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Remarks on the Asymptotic Relationships between Solutions of Two Systems of Ordinary Differential Equations

Yuichi KITAMURA

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1. Introduction

Consider the system of ordinary differential equations

$$x' = A(t)x + f(t, x), \qquad t \geqq t_0, \tag{1}$$

where x is an n-vector, $A(t)$ is a continuous $n \times n$ matrix function on $I = [t_0, \infty)$, and $f(t, x)$ is a continuous n-vector function of t and x on $I \times R^n$. Recently, Rab [7] has taken up the case where all components f_j of f depend only on t and some of the components of x, say, $x_{i_1},\dots, x_{i_q}$, $1 \leqq i_1 < \cdots < i_q \leqq n$, and has presented conditions which lead to an equivalence between certain components of the solutions of the system (1) and certain components of the solutions of the unperturbed system

$$y' = A(t)y, \qquad t \geqq t_0. \tag{2}$$

He has shown in particular that the first theorem of Hallam [6] concerning the second order scalar differential equations

$$x'' = a(t)x + f(t, x), \qquad y'' = a(t)y \tag{3}$$

follows from his theorem as a corollary.

The purpose of this note is to establish a theorem which improves considerably the above mentioned results of Rab and to provide some examples demonstrating its application to specific classes of differential equations. In particular it is shown that our result, when applied to (3), yields the second theorem of Hallam [6] which is not covered by Ráb's result.

2. Main result

We assume that the components f_j of f depend essentially on t and the q components $x_{i_1},\dots, x_{i_q}$ $(1 \leqq i_1 < \cdots < i_q \leqq n)$ of x in the sense that

$$|f_j(t, x_1,\dots, x_n)| \leqq \omega_j(t, |x_{i_1}|,\dots, |x_{i_q}|) \tag{4}$$

for $(t, x) \in I \times R^n$ and $j = 1,\dots, n$, where each $\omega_j(t, r_1,\dots, r_q)$ is continuous on I

$\times R_+^q$ ($R_+ = [0, \infty)$) and nondecreasing in $(r_1, \dots, r_q)$ for each fixed $t \in I$.

We are interested in some asymptotic relationships between the p components $x_{i_1}(t), \dots, x_{i_p}(t)$ $(1 \leqq i_1 < \cdots < i_p, q \leqq p \leqq n)$ of the solutions $x(t)$ of (1) and the corresponding components of the solutions $y(t)$ of (2).

Let $Y(t) = (y_{ij}(t))$ be a fundamental matrix for the system (2) and $Y^{-1}(t) = (y^{ji}(t))$ the inverse matrix of $Y(t)$; obviously, $y^{ji}(t) = Y^{ji}(t)/\det Y(t)$, where $Y^{ji}(t)$ is the cofactor of $y_{ji}(t)$. Let $N = \{1, \dots, n\}$. Suppose that there exist subsets N_0, M of N such that $N_0 \subset M$ and positive continuous functions $\mu_i(t)$, $m_i(t)$, $i = i_1, \dots, i_p$, satisfying

$$\mu_i(t) \geq \max_{j \in N_0} |y_{ij}(t)|, \qquad t \in I, \quad i = i_1, \dots, i_p,$$

$$m_i(t) \geqq \max\{\max_{j \in M} |y_{ij}(t)|, \mu_i(t)\}, \qquad t \in I, \quad i = i_1, \dots, i_p.$$

Suppose moreover that there exist a constant $K > 0$ and a subset B (possibly empty) of N_0 such that

$$\int_{t_0}^{\infty} |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds < \infty, \qquad j \in N, \quad k \in A, \tag{5}$$

$$\int_t^{\infty} \Big| \sum_{k \in A \setminus N_0} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds = o(\mu_i(t)) \quad \text{as} \quad t \to \infty, \tag{6}$$

$$j \in N, \quad i = i_1, \dots, i_p,$$

$$\int_{t_0}^{t} \Big| \sum_{k \in B} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds = o(\mu_i(t)) \quad \text{as} \quad t \to \infty, \tag{7}$$

$$j \in N, \quad i = i_1, \dots, i_p,$$

where $A = N \backslash B$ and $m(s) = (m_{i_1}(s), \dots, m_{i_q}(s))$.

Our main result is the following

THEOREM 1. *Suppose that the conditions* (4)-(7) *hold. Then, to any constant vector* $(\gamma_1, \dots, \gamma_n)$ *with* $\sum_{j \in M} |\gamma_j| < \kappa$, *there exists a solution* $x(t) = (\xi_1(t), \dots, \xi_n(t))$ *of* (1) *such that*

$$\Big| \xi_i(t) - \sum_{j \in M} y_{ij}(t) \gamma_j \Big| = o(\mu_i(t)) \quad as \quad t \to \infty, \qquad i = i_1, \dots, i_p. \tag{8}$$

In addition, if $x(t) = (\xi_1(t), \dots, \xi_n(t))$ *is a solution of* (1) *which satisfies* $|\xi_i(t)| \leqq \kappa m_i(t)$ *for* $t \in I$ *and* $i = i_1, \dots, i_q$, *then there exists a constant vector* $(\gamma_1, \dots, \gamma_n)$ *such that*

$$\Big| \xi_i(t) - \sum_{j \in N} y_{ij}(t) \gamma_j \Big| = o(\mu_i(t)) \quad as \quad t \to \infty, \qquad i = i_1, \dots, i_p. \tag{9}$$

