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# *Some Differentialsin the mod p Adams Spectral Sequence* $(p \geqq 5)$

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## Introduction

Let $p$ be an odd prime. Let $A$ denote the Steenrod algebra mod $p$ and $\pi_k(S;p)$ the $p$-primary component of the $k$th stable homotopy group of spheres.

J. F. Adams [1] introduced a spectral sequence having $H^{**}(A)=\mathrm{Ext}_A^{**}(Z_p, Z_p)$ as its $E_2$ term and a bi-graded algebra associated to $\pi_*(S;p)$ as its $E_\infty$ term. In his thesis [5], J. P. May constructed another spectral sequence which has as its $E_\infty$ term an algebra $E^0H^{**}(A)$, i.e., a tri-graded algebra associated to $H^{**}(A)$, and he made extensive computations of $H^{**}(A)$.

In [10], we extended May's computations in his techniques, and obtained complete informations on the module $H^{s\ t}(A)$ in the range $t-s \leqq (3p^2+3p+4)q$ -2, $q=2(p-1)$. For the case $p=3$, we also determined in [11] all differentials in the Adams spectral sequence $E_r^{s,t}$ in the range $t-s \leqq 104$, and obtained the complete group structure of $\pi_k(S;3)$ for fcg!03. On the other hand, for the primes $p \geqq 5$, we determined in [12] all differentials in the range $t-s \leqq (2p^2+p)q$ $-3$ from our results on the groups $\pi_k(S;p)$, which were obtained without any information on $H^{**}(A)$, together with our results on $H^{**}(A)$ [10].

In this paper, we shall always treat the case $p \geqq 5$. We shall determine differentials in the mod $p$ Adams spectral sequence by the same techniques as in [11], and by making use of known results on $\pi_*(S\ p)$. Our main results on differentials are Theorems 2.1 and 2.4, where all differentials $d_r$ on $E_r^{s,t}$ for $t-s \leqq (2p^2+4p+1)q$ $-6$ and some ones for greater $t-s$ will be computed. From these results, we shall determine the $E_\infty^{s,t}$ term and the group $\pi_k(S;p)$ for $t-s$ and fc^$(2p^2\ +4p$ $+1)q-7$ in Theorems 3.1 and 4.1, respectively. Several partial results for $E_\infty$ and $\pi_*(S;p)$ in higher degrees will be also obtained.

In §1, we shall compute several products in the algebra $H^{**}(A)$ which differ from ones in the algebra $E^0H^{**}(A)$ (Theorem 1.1). We shall compute the differentials in §2 and the $E_\infty$ term in §3. The group $\pi_k(S;p)$, together with its generator, will be determined in § 4. In § 5, several Toda bracket formulas in $\pi_*(S;p)$ will be obtained and the group extension in $\pi_{(2p^2+p+1)q-3}(S;p)$ will be determined.

## § 1. Algebra structure of $\mathrm{Ext}_A^{**}(Z_p, Z_p)$

Throughout this paper, $p$ will denote a prime integer $\geqq 5$, and we will set $q=2(p-1)$ and write $H^{**}(A)$ instead of $\mathrm{Ext}_A^{**}(Z_p, Z_p)$ for the cohomology of the mod $p$ Steenrod algebra $A$.

The algebra $H^{**}(A)$ is naturally isomorphic as a module over $Z_p$ to an associated graded algebra $E^0H^{**}(A)$, and the structure of the algebra $E^0H^{**}(A)$ (and of the module $H^{**}(A)$ also) has been computed by A. Liulevicius [4], J. P. May [5] and the first-named author [10]. We shall use the same notations for the elements of $E^0H^{**}(A)$ (and corresponding ones of $H^{**}(A)$) as in the previous paper [10]. Some of these notations differ from May's original ones, i.e., we write $b_{ij}, g_{i,l}$ and $k_{i,l}$ in place of his ftj, $g_i^l$ and $k_i^l$, respectively.

Now we first correct errors of our paper [10].

CORRECTIONS TO THEOREMS 3.3 AND 4.4 OF [10]. (i) The relation 11. g) in Theorem 3.3 should be replaced by

11. g). $k_{1,0}w = g_{2,0}x \quad$ *if* $p = 3$,

$\mathrm{fc}_{\mathrm{u}}\mathrm{w}=0,\ 0 \leqq l \leqq p-3, \quad$ *if* $p \geqq 5$.

(ii) The elements 17. h) and 44. c) in Theorem 4.4 should respectively be replaced by

17. h). $h_0 b_{01}^k k_{1,l} b_{11}^2 \in (2k+l+7,\ (2p^2+kp+lp+2p+l+2)q-2k-7)$,

$1 \leqq l \leqq p-3$,

44. c). fcu/c$_{2>/}$e(/ $+5,\ (2p^2+3p+lp+l+1)q-5)$, $\quad 0 \leqq / \leqq p-4$.

In our computations of $H^{**}(A)$ [10] by May's techniques, the product obtained from Theorem 3.3 and Proposition 4.3 of [10], which is actually the product with respect to the algebra structure of $E^0H^{**}(A)$, may not be the one in the algebra $H^{**}(A)$. A product in $H^{**}(A)$ of two elements always contains as a summand their product in $E^0H^{**}(A)$ but may possibly contain also other terms of the same bi-grading but of lower weight in the sense of J. P. May [5]. The following relations in $H^{**}(A)$ differ from ones in $E^0H^{**}(A)$. This list is by no means complete.

THEOREM 1.1. *In the algebra* $H^{**}(A), p \geqq 5$, *the following relations hold.*

(i) $h_2 \cdot u = a_0^{p-1} c$.

(ii) $g_{2,1} \cdot h_1 b_{02} = h_0 k_{1,1} b_{11},\quad h_1\ g_{2,1} b_{02} = 2h_0 k_{1,1} b_{11}$,

$h_1 b_{02} \cdot g_{2,1} b_{02} = 3h_0 k_{1,1} b_{11} b_{02}$

(iii) $h_0 \cdot h_1 b_{02} = -k_{1,0} b_{11}$

(iv) $h_1 \cdot h_1 b_{02} = b_{01} c$.

(v) $h_2 \cdot h_1 b_{02} = -b_{11} c - 2b_{01} d,\quad h_1\ h_2 b_{02} = \mathrm{fc}_{01} d + 2b_{11} c$.

(vi) $h_2\ h_2 b_{02} = -b_{11} d$.

(vii) $k_{1,l}\ h_1 b_{02} = -b_{01}k_{2,l},\quad 0 \leqq / \leqq p-4,$

$h_1 \cdot$ fci $_{,l}b_{02} = 1/(/+2)b_{01}k_{2,l},\quad 0 \leqq / \leqq p-4.$

(viii) fc$_{u}$ $h_2 b_{02} = -(l+3)/(/+2)b_{11}k_{2,l},\quad 0 \leqq / \leqq p-4,$

$h_2 \cdot k_{1,l}b_{02} = -1/(l+2)b_{11}k_{2,l},\quad 0 \leqq / \leqq p-4.$

(ix) $h_2 \cdot g_{2,l}b_{02} = 2b_{01}v_l,\quad 1 \leqq l \leqq p-3,$

$g_{2,l} \cdot h_2 b_{02} = b_{01}v_l,\quad 0 \leqq l \leqq p-3$

( x ) $h_2\ u b_{02} = -2b_{01}G.$

The results for $p=3$ corresponding to the above are seen in Proposition 1.1 of [11]. In particular, (i), (iii), (v) and the second of (ix) also hold for $p=3$, and (iv) and (vi) differ from the case $p=3$: $h_1\ h_1 b_{02} = b_{11}^2$ *and* $h_2 \cdot h_2 b_{02} = -b_{01}b_{21} - b_{11}d$ *for* $p=3$. We obtained the results for $p=3$ by computations in the *cobar construction* $F^*(A^*)$, but some of the computations are too long for the case $p \geqq 5$. So we shall also make use of May's *imbedding method* [5] and *matric Massey products* [6]; (i) and (ii) in Theorem 1.1 are proved by the imbedding method, (iii)-(vi) are proved by computations in the cobar construction, and (vii)-(x) are proved by computations of matric Massey products.

