_{n+1}=\beta$.

Given a strictly positive function r on Y, the quartet $<X, Y, K, r>$ is then called an infinite network. For simplicity denote by $<X, Y>$ a network $<X, Y, K, r>$ if there is no confusion from the context.

Let X' and Y' be subsets of X and Y respectively and let K' and r' be the restrictions of K and r onto $X' \times Y'$ and Y' respectively. We say that $<X', Y'> = <X', Y', K', r'>$ is a subnetwork of $<X, Y, K, r>$ if it is a network in itself. In case X' (or Y') is a finite set, we call $<X', Y'>$ a finite subnetwork of $<X, Y>$.

We say that a sequence $\{<X_n, Y_n>\}$ of finite subnetworks of $<X, Y>$ is an exhaustion of $<X, Y>$ if

$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n, \tag{1.4}$$

$$\{j \in Y;\ K_{\nu j} \neq 0\} \subset Y_{n+1} \qquad \text{for each} \quad \nu \in X_n. \tag{1.5}$$

Let p and q be positive numbers such that

$$1/p + 1/q = 1 \quad \text{and} \quad p > 1. \tag{1.6}$$

Let $L(X)$ and $L(Y)$ be the sets of all real functions on X and Y respectively. For $u \in L(X)$ and $w \in L(Y)$, we put

$$u_\nu = u(\nu), \quad w_j = w(j),$$

$$Su = \{\nu \in X;\ u_\nu \neq 0\}, \quad Sw = \{j \in Y;\ w_j \neq 0\},$$

$$D_p(u) = \sum_{j=1}^{\infty} r_j^{1-p} \Big| \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu \Big|^p, \tag{1.7}$$

$$H_p(w) = \sum_{j=1}^{\infty} r_j |w_j|^p. \tag{1.8}$$

We shall use the following classes of functions on X and Y:

$$L_0(X) = \{u \in L(X);\ Su \text{ is a finite set}\},$$

$$L_0(Y) = \{w \in L(Y);\ Sw \text{ is a finite set}\},$$

$$L^+(Y) = \{w \in L(Y);\ w_j \geqq 0 \quad \text{on} \quad Y\},$$

$$L_p(Y; r) = \{w \in L(Y);\ H_p(w) < \infty\},$$

$$L_p^+(Y; r) = \{w \in L^+(Y);\ H_p(w) < \infty\}.$$

Note that $L_p(Y; r)$ is a reflexive Banach space with respect to the norm $[H_p(w)]^{1/p}$. If $H_p(w - w^{(n)}) \to 0$ as $n \to \infty$, then $w_j^{(n)} \to w_j$ as $n \to \infty$ for each j.

For a nonempty subset A of X, let us put

$$\boldsymbol{D}^{(p)} = \boldsymbol{D}^{(p),A} = \{u \in L(X);\ D_p(u) < \infty \text{ and } u = 0 \text{ on } A\}.$$

We have

LEMMA 1.1.[1] *For any n, there exists a constant M_n such that*

$$\sum_{i=0}^{n} |u_i| \leqq M_n[D_p(u)]^{1/p}$$

for all $u \in \boldsymbol{D}^{(p)}$.

PROPOSITION 1.1. *$\boldsymbol{D}^{(p)}$ is a reflexive Banach space with respect to the norm $[D_p(u)]^{1/p}$.*

PROOF. It follows from Lemma 1.1 and the Minkowski inequality that $[D_p(u)]^{1/p}$ is a norm on $\boldsymbol{D}^{(p)}$. We can prove by a standard argument that $\boldsymbol{D}^{(p)}$ is a Banach space. Let E be the linear transformation from $L(X)$ into $L(Y)$ defined by

$$w_j = (Eu)_j = r_j^{-1} \sum_{\nu=0}^{\infty} K_{\nu j} u_\nu$$

and denote by $E(\boldsymbol{D}^{(p)})$ the image of $\boldsymbol{D}^{(p)}$ under E. From the relation $H_p(Eu) = D_p(u)$, it follows that E is a Banach space isomorphism from $\boldsymbol{D}^{(p)}$ onto $E(\boldsymbol{D}^{(p)})$. It is easily seen that $E(\boldsymbol{D}^{(p)})$ is a closed linear subspace of $L_p(Y; r)$. Since $L_p(Y; r)$ is a reflexive Banach space, $E(\boldsymbol{D}^{(p)})$ is also a reflexive Banach space (cf. [2], p. 116, Proposition 11). Therefore $\boldsymbol{D}^{(p)}$ is reflexive.

LEMMA 1.2.[2] *Let T be a normal contraction of the real line R and $u \in \boldsymbol{D}^{(p)}$. Then $Tu \in \boldsymbol{D}^{(p)}$ and $D_p(Tu) \leqq D_p(u)$.*

We often use the following theorem to assure the existence of an optimal solution of an extremum problem.

THEOREM A.[3] *Let Z be a reflexive Banach space with the norm $\|z\|$ and C be a nonempty closed convex set in Z. Then there exists a point $\hat{z} \in C$ such that $\|\hat{z}\| = \min\{\|z\|;\ z \in C\}$. This minimizing point is unique if every boundary point of the ball $\|z\| \leqq 1$ is an extreme point.*

§2. Generalized extremal length of a network

A path P from node α to node β is the triple $(C_X(P), C_Y(P), p(P))$ of a finite

1) Cf. Lemma 1 in [7].
2) Cf. Lemma 2 in [7].
3) [2], p. 117, Exercise 1.

ordered set $C_X(P)=\{v_0, v_1,..., v_n\}$ of nodes, a finite ordered set $C_Y(P)=\{j_1, j_2,..., j_n\}$ of arcs and a function $p(P)$ on Y called the index of P such that

$$(P)\qquad \begin{aligned} &v_0 = \alpha,\ v_n = \beta,\ v_i \neq v_k \qquad (i \neq k), \\ &e(j_i) = \{v_{i-1}, v_i\} \qquad \text{if} \quad j \in C_Y(P), \\ &p_j(P) = 0 \qquad \text{if} \quad j \notin C_Y(P), \\ &p_j(P) = -K_{vj} \text{ with } v = v_{i-1} \qquad \text{if} \quad j = j_i. \end{aligned}$$

A path P from node α to the ideal boundary ∞ of $<X, Y>$ is the triple $(C_X(P), C_Y(P), p(P))$ of an infinite ordered set $C_X(P)=\{v_0, v_1,...\}$ of nodes, an infinite ordered set $C_Y(P)=\{j_1, j_2,...\}$ of arcs and a function $p(P)$ on Y called the index of P which satisfy condition (P) except the terminal condition $v_n=\beta$.