PROOF. The proof of the first half of the theorem proceeds as in Ráb [7] with necessary modifications. Without loss of generality we may suppose that $i_1=1,\dots, i_p=p$. Let $(\gamma_1,\dots,\gamma_n)$ be a constant vector such that $\sum_{j\in M}|\gamma_j|<\kappa$ and take a number δ satisfying $0<\delta<\kappa-\sum_{j\in M}|\gamma_j|$ In view of (5), (6), (7) we can choose t_0 so large that the following inequalities hold:

$$(10)\qquad \sum_{k\in A\cap N_0}\int_{t_0}^{\infty}|y^{jk}(s)|\,\omega_j(s\kappa m(s))\,ds < \mathrm{A}_{ta},$$

$$(11)\qquad \int_t^{\infty}\Big|\sum_{k\in A\setminus N_0} y_{ik}(t)y^{jk}(s)\Big|\,\omega_j(s,\kappa m(s))\,ds < \frac{\delta}{3n}\mu_i(t),$$

$$(12)\qquad \int_{t_0}^{t}\Big|\sum_{k\in B} y_{ik}(t)y^{jk}(s)\Big|\,\omega_j(s,\kappa m(s))\,ds < \frac{\delta}{3n}\mu_i(t),$$

for $t\in I, j\in N$ and $i=1,\dots,p$. For $i=p+1,\dots,n$ put

$$\rho_i(t)=\kappa\sum_{j\in M}|y_{ij}(t)|+\sum_{\substack{j\in N\\ k\in B}}|y_{ik}(t)|\int_{t_0}^{t}|y^{jk}(s)|\omega_j(s,\kappa m(s))ds$$
$$+\sum_{\substack{j\in N\\ k\in A}}|y_{ik}(t)|\int_t^{\infty}|y^{jk}(s)|\omega_j(s,\kappa m(s))ds,\qquad t\in I.$$

Let F denote the set of all vector functions $x(t)=(\xi_1(t),\dots\xi_n(t))$ which are continuous on I and satisfy

$$|\xi_i(t)|\leqq \kappa m_i(t),\quad i=1,\dots,p;\qquad |\xi_i(t)|\leqq\rho_i(t),\quad i=p+1,\dots,n.$$

We now define the operator Φ acting on F by

$$(13)\qquad (\Phi x)_i(t)=\sum_{j\in M} y_{ij}(t)\gamma_j+\int_{t_0}^{t}\sum_{\substack{j\in N\\ k\in B}} y_{ik}(t)y^{jk}(s)f_j(s,x(s))ds$$
$$-\int_t^{\infty}\sum_{\substack{j\in N\\ k\in A}} y_{ik}(t)y^{jk}(s)f_j(s,x(s))ds,\qquad i=1,\dots,n.$$

a) *Φ maps F into F. If $i\in\{1,\dots,p\}$, then by* (12) we *have*

$$\Big|\int_{t_0}^{t}\sum_{\substack{j\in N\\ k\in B}} y_{ik}(t)y^{jk}(s)f_j(s,x(s))ds\Big|$$
$$\leqq\sum_{j\in N}\int_{t_0}^{t}\Big|\sum_{k\in B}y_{ik}(t)y^{jk}(s)\Big|\,\omega_j(s,\kappa m(s))ds\leqq\frac{\delta}{3}\mu_i(t),$$

and using (10) and (11) we see that

$$\Big|\int_t^{\infty}\sum_{\substack{j\in N\\ k\in A\cap N_0}} y_{ik}(t)y^{jk}(s)f_j(s,x(s))ds\Big|$$

$$\leqq \sum_{\substack{j\in N\\ k\in A\cap N_0}} |y_{ik}(t)|\int_t^\infty |y^{jk}(s)|\omega_j(s,\ \kappa m(s))ds$$

$$\leqq \mu_i(t) \sum_{\substack{j\in N\\ k\in A\cap N_0}} \int_t^\infty |y^{jk}(s)|\,\omega_j(s,\ \kappa m(s))\,ds \leqq \frac{\delta}{3}\mu_i(t)\,,$$

$$\left|\int_t^\infty \sum_{\substack{j\in\\ k\in A\setminus N_0}} \qquad {}_j(s,\ x(s))ds\right|$$

$$\leq \sum_{j\in N}\int_t^\infty \Big|\sum_{k\in A\setminus N_0} y_{ik}(t)y^{jk}(s)\Big|\,\omega_j(s,\ \kappa m(s))ds \leqq \frac{\delta}{3}\mu_i(t)\,.$$

It follows that for $i=1,\dots, p$

$$|(\Phi x)_i(t)| \leqq \Big(\sum_{j\in M}|\gamma_j|\Big)m_i(t)+\delta\mu_i(t)$$

$$\leqq \Big(\sum_{j\in M}|\gamma_j|+\delta\Big)m_i(t) \leqq \kappa m_i(t), \qquad t\in I.$$

From the definition of $\rho_i(t)$ it is easy to see that $|(\Phi x)_i(t)|\leqq\rho_i(t)$, $t\in I$, for $i=p+1,\dots, n$. Therefore, Φ maps F into itself.

b) Φ *is continuous.* Suppose that $x_l\in F$ and, as $l\to\infty$, $x_l(t)\to x(t)$ uniformly on any finite subinterval of I. Consider an interval of the form $[t_0,\ T]$ Given an $\varepsilon>0$, there is $t_1\geqq T$ such that

$$\int_{t_1}^\infty |y^{jk}(s)|\,\omega_j(s,\ \kappa m(s))\,ds < \frac{\varepsilon}{3n^2K},\ j\in N,\quad k\in A, \tag{14}$$

where $K=\max_{i,k\in N}\{\max_{t\in[t_0,T]}|y_{ik}(t)|\}$. Choose $l_0>0$ so large that $l\geqq l_0$ implies

$$|y^{jk}(s)|\,|f_j(s,\ x_l(s))-f_j(s,\ x(s))| < \frac{\varepsilon}{3n^2K(t_1-t_0)} \tag{15}$$

for $s\in[t_0,\ t_1]$ and $j,\ k\in N$. This is possible since f is continuous and $x_l(t)$ converges uniformly to $x(t)$ on $[t_0,\ t_1]$. Now we have for $t\in[t_0,\ T]$

$$\begin{aligned}|(\Phi x_l)_i(t)-(\Phi x)_i(t)| &\leqq \int_{t_0}^t \sum_{\substack{j\in N\\ k\in B}} |y_{ik}(t)|\,|y^{jk}(s)|\,|f_j(s,\ x_l(s))-f_j(s,\ x(s))|ds\\ &+\int_t^{t_1} \sum_{\substack{j\in N\\ k\in A}} |y_{ik}(t)|\,|y^{jk}(s)|\,|f_j(s,\ x_l(s))-f_j(s,\ x(s))|ds\\ &+\int_{t_1}^\infty \sum_{\substack{j\in N\\ k\in A}} |y_{ik}(t)|\,|y^{jk}(s)|\,|f_j(s,\ x_l(s))-f_j(s,\ x(s))|ds.\end{aligned} \tag{16}$$