PROOF OF THEOREM 1.1. (i) For any element $a$ in $H^{**}(A)$, we denote its dual in $H_{**}(A) = \mathrm{Tor}^A_{**}(Z_p Z_p)$ by $a^*$. By dimensional and filtrational considerations, we have $h_2\ u = \alpha a_0^{p-1}c$ for some $\alpha \in Z_p$. To determine the coefficient α, we compute the comultiplication of the dual $(a_0^{p-1}c)^*$. By routine computations, $(a_0^{p-1}c)^*$ is represented in the bar construction $\bar{B}(A)$ of $A$ by the element

$$\begin{aligned}
\xi = {} & \{P_2^1\}*\{P_1^1\}*\{Q_0\}^{p-1} - \{P_1^2\}*\{P_2^0\}*\{Q_1\}^{p-1} \\
& + \textstyle\sum_{i=1}^{p-1}(-1)^i(i-1)!\{P_2^1\}*\{[Q_1]^i|(P_1^0)^{p-i}\}*\{Q_0\}^{p-i-1} \\
& + \textstyle\sum_{i=0}^{p-2}(-1)^i i!\{[P_3^0]*[Q_1]^i|(P_1^0)^{p-i-1}\}*\{Q_0\}^{p-i-1} \\
& + \textstyle\sum_{i=0}^{p-2}(-1)^{i+1}\ i!\{[Q_2]^{i+1}|(P_2^0)^{p-i-1}\}*\{P_1^1\}*\{Q_0\}^{p-i-2} \\
& + \textstyle\sum_{i=0}^{p-2}(-1)^i i!\{[Q_2]^{i+1}|(P_1^1)^{p-i-1}\}*\{P_2^0\}*\{Q_1\}^{p-i-2} \\
& + \textstyle\sum_{i=0}^{p-3}\sum_{j=0}^{p-i-3}(-1)^{i+j}i!j!\{[Q_2]^{i+1}|(P_2^0)^{p-i-1}\} \\
& \qquad * \{[Q_1]^{j+1}|(P_1^0)^{p-j-1}\}*\{Q_0\}^{p-i-j-3} \\
& + \textstyle\sum_{k=2}^{p-1}\sum_{i=0}^{p-k-1}(-1)^{k+i+1}(k+i)!/k!(p-k+1)! \\
& \qquad \{[Q_2]^{i+1}|(P_2^0)^{k-1}(P_1^0)^{p-k+1}(P_1^1)^{p-k-i-1}\}*\{P_2^0\}*\{Q_1\}^{p-i-2}.
\end{aligned}$$

Here we use the same notations as in [5]; $Q_i$ and $P_j^i = P^R$, $R = (0,\ldots,0,p^i,0,\ldots)$, are the Milnor basis elements [8] and * denotes the **shuffle** product in $\bar{B}(A)$. For convenience we use the following abbreviated **notations:**

$$[x]^i = [x|\cdots|x]\,(i\text{-times}),$$

$$[x]^i*[y] = \textstyle\sum_{j=0}^{i}(-1)^{\deg[y]\deg[x]^{i-j}}[x]^j|y|[x]^{i-j}.$$

Let $D$ denote the comultiplication of $\bar{B}(A)$. Then the summand in $D\xi$ having the left component $\{P_1^2\}$ is

$$-\{P_1^2\}\otimes\{P_2^0\}*\{Q_1\}^{p-1}.$$

Since $h_2^*$ and $u^*$ are represented by $\{P_1^2\}$ and $-\{P_2^0\}*\{Q_1\}^{p-1}$ respectively, we have $h_2\ u = a_0^{p-1}c$ as desired.

(ii) For dimensional and filtrational reason, we see that in each equality the element in the left is a multiple of the one in the right. The dual element $(h_0k_{1,1}b_{11})^*$ is represented in $\bar{B}(A)$ by

$$\begin{aligned}\eta = &-\{P_1^0\}*\{P_1^1\}*\{P_2^0\}*\{P_1^1|(P_1^1)^{p-1}\}*\{Q_2\}\\ &-\{P_1^1\}*\{P_2^0\}*\{[P_1^1]*[P_2^0]|(P_1^1)^{p-2}\}*\{Q_2\}\\ &+\{P_1^0\}*\{P_1^1\}*\{P_2^0\}*\{P_1^0|(P_2^0)^{p-1}\}*\{Q_2\}\\ &+\{P_1^1\}*\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\},\end{aligned}$$

and the summands with the left components $-\{P_1^1\}*\{P_2^1|(P_2^1)^{p-1}\}$ and $\{P_1^1\}$ in $D\eta$ are, respectively,

$$-\{P_1^1\}*\{P_2^0|(P_2^0)^{p-1}\}\otimes-\{P_1^0\}*\{P_2^0\}*\{Q_1\}$$

and

$$\{P_1^1\}\otimes A,$$

where

$$\begin{aligned}A = &\ \{P_1^0\}*\{P_2^0\}*\{P_1^1|(P_1^1)^{p-1}\}*\{Q_2\}\\ &-\{P_2^0\}*\{[P_1^1]*[P_2^0]|(P_1^1)^{p-2}\}*\{Q_2\}\\ &-\{P_1^0\}*\{P_2^0\}*\{P_1^0|(P_2^0)^{p-1}\}*\{Q_2\}\\ &+\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\}\\ = &\ 2\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\}+\delta A',\\ A' = &-\{P_1^1\}*\{P_1^0\}*\{P_2^0\}*\{P_1^1|(P_1^1)^{p-1}\}*\{Q_1\}\\ &+\{P_1^1\}*\{P_2^0\}*\{[P_1^1]*[P_2^0]|(P_1^1)^{p-2}\}*\{Q_1\}\\ &+\{P_1^1\}*\{P_1^0\}*\{P_2^0\}*\{[P_1^1]*[Q_2]|(P_1^1)^{p-2}\}\end{aligned}$$

$$-2\{P_1^1\}*\{P_2^0\}*\{[P_1^1]*[P_2^0]*[Q_2]|(P_1^1)^{p-3}\}$$

$$+\{P_1^1\}*\{P_1^0\}*\{P_2^0\}*\{P_1^0|(P_2^0)^{p-1}\}*\{Q_1\}.$$

Since $(h_1b_{02})^*$ and $h_1^*$ are represented by $-\{P_1^1\}*\{P_2^0|(P_2^0)^{p-1}\}$ and $\{P_1^1\}$, we have $h_1b_{02}\cdot g_{2,1}=h_0k_{1,1}b_{11}$ and $h_1\cdot g_{2,1}b_{02}=2h_0k_{1,1}b_{11}$.