Denote by $\boldsymbol{P}_{\alpha\beta}$ (resp. $\boldsymbol{P}_{\alpha\infty}$) the set of all paths from node α to node β (resp. ∞). Note that condition (1.3) means $\boldsymbol{P}_{\alpha\beta}\neq\phi$ for any α, $\beta\in X$. For mutually disjoint nonempty subsets A and B of X, denote by $\boldsymbol{P}_{A,B}$ the set of all paths P such that $P\in\boldsymbol{P}_{\alpha\beta}$, $C_X(P)\cap A=\{\alpha\}$ and $C_X(P)\cap B=\{\beta\}$ for some $\alpha\in A$ and $\beta\in B$. Let $\boldsymbol{P}_{A,\infty}$ be the set of all paths P such that $P\in\boldsymbol{P}_{\alpha\infty}$ and $C_X(P)\cap A=\{\alpha\}$ for some $\alpha\in A$.

Let Γ be a set of paths in an infinite network $<X, Y, K, r>$. For every $W\in L^+(Y)$, a value $t(W; \Gamma)$ is defined by

$$t(W; \Gamma) = \inf\{\textstyle\sum_P r_j W_j;\ P\in\Gamma\}, \tag{2.1}$$

where $\sum_P r_j W_j$ is an abbreviation of $\sum_{j\in C_Y(P)} r_j W_j$.

We define the extremal length $\lambda_p(\Gamma)$ of Γ of order p by

$$\lambda_p(\Gamma)^{-1} = \inf\{H_p(W);\ W\in E_p(\Gamma)\}, \tag{2.2}$$

where $\qquad E_p(\Gamma) = \{W\in L_p^+(Y; r);\ t(W; \Gamma)\geqq 1\}.$

We use the convention in this paper that the infimum of a real function on the empty set ϕ is equal to ∞. We shall study some properties of the extremal length which are analogous to the continuous case (cf. [6]).

Let Γ_1 and Γ_2 be sets of paths in $<X, Y>$. We shall write $\Gamma_1<\Gamma_2$ if for any $P^{(2)}\in\Gamma_2$ there is a $P^{(1)}\in\Gamma_1$ such that $C_Y(P^{(1)})\subset C_Y(P^{(2)})$.

We easily obtain

LEMMA 2.1. *If Γ_1 and Γ_2 are sets of paths in $<X, Y>$ such that $\Gamma_1<\Gamma_2$, then $\lambda_p(\Gamma_1)\leqq\lambda_p(\Gamma_2)$.*

PROPOSITION 2.1. *Let P be a path and set $R(P)=\sum_P r_j$. Then $\lambda_p(\{P\})=R(P)^{p-1}$.*

PROOF. Let $W\in E_p(\{P\})$. Then $\sum_P r_jW_j\geqq 1$. It follows from Hölder's inequality that $1\leqq R(P)^{1/q}H_p(W)^{1/p}$. Thus we have $\lambda_p(\{P\})\leqq R(P)^{p-1}$. Next we show the converse inequality. Let $\{<X_n, Y_n>\}$ be an exhaustion of $<X, Y>$ such that $C_Y(P)\cap Y_1\neq\phi$. Set $Y'_n=C_Y(P)\cap Y_n$ and define $W^{(n)}\in L(Y)$ by $W_j^{(n)}=(\sum_{Y'_n} r_j)^{-1}$ if $j\in Y'_n$ and $W_j^{(n)}=0$ if $j\notin Y'_n$. Then $W^{(n)}\in E_p(\{P\})$ and

$$\lambda_p(\{P\})\geqq H_p(W^{(n)})^{-1}=(\sum_{Y'_n} r_j)^{p-1}.$$

By letting $n\to\infty$, we conclude that $\lambda_p(\{P\})\geqq R(P)^{p-1}$. This completes the proof.

Let Γ_1 and Γ_2 be sets of paths in $<X, Y>$. We say that Γ_1 and Γ_2 are mutually disjoint if $C_Y(P^{(1)})\cap C_Y(P^{(2)})=\phi$ for every $P^{(1)}\in\Gamma_1$ and $P^{(2)}\in\Gamma_2$.

LEMMA 2.2.[4]) *Let $\{\Gamma_n;\ n=1, 2,\dots\}$ be mutually disjoint sets of paths and Γ be a set of paths. If $\Gamma_n<\Gamma$ for each n, then*

$$\lambda_p(\Gamma)^{q-1}\geqq\sum_{n=1}^{\infty}\lambda_p(\Gamma_n)^{q-1}.$$

PROOF. If $\lambda_p(\Gamma_n)=\infty$ for at least one n, our inequality is valid by Lemma 2.1. Therefore we may assume that $\lambda_p(\Gamma_n)<\infty$ for each n. Moreover we may assume that $\lambda_p(\Gamma_n)>0$, i.e., $E_p(\Gamma_n)\neq\phi$ for each n. Let $Y_n=\cup\{C_Y(P);\ P\in\Gamma_n\}$. Then

$$\lambda_p(\Gamma_n)^{-1}=\inf\{H_p(W);\ W\in E_p(\Gamma_n)\text{ and }W=0\text{ on }Y-Y_n\}.$$

Choose any positive integer m and fix it. Let $t_1, t_2,\dots, t_m$ be non-negative numbers such that $\sum_{n=1}^{m} t_n=1$; they will be determined below. Taking $W_j=\sum_{n=1}^{m} t_nW_j^{(n)}$ with $W^{(n)}\in E_p(\Gamma_n)$ such that $W^{(n)}=0$ on $Y-Y^{(n)}$, we have $W_j=t_nW_j^{(n)}$ for each $j\in Y_n$ and

$$\sum_P r_jW_j=\sum_{n=1}^{m} t_n\sum_P r_jW_j^{(n)}\geqq\sum_{n=1}^{m} t_n=1$$

for every $P\in\Gamma$, so that $W\in E_p(\Gamma)$. Therefore

$$\lambda_p(\Gamma)^{-1}\leqq\sum_{j=1}^{\infty} r_j|\sum_{n=1}^{m} t_nW_j^{(n)}|^p=\sum_{n=1}^{m} t_n^pH_p(W^{(n)}).$$

It follows that

$$\lambda_p(\Gamma)^{-1}\leqq\sum_{n=1}^{m} t_n^p\lambda_p(\Gamma_n)^{-1}.$$

4) Cf. [6], p. 79, Theorem 2.10.