Using (14) we see easily that the last integral in (16) does not exceed

$$2K \sum_{\substack{j \in N \\ k \in A}} \int_{t_1}^{\infty} |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds < \frac{2}{3} \varepsilon .$$

The sum of the first two integrals in (16) is bounded from above by

$$\sum_{\substack{j \in N \\ k \in N}} |y_{ik}(t)| \int_{t_0}^{t_1} |y^{jk}(s)| \, |f_j(s, x_l(s)) - f_j(s, x(s))| ds,$$

which in turn is bounded by $\varepsilon/3$ on account of (15) provided $l \geq l_0$. Consequently, we obtain $|(\Phi x_l)_i(t) - (\Phi x)_i(t)| < \varepsilon$ for $t \in [t_0, T]$ and $i \in N$. Therefore, $\Phi x_l(t) \to \Phi x(t)$ as $l \to \infty$ uniformly on every finite subinterval of I. This means that Φ is continuous.

c) ΦF *is uniformly bounded and equicontinuous at every point of* I. The uniform boundedness of ΦF is obvious. Differentiating (13) and using the equations

$$y'_{ij}(t) = \sum_{h \in N} a_{ih}(t) y_{hj}(t), \qquad \sum_{k \in N} y_{ik}(t) y^{jk}(t) = \delta_{ij},$$

where $a_{ih}(t)$ are the entries of the matrix $A(t)$, we obtain

$$\begin{aligned} |(\Phi x)'_i(t)| \leqq \sum_{h \in N} |a_{ih}(t)| \Big[& \sum_{j \in M} |y_{hj}(t)| \, |\gamma_j| \\ & + \sum_{j \in N} \int_{t_0}^{t} \Big| \sum_{k \in B} y_{hk}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds \\ & + \sum_{j \in N} \int_{t}^{\infty} \Big| \sum_{k \in A} y_{hk}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds \Big] + \omega_i(t, \kappa m(t)), \end{aligned}$$

which implies that, on any finite subinterval of I, the functions $(\Phi x)'_i(t)$, $i \in N$, are bounded by a constant independent of $x \in F$. Hence, ΦF is equicontinuous on every finite subinterval of I.

From the above observation we are able to apply the Schauder-Tychonoff fixed point theorem as formulated in Coppel [4, p. 9] to conclude that Φ has a fixed point $x = x(t) = (\xi_1(t), \ldots, \xi_n(t)) \in F$. Clearly, this $x(t)$ is a solution of (1) on $[t_0, \infty)$. Using (13) we see that

$$\begin{aligned} \Big|\xi_i(t) - \sum_{j \in M} y_{ij}(t) \gamma_j\Big| \leqq & \sum_{j \in N} \int_{t_0}^{t} \Big| \sum_{k \in B} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds \\ & + \sum_{j \in N} \int_{t}^{\infty} \Big| \sum_{k \in A \setminus N_0} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds \\ & + \mu_i(t) \sum_{\substack{j \in N \\ k \in A \cap N_0}} \int_{t}^{\infty} |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds \end{aligned}$$

for $i=1,\dots,p$. This together with (5)-(7) shows that the solution $x(t)$ has the required asymptotic property (8).

To prove the second assertion of the theorem, let $x(t)=(\xi_1(t),\dots,\xi_n(t))$ be a solution of (1) satisfying $|\xi_i(t)|\leqq \kappa m_i(t)$ for $t\in I$, $i=1,\dots,q$. Define the vector function $y(t)=(\eta_1(t),\ \ ,\eta_n(t))$ by

$$\eta_i(t) = \xi_i(t)-\int_{t_0}^{t}\sum_{\substack{j\in N\\ k\in B}} y_{ik}(t)y^{jk}(s)f_j(sx(s))ds \tag{17}$$

$$+\int_t^{\infty}\sum_{\substack{j\in N\\ k\in A}} y_{ik}(t)y^{jk}(s)f_j(sx(s))ds, \qquad i\in N.$$

It is easy to see that $y(t)$ is a solution of (2) on I. Put $\gamma_j=\sum_{i\in N} y^{ij}(t_0)\eta_i(t_0)$, $j\in N$, and consider the function $z(t)=(\zeta_1(t),\dots,\zeta_n(t))$, where $\zeta_i(t)=\sum_{j\in N} y_{ij}(t)\gamma_j$, $i\in N$. Since $z(t)$ is a solution of (2) and

$$\sum_{j\in N} y_{ij}(t_0)\gamma_j=\sum_{k\in N}\Big(\sum_{j\in N} y_{ij}(t_0)y^{kj}(t_0)\Big)\eta_k(t_0)=\eta_i(t_0),$$

for $i\in N$, $y(t)$ and $z(t)$ must coincide on I, i.e.,

$$\eta_i(t)=\sum_{j\in N} y_{ij}(t)\gamma_j \qquad \text{for} \quad t\in I, \quad i\in N. \tag{18}$$

From (17) and (18) it follows that $y(t)$ satisfies the asymptotic relationship (9). This completes the proof of Theorem 1.

REMARK. In the particular case where $N_0=M=N$ and $A=\{i_1,\dots,i_q\}$ Theorem 1 reduces to Ráb's theorem obtained in [7].

3. Applications

A) We first consider the scalar second order differential equations

$$(p(t)u')'+q(t)u=\phi(t,u,u'), \tag{19}$$

$$(p(t)v')'+q(t)v=0, \tag{20}$$

where $p(t)>0$ and $q(t)$ are continuous on $I=[t_0,\infty)$, and $\phi(t,u,u')$ is continuous on $I\times R^2$. The equations (19), (20) can be written as the vector equations

$$x'=A(t)x+f(t,x), \tag{21}$$

$$y'=A(t)y, \tag{22}$$

where $x=(x_1,x_2)\equiv(u,p(t)u')$, $y=(y_1,y_2)\equiv(v,p(t)v')$,

$$A(t)=\begin{pmatrix} 0 & 1/p(t) \\ -q(t) & 0 \end{pmatrix} \quad \text{and} \quad f(t, x)=\begin{pmatrix} 0 \\ \phi(t, x_1, x_2/p(t)) \end{pmatrix}.$$