Similarly, $(h_0k_{1,1}b_{11}b_{02})^*$ is represented by

$$C=\{P_1^0\}*\{P_1^1\}*\{P_2^0\}*\{P_1^1|(P_1^1)^{p-1}\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_2\}$$

$$+\{P_1^1\}*\{P_2^0\}*\{[P_1^1]*[P_2^0]|(P_1^1)^{p-2}\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_2\}$$

$$-\{P_1^0\}*\{P_1^1\}*\{P_2^0\}*\{P_1^0|(P_2^0)^{p-1}\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_2\}$$

$$-\{P_1^1\}*\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\},$$

and the summand with the left component $-\{P_1^1\}*\{P_2^1|(P_2^1)^{p-1}\}$ in $D\zeta$ is $-\{P_1^1\}*\{P_2^0|(P_2^0)^{p-1}\}\otimes B$, where

$$\begin{aligned}B=\ &\{P_1^0\}*\{P_2^0\}*\{P_1^1|(P_1^1)^{p-1}\}*\{Q_2\}+\{P_1^0\}*\{P_1^1\}*\{P_2^0\}*\{(P_1^1)^{p-1}\}*\{Q_2\}\\&-\{P_2^0\}*\{[P_1^1]*[P_2^0]|(P_1^1)^{p-2}\}*\{Q_2\}-\{P_1^1\}*\{P_2^0\}*\{P_2^0|(P_1^1)^{p-2}\}*\{Q_2\}\\&-\{P_1^0\}*\{P_2^0\}*\{P_1^0|(P_2^0)^{p-1}\}*\{Q_2\}+2\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\}\\=\ &3\{P_1^0\}*\{P_2^0\}*\{P_2^0|(P_2^0)^{p-1}\}*\{Q_1\}+\delta B'\end{aligned}$$

for some $B'$. Hence we have $h_1b_{02}\cdot g_{2,1}b_{02}=3h_0k_{1,1}b_{11}b_{02}$.

(iii) We consider the cobar construction ([4], [5]) $F^*(A^*)$. Let $\bar{b}_{02}$ be a cochain

$$-\textstyle\sum_{j=1}^{p-1}1/(p-j)!j![\xi_2^{p-j}|\xi_2^j]$$

$$-\textstyle\sum_{i=1}^{p-1}\sum_{k=0}^{p-i}1/(p-k-i)!k!i![\xi_1^{pi}\xi_2^{p-i-k}|\xi_1^i\xi_2^k]$$

$$+\textstyle\sum_{j=1}^{p-2}1/(p-j)!(j+1)![\xi_1^{p(p-j)}|\xi_1^{p(j+1)}].$$

Here $\xi$'s are the Milnor basis elements [8] in the dual Hopf algebra $A^*$ of $A$. Put

$$\bar{b}_{01}=-\textstyle\sum_{j=1}^{p-1}1/(p-j)!j![\xi_1^{p-j}|\xi_1^j],$$

$$\bar{b}_{11}=-\textstyle\sum_{j=1}^{p-1}1/(p-j)!j![\xi_1^{p(p-j)}|\xi_1^{pj}].$$

These cochains in $F^*(A^*)$ represent $b_{01}$ and fc$_n$, respectively. Then we have

$$\delta\bar{b}_{02}=[\xi_1^p]\bar{b}_{11}-[\xi_1^{p^2}]\bar{b}_{01},$$

and hence

$$\delta\{-[\xi_1^{p+1}]\bar{b}_{02}+[\xi_2]\bar{b}_{02}-[\xi_1^{p^2+p+1}]\bar{b}_{01}+[\xi_1\xi_2^p]\bar{b}_{01}+[\xi_1^{p^2}\xi_2]\bar{b}_{01}-[\xi_3]\bar{b}_{01}\}$$
$$=[\xi_1](h_1b_{02})^-+\bar{k}_{1,0}\bar{b}_{11},$$

where

$$(h_1b_{02})^- = [\xi_1^p]\bar{b}_{02}-1/2[\xi_1^{2p}]\bar{b}_{11}+[\xi_1^{p^2+p}]\bar{b}_{01}-[\xi_2^p]\bar{b}_{01}$$

and

$$\bar{k}_{1,0} = -[\xi_2|\xi_1^p]+[\xi_1^{p+1}|\xi_1^p]+1/2[\xi_1|\xi_1^{2p}].$$

Since the cochains $[\xi_1]$, $(h_1b_{02})^-$ and $\bar{k}_{1,0}$ represent $h_0$, $h_1b_{02}$ and $k_{1,0}$ respectively, we have $h_0\cdot h_1b_{02}$ =—fei.o^n

(iv) We have

$$\delta\{-1/2[\xi_1^{2p}]\bar{b}_{02}+1/6[\xi_1^{3p}]\bar{b}_{11}+[\xi_1^p\xi_2^p]\bar{b}_{01}-1/2[\xi_1^{p^2+2p}]\bar{b}_{01}\}$$
$$=[\xi_1^p](h_1b_{02})^- -\bar{c}\bar{b}_{01},$$

where $\bar{c}=[\xi_2^p|\xi_1^p]+1/2[\xi_1^{p^2}|\xi_1^{2p}]$. Since $[\xi_1^p]$ and $c$ are representatives of $h_1$ and c respectively, we have $h_1\cdot h_1b_{02}=cb_{01}$.

(v) Since we have

$$\delta\{-[\xi_2^p]\bar{b}_{02}-[\xi_1^{p^2}\xi_2^p]\bar{b}_{01}\} = [\xi_1^{p^2}](h_1b_{02})^-+\bar{c}\bar{b}_{11}+2\bar{d}\bar{b}_{01}$$

and

$$\delta\{[\xi_2^p]\bar{b}_{02}-[\xi_1^{p^2+p}]\bar{b}_{02}+[\xi_1^p\xi_2^p]\bar{b}_{11}+[\xi_1^{p^2}\xi_2^p]\bar{b}_{01}-1/2[\xi_1^{2p^2+p}]\bar{b}_{01}\}$$
$$=[\xi_1^p](h_2b_{02})^- -2\bar{c}\bar{b}_{11}-\bar{d}\bar{b}_{01},$$

where $\bar{d}=[\xi_1^{p^2}|\xi_2^p]+1/2[\xi_1^{2p^2}|\xi_1^p]$ and $(h_2b_{02})^-=[\xi_1^{p^2}]\bar{b}_{02}-[\xi_2^p]\bar{b}_{11}+1/2[\xi_1^{2p^2}]\bar{b}_{01}$ are representatives of $d$ and $h_2b_{02}$ respectively, the desired results $h_2\cdot h_1b_{02}=-cb_{11}-2b_{01}d$ and $h_1\ h_2b_{02}=2cb_{11}+db_{01}$ follow.

(vi) This follows from $\delta\{-1/2[\xi_1^{2p^2}]\bar{b}_{02}-1/6[\xi_1^{3p^2}]\bar{b}_{01}\}=[\xi_1^{p^2}](h_2b_{02})^- +\bar{d}\bar{b}_{11}$.

Before proving the rests of the theorem, we prepare some results on matric Massey products in $H^{**}(A)$. We recall the May spectral sequence $(\tilde{E}_r, \delta_r)$[5] and Priddy's one $(\tilde{\tilde{E}}_r, \delta_r)$[16], whose initial and terminal terms are given as follows:

$$\tilde{E}_2 = H^{**}\ (E^0A),\quad \tilde{E}_\infty = E^0H^{**}(A)$$
$$\tilde{\tilde{E}}_1 = H^{****}(E^0E^0A),\quad \tilde{\tilde{E}}_\infty = E^0H^{***}(E^0A).$$

The algebra $H^{****}(E^0E^0A)$, from which our calculations of $H^{**}(A)$[10]started, is equal to

$$E(R_j^i | i \geqq 0, j \geqq 1) \circledR P(S_k | k \geqq 0) \circledR P(\tilde{R}_j^i | i \geqq 0, j \geqq 1),$$

where $E$ and P denote an exterior and a polynomial algebras, respectively (cf. [10; pp. 10–11]).