Now we choose $t_n=\lambda_p(\Gamma_n)^{q-1}\left(\sum_{n=1}^{m}\lambda_p(\Gamma_n)^{q-1}\right)^{-1}$ and obtain

$$\lambda_p(\Gamma)^{-1} \leqq \left[\sum_{n=1}^{m}\lambda_p(\Gamma_n)^{q-1}\right]^{1-p},$$

which leads to the desired inequality.

Let A and B be mutually disjoint nonempty subsets of X. We define the extremal distance $EL_p(A, B)$ (resp. $EL_p(A, \infty)$) of order p of an infinite network $<X, Y, K, r>$ relative to A and B (resp. A and ∞) by

$$EL_p(A, B) = \lambda_p(\boldsymbol{P}_{A,B}), \tag{2.3}$$

$$EL_p(A, \infty) = \lambda_p(\boldsymbol{P}_{A,\infty}). \tag{2.4}$$

Next we consider the following extremum problem:

(2.5) Find

$$d_p(A, B) = \inf\{D_p(u);\ u\in L(X),\ u = 0 \text{ on } A \text{ and } u = 1 \text{ on } B\}.$$

We have

LEMMA 2.3.[5] *Let $V\in L^+(Y)$. There exists $u\in L(X)$ such that $u=0$ on A,*

$$\left|\sum_{\nu=0}^{\infty}K_{\nu j}u_\nu\right| \leqq V_j \qquad \textit{for each} \quad j\in Y, \tag{2.6}$$

and

$$\inf\{\sum_P V_j;\ P\in\boldsymbol{P}_{A,B}\} = \inf\{u_\nu;\ \nu\in B\}. \tag{2.7}$$

THEOREM 2.1. *$d_p(A, B)=EL_p(A, B)^{-1}$.*

PROOF. We set $d_p=d_p(A, B)$ and $EL_p=EL_p(A, B)$. First we shall prove $d_p\leqq EL_p^{-1}$ in case $EL_p^{-1}<\infty$. Let $W\in E_p(\boldsymbol{P}_{A,B})$ and put $V_j=r_jW_j$. Then $\inf\{\sum_P V_j;\ P\in\boldsymbol{P}_{A,B}\}=t(W;\boldsymbol{P}_{A,B})\geqq 1$. We can find $u\in L(X)$ by Lemma 2.3 such that $u=0$ on A and u satisfies (2.6) and (2.7). Then $u\geqq 1$ on B and

$$D_p(u) = \sum_{j=1}^{\infty} r_j^{1-p}\left|\sum_{\nu=0}^{\infty}K_{\nu j}u_\nu\right|^p \leqq \sum_{j=1}^{\infty} r_j^{1-p}V_j^p = H_p(W) < \infty.$$

Let $v=\min(u, 1)$. Then $v=0$ on A and $v=1$ on B, so that

$$d_p \leqq D_p(v) \leqq D_p(u) \leqq H_p(W)$$

5) Cf. Theorem 3 in [7].

by Lemma 1.2. By the arbitrariness of W, we obtain $d_p \leqq EL_p^{-1}$. Next we shall show that $EL_p^{-1} \leqq d_p$ in case $d_p < \infty$. Let $u \in L(X)$ satisfy $u=0$ on A, $u=1$ on B and $D_p(u) < \infty$. Define $W \in L^+(Y)$ by $W_j = r_j^{-1} | \sum_{v=0}^{\infty} K_{vj} u_v |$. Then it is easily seen that $W \in E_p(\boldsymbol{P}_{A,B})$ (cf. the proof of Theorem 4 in [7]). Hence $EL_p^{-1} \leqq H_p(W) = D_p(u)$ and $EL_p^{-1} \leqq d_p$. Thus we have $d_p = EL_p^{-1}$.

By the aid of Theorem A, we have

PROPOSITION 2.2. *In case $E_p(\boldsymbol{P}_{A,B}) \neq \phi$, there exists a unique $\hat{W} \in E_p(\boldsymbol{P}_{A,B})$ such that $EL_p(A, B)^{-1} = H_p(\hat{W})$.*

PROPOSITION 2.3. *In case $\{u \in \boldsymbol{D}^{(p),A}; u=1$ on $B\} \neq \phi$, there exists a unique optimal solution $\hat{u}$ of problem* (2.5), *i.e., $\hat{u} \in \{u \in \boldsymbol{D}^{(p),A}; u=1$ on $B\}$ such that $d_p(A, B) = D_p(\hat{u})$.*

Hereafter in this section, we always assume that A is a nonempty finite subset of X and that $\{<X_n, Y_n>\}$ is an exhaustion of $<X, Y>$ such that $A \subset X_1$. We shall be concerned with the relation between $EL_p(A, X - X_n)$ and $EL_p(A, \infty)$.

We prepare

LEMMA 2.4. *Let $W \in L^+(Y)$ and set $t_n(W) = t(W; \boldsymbol{P}_{A,X-X_n})$ and $t(W) = t(W; \boldsymbol{P}_{A,\infty})$. Then $t_n(W) \leqq t_{n+1}(W) \leqq t(W)$ and $t_n(W) \to t(W)$ as $n \to \infty$. Furthermore there exists $P \in \boldsymbol{P}_{A,\infty}$ such that $t(W) = \sum_P r_j W_j$.*

PROOF. Since $\boldsymbol{P}_{A,X-X_n} < \boldsymbol{P}_{A,X-X_{n+1}} < \boldsymbol{P}_{A,\infty}$, we have $t_n(W) \leqq t_{n+1}(W) \leqq t(W)$. For each n there exists $P^{(n)} \in \boldsymbol{P}_{A,X-X_n}$ such that $t_n(W) = \sum_{P^{(n)}} r_j W_j$. Since A is a finite set, there is $\alpha_0 \in A$ such that $\alpha_0 \in C_X(P^{(n)})$ for infinitely many n. For each $\alpha \in X$, we put

$$Y(\alpha) = \{j \in Y;\ K_{\alpha j} \neq 0\},$$

$$X(\alpha) = \{v \in X;\ v \neq \alpha \text{ and } K_{vj} \neq 0 \text{ for some } j \in Y(\alpha)\}.$$