Let $v_1(t)$, $v_2(t)$ be linearly independent solutions of (20) such that

$$Y(t)=\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \equiv \begin{pmatrix} v_1 & v_2 \\ p(t)v_1' & p(t)v_2' \end{pmatrix}$$

is a fundamental matrix for (22) with det $Y(t)\equiv 1$ on I. Then,

$$Y^{-1}(t)=\begin{pmatrix} y^{11} & y^{21} \\ y^{12} & y^{22} \end{pmatrix} \equiv \begin{pmatrix} p(t)v_2' & -v_2 \\ -p(t)v_1' & v_1 \end{pmatrix}.$$

Suppose that there exist positive continuous functions $v_1^*(t)$, $v_2^*(t)$ satisfying

$$(23) \qquad |v_1(t)| \leqq v_1^*(t), \quad |v_2(t)| \leqq v_2^*(t), \qquad t \in I.$$

Let $\phi(t, u, u')$ satisfy the inequality

$$(24) \qquad |\phi(t, u, u')| \leqq \omega(t, |u|), \qquad (t, u, u') \in I \times R^2,$$

where $\omega(t, r)$ is a continuous function on $I \times R_+$ which is nondecreasing in r for each fixed $t \in I$.

THEOREM 2. *Let* (23) *and* (24) *hold. Assume that*

$$(25) \qquad \int_{t_0}^{\infty} v_1^*(s)\omega(s, \kappa v_2^*(s))ds < \infty$$

for some constant $\kappa>0$, *and that*

$$(26) \qquad \lim_{t\to\infty} \frac{v_1^*(t)}{v_2^*(t)} \int_{t_0}^{t} v_2^*(s)\omega(s, \kappa v_2^*(s))ds = 0.$$

Then, for any constant γ *with* $|\gamma|<\kappa$, *there exists a solution* $u(t)$ *of* (19) *such that*

$$(27) \qquad u(t)=\gamma v_2(t)+o(v_2^*(t)) \quad as \quad t\to\infty.$$

In addition, if $u(t)$ *is any solution of* (19) *satisfying* $|u(t)| \leqq \kappa v_2^*(t)$, *then there exists a constant* γ *for which* (27) *holds.*

PROOF. In this case, $q=1$ and $i_1=1$. In view of (26) we may suppose that $v_2^*(t) \geqq v_1^*(t)$ on I. We want to apply Theorem 1 to the systems (21) and (22) by putting

$$p=1, \quad N_0=M=\{1, 2\}, \quad A=\{2\}, \quad B=\{1\},$$

$$\mu_1(t) = m_1(t) = v_2^*(t).$$

Condition (5) is satisfied, since by (25)

$$\int_{t_0}^{\infty} |y^{22}(s)|\omega(s, \kappa m_1(s))ds \leqq \int_{t_0}^{\infty} v_1^*(s)\omega(s, \kappa v_2^*(s))ds < \infty.$$

Since $A\backslash N_0 = \phi$, condition (6) holds trivially true. Using (26) we have

$$\frac{1}{\mu_1(t)}\int_{t_0}^{t} |y_{11}(t)y^{21}(s)|\,\omega(s, \kappa m_1(s))ds$$

$$\leqq \frac{v_1^*(t)}{v_2^*(t)}\int_{t_0}^{t} v_2^*(s)\omega(s, \kappa v_2^*(s))ds \longrightarrow 0 \quad as\ t \longrightarrow \infty,$$

which implies (7). Therefore, it follows from Theorem 1 that, for any constant γ with $|\gamma| < \kappa$, there exists a solution $x(t) = (x_1(t), x_2(t))$ of (21) such that

$$x_1(t) = \gamma y_{12}(t) + o(v_2^*(t)) \quad \text{as} \quad t \to \infty.$$

This means that equation (19) has a solution $u(t)$ such that (27) holds.

The opposite relationship between the solutions of (19) and (20) follows readily from the second half of Theorem 1.

THEOREM 3. *Let* (23) *and* (24) *hold. Assume that*

(28) $$\int_{t_0}^{\infty} v_1^*(s)\omega(s, \kappa v_1^*(s))ds < \infty,$$

(29) $$\int_{t_0}^{\infty} v_2^*(s)\omega(s, \kappa v_1^*(s))ds < \infty,$$

(30) $$\lim_{t\to\infty} \frac{v_2^*(t)}{v_1^*(t)}\int_{t}^{\infty} v_1^*(s)\omega(s, \kappa v_1^*(s))ds = 0.$$

Then, for any constant γ *with* $|\gamma| < \kappa$, *there exists a solution* $u(t)$ *of* (19) *such that*

(31) $$u(t) = \gamma v_1(t) + o(v_1^*(t)) \quad as \quad t \to \infty.$$

In addition, if $u(t)$ *is a solution of* (19) *satisfying* $|u(t)| \leqq \kappa v_1^*(t)$, *then there exists a constant* γ *such that* (31) *holds.*

PROOF. In this case, $q = 1$ and $i_1 = 1$. Put

$$p = 1, \quad N_0 = M = \{1\}, \quad A = \{1, 2\}, \quad 5 = \phi,$$

$$\mu_1(t) = m_1(t) = v_1^*(t).$$

From (28) and (29) we find

$$\int_{t_0}^{\infty} |y^{21}(s)|\omega(s,\kappa m_1(s))ds \leqq \int_{t_0}^{\infty} v_2^*(s)\omega(s, \kappa v_1^*(s))ds < \infty ,$$

$$\int_{t_0}^{\infty} |y^{22}(s)|\omega(s,\kappa m_1(s))ds \leqq \int_{t_0}^{\infty} v_1^*(s)\omega(s, \kappa v_1^*(s))ds < \infty ,$$

which guarantee that condition (5) holds. Condition (6) is satisfied, since by (30)

$$\frac{1}{\mu_1(t)}\int_t^{\infty} |y_{12}(t)y^{22}(s)|\,\omega(s, \kappa m_1(s))\,ds$$

$$\leqq \frac{v_2^*(t)}{v_1^*(t)}\int_t^{\infty} v_1^*(s)\omega(s, \kappa v_1^*(s))ds \longrightarrow 0 \ as\ t \longrightarrow \infty .$$

The conclusion of Theorem 3 now follows from Theorem 1.