In the above two spectral sequences $\tilde{E}_r$, $\tilde{\tilde{E}}_r$ and Adams' one $E_r$, (matric) Massey products [6] can be formed, and we make often use of May's *convergence theorem* [6; Th. 4.1] for Massey products in $\tilde{E}_r$ or $\tilde{\tilde{E}}_r$ and Moss' one [9; (1.2)] in $E_r$. Roughly speaking, these convergence theorems are stated as follows: if $v_i$ ($\in \tilde{\tilde{E}}_r$, $\tilde{E}_r$ or $E_r$) converges to $w_i$ ($\in H^{***}(E^0A), H^{**}(A)$ or $\pi_*(S; p)$), $1 \leqq i \leqq 3$, Massey products $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$ (in $E_r$, w is a Toda bracket) are defined, and if some condition ([6; Th. 4.1, (*)] or [9; (1.3)]) on permanent (co)cycles is satisfied, then $v$ converges to w, (cf. [11; Th. 2.6, Th. 2.8]).

CONVENTION. R. M. F. Moss also defined a triple (matric) Massey product [9; pp. 294–295]. His definition slightly differs from May's one [6]. In this paper, we adopt Moss' definition for triple products. In particular the usual triple product $\langle a, b, c \rangle$ is defined as follows: *if* $ab = 0$ *and* $bc = 0$, *then* $\langle a, b, c \rangle$ *consists of all elements represented by (co)cycles* $(-1)^{\deg a+1} a\eta + \xi c$, where $ab = \delta(\xi)$ and $bc = \delta(\eta)$ at (co)chain level ($\delta = \delta_r$ or $d_r$), that is,

$$\begin{pmatrix} * & \eta & c \\ \xi & b & \\ a & & \end{pmatrix}$$

is a *defining system* for $\langle a, b, c \rangle$. Then the *associative law* for this triple product is given by the formulas [9; (3.4)] and

$$a\langle b, c, d \rangle = (-1)^{\deg a+1} \langle a, b, c \rangle d,$$

which also differ from May's ones [6; Th. 3.1, Cor. 3.2].

LEMMA 1.2. *The following matric Massey product formulas hold in* $H^{**}(A)$.

(i) $$\left\langle h_1, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle = h_1 b_{02}.$$

(ii) $$\left\langle h_2, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle = h_2 b_{02}.$$

(iii) $\left\langle \mathrm{fc}_u, (h_1,h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle = -k_{1,l}b_{02}, \quad 0 \leqq l \leqq p-4.$

(iv) $\left\langle g_{2,l}, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle = -g_{2,l}b_{02}, \quad 1 \leqq l \leqq p-3.$

(v) $\left\langle h_0u, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle = -h_0ub_{02}.$

( vi ) $\langle k_{1,l}, h_1, (h_1, h_2)\rangle = (0, -k_{2,l}), \quad 0 \leqq / \leqq p-4.$

(vii) $\langle h_1, \mathrm{fc}_u, (h_1, h_2)\rangle = (0, 1/(/+2)k_{2,l}), \quad 0 \leqq / \leqq p-4.$

(viii) $\langle k_{1,l}, h_2, (h_1, h_2)\rangle = ((l+3)/(/+2)k_{2,l}, 0), \quad 0 \leqq / \leqq p-4.$

(ix) $\langle h_2, \mathrm{fc}_u, (h_1, h_2)\rangle = (1/(l+2)k_{2,l}, 0), \quad 0 \leqq / \leqq p-4.$

( x ) $\langle h_2, g_{2,l}, (h_1, h_2)\rangle = (0, 2v_l), \quad 1 \leqq / \leqq p-3.$

(xi) $\langle g_{2,l}, h_2, (h_1, h_2)\rangle = (0, v_l), \quad 0 \leqq / \leqq p-3.$

**(xii)** $\langle h_2, h_0u, (h_1, h_2)\rangle = (0, 2h_0G).$

PROOF. Since $h_1^2=0$, $h_1h_2=0$ and $\delta_p b_{02}=h_1b_{11}-h_2b_{01}$ in the $\tilde{E}_p$ term [10; Th. 3.3, Prop. 4.3], the matric Massey product in (i) is defined and equal to $h_1b_{02}$ in the $\tilde{E}_{p+1}$ term by definition. The condition (*) in May's convergence theorem is satisfied for this Massey product in $\tilde{E}_{p+1}$, and hence (i) holds in $H^{**}(A)$. In the same way, (ii)-(v) are proved. The elements $\mathrm{fc}_u$, $h_1$, $h_2$, $k_{2,l}$ are represented by $R_1^1R_2^0S_2^l$, $R_1^1$,$R_1^2$, $R_2^1R_1^1R_2^0S_2^l$, respectively, and $(R_1^1)^2 = 0$ and $\delta_1R_2^1 = -R_1^2R_1^1$ in the $\tilde{\tilde{E}}_1$ term [10; Th. 3.3, Prop. 1.3]. So the matric Massey product $\langle k_{1,l}, h_1, (h_1, h_2)\rangle$ is defined and equal to (0, — $\mathrm{fc}_{2f/}$) in the $\tilde{\tilde{E}}_2$ term. By iterated use of the May convergence theorem, we obtain the relation **(vi)** in $H^{**}(A)$. In the same way, (vii)-(xii) are proved. *q.e.d.*

PROOF OF THEOREM 1.1 (CONTINUED). (vii) By Lemma 1.2 (i), (vi) and the associativity of the matric Massey products, we have

$$k_{1,l}\cdot h_1b_{02} = k_{1,l}\left\langle h_1, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} \right\rangle$$

$$= -\langle k_{1,l}, h_1, (h_1, h_2)\rangle \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix}$$

$$= -(0, -k_{2,l})\begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix} = -k_{2,l}b_{01}$$

Similarly we have

$$h_1 \cdot k_{1,l}b_{02} = -h_1\left\langle k_{1,l}, (h_1, h_2), \begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix}\right\rangle$$

$$= -\langle h_1, k_{1,l}, (h_1, h_2)\rangle\begin{pmatrix} b_{11} \\ -b_{01} \end{pmatrix}$$

$$= 1/(l+2)k_{2,l}b_{01}$$

by Lemma 1.2 (iii), (vii).

(viii) Similarly the first and the second formulas follow from Lemma 1.2 (ii), (viii) and (iii), (ix), respectively.

(ix) This follows from Lemma 1.2 (iv), (x), (ii) and (xi).

(x) By dimensional consideration, $h_2 \cdot ub_{02} = \alpha Gb_{01}$ for some $\alpha \in Z_p$. By Lemma 1.2 (v) and (xii), we have $h_2 \cdot h_0ub_{02} = 2h_0Gb_{01} \neq 0$, and hence $\alpha = -2$.

*q.e.d.*

## %2. Differentials in the Adams spectral sequence

From now on, we shall write $(E_r, d_r)$ and $(\tilde{E}_r, \delta_r)$ for the mod $p$ Adams spectral sequence and May's, respectively. Since Toda's first nontrivial differential on $E_2^{1,pq}$[21–II], the differentials $d_r$ have been computed by the several authors, e.g., A. Liulevicius [4], N. Shimada and T. Yamanoshita [17], H. H. Gershenson [3], J. P. May [5], R. J. Milgram [7], H. Toda [21, 22, 23, 24], E. Thomas and R. S. Zahler [19], and S. Oka and H. Toda [15].

In our case $p \geqq 5$, from these works and our results on $E_2$[10] and on $\pi_*(S; p)$[12], we have recently determined all differentials $d_r$ and elements surviving to $E_\infty$ in the range $t-s \leqq (2p^2+p)q-4$ [12–III]. In this section, we shall compute them and obtain complete information in the range $t-s \leqq (2p^2+4p+1)q-7$.

THEOREM 2.1. *In the $E_2$ term of the* mod $p$ *Adams spectral sequence, $p \geqq 5$, the following equalities are satisfied up to nonzero coefficients. All nontrivial differentials $d_2$ on $E_2^{s,t}$ in the range $t-s \leqq (2p^2+4p+2)q-8$ are given by Theorem* 21.1. I *of* [12] *and by the following* I.