Since $X(\alpha_0)$ is a finite subset of X, there are $\alpha_1 \in X(\alpha_0)$ and $j_1 \in Y(\alpha_0)$ such that $e(j_1) = \{\alpha_0, \alpha_1\}$ and $j_1 \in C_Y(P^{(n)})$ for infinitely many n. Similarly there are $\alpha_2 \in X(\alpha_1)$ and $j_2 \in Y(\alpha_1)$ such that $e(j_2) = \{\alpha_1, \alpha_2\}$ and $\{j_1, j_2\} \subset C_Y(P^{(n)})$ for infinitely many n. Repeating this process, we can define ordered sets $C_X(P)$ and $C_Y(P)$ by

$$C_X(P) = \{\alpha_0, \alpha_1, \alpha_2, \ldots\} \quad \text{and} \quad C_Y(P) = \{j_1, j_2, \ldots\}.$$

Define $p(P) \in L(Y)$ by $p_j(P) = -K_{vj}$ with $v = \alpha_{i-1}$ if $j = j_i$ and $p_j(P) = 0$ if $j \notin C_Y(P)$. Then $P \in \boldsymbol{P}_{\alpha_0 \infty}$. For any m, there are infinitely many n such that $\{j_1, j_2, \ldots, j_m\} \subset C_Y(P^{(n)})$. Thereby we have

$$\sum_{k=1}^{m} r_{j_k} W_{j_k} \leqq \sum_{P^{(n)}} r_j W_j = t_n(W) \leqq \lim_{n\to\infty} t_n(W).$$

By letting $m \to \infty$, we have

$$t(W) \leqq \sum_{P} r_j W_j \leqq \lim_{n\to\infty} t_n(W).$$

This completes the proof.

We have

THEOREM 2.2. $\lim_{n\to\infty} EL_p(A, X-X_n) = EL_p(A, \infty)$.

PROOF. Since $\boldsymbol{P}_{A,X-X_n} < \boldsymbol{P}_{A,X-X_{n+1}} < \boldsymbol{P}_{A,\infty}$, we have $EL_p(A, X-X_n) \leqq EL_p(A, X-X_{n+1}) \leqq EL_p(A, \infty)$ by Lemma 2.1. Therefore

$$\lim_{n\to\infty} EL_p(A, X-X_n) \leqq EL_p(A, \infty).$$

Let $W \in E_p(\boldsymbol{P}_{A,\infty})$. Then $t(W) = t(W; \boldsymbol{P}_{A,\infty}) \geqq 1$. Since $t_n(W) = t(W; \boldsymbol{P}_{A,X-X_n}) \to t(W)$ as $n \to \infty$ by Lemma 2.4, we may assume that $t_n(W) > 0$ for all n. Writing $W^{(n)} = W/t_n(W)$, we see that $W^{(n)} \in E_p(\boldsymbol{P}_{A,X-X_n})$ and $EL_p(A, X-X_n) \geqq H_p(W^{(n)})^{-1} = t_n(W)^p(H_p(W))^{-1}$. It follows that

$$\lim_{n\to\infty} EL_p(A, X-X_n) \geqq t(W)^p(H_p(W))^{-1} \geqq H_p(W)^{-1}$$

for all $W \in E_p(\boldsymbol{P}_{A,\infty})$. Hence $\lim_{n\to\infty} EL_p(A, X-X_n) \geqq EL_p(A, \infty)$. This completes the proof.

We shall give upper and lower bounds for $EL_p(A, \infty)$.

PROPOSITION 2.4. *$EL_p(A, \infty) \leqq R(P)^{p-1}$ for every $P \in \boldsymbol{P}_{A,\infty}$.*

PROOF. Let $P \in \boldsymbol{P}_{A,\infty}$. Then

$$EL_p(A, \infty) \leqq \lambda_p(\{P\}) = R(P)^{p-1}$$

by Lemma 2.1 and Proposition 2.1.

By taking $\Gamma_n = \boldsymbol{P}_{X_n, X_{n+1}-X_n}$ and $\Gamma = \boldsymbol{P}_{A,\infty}$ in Lemma 2.2, we obtain

PROPOSITION 2.5. $EL_p(A, \infty)^{q-1} \geqq \sum_{n=1}^{\infty} EL_p(X_n, X_{n+1}-X_n)^{q-1}$.

We have

PROPOSITION 2.6. *Let $Z_n = Y_{n+1} - Y_n$ and $\mu_n = \sum_{Z_n} r_j^{1-p}$. Then*

$$EL_p(A, \infty)^{q-1} \geqq \sum_{n=1}^{\infty} \mu_n^{1-q}.$$

PROOF. In view of Proposition 2.5, it suffices to show that $\lambda_p(\Gamma_n)^{-1} \leqq \mu_n$ for all n, where $\Gamma_n = \boldsymbol{P}_{X_n, X_{n+1}-X_n}$. Put $U_n = \cup \{C_Y(P);\, P \in \Gamma_n\}$. Then $U_n \subset Z_n$. Define $W^{(n)} \in L(Y)$ by $W_j^{(n)} = r_j^{-1}$ if $j \in Z_n$ and $W_j^{(n)} = 0$ if $j \notin Z_n$. Then $W^{(n)} \in E_p(\Gamma_n)$ and

$$\lambda_p(\Gamma_n)^{-1} \leqq H_p(W^{(n)}) = \sum_{Z_n} r_j^{1-p} = \mu_n.$$

§3. Max-flows and min-cuts

Let A and B be mutually disjoint nonempty subsets of X. We say that a subset Q of Y is a cut between A and B if there exist mutually disjoint subsets $Q(A)$ and $Q(B)$ of X such that $A \subset Q(A)$, $B \subset Q(B)$, $X = Q(A) \cup Q(B)$ and the set

$$Q(A) \ominus Q(B) = \{j \in Y;\, K_{aj}K_{bj} = -1 \text{ for some } a \in Q(A) \text{ and } b \in Q(B)\}$$

is equal to Q.

Let A be a nonempty finite subset of X. We say that a subset Q of Y is a cut between A and the ideal boundary ∞ of $<X, Y>$ if there exist mutually disjoint subsets $Q(A)$ and $Q(\infty)$ such that $A \subset Q(A)$, $Q(\infty) = X - Q(A)$, $Q(A)$ is a finite set and $Q = Q(A) \ominus Q(\infty)$. Denote by $\boldsymbol{Q}_{A,B}$ (resp. $\boldsymbol{Q}_{A,\infty}$) the set of all cuts between A and B (resp. ∞). We define the characteristic function $u = u(Q) \in L(X)$ of $Q \in \boldsymbol{Q}_{A,B}$ and the index $s = s(Q) \in L(Y)$ of Q by

$$u_v = 0 \text{ if } v \in Q(A) \quad \text{and} \quad u_v = 1 \text{ if } v \in Q(B),$$

$$s_j = \sum_{v=0}^{\infty} K_{vj} u_v.$$

We have $s_j = 0$ if $j \notin Q$ and $|s_j| = 1$ if $j \in Q$.