THEOREM 4. *Let* (23) *and* (24) *hold. Assume that* $v_1^*(t) \leqq v_2^*(t)$ *for* $t \in I$,

$$\int_{t_0}^{\infty} v_2^*(s)\omega(s, \kappa v_2^*(s))ds < \infty ,$$

and

$$\lim_{t\to\infty} \frac{v_2^*(t)}{v_1^*(t)}\int_t^{\infty} v_1^*(s)\omega(s, \kappa v_2^*(s))ds = 0 .$$

Then, for any constants γ_1, γ_2 *with* $|\gamma_1| + |\gamma_2| < \kappa$, *there exists a solution* $u(t)$ *of* (19) *such that*

$$u(t) = \gamma_1 v_1(t) + \gamma_2 v_2(t) + o(v_1^*(t)) \quad as \quad t\to\infty. \tag{32}$$

In addition, if $u(t)$ *is any solution of* (19) *satisfying* $|u(t)| \leqq \kappa v_2^*(t)$, *then there are constants* γ_1, γ_2 *for which* (32) *holds.*

PROOF. Put
$N_0 = \{1\}$, $M = \{1, 2\}$, $A = \{1, 2\}$, $B = \phi$, $\mu_1(t) = v_1^*(t)$, $m_1(t) = v_2^*(t)$,
and apply Theorem 1.

REMARK. Theorems 2 and 3 generalize slightly Theorems 1 and 2 of Hallam [6], respectively.

EXAMPLE. Consider the differential equation

$$(t^{\alpha+1}u')' + \beta t^{\alpha-1}u = a(t)u^r, \tag{33}$$

where α, β, $r>0$, are constants, and $a(t)$ is a continuous function for $t \geqq 1$. We suppose $\alpha^2 - 4\beta \leqq 0$. The associated homogeneous equation

$$(t^{\alpha+1}v')' + \beta t^{\alpha-1}v = 0$$

has linearly independent solutions $v_1(t), v_2(t)$ given by

$$v_1(t) = t^{-\alpha/2},\ v_2(t) = t^{-\alpha/2} \log t,\ (\alpha^2 - 4\beta = 0),$$

$$v_1(t) = t^{-\alpha/2} \cos \frac{\sqrt{4\beta - \alpha^2}}{2} \log t,\ v_2(t) = t^{-\alpha/2} \sin \frac{\sqrt{4\beta - \alpha^2}}{2} \log t,$$

$$(\alpha^2 - 4\beta < 0).$$

Let $\alpha^2 - 4\beta = 0$. We take $v_i^*(t) = v_i(t)$, $i = 1, 2$, and apply Theorems 2 and 4. From Theorem 2 it follows that if

$$\int_1^\infty s^{-\alpha(1+r)/2} (\log s)^r |a(s)| ds < \infty,$$

and

$$\lim_{t \to \infty} (\log t)^{-1} \int_1^t s^{-\alpha(1+r)/2} (\log s)^{1+r} |a(s)| ds = 0,$$

then, for any constant 7, there is a solution $u(t)$ of (33) which satisfies

$$u(t) = \gamma t^{-\alpha/2} \log t + o(t^{-\alpha/2} \log 0 \quad \text{as} \quad t \to \infty.$$

Theorem 4 implies that if

$$\int_1^\infty s^{-\ (1+r)/2} (\log s)^{1+r} |a(s)| ds < \infty,$$

then for any constants γ_1, γ_2, there is a solution $u(t)$ of (33) such that

$$u(t) = t^{-\alpha/2} (\gamma_1 + \gamma_2 \log t) + o(t^{-\alpha/2}) \quad \text{as} \quad t \to \infty.$$

Let $\alpha^2 - 4\beta < 0$. Taking $v_1^*(t) = v_2^*(t) = t^{-\alpha/2}$ and applying Theorem 4, we conclude that if

$$\int_1^\infty s^{-\alpha(1+r)/2} |a(s)| ds < \infty,$$

then, for any constants γ_1, γ_2, there is a solution $u(t)$ of (33) such that

$$u(t) = t^{-\alpha/2} \left(\gamma_1 \cos \frac{\sqrt{4\beta - \alpha^2}}{2} \log t + \gamma_2 \sin \frac{\sqrt{4\beta - \alpha^2}}{2} \log t \right)$$

$$+ o(t^{-\alpha/2}) \quad \text{as} \quad t \to \infty.$$

B) Next we examine systems of differential equations

(34) $$x' = Ax + f(t, x),$$

$$(35) \qquad y' = Ay,$$

for $t \geqq t_0$, where A is a constant $n \times n$ matrix and $f(t, x)$ is a continuous n-vector function on $I \times R^n, I = [t_0, \infty)$. We assume that A is in Jordan canonical form:

$$A = \operatorname{diag}[J_1, J_2, \dots, J_l],$$

where J_h is a square matrix of order n_h with λ_h on the diagonal, 1 on the sub-diagonal, and 0 elsewhere. A fundamental matrix $Y(t) = (y_{ij}(t)) = e^{tA}$ of (35) is given explicitly by

$$Y(t) = e^{tA} = \operatorname{diag}[e^{tJ_1}, e^{tJ_2}, \dots, e^{tJ_l}],$$

where

$$e^{tJ_h} = e^{\lambda_h t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_h-1}}{(n_h-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_h-2}}{(n_h-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots\cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad h = 1, \dots, l.$$

Take real numbers α and α_0, $\alpha_0 \leqq \alpha$, from $\{\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_l\}$ and let ν and ν_0 be integers such that

$$1 \leqq \nu \leqq \max\{n_h : \operatorname{Re}\lambda_h = \alpha\},$$

$$1 \leqq \nu_0 \leqq \max\{n_h : \operatorname{Re}\lambda_h = \alpha_0\}$$

we assume that $\nu_0 \leqq \nu$ if $\alpha = \alpha_0$. We need the following notation:

$$H_- = \{h : \operatorname{Re}\lambda_h < \alpha_0\}, \; H_0 = \{h : \operatorname{Re}\lambda_h = \alpha_0\}, \; H_+ = \{h : \operatorname{Re}\lambda_h > \alpha_0\},$$

$$K_- = \{h : \operatorname{Re}\lambda_h < \alpha\}, \; K_0 = \{h : \operatorname{Re}\lambda_h = \alpha\},$$