I. (i) $d_2(b_{01}^k g_{2,l} b_{11}^2) = b_{01}^{k+1} g_{1,l+1} b_{11}^2$, $\quad 0 \leqq l \leqq p-5, \quad k \geqq 0.$

(ii) $d_2(a_0^i d) = a_0^{i+1} h_2 b_{02}$, $\quad 0 \leqq i \leqq p-2,$

$$d_2(a_0^i b_{01} d) = a_0^{i+1} b_{01} h_2 b_{02}, \qquad 0 \leqq i \leqq p-3,$$

$$d_2(a_0^i b_{01}^k d) = a_0^{i+1} b_{01}^k h_2 b_{02}, \qquad 0 \leqq i \leqq p-4, \quad k \geqq 2.$$

(iii) $d_2(a_0^i b_{11} c) = a_0^{i+1} h_2 b_{01} b_{02}, \qquad 0 \leqq i \leqq p-3.$

(iv) $d_2(v_l) = g_{1,l+1} h_2 b_{02}, \qquad 0 \leqq l \leqq p-3.$

(v) $d_2(b_{01}^k g_{3,l} b_{11}) = b_{01}^k g_{2,l+1} b_{11} b_{02}, \qquad 0 \leqq l \leqq p-4, \quad k \geqq 0.$

(vi) rf$_2$(*S$^+{}_1$'β|,A2β2) $= a_0^{i+1} b_{01}^{k+1} b_{11} a_2, \ 0 \leqq i \leqq p-4, k \geqq 0.$

(vii) rf$_2$(feδι32Λι«2) $= g_{1,l+1} b_{01}^{k+1} b_{11} a_2, \qquad 0 \leqq l \leqq p-4, \quad k \geqq 0.$

(viii) $d_2(a_0^i h_1 d) = a_0^{i+1}(b_{01} d + c b_{11}), \qquad 0 \leqq i \leqq p-2.$

(ix) rf$_2$(βS$^{+\prime}$-$^3$c6$_{02}$) $= a_0^{p+i-2} h_1 b_{02}^2, \ 0 \leqq i \leqq p.$

(x) rfa^LjC) $= g_{1,l} b_{01} h_2 a_2, \qquad 0 \leqq l \leqq p-2,$

$$d_2(b_{01} G) = a_0^p h_1 b_{02}^2.$$

(xi) rf$_2$(/J$_2$w) $= a_0^{p-1} c b_{02} - b_{01} G,$

$$d_2(g_{1,l} h_2 w) = g_{1,l} b_{01} G, \qquad 0 \leqq / \leqq 1.$$

(xii) rf$_2$(flί,(ί>οιC«2-αowί'O2)) $= a_0^{i+1}(b_{01} h_1 b_{02} a_2 - a_0 u b_{02}^2), \ 0 \leqq i \leqq p-2,$

$$d_2(a_0^i b_{01}^k (b_{01} c a_2 - a_0^2 w b_{02})) = a_0^{i+1} b_{01}^k (b_{01} h_1 b_{02} a_2 - a_0 u b_{02}^2),$$

$$0 \leqq i \leqq p-4, \quad k \geqq 1.$$

(xiii) $d_2(h_2 g_{3,l}) = b_{01} v_{l+1}, \qquad 0 \leqq l \leqq p-4,$

$$d_2(b_{01}^k h_2 g_{3,0}) = b_{01}^{k+1} v_1, \qquad k \geqq 1.$$

II. (i) $d_2(a_0^i b_{11} d) = a_0^{i+1} h_2 b_{11} b_{02}, \qquad 0 \leqq i \leqq p-2,$

$$d_2(b_{01}^k a_0^i b_{11} d) = a_0^{i+1} b_{01}^k h_2 b_{11} b_{02}, \qquad 0 \leqq i \leqq p-5.$$

(ii) $d_2(b_{01}^k h_1 g_{3,0} b_{02}) = b_{01}^k h_0 k_{1,1} b_{11} b_{02}, \qquad k \geqq 0.$

(iii) $d_2(b_{01}^k h_1 g_{3,0} a_2) = b_{01}^k h_0 k_{1,1} b_{11} a_2, \qquad k \geqq 0$

(iv) $d_2(b_{01}^k j_l a_2) = b_{01}^k h_0 k_{1,l+1} b_{11} a_2, \qquad 1 \leqq l \leqq p-4, \quad k \geqq 0.$

(v) $d_2(b_{01}^k g_{2,l} b_{11}^3) = b_{01}^{k+1} g_{1,l+1} b_{11}^3, \qquad 0 \leqq l \leqq p-5, \quad k \geqq 0.$

To prove the theorem, we prepare some lemmas.

LEMMA 2.2. *For the same nonzero coefficient* $\alpha$, *the identities* $d_2(h_1)=\alpha a_0 b_{01}$ *and* $d_2(h_2)=\alpha a_0 b_{11}$ *hold.*

PROOF. This follows easily from $0=d_2(h_1 h_2)=d_2(h_1)h_2-h_1 d_2(h_2)$ and $a_0 b_{01} h_2=a_0 h_1 b_{11}\neq 0$. *q.e.d.*

The coefficient $\alpha$ above will often appear in the proof of Theorem 2.1 below.

LEMMA 2.3. *The following Massey product formulas hold in* $H^{**}(A)$.

(i) $\langle h_2, -h_2, h_1\rangle = d$.

(ii) $\langle b_{01}, -a_0^p, d\rangle = b_{01}h_2 a_2$.

(iii) $\langle b_{01}, -a_0^p, h_2 b_{02}\rangle = b_{01}b_{11}a_2$.

(iv) $\langle v_l, -a_0, a_0^{p-1}h_1\rangle = a_1 v_l, \qquad 0\leqq l\leqq p-3.$

(v) $\langle g_{1,l}h_2 b_{02}, -a_0, a_0^{p-1}h_1\rangle = g_{1,l}b_{01}h_2 a_2, \qquad 0\leqq \iota\leqq p-2.$

(vi) $\langle a_0^{p-1}, b_{01}, h_2 w\rangle = a_0^{p-1}wb_{02}$.

(vii) $\langle a_0^p, \mathrm{ft}_0\iota, a_0^{p-1}cb_{02}-b_{01}G\rangle = -1/2a_0^p u b_{02}^2$.

**PROOF.** (i) The elements $h_1, h_2$ and $d$ are represented in the cobar construction $F^*(A^*)$ by the elements $[\xi_1^p], [\xi_1^{p^2}]$ and $\bar{d}=[\xi_1^{p^2}|\xi_2^p]+(1/2)[\xi_1^{2p^2}|\xi_1^p]$, (cf. [4]). Since $\delta[\xi_2^p]=-[\xi_1^{p^2}|\xi_1^p]$ and $\delta[\xi_1^{2p^2}]=-2[\xi_1^{p^2}|\xi_1^{p^2}]$, $\langle h_2, -h_2, h_1\rangle$ contains an element represented by $\bar{d}$. By dimensional consideration, we obtain $\langle h_2, -h_2, h_1\rangle=d$.

(ii)-(iii) Since $\delta_{2p-1}(h_2a_2)=a_0^p d$ in $\tilde{E}_{2p-1}$ [5, Th. II. 6.9], (cf. [10, Th. 4.2]), the Massey product in (ii) is defined and equal to $b_{01}h_2a_2$ in $\tilde{E}_{2p}$. By May's convergence theorem, (ii) holds in $H^{**}(A)$. By using the relation $\delta_p(b_{11}a_2)=a_0^p h_2 b_{02}$ in $\tilde{E}_p$ [10], (iii) is proved similarly.