Let A and B be mutually disjoint nonempty finite subsets of X. We say that $w \in L(Y)$ is a flow from A to B of strength $I(w)$ if

$$\sum_{j=1}^{\infty} K_{vj} w_j = 0 \qquad (v \notin A \cup B), \tag{3.1}$$

$$I(w) = -\sum_{v \in A} \sum_{j=1}^{\infty} K_{vj} w_j = \sum_{v \in B} \sum_{j=1}^{\infty} K_{vj} w_j. \tag{3.2}$$

Denote by $F(A, B)$ the set of all flows from A to B and set

$$G(A, B) = F(A, B) \cap L_0(Y).$$

Let $F_q(A, B)$ be the closure of $G(A, B)$ in $L_q(Y; r)$. Thus for any $w \in F_q(A, B)$, there exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $H_q(w - w^{(n)}) \to 0$ as $n \to \infty$. It follows that $w \in F(A, B)$ and $I(w^{(n)}) \to I(w)$ as $n \to \infty$.

Let $g_p(t)$ be the real function on the real line R defined by

$$g_p(t) = |t|^{p-1} \operatorname{sign}(t).$$

It is clear that

$$tg_p(t) = |t|^p \quad \text{and} \quad \frac{d}{dt}|t|^p = pg_p(t).$$

We say that $w \in L(Y)$ is a p-flow from A to B of strength $I_p(w)$ if $g_p \circ w$ is a flow from A to B and $I_p(w) = I(g_p \circ w)$. Denote by $F^{(p)}(A, B)$ the set of all p-flows from A to B and set

$$G^{(p)}(A, B) = F^{(p)}(A, B) \cap L_0(Y).$$

It is clear that $F^{(2)}(A, B) = F(A, B)$ and $I_2(w) = I(w)$. We remark that a p-flow is a non-linear flow in the sense of Birkhoff [1] and Duffin [3].

REMARK 3.1. $w \in G^{(p)}(A, B)$ if and only if $g_p \circ w \in G(A, B)$.

REMARK 3.2. Let A and B be mutually disjoint nonempty finite subsets of X and let $\hat{u}$ be the optimal solution of problem (2.5). Define $\hat{w} \in L(Y)$ by

$$\hat{w}_j = r_j^{-1} \sum_{\nu=0}^{\infty} K_{\nu j} \hat{u}_\nu .$$

Then it can be shown that $\hat{w} \in F^{(p)}(A, B)$.

We prepare

LEMMA 3.1. *Let $u \in L(X)$ and $w \in L(Y)$. Then*

$$\sum_{\nu=0}^{\infty} u_\nu \left(\sum_{j=1}^{\infty} K_{\nu j} w_j \right) = \sum_{j=1}^{\infty} w_j \left(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu \right) \tag{3.3}$$

holds if any one of the following conditions is fulfilled:

(i) $\qquad u \in L_0(X) \quad or \quad w \in L_0(Y).$

(ii) $\qquad D_p(u) < \infty \quad and \quad w \in F_q(A, B).$

PROOF. If condition (i) is satisfied, then (3.3) is clear. Assume condition (ii). Then there exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $H_q(w - w^{(n)}) \to 0$ as $n \to \infty$. We have

$$\sum_{j=1}^{\infty} w_j^{(n)} \left(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu \right) = \sum_{\nu=0}^{\infty} u_\nu \left(\sum_{j=1}^{\infty} K_{\nu j} w_j^{(n)} \right)$$

$$= \sum_{\nu\in A\cup B} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j} w_j^{(n)}\Big)$$

$$\to \sum_{\nu\in A\cup B} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j} w_j\Big) = \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j} w_j\Big)$$

as $n\to\infty$, since $w_j^{(n)}\to w_j$ as $n\to\infty$ for each $j\in Y$. On the other hand, we have

$$\sum_{j=1}^{\infty} |w_j - w_j^{(n)}|\,|\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu| \leqq [H_q(w-w^{(n)})]^{1/q}[D_p(u)]^{1/p}$$

by Hölder's inequality, so that

$$\sum_{j=1}^{\infty} w_j\Big(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu\Big) = \lim_{n\to\infty} \sum_{j=1}^{\infty} w_j^{(n)}\Big(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu\Big) = \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j} w_j\Big).$$

This completes the proof.

Let $W\in L^+(Y)$. Let us consider the following extremum problems which are generalizations of the max-flow problem in network theory on a finite graph.

(3.4) Find
$M(W; F_q(A, B)) = \sup\{I(w);\ w\in F_q(A, B) \text{ and } |w_j| \leqq W_j \text{ on } Y\}$.

(3.5) Find
$M(W; G(A, B)) = \sup\{I(w);\ w\in G(A, B) \text{ and } |w_j| \leqq W_j \text{ on } Y\}$.

(3.6) Find
$M_p(W; G^{(p)}(A, B)) = \sup\{I_p(w);\ w\in G^{(p)}(A, B) \text{ and } |w_j| \leqq W_j \text{ on } Y\}$.

For $W\in L^+(Y)$ let us denote by W^p the function $V\in L(Y)$ defined by $V_j = W_j^p$ for each $j\in Y$.

On account of Remark 3.1, we have

PROPOSITION 3.1. $M_p(W; G^{(p)}(A, B)) = M(W^{p-1}; G(A, B))$.

We shall prove

LEMMA 3.2. *Let $W\in L_p^+(Y; r)$. Then there exists $\hat{w}\in F_q(A, B)$ such that $|\hat{w}_j|\leqq W_j^{p-1}$ on Y and $I(\hat{w}) = M(W^{p-1}; G(A, B))$.*

PROOF. There exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $|w_j^{(n)}|\leqq W_j^{p-1}$ on Y and $I(w^{(n)})$ converges to $M(W^{p-1}; G(A, B))$. Since $L_q(Y; r)$ is a reflexive Banach space and $\{w\in F_q(A, B);\ |w_j|\leqq W_j^{p-1} \text{ on } Y\}$ is a bounded closed convex set in $L_q(Y; r)$, we may assume that $\{w^{(n)}\}$ converges weakly to $\hat{w}\in L_q(Y; r)$. Then $w_j^{(n)}\to\hat{w}_j$ as $n\to\infty$ for each j. Hence $\hat{w}\in F_q(A, B)$, $|\hat{w}_j|\leqq W_j^{p-1}$ on Y and

$$I(\hat{w}) = \sum_{\nu\in B}\sum_{j=1}^{\infty} K_{\nu j}\hat{w}_j = \lim_{n\to\infty} I(w^{(n)}) = M(W^{p-1}; G(A, B)).$$

This completes the proof.