$$\sigma_0 = 0, \; \sigma_h = n_1 + \cdots + n_h, \qquad h = 1, \dots, l,$$

$$L_h = \{\sigma_{h-1} + 1, \dots, \sigma_h\}, \qquad h = 1, \dots, l,$$

$$L(S) = \bigcup_{h \in S} L_h \quad \text{for} \quad S \subset \{1, \dots, l\}, \qquad (L(\phi) = \phi),$$

$$M = L(K_-) \cup M^*, \quad \text{where} \quad M^* = \bigcup_{h \in K_0} [\{\sigma_{h-1} + 1, \dots, \sigma_{h-1} + \nu\} \cap L_h]$$

$$N = \{1 \, \dots\dots \, n\}.$$

Define the functions $\mu_i(t)$, $m_i(t)$ by

$$\mu_i(t)=\begin{cases} e^{\alpha_0 t} & \text{if } i\in L(H_-\cup H_+),\\ t^{\sigma_{h-1}+\nu_0-i}e^{\alpha_0 t} & \text{if } i\in L_h,\ h\in H_0,\end{cases}$$

$$m_i(t)=\max_{j\in M}\ |y_{ij}(t)|,\ \mu_i(t)\},\qquad i=1,\dots,n.$$

THEOREM 5. *Let $f(t, x)$ foe a continuous n-vector function which satisfies* (4) *on $I\times R^n$. Suppose that*

$$\int_{t_0}^{\infty}\frac{\omega_j(s,\kappa m(s))}{\mu_j(s)}ds<\infty,\qquad j=1,\dots,n, \tag{36}$$

where $\kappa>0$ is α $m(s)=(m_{i_1}(s),\dots,m_{i_q}(s))$. for any constant vector $(\gamma_1,\dots,\gamma_n)$ $\gamma_j=0$ for $j\in N\backslash M$ $\sum_{j\in M}|\gamma_j|<\kappa$, there exists a solution $x(t)=(\xi_1(t)$, , of (34) *such that*

$$\sum_{h=1}^{l}\sum_{j=i}^{\sigma_h}\frac{\gamma_j}{(j-i)!}t^{j-i}e^{\lambda_h t}+o(\mu_i(t))\quad as\quad t\longrightarrow\infty, \tag{37}$$

$$i=1,\dots,n.$$

In If $x(t)=(\xi_1(t),\dots,\xi_n(t))$ *is a of* (34) *satisfying $|\xi_i(t)|\leqq\kappa m_i(t)$, $i=1,\dots$,* n, α $(\gamma_1, ., \gamma_n)$ *such* $\gamma_j=0$ *for $j\in N\backslash M$ and* (37) *holds.*

REMARK. It Is the $\sum_{j=i}^{\sigma_h}$ if $i>\sigma_h$.

PROOF. We the of the $Y(t)=(y_{ij}(t))$ $Y^{-1}(t)=(y^{ji}(t))$ are by

$$y_{ij}(t)=\begin{cases}\frac{t^{j-i}}{(j-i)!}e^{\lambda_h t} & \text{if } i\leqq j,\ i,j\in L_h,\ h=1,\dots,l,\\ 0 & \text{if } j<i,\ i,j\in L_h,\ h=1,\dots,l,\\ & \text{or if } \quad j\in L_{h'},\ h\neq h',\end{cases}$$

$$y^{jk}(t)=\begin{cases}\frac{(-t)^{j-k}}{(j-k)!}e^{-\lambda_h t} & \text{if } \qquad h=1,\dots,l,\\ 0 & \text{if } j<k,\ k,j\in L_h,\ h=1,\dots,l,\\ & \text{or if } k\in L_h,\ j\in L_{h'},\ h\neq h'.\end{cases}$$

We the N_0, A, B of N as follows:

$$N_0=L(H_-)\cup N_0^*,\ N_0^*=\bigcup_{h\in H_0}[\{\sigma_{h-1}+1,\dots,\sigma_{h-1}+\nu_0\}\cap L_h]$$

$$A = L(H_+) \cup A^*,\ A^* = \bigcup_{h\in H_0} A_h,\ A_h = \begin{cases} \{\sigma_{h-1}+\nu_0, \dots, \sigma_h\} & (\nu_0 \leqq n_h) \\ \phi & (\nu_0 > n_h) \end{cases}$$

$$B = L(H_-) \cup B,\ B^* = \bigcup_{h\in H_0} B_h,\ B_h = L_h \backslash A_h.$$

It is obvious that $A \cup B = N, A \cap B = \phi$ and $N_0 \subset M$.

It can be shown without difficulty that $\mu_i(t) \geqq \max_{j\in N_0} |y_{ij}(t)|$ for $i=1,\dots, n$ We shall show that conditions (5), (6), (7) of Theorem 1 are satisfied.

Condition (5). Note that $k \in A$ implies $k \in L(H_+ \cup H_0)$. Let j, $k \in L_h$ for some $h \in H_+$. Then,

$$|y^{jk}(t)| < t^{j-k} e^{-(\mathrm{Re}\lambda_h)t} \leqq e^{-\alpha_0 t} \leqq 1/\mu_j(t)$$

for all $t \geqq t_0$, provided t_0 is taken sufficiently large. Let j, $k \in L_h$ for some $h \in H_0$. Then, $k \geqq \sigma_{h-1} + \nu_0$, and so

$$|y^{jk}(t)| \leqq t^{j-\sigma_{h-1}-\nu_0}\, e^{-\alpha_0 t} \leqq 1/\mu_j(t).$$

In any case we have by (36)

$$\int_{t_0}^{\infty} |y^{jk}(s)|\, \omega_j(s, \kappa m(s)) ds \leqq \int_{t_0}^{\infty} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds < \infty.$$

Condition (6). We obtain

$$\int_t^{\infty} \Big| \sum_{k\in A\backslash N_0} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s\kappa m(s)) ds$$

$$\leqq \sum_{h\in H_+} \int_t^{\infty} \Big| \sum_{k\in L_h} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds$$

$$+ \sum_{h\in H_0} \sum_{k\in A_h} |y_{ik}(t)| \int_t^{\infty} |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds.$$