(iv)-(v) Since $H^{p+1,(2p^2+2p-1)q+p-2}(A)$ is generated by $h_1x$ with May's weight $2p-2$ and $a_0v_{p-3}$ has May's weight $p-1$, we have $a_0v_{p-3}=0$ in $H^{**}(A)$, and hence these Massey products are defined in $H^{**}(A)$. Then, by the relations $\delta_p(a_1)=-a_0^p h_1$, $a_0v_l=0$ and $a_0g_{1,l}h_2b_{02}=0$ in $\tilde{E}_p$[10] and by May's convergence theorem, the desired results are obtained.

(vi)-(vii) These follow from $\delta_p(wb_{02})=-b_{01}h_2w$ in $\tilde{E}_p$ and $\delta_{2p-1}(h_1b_{02}a_2)=2a_0b_{01}G-2a_0^p cb_{02}$ in $\tilde{E}_{2p-1}$[10; Prop. 4.2]. *q.e.d.*

PROOF OF THEOREM 2.1. I. (i), (iii), (v), (vii) and II. (v) are easily derived from Theorem 21.2. I of [12–III].

I. (ii) By Moss' Leibnitz formula [9; (1.1)] and Lemmas 2.2, 2.3 (i), we have

$$d_2(d) = d_2\langle h_2,\ -h_2,\ h_1\rangle$$

$$\in \left\langle (\alpha a_0 b_{11}, h_2),\ \begin{pmatrix} -h_2 & 0 \\ \alpha a_0 b_{11} & -h_2 \end{pmatrix},\ \begin{pmatrix} h_1 \\ \alpha a_0 b_{01} \end{pmatrix} \right\rangle = -P.$$

The last **matric** Massey product $P$ has a defining system

$$\begin{pmatrix} * & \begin{pmatrix} 0 \\ \alpha a_0 b_{02} \end{pmatrix} & \begin{pmatrix} h_1 \\ \alpha a_0 b_{01} \end{pmatrix} \\ (0, 0) & \begin{pmatrix} -h_2 & 0 \\ \alpha a_0 b_{11} & -h_2 \end{pmatrix} & \\ (\alpha a_0 b_{11}, h_2) & & \end{pmatrix}$$

and so P is defined and equal to

$$(-\alpha a_0 b_{11},\ h_2)\begin{pmatrix} 0 \\ \alpha a_0 b_{02} \end{pmatrix} + (0, 0)\begin{pmatrix} h_1 \\ \alpha a_0 b_{01} \end{pmatrix} = \alpha a_0 h_0 b_{02}$$

in the $\tilde{E}_{p+1}$ term of the May spectral sequence. Hence we have

$$(2.1) \qquad d_2(d) = -\alpha a_0 h_2 b_{02}.$$

The desired results follow from this and

(2.2) *$a_0$ and $b_{01}$ are permanent cycles.*

I. (iv) By the relations 4. k) and 4. a) of Theorem 3.3 [10],

$$g_{1,p-3-l} v_l = 1/(l+2) a_0^{p-3} b_{01} d,$$

$$g_{1,p-3-l} g_{1,l+1} = 1/(l+2) a_0^{p-2} b_{01},$$

for $0 \leqq l \leqq p-3$. Then, by (2.1), $-g_{1,p-3-l} d_2(v_l) = -\alpha/(l+2) a_0^{p-2} b_{01} h_2 b_{02} = -\alpha g_{1,p-3-l} \cdot g_{1,l+1} h_2 b_{02}$. For dimensional reason, we see that $d_2(v_l)$ is a multiple of $g_{1,l+1} h_2 b_{02}$ ($\neq 0$). Therefore

$$(2.3) \qquad d_2(v_l) = \alpha g_{1,l+1} h_2 b_{02}, \qquad 0 \leqq l \leqq p-3,$$

as desired.

I. (vi) By Lemma 2.3 (ii), **Moss'** formula [9; **(1.1)]**, (2.1) and Lemma 2.3 (iii), we have $d_2(b_{01} h_2 a_2) = \alpha a_0 b_{01} b_{11} a_2$ Then the desired results follow from (2.2).

I. (viii) $d_2(h_1 d) = \alpha a_0 b_{01} d + \alpha a_0 h_1\ h_2 b_{02} = 2\alpha(a_0 b_{01} d + a_0 c b_{11})$ by Lemma 2.2 and Theorem 1.1 (v).

I. (ix)-(x) By [10; Th. 3.3] and Lemma 2.3 (iv), $g_{1,l} G = a_1 v_{l-1} = \langle v_{l-1},$

$-a_0, a_0^{p-1}h_1\rangle$. Then we have $d_2(g_{1,l}G) = \alpha g_{1,l}b_{01}h_2a_2$ for $l \geqq 1$ by [9; (1.1)], (2.3) and Lemma 2.3 (v). By the relation 14 of [10; Prop. 4.1], $a_0b_{01}G = a_0^p cb_{02}$ in $H^{**}(A)$ and so

$$g_{1,p-2}g_{1,0}G = a_0^{p-2}b_{01}G = a_0^{2p-3}cb_{02} = -g_{1,0}g_{1,p-2}G.$$

Then we have

$$g_{1,p-2}d_2(g_{1,0}G) = \alpha g_{1,p-2}\cdot g_{1,0}b_{01}h_2a_2 = \alpha a_0^{2p-2}h_1b_{02}^2 \neq 0,$$

$$a_0^p d_2(a_0^{p-3}cb_{02}) = -\alpha a_0^{2p-2}h_1b_{02}^2 \neq 0.$$

By dimensional consideration, we obtain

$$d_2(g_{1,0}G) = \alpha g_{1,0}b_{01}h_2a_2, \quad d_2(a_0^{p-3}cb_{02}) = -\alpha a_0^{p-2}h_1b_{02}^2.$$

We have also $a_0d_2(b_{01}G) = d_2(a_0^p cb_{02}) = -\alpha a_0^{p+1}h_1b_{02}^2$ and so $d_2(b_{01}G) = -\alpha a_0^p h_1b_{02}^2$.

I. (xi) We have $-h_0\ d_2(h_2w) = h_2\ d_2(h_0w) = h_2\ h_0ub_{02} = 2h_0b_{01}G \neq 0$ by Theorem 1.1 (x). Since $a_0^{p-1}cb_{02} - b_{01}G$ is the only $d_2$-cycle in $E_2^{p+3,(2p^2+3p)q+p-1}$, we obtain $d_2(h_2w) = 2(a_0^{p-1}cb_{02} - b_{01}G)$. Since $a_0g_{1,l} = 0$, $d_2(g_{1,l}h_2w) = 2g_{1,l}b_{01}G$ for $0 \leqq l \leqq 1$.

I. (xii) By Lemma 2.3 (vi)-(vii), we have $d_2(a_0^p wb_{02}) = (1/2)a_0^p ub_{02}^2$. By dimensional consideration and by (2.2), we obtain (xii).