Let $W \in L^+(Y)$ and consider the following extremum problem which is a generalization of the min-cut problem in (finite) network theory:

(3.7) Find
$$M^*(W; \boldsymbol{Q}_{A,B}) = \inf\{\sum_Q W_j;\ Q \in \boldsymbol{Q}_{A,B}\}.$$

We have

LEMMA 3.3.[6] $M(W; G(A, B)) = M^*(W; \boldsymbol{Q}_{A,B})$.

By Lemma 3.3 and Proposition 3.1, we have

COROLLARY. $M_p(W; G^{(p)}(A, B)) = M^*(W^{p-1}; \boldsymbol{Q}_{A,B})$.

§4. Generalized extremal width of a network

Let A and B be mutually disjoint nonempty subsets of X. We define the extremal width $EW_p(A, B)$ of order p of an infinite network $<X, Y, K, r>$ relative to two sets A and B by the value of the following extremum problem.

(4.1) Find
$$EW_p(A, B)^{-1} = \inf\{H_p(W);\ W \in E_p^*(\boldsymbol{Q}_{A,B})\},$$
where $E_p^*(\boldsymbol{Q}_{A,B}) = \{W \in L_p^+(Y; r);\ \sum_Q W_j^{p-1} \geqq 1 \text{ for all } Q \in \boldsymbol{Q}_{A,B}\}$.

Hereafter in this section we always assume that A and B are finite subsets of X. In connection with the above problem, we consider the following extremum problems.

(4.2) Find
$$d_q^*(A, B) = \inf\{H_q(w);\ w \in F_q(A, B) \text{ and } I(w) = 1\}.$$

(4.3) Find
$$\hat{d}_p^*(A, B) = \inf\{H_p(w);\ w \in G^{(p)}(A, B) \text{ and } I_p(w) = 1\}.$$

We shall prove

PROPOSITION 4.1. *$\hat{d}_p^*(A, B) = d_q^*(A, B) = \inf\{H_q(w);\ w \in G(A, B)$ and $I(w) = 1\}$.*

PROOF. We set $\hat{d}_p^* = \hat{d}_p^*(A, B)$ and $d_q^* = d_q^*(A, B)$. By Remark 3.1 and by the relations $I(g_p \circ w) = I_p(w)$ and $H_q(g_p \circ w) = H_p(w)$, we have

6) Cf. Theorem 6 in [7].

$$(4.4) \qquad \hat{d}_p^* = \inf\{H_q(z);\ z\in G(A,B) \text{ and } I(z)=1\},$$

so that $\hat{d}_p^* \geqq d_q^*$. On the other hand, let $w\in F_q(A, B)$ and $I(w)=1$. There exists a sequence $\{w^{(n)}\}$ in $G(A, B)$ such that $H_q(w-w^{(n)})\to 0$ as $n\to\infty$. Since $I(w^{(n)}) \to I(w)$ as $n\to\infty$, we may suppose that $I(w^{(n)})>0$ for all n. It follows from (4.4) that

$$\hat{d}_p^* \leqq H_q(w^{(n)}/I(w^{(n)})) = H_q(w^{(n)})/(I(w^{(n)}))^q.$$

By letting $n\to\infty$, we have $\hat{d}_p^* \leqq H_q(w)$, so that $\hat{d}_p^* \leqq d_q^*$. Hence $\hat{d}_p^* = d_q^*$.

THEOREM 4.1. *$EW_p(A, B)^{-1} = d_q^*(A, B)$.*

PROOF. We set $EW_p = EW_p(A, B)$ and $d_q^* = d_q^*(A, B)$. For each $w\in G(A, B)$ such that $I(w)=1$, consider $W\in L^+(Y)$ defined by $W_j=|w_j|^{1/(p-1)}$ on Y. Then we show that $W\in E_p^*(\boldsymbol{Q}_{A,B})$. Let $u=u(Q)$ be the characteristic function of $Q\in \boldsymbol{Q}_{A,B}$. We have by Lemma 3.1

$$1 = I(w) = \sum_{\nu=0}^{\infty} u_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j} w_j\Big) = \sum_{j=1}^{\infty} w_j\Big(\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu\Big)$$

$$\leqq \sum_{j=1}^{\infty} |w_j|\,\Big|\sum_{\nu=0}^{\infty} K_{\nu j} u_\nu\Big| = \sum_{Q} W_j^{p-1}.$$

Therefore $W\in E_p^*(\boldsymbol{Q}_{A,B})$ and

$$EW_p^{-1} \leqq H_p(W) = \sum_{j=1}^{\infty} r_j |w_j|^{p/(p-1)} = H_q(w).$$

Thus we have $EW_p^{-1} \leqq d_q^*$ by Proposition 4.1. On the other hand, let $W\in E_p^*(\boldsymbol{Q}_{A,B})$, i.e., $W\in L_p^+(Y; r)$ and $M^*(W^{p-1}; \boldsymbol{Q}_{A,B})\geqq 1$. We can find $w\in F_q(A, B)$ such that $|w_j|\leqq W_j^{p-1}$ on Y and $M(W^{p-1}; G(A, B)) = I(w)$ by Lemma 3.2. It follows from Lemma 3.3 that $I(w)\geqq 1$. We have

$$d_q^* \leqq H_q(w/I(w)) \leqq H_q(w) = \sum_{j=1}^{\infty} r_j |w_j|^q$$

$$\leqq \sum_{j=1}^{\infty} r_j W_j^{q(p-1)} = H_p(W),$$

so that $d_q^* \leqq EW_p^{-1}$. Therefore $d_q^* = EW_p^{-1}$.

By the aid of Theorem A, we have

PROPOSITION 4.2. *There exists a unique $\hat{w}\in F_q(A, B)$ such that $I(\hat{w})=1$ and $d_q^*(A, B) = H_q(\hat{w})$, i.e., $\hat{w}$ is the optimal solution of problem* (4.2).