Let $h \in H_+$. We need only to consider those i, j which lie in L_h. Since $\mu_i(t) = \mu_j(t) = e^{\alpha_0 t}$ and

$$\text{(38)} \quad \sum_{k\in L_h} y_{ik}(t) y^{jk}(s) = \begin{cases} \dfrac{(t-s)^{j-i}}{(j-i)!} e^{\lambda_h(t-s)} & \text{if } \sigma_{h-1}+1 \leqq i \leqq j \leqq \sigma_h, \\ 0 & \text{if } \sigma_{h-1}+1 \leqq j < i \leqq \sigma_h, \end{cases}$$

we see with the use of (38) that

$$\frac{1}{\mu_i(t)} \int_t^{\infty} \Big| \sum_{k\in L_h} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds$$

$$\leqq \frac{1}{\mu_i(t)} \int_t^\infty \Big| \sum_{k \in L_h} y_{ik}(t) y^{jk}(s) \Big| \mu_j(s) \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds$$

$$\leqq \int_t^\infty (s-t)^{j-i} e^{-(\mathrm{Re}\lambda_h - \alpha_0)(s-t)} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds$$

$$\leqq C_1 \int_t^\infty \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds \longrightarrow 0 \quad as \quad t \longrightarrow \infty ,$$

where $C_1 = \sup_{z \geqq 0} z^{j-i} e^{-(\mathrm{Re}\lambda_h - \alpha_0)z} < \infty$. Let $h \in H_0$. It suffices to consider only those i, j which lie in L_h and satisfy $i \leqq k \leqq j$. Observe that

$$\frac{|y_{ik}(t)|}{\mu_i(t)} \leqq \frac{t^{k-i} e^{\alpha_0 t}}{t^{\sigma_{h-1}+\nu_0-i} e^{\alpha_0 t}} = t^{k-\sigma_{h-1}-\nu_0},$$

$$|y^{jk}(s)| \mu_j(s) \leqq s^{j-k} e^{-\alpha_0 s} \cdot s^{\sigma_{h-1}+\nu_0-j} e^{\alpha_0 s} = s^{\sigma_{h-1}+\nu_0-k} .$$

Using these and the inequality $k \geqq \sigma_{h-1} + \nu_0$, we have

$$\frac{1}{\mu_i(t)} \sum_{k \in A_h} |y_{ik}(t)| \int_t^\infty |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds$$

$$\leqq \sum_{k \in A_h} \frac{|y_{ik}(t)|}{\mu_i(t)} \int_t^\infty |y^{jk}(s)| \mu_j(s) \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds$$

$$\leqq \sum_{k \in A_h} t^{k-\sigma_{h-1}-\nu_0} \int_t^\infty s^{\sigma_{h-1}+\nu_0-k} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds$$

$$\leqq n_h \int_t^\infty \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds \longrightarrow 0 \quad as \quad t \longrightarrow \infty .$$

Thus the condition (6) is satisfied.

Condition (7). It holds that

$$\int_{t_0}^t \Big| \sum_{k \in B} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds$$

$$\leqq \sum_{h \in H_-} \int_{t_0}^t \Big| \sum_{k \in L_h} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds$$

$$+ \sum_{h \in H_0} \sum_{k \in B_h} |y_{ik}(t)| \int_{t_0}^t |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds.$$

Suppose $h \in H_-$ and consider those $i, j, k \in L_h$ such that $i \leqq k \leqq j$. Since $\mu_i(t) = \mu_j(t) = e^{\alpha_0 t}$, we obtain, using (38),

$$\frac{1}{\mu_i(t)} \int_{t_0}^t \Big| \sum_{k \in L_h} y_{ik}(t) y^{jk}(s) \Big| \omega_j(s, \kappa m(s)) ds$$

$$\leqq \int_{t_0}^{t} (t-s)^{j-i} e^{-(\alpha_0 - \mathrm{Re}\lambda_h)(t-s)} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds$$

$$\leqq C_2 \int_{t_0}^{t} e^{-(\alpha_0 - \mathrm{Re}\lambda_h)(t-s)/2} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds ,$$

where $C_2 = \sup_{z \geqq 0} z^{j-i} e^{-(\alpha_0 - \mathrm{Re}\lambda_h)z/2} < \infty$ Taking (36) into account and applying a lemma of Brauer [2], we conclude that the last integral tends to zero as $t \to \infty$. Suppose now that $h \in H_0$. If $\nu_0 = 1$, then $B_h = \phi$. If $\nu_0 \neq 1$, then $k - \sigma_{h-1} - \nu_0 < 0$, and we have

$$\frac{1}{\mu_i(t)} \sum_{k \in B_h} |y_{ik}(t)| \int_{t_0}^{t} |y^{jk}(s)| \omega_j(s, \kappa m(s)) ds$$

$$\leqq \sum_{k \in B_h} t^{k-\sigma_{h-1}-\nu_0} \int_{t_0}^{t} s^{\sigma_{h-1}+\nu_0-k} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds .$$

By a lemma of Hallam [5] the last sum tends to zero as $t \to \infty$. It follows that the condition (7) is satisfied.

The above observation enables us to apply the first half of Theorem 1 (with $p = n$) to (34) and (35) to conclude that there exists a solution $x(t) = (\xi_1(t), \ldots, \xi_n(t))$ of (34) which satisfies the relation (37). The second half of Theorem 1 yields the opposite relationship between the solutions of (34) and (35). The proof is thus complete.

REMARK. It can be shown that a result of Brauer and Wong [3, Theorem 2] follows from Theorem 5 as a corollary.