I. (xiii) $d_2(h_2g_{3,l}) = h_2\ g_{2,l+1}b_{02}$, which is equal to $2b_{01}v_{l+1}$ by Theorem 1.1 (iv).

II. (i) This follows immediately from I. (ii) and (2.2).

II. (ii) By Theorem 1.1 (ii), $d_2(g_{3,0}h_1b_{02}) = g_{2,1}b_{02}\cdot h_1b_{02} = 3h_0k_{1,1}b_{11}b_{02}$.

II. (iii)-(iv) By Theorem 3.3 of [10], we have $f\cdot j_l = -b_{01}^2 j_la_2$, $f\cdot h_1 = a_0ub_{01}b_{02} - b_{01}^2h_1a_2$ and $f\cdot h_0b_{11} = -h_0b_{01}^2b_{11}a_2$ for $f = b_{01}(a_1b_{02} - a_2b_{01})$. Since $d_2(b_{01}^2 j_la_2) = -fd_2(j_l) = b_{01}^2h_0k_{1,l+1}b_{11}a_2$ we have $d_2(j_la_2) = h_0k_{1,l+1}b_{11}a_2$ and (iv) follows. By Theorem 1.1 (ii), $b_{01}^2h_1a_2g_{2,1}b_{02} = -fh_1\ g_{2,1}b_{02} = 2h_0b_{01}^2k_{1,1}b_{11}a_2$, and so $b_{01}h_1a_2\cdot g_{2,1}b_{02} = 2h_0b_{01}k_{1,1}b_{11}a_2$. Then $b_{01}\cdot d_2(h_1g_{3,0}a_2) = 2h_0b_{01}k_{1,1}b_{11}a_2$ and (iii) follows. *q.e.d.*

The first unknown differential $d_2$ after Theorem 2.1 is $d_2(h_0k_{1,0}b_{02}^2) \in E_2^{9,(2p^2+4p+2)q+1}$, where $h_0k_{1,0}b_{02}^2$ and $b_{01}k_{1,1}b_{11}^2$ generate $E_2^{7,(2p^2+4p+2)q} = Z_p$ and $E_2^{9,(2p^2+4p+2)q+1} = Z_p$, respectively, and the determination of this $d_2$ is equivalent to determine the product $h_0\ h_0k_{1,0}b_{02}^2$, which is a multiple of $b_{01}^2h_2g_{3,0}$.

THEOREM 2.4. *In the $E_r$ terms, $r \geqq 3$, of the* mod $p$ *Adams spectral sequence, $p \geqq 5$, the following equalities are satisfied up to nonzero coefficients, and all nontrivial differentials $d_r$, $r \geqq 3$, on $E_r^{s,t}$ in the range $t-s \leqq (2p^2+4p+1)q-6$*

*are given by Theorem* 21.1. II, III *and Proposition* 21.4 *of* [12] *and by the following* I.

I. (i) $d_{p+1}(b_{01}^k g_{2,p-4} b_{11}^2) = h_0 b_{01}^{k+2} k_{1,p-4} a_2, \quad k \geqq 0.$

(ii) $d_{2p-1}(b_{01}^k g_{1,p-3} b_{11} b_{02}) = b_{01}^{p+k} g_{2,p-3} b_{11}, \quad k \geqq 0.$

(iii) $d_{2p-1}(b_{01}^k g_{2,p-3} b_{11}^2) = a_0^{p-3} b_{01}^{p+k+2} h_2, \quad k \geqq 0.$

(iv) $d_{2p+1}(b_{01}^k h_1 x) = b_{01}^{p+k-1} f, \quad k \geqq 0.$

(v) $d_p(b_{01}^k k_{1,0} x) = a_0^{p-3} b_{01}^{k+2} h_2 a_2, \quad k \geqq 0.$

(vi) $d_{2p}(a_0 h_2 w) = b_{01}^p k_{1,0} a_2.$

(vii) $d_{p+1}(b_{01}^k g_{2,0} x) = b_{01}^{p+k-1} h_0 k_{1,0} b_{02}$, fc^O.

(viii) $d_{p+1}(b_{01}^k k_{1,0} b_{11} b_{02}) = b_{01}^{k+2} h_0 b_{11} a_2, \quad k \geqq 0.$

(ix) $d_{2p-1}(b_{01}^k (b_{01} h_1 b_{02} a_2 - a_0 u b_{02}^2)) = b_{01}^{p+k+1} k_{1,0} a_2, \quad k \geqq 1.$

II. (i) $d_{p+1}(b_{01}^k k_{1,0} b_{02} a_2) = b_{01}^k h_0 (Z), \quad k \geqq 0,$

*where* $(Z) = a_1^2 b_{02}^2 - 2 b_{01} a_1 b_{02} a_2 + b_{01}^2 a_2^2.$

(ii) $d_{2p}(b_{01}^k k_{2,l} a_2) = b_{01}^{p+k} k_{1,l+1} a_2, \quad 0 \leqq l \leqq p-4, \quad k \geqq 0.$

(iii) $d_{2p-1}(b_{01}^k k_{1,l} b_{11} b_{02}) = b_{01}^{p+k} h_0 k_{1,l} b_{02}, \quad 1 \leqq l \leqq p-4, \quad k \geqq 0.$

(iv) $d_{2p-1}(b_{01}^k b_{11}^3) = h_0 b_{01}^{p+k} b_{11}^2, \quad k \geqq 0.$

(v) $d_p(b_{01}^k k_{1,l} b_{02} a_2) = b_{01}^{k+1} h_0 k_{1,l-1} a_2^2, \quad 1 \leqq l \leqq p-4, \quad k \geqq 0.$

(vi) $d_{2p-1}(b_{01}^k b_{11} x) = h_0 b_{01}^{p+k} x, \quad k \geqq 0.$

(vii) $d_{2p-1}(b_{01}^k h_2 b_{11} b_{02}) = b_{01}^{p+k-1} k_{1,0} b_{11}^2, \quad k \geqq 0.$

PROOF. I. (i) By III. (i) of [12; Th. 21.2], we have $d_{p+1}(k_{1,p-4} \cdot k_{1,0} b_{02}) = k_{1,p-4} \cdot b_{01} e_1 = h_0 b_{01}^2 k_{1,p-4} a_2 \neq 0$ in $E_{p+1}$. Hence $k_{1,p-4}\, k_{1,0} b_{02} \neq 0$ and $k_{1,p-4}$fcι,o^θ2 $= g_{2,p-4} b_{11}^2$ up to a nonzero coefficient. Then we have the result.

I. (ii) By 7. e) and 11. a) of [10; Th. 3.3], we have $h_1\, k_{1,p-3} = -g_{1,p-3} b_{11}$ and fcι.0*ι,,-3 $= 1/2 g_{2,p-3} b_{11}$ By HI. (iii) of [12; Th. 12.1], we have $d_{2p-1}(g_{1,p-3} b_{11} b_{02}) = d_{2p-1}(-k_{1,p-3} \cdot h_1 b_{02}) = -k_{1,p-3} \cdot b_{01}^p k_{1,0} = -1/2 b_{01}^p g_{2,p-3} b_{11}$.

I. (iii) By III. (ii) of [12; Th. 21.1], $d_{2p-1}(g_{2,p-3} b_{11}^2) = d_{2p-1}(2 k_{1,p-3}\, k_{1,0} b_{11}) = 2 k_{1,p-3} h_0 b_{01}^p k_{1,0} = h_0 g_{2,p-3} b_{01}^p b_{11} = -a_0^{p-3} h_1 b_{01}^{p+1} b_{11} = a_0^{p-3} h_2 b_{01}^{p+2}$.

I. (iv) By [12; Th. 21.2], the element $f \in H^{**}(A)$ survives to $E_\infty$ and corresponds to the element $\beta_1 \beta_{p+1} \in \pi_*(S;p)$. By [24; Th. 5.8], $\beta_1^p \beta_{p+1} = 0$, and hence the permanent cycle $b_{01}^{p-1}f$ is killed by some differential. For dimensional reason, there is only one possible differential $d_{2p+1}(h_1x) = b_{01}^{p-1}f$.

I. (v) By Proposition 21.4 of [12] and Theorem 3.3, 11. a) of [10], $d_p(k_{1,0}x) = h_0k_{1,0}k_{1,p-3}a_2 = 1/2h_0g_{2,p-3}b_{11}a_2 = -1/2a_0^{p-3}b_{01}^2h_2a_2$.