Let A be a nonempty finite subset of X. We define the extremal width

$EW_p(A, \infty)$ of order p of an infinite network relative to A and ∞ by the value of the following extremum problem.

(3.5) Find

$$EW_p(A, \infty)^{-1} = \inf\{H_p(W);\ W \in E_p^*(\boldsymbol{Q}_{A,\infty})\},$$

where $E_p^*(\boldsymbol{Q}_{A,\infty}) = \{W \in L_p^+(Y; r);\ \sum_Q W_j^{p-1} \geqq 1 \text{ for all } Q \in \boldsymbol{Q}_{A,\infty}\}$.

Let $\{<X_n, Y_n>\}$ be an exhaustion of $<X, Y>$ such that $A \subset X_1$. We shall be concerned with the relation between $EW_p(A, X-X_n)$ and $EW_p(A, \infty)$.

We shall prove

THEOREM 4.2. $\lim_{n\to\infty} EW_p(A, X-X_n) = EW_p(A, \infty)$.

PROOF. Since $\boldsymbol{Q}_{A,X-X_n} \subset \boldsymbol{Q}_{A,X-X_{n+1}} \subset \boldsymbol{Q}_{A,\infty}$, we have $EW_p(A, \infty) \leqq EW_p(A, X-X_{n+1}) \leqq EW_p(A, X-X_n)$, and hence

$$\lim_{n\to\infty} EW_p(A, X-X_n) \geqq EW_p(A, \infty).$$

To prove the converse inequality we may assume that $\lim_{n\to\infty} EW(A, X-X_n) > 0$. For each n, there is $W^{(n)} \in E_p^*(\boldsymbol{Q}_{A,X-X_n})$ such that $EW_p(A, X-X_n) = H_p(W^{(n)})^{-1}$. Since $\{H_p(W^{(n)})\}$ is a bounded sequence and $L_p(Y; r)$ is a reflexive Banach space, we can choose a weakly convergent subsequence of $\{W^{(n)}\}$. Denote by $\{W^{(n)}\}$ the subsequence again and let $\hat{W}$ be the weak limit. We show that $\hat{W} \in E_p^*(\boldsymbol{Q}_{A,\infty})$. Let $Q \in \boldsymbol{Q}_{A,\infty}$ with $Q = Q(A) \ominus Q(\infty)$. Since $Q(A)$ is a finite set, there is a number n_0 such that $Q(A) \subset X_{n_0}$. Then $X - X_n \subset Q(\infty)$ and hence $Q \in \boldsymbol{Q}_{A,X-X_n}$ for all $n \geqq n_0$. Therefore $\sum_Q [W_j^{(n)}]^{p-1} \geqq 1$ for all $n \geqq n_0$. Since $\{W^{(n)}\}$ converges weakly to $\hat{W}$ and Q is a finite set, we obtain $\sum_Q \hat{W}_j^{p-1} \geqq 1$. Thus $\hat{W} \in E_p^*(\boldsymbol{Q}_{A,\infty})$. Since $[H_p(w)]^{1/p}$ is weakly lower semicontinuous in $L_p(Y; r)$, we have

$$\begin{aligned}\lim_{n\to\infty} [EW_p(A, X-X_n)]^{-1} &= \lim_{n\to\infty} H_p(W^{(n)}) \\ &\geqq H_p(\hat{W}) \geqq [EW_p(A, \infty)]^{-1}.\end{aligned}$$

This completes the proof.

§5. A reciprocal relation between EL_p and EW_p

Let A and B be mutually disjoint nonempty finite subsets of X.

We prepare

LEMMA 5.1. *Let $\hat{w}$ be the optimal solution of problem* (4.2). *If $w' \in F_q(A, B)$ and $I(w') = 0$, then*

$$(5.1)\qquad \sum_{j=1}^{\infty} r_j w'_j g_q(\hat{w}_j) = 0.$$

PROOF. For any real number t, we have $\hat{w}+tw' \in F_q(A, B)$ and $I(\hat{w}+tw')=1$, so that $d_q^*(A, B)=H_q(\hat{w}) \leqq H_q(\hat{w}+tw')$. Thus the derivative of $H_q(\hat{w}+tw')$ with respect to t vanishes at $t=0$. Since $H_q(\hat{w}+tw')$ can be differentiated term by term at $t=0$, we obtain (5.1).

COROLLARY 1. *Let $\hat{w}$ be the optimal solution of problem* (4.2) *and P be a path from node $\alpha \in A$ to node $\beta \in B$. Then*

$$(5.2)\qquad d_q^*(A, B) = \sum_{j=1}^{\infty} r_j p_j(P) g_q(\hat{w}_j).$$

PROOF. Note that $p(P)$ is a flow from $\{\alpha\}$ to $\{\beta\}$ such that $I(p(P))=1$. Taking $w'=\hat{w}-p(P)$, we see that $w' \in F_q(A, B)$ and $I(w')=0$. Thus we have by (5.1)

$$\sum_{j=1}^{\infty} r_j(\hat{w}_j - p_j(P)) g_q(\hat{w}_j) = 0.$$

Therefore

$$d_q^*(A, B) = H_q(\hat{w}) = \sum_{j=1}^{\infty} r_j \hat{w}_j g_q(\hat{w}_j) = \sum_{j=1}^{\infty} r_j p_j(P) g_q(\hat{w}_j).$$

COROLLARY 2. *Let $\hat{w}$ be the optimal solution of problem* (4.2) *and let $\alpha, \nu \in X$ $(\alpha \neq \nu)$. If P and P' are paths from node α to node ν, then*

$$(5.3)\qquad \sum_{j=1}^{\infty} r_j p_j(P) g_q(\hat{w}_j) = \sum_{j=1}^{\infty} r_j p_j(P') g_q(\hat{w}_j).$$

PROOF. Taking $w'=p(P)-p(P')$, we see that $w' \in F_q(A, B)$ and $I(w')=0$. Then (5.3) follows from (5.1).