COROLLARY. *In the Jordan canonical form of A, let $m > 0$ be the maximum order of those blocks which corresponds to eigenvalues of A with real part equal to ρ. Let $y(t)$ be a solution of (35) which is not identically zero and satisfies*

$$\limsup_{t \to \infty} t^{-g} e^{-\rho t} \| y(t) \| < \infty,$$

where g is an integer, $0 \leqq g < m$. Suppose that $f(t, x)$ satisfies

$$\| f(t, x) \| \leqq \phi(t, \|x\|),$$

where $\phi(t, r)$ is a continuous function on $I \times R_+$ which is nondecreasing in r for each fixed t. Here $\| \cdot \|$ denotes any convenient vector norm, say $\|x\| = \max_{i \in N} |x_i|$. If

$$\int^{\infty} t^{m-g-1} e^{-\rho t} \phi(t, \kappa t^g e^{\rho t}) dt < \infty \qquad \textit{for every} \quad \kappa > 0,$$

then there exists a solution $x(t)$ of (34) such that

$$x(t) = y(t) + o(t^g e^{\rho t}) \quad as \quad t \to \infty.$$

PROOF. We put

$$\omega_j(t, r_1, \dots, r_n) = \phi(t, \max_{i \in N} r_i), \qquad j = 1, \dots, n,$$

$$\alpha = \alpha_0 = \rho, \quad \nu = \nu_0 = g+1,$$

$$\mu_i(t) = m_i(t) = \begin{cases} e^{\rho t} & \text{if } \ i \in L(H_- \cup H_+), \\ t^{\sigma_{h-1}+g+1-i} e^{\rho t} & \text{if } \ i \in L_h, \ h \in H_0, \end{cases}$$

$$m(t) = (m_1(t), \dots, m_n(t)).$$

Then,

$$\max_{i \in N} |m_i(t)| = \max_{h \in H_0} \{\max_{i \in L_h} t^{\sigma_{h-1}+g+1-i} e^{\rho t}\} = t^g e^{\rho t},$$

and

$$\max_{j \in N} \frac{1}{\mu_j(t)} = \max_{h \in H_0} \{\max_{j \in L_h} t^{j-\sigma_{h-1}-g-1} e^{-\rho t}\} = t^{m-g-1} e^{-\rho t}.$$

Therefore, we have

$$\int_{t_0}^{\infty} \frac{\omega_j(s, \kappa m(s))}{\mu_j(s)} ds = \int_{t_0}^{\infty} \frac{1}{\mu_j(s)} \phi(s, \kappa \max_{i \in N} m_i(s)) ds$$
$$\leq \int_{t_0}^{\infty} s^{m-g-1} e^{-\rho s} \phi(s, \kappa s^g e^{\rho s}) ds < \infty$$

for $j = 1, \dots, n$. Now the desired conclusion follows from Theorem 5.

REMARK. For other related results we refer to Bihari [1].

EXAMPLE. Consider the fourth order scalar equation

(39) $$u^{(iv)} + u''' + u'' = \phi(t, u, u', u'', u'''), \qquad t \in I = [t_0, \infty)$$

where $\phi(t, u, u', u'', u''')$ is continuous on $I \times R^4$ and satisfies

$$|\phi(t, u, u', u'', u''')| \leqq a(t) |u'|^r$$

for some nonnegative continuous function $a(t)$ on I and some constant $r > 0$. We compare (39) with the unperturbed equation

$$v^{(iv)} + v''' + v'' = 0.$$

Let $w = \mathrm{col}(u, u', u'', u''')$, $x = \mathrm{col}(x_1, x_2, x_3, x_4)$, and make the change of variables $w = Px$, where

$$P=\begin{bmatrix}1&1&1&0\\ \omega&\omega&0&1\\ \omega&\omega&0&0\\ 1&1&0&0\end{bmatrix},\quad \omega=(-1+\sqrt{3}\,i)/2\,.$$

Then, equation (39) is put into the system $x'=Ax+f(t,x)$ with

$$A=\begin{bmatrix}\omega&0&0&0\\ 0&\omega&0&0\\ 0&0&0&1\\ 0&0&0&0\end{bmatrix},\quad \text{and}\quad f(t,x)=\phi(t,Px)P^{-1}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

If we choose a set of functions $\{\mu_i(t), m_i(t)\}$ appropriately, then we are able to apply Theorem 5 to deduce a corresponding result. Below, we list four propositions which are obtained in this manner and which, except the third one, do not follow from Theorem 2 of Brauer and Wong [3].

1) Take $\alpha_0=\alpha=0$ and $\nu_0=\nu=2$. In this case, $N_0=M=\{1,2,3,4\}$, $\mu_i(t)=1$ $(i=1,2,4)$, $\mu_3(t)=t$, and $m_i(t)=1$ 0 = 1,2, 4). If $\int^\infty a(t)dt<\infty$, then for any γ, there is a solution $u(t)$ of (39) such that $u(t)=\gamma t+o(t)$ as $t\to\infty$.

2) Take $\alpha_0=\alpha=0$ and $\nu_0=1$, $\nu=2$. In this case, $N_0=\{1,2,3\}$, $M=\{1,2,3,4\}$, $\mu_i(t)=1$ $(i=1,2,3)$, $\mu_4(t)=1/t$, and $m_i(t)=1$ $(i=1,2,4)$. If $\int^\infty ta(t)dt<\infty$, then for any γ_1,γ_2, there is a solution $u(t)$ of (39) such that $u(t)=\gamma_1+\gamma_2 t+o(1)$ as $t\to\infty$.

3) Take $\alpha_0=\alpha=-1/2$ and $\nu_0=\nu=1$. In this case, $N_0=M=\{1,2\}$, $\mu_i(t)=e^{-t/2}$ $(i=1,2,3,4)$, and $m_i(t)=e^{-t/2}$ $(i=1,2,4)$. If $\int^\infty e^{(1-i)t/2}\,a(t)dt<\infty$, then for any γ_1,γ_2, there exists a solution $u(t)$ of (39) such that $u(t)=e^{-t/2}(\gamma_1\cos(\sqrt{3}\,t/2)+\gamma_2\sin(\sqrt{3}\,t/2)+o(1))$ as $t\to\infty$.

4) Take $\alpha_0=-1/2$, $\alpha=0$ and $\nu_0=1$, $\nu=2$. In this case, $N_0=\{1,2\}$, $M=\{1,2,3,4\}$, $\mu_i(t)=e^{-t/2}$ $(i=1,2,3,4)$, $m_i(t)=e^{-t/2}$ $(i=1,2)$, $m_4(t)=1$. If $\int^\infty e^{t/2}a(t)dt<\infty$, then for any $\gamma_1,\gamma_2,\gamma_3,\gamma_4$, there exists a solution $u(t)$ of (39) such that

$$u(t)=e^{-t/2}(\gamma_1\cos(\sqrt{3}\,t/2)+\gamma_2\sin(\sqrt{3}\,t/2))+\gamma_3+\gamma_4 t$$
$$+o(e^{-t/2})\quad\text{as}\quad t\to\infty.$$

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Department of Mathematics,
Faculty of Science,
Hiroshima University