Let $\hat{w}$ be the optimal solution of problem (4.2). For any $\alpha \in A$, we define $v^{(\alpha)} \in L(X)$ by

$$(5.4)\qquad v_\alpha^{(\alpha)} = 0, \quad v_\nu^{(\alpha)} = \sum_{j=1}^{\infty} r_j p_j(P) g_q(\hat{w}_j) \qquad (\nu \neq \alpha)$$

for some path P from node α to node ν. It follows from Corollary 2 of Lemma 5.1 that $v^{(\alpha)}$ is uniquely determined by $\hat{w}$. Define $\hat{v} \in L(X)$ by

$$(5.5)\qquad \hat{v}_\nu = \inf\{|v_\nu^{(\alpha)}|;\ \alpha \in A\}.$$

We have

LEMMA 5.2. *Let $\hat{v}$ be the function defined by* (5.4) *and* (5.5). *Then $\hat{v}$*

$=0$ *on* A, $\hat{v}=d_q^*(A, B)$ *on* B *and*

$$(5.6) \qquad |\sum_{\nu=0}^{\infty} K_{\nu j}\hat{v}_\nu| \leqq r_j|\hat{w}_j|^{q-1} \qquad on \quad Y.$$

PROOF. Since $v_\alpha^{(\alpha)}=0$ for any $\alpha \in A$, we have $\hat{v}=0$ on A. We have $\hat{v}=d_q^*(A, B)$ on B by Corollary 1 of Lemma 5.1. The proof of (5.6) is carried out by the same reasoning as in the proof of Lemma 12 in [7].

We shall prove

THEOREM 5.1. $[d_p(A, B)]^{1/p}[d_q^*(A, B)]^{1/q}=1$.

PROOF. We set $d_p=d_p(A, B)$ and $d_q^*=d_q^*(A, B)$. First we show that $1 \leqq (d_p)^{1/p}(d_q^*)^{1/q}$. For any $v \in L(X)$ such that $v=0$ on A, $v=1$ on B and $D_p(v)<\infty$ and any $w \in F_q(A, B)$ such that $I(w)=1$, we have by Lemma 3.1

$$1 = I(w) = \sum_{\nu=0}^{\infty} v_\nu\Big(\sum_{j=1}^{\infty} K_{\nu j}w_j\Big) = \sum_{j=1}^{\infty} w_j\Big(\sum_{\nu=0}^{\infty} K_{\nu j}v_\nu\Big)$$
$$\leqq [D_p(v)]^{1/p}[H_q(w)]^{1/q},$$

which leads to the desired inequality. Next we show that $(d_p)^{1/p}(d_q^*)^{1/q} \leqq 1$. Let $\hat{w}$ be the optimal solution of problem (4.2) and define $\hat{v} \in L(X)$ by (5.4) and (5.5). Then we have by (5.6)

$$D_p(\hat{v}) = \sum_{j=1}^{\infty} r_j^{1-p} |\sum_{\nu=0}^{\infty} K_{\nu j}\hat{v}_\nu|^p \leqq \sum_{j=1}^{\infty} r_j|\hat{w}_j|^{p(q-1)} = H_q(\hat{w}) = d_q^*.$$

Writing $\hat{u}=\hat{v}/d_q^*$, we see by Lemma 5.2 that $\hat{u}=0$ on A and $\hat{u}=1$ on B, so that

$$d_p \leqq D_p(\hat{u}) = D_p(\hat{v})(d_q^*)^{-p} \leqq (d_q^*)^{1-p} = (d_q^*)^{-p/q},$$

or $(d_p)^{1/p}(d_q^*)^{1/q} \leqq 1$.

By Proposition 4.1 and Theorem 5.1, we have

COROLLARY. $[d_p(A, B)]^{1/p}[\hat{d}_p^*(A, B)]^{1/q}=1$.

By Theorems 2.1, 4.1 and 5.1, we have

THEOREM 5.2. $[EL_p(A, B)]^{1/p}[EW_p(A, B)]^{1/q}=1$.

Next we shall be concerned with the reciprocal relation between $EL_p(A, \infty)$ and $EW_p(A, \infty)$. Henceforth let A be a nonempty finite subset of X and $\{<X_n, Y_n>\}$ be an exhaustion of $<X, Y>$ such that $A \subset X_1$.

We prepare

LEMMA 5.3. *For every* $Q \in \boldsymbol{Q}_{A,X_{n+1}-X_n}$, *there exists* $Q' \in \boldsymbol{Q}_{A,X-X_n}$ *such that* $Q' \subset Q$.

PROOF. Let $Q \in \boldsymbol{Q}_{A,X_{n+1}-X_n}$ and $Q = Q(A) \ominus Q(X_{n+1}-X_n)$. Let us define $Q'(A)$ and $Q'(X-X_n)$ by

$$Q'(A) = Q(A) - (X - X_n) \quad \text{and} \quad Q'(X - X_n) = X - Q'(A).$$

Since $A \cap (X - X_n) = \phi$ and $Q'(A) \cap (X - X_n) = \phi$, we see that $A \subset Q'(A)$ and $X - X_n \subset Q'(X - X_n)$, so that $Q' = Q'(A) \ominus Q'(X - X_n) \in \boldsymbol{Q}_{A,X-X_n}$. It can be easily shown that $Q' \subset Q$.

We have

THEOREM 5.3. $EW_p(A, \infty) = EL_p(A, \infty)^{1-q}$.

PROOF. Since $\boldsymbol{P}_{A,X-X_n} = \boldsymbol{P}_{A,X_{n+1}-X_n}$, we have

$$EL_p(A, X - X_n) = EL_p(A, X_{n+1} - X_n).$$

It follows from Lemma 5.3 that

$$EW_p(A, X - X_n) = EW_p(A, X_{n+1} - X_n).$$

We have by Theorem 5.2

$$EW_p(A, X - X_n) = EL_p(A, X - X_n)^{1-q}.$$

Our assertion follows from Theorems 2.2 and 4.2.

References

[1] G. Birkhoff: A variational principle for nonlinear networks, Quart. Appl. Math., **21** (1963), 160–162.
[2] N. Bourbaki: Espaces vectoriels topologiques, Chap. III–V, Paris, 1955.
[3] R. J. Duffin: Nonlinear networks, IIa, Bull. Amer. Math. Soc., **53** (1947), 963–971.
[4] R. J. Duffin: The extremal length of a network, J. Math. Anal. Appl., **5** (1962), 200–215.
[5] M. Ohtsuka: On Duffin's problem concerning extremal length, An abstract of a talk given at the Fall Meeting of the Math. Soc. Japan, 1970.
[6] M. Ohtsuka: Dirichlet problem, extremal length and prime ends, Van Nostrand, 1970.
[7] M. Yamasaki: Extremum problems on an infinite network, Hiroshima Math. J., **5** (1975), 223–250.

Kawasaki Medical College
and
School of Engineering,
Okayama University