I. (vi) By [12; Th. 21.2], the element $k_{1,0}a_2$ survives to $E_\infty$ and corresponds to $\beta_{p+2} \in \pi_*(S; p)$. By the relation $\beta_1^p\beta_{p+2} = 0$ [24; Th. 5.8], $b_{01}^pk_{1,0}a_2$ is killed by some differential. Hence we have (vi).

I. (viii) By III. (i)–(ii) of [12; Th. 21.2], $d_{p+1}(b_{11}) = 0$ and $d_{p+1}(k_{1,0}b_{02}) = b_{01}e_1 = h_0b_{01}^2a_2$. Hence $d_{p+1}(k_{1,0}b_{11}b_{02}) = h_0b_{01}^2b_{11}a_2$.

I. (vii) and II. (iii) The survivor $h_0k_{1,l}b_{02}$, $0 \leqq l \leqq p-4$, corresponds to the element $\kappa_{l+1} \in \pi_*(S;p)$, by Theorem 21.2 of [12]. We have

$$\begin{aligned}
\beta_1^p\kappa_{l+1} &= \langle \beta_1\beta_{p+l+1}, \alpha_1, \alpha_1 \rangle \beta_1^p && \text{by [12; (19.1)]} \\
&= -\beta_1\beta_{p+l+1}\langle \alpha_1, \alpha_1, \beta_1^p \rangle = -\beta_{p+l+1}\varepsilon' && \text{by [12; (6.2)]} \\
&= (l+1)/(\mathrm{I}+3)\beta_{l+3}\beta_{p-1}\varepsilon' && \text{by [24; (5.7)]} \\
&= 0 && \text{by [12; (23.8)]}.
\end{aligned}$$

Hence $b_{01}^ph_0k_{1,l}b_{02}, 0 \leqq l \leqq p-4$, is killed by some differential, and so $d_{p+1}(b_{01}g_{2,0}x) = b_{01}^ph_0k_{1,0}b_{02}$ and $d_{2p-1}(k_{1,l}b_{11}b_{02}) \neq b_{01}^ph_0k_{1,l}b_{02}$ for $1 \leqq l \leqq p-4$. By dimensional consideration, $d_{p+1}(g_{2,0}x) \neq b_{01}^{p-1}h_0k_{1,0}b_{02}$.

I. (ix) Since $b_{01}(b_{01}h_1b_{02}a_2 - a_0ub_{02}^2) = -h_1b_{02}\cdot f$, $b_{01}^2k_{1,0}a_2 = -k_{1,0}\cdot f$ and $d_{2p-1}(h_1b_{02}) = b_{01}^pk_{1,0}$, we have $b_{01}d_{2p-1}(b_{01}h_1b_{02}a_2 - a_0ub_{02}^2) = b_{01}^{p+1}k_{1,0}a_2$. By dimensional consideration, we obtain (ix).

II. (i), (ii) and (v) In a manner similar to the above, we have $d_{p+1}(b_{01}^2k_{1,0}b_{02}a_2) = -d_{p+1}(fk_{1,0}b_{02}) = -fb_{01}e_1 = -h_0b_{01}^2(\mathrm{Z})$, $d_{2p}(b_{01}^2k_{2,l}a_2) = -d_{2p}(k_{2,l}f) \neq -b_{01}^pk_{1,l+1}f = b_{01}^{p+2}k_{1,l+2}a_2$, $0 \leqq l \leqq p-4$, and $d_p(b_{01}^2k_{1,l}b_{02}a_2) = -d_p(k_{1,l}b_{02}\cdot f) = -h_0b_{01}k_{1,l-1}a_2\cdot f = h_0b_{01}^3k_{1,l-1}a_2^2 1 \leqq l \leqq p-4$, by III of [12; Th. 21.1]. Therefore the desired results follow from dimensional considerations.

II. (iv) This follows immediately form III. (ii) of [12; Th. 21.1].

II. (vii) By Theorem 1.1 (iii) and III. (ii) of [12; Th. 21.1], we have $d_{2p-1}(h_2b_{11}b_{02}) = h_0b_{01}^p\ h_2b_{02} = h_0b_{01}^{p-1}\cdot h_1b_{02}b_{11} = -b_{01}^{p-1}k_{1,0}b_{11}^2$

II. (vi) By the discussions in [12; §21] and I. (ii) of Theorem 2.1, we see that the element $h_0x \in E_2^{p+1,(2p^2+p)q+p-2}$ survives to $E_\infty$. By Moss' convergence theorem [9; (1.2)], we also see that the Toda bracket $\mathrm{v} = \langle \beta_{2p-1}, \alpha_1, \alpha_1 \rangle$ is defined (in $\pi_*(S; p)$) and corresponds to $h_0x$ (cf. Proposition 5.1 in the below). An argument similar to I. (vii) and II. (iii) shows the relation $\beta_1^{p+1}\nu = 0$. Then ftgy^o* is killed by some differential, and we have the result. *q.e.d.*

The first unknown differential after Theorems 2.1 and 2.4 is $d_{p+1}(k_{2,0}b_{02})$, which is a multiple of fc$_0$ι$^{fc}$ι,o$^b$ιι$^\beta$2 Here $k_{2,0}b_{02} \in E_{p+1}^{5,(2p^2+4p+1)q} = Z_p$ and

$b_{01}k_{1,0}b_{11}a_2 \in E_{p+1}^{p+6,(2p^2+4p+1)q+lp} = Z_p$. We shall propose in (20) of the next section a problem equivalent to the above.

## §3. $E_\infty$ term of the Adams spectral sequence

We now immediately obtain information on the $E_\infty$ term from Theorems 2.1 and 2.4. In [12; Th. 21.2 and Prop. 21.3] we listed all elements of $E_\infty^{s,t}$ for $t-s \leqq (2p^2+p)q-4$. So, in the following results, we shall omit almost all survivors in this range.

THEOREM 3.1. *In the* mod $p$ *Adams spectral sequence,* $p \geqq 5$, *the following elements* **survive** *to the* $E_\infty$ *term, and give, at least in the range* $(2p^2+p)q-3 \leqq t-s \leqq (2p^2+4p+l)q-l$, $q=2(p-1)$, *a* $Z_p$*-basis for* $E_\infty^{s,t}$ *(Following* [5] *and* [10], *we write simply* $a \in (s, t-s)$ *instead of* $a \in E_\infty^{s,t}$*).*

(1) $g_{1,l}a_1^j \in (jp+l+1, (jp+l+1)q-1)$, $0 \leqq l \leqq p-2$, $j \geqq 0$.

(2) $a_0^i a_1^j u \in ((j+1)p+i, (j+2)pq-1)$, $j \not\equiv -2 \bmod p$, $i = p-1, p$.

(3) $a_0^i a_1^{2p-3} u a_2 \in (2p^2-p+i, 3p^2q-1)$, $p^2+p-2 \leqq i \leqq p^2+p$.

(4) $b_{01}^k \in (2k, kpq-2k)$, $k \leqq 3p+2$.

(5) $b_{01}^k k_{1,l} a_2 \in (2k+l+p+2, (p^2+kp+lp+3p+l+1)q-2k-2)$,

$0 \leqq l \leqq p-3$, $0 \leqq k < p$,

$b_{01}^k k_{1,l} a_2^2 \in (2k+l+2p+2, (2p^2+kp+lp+4p+l+1)q-2k-2)$,

$0 \leqq l \leqq p-3$, $0 \leqq k < p$, $0 \leqq k+l \leqq p-3$.

(6) $a_1 G - 2a_0^{p-1} c a_2 \in (2p+1, (2p^2+3p)q-2)$,

$a_0^{p+1} w b_{02} \in (2p+3, (2p^2+4p)q-4)$,

$b_{01}^k(Z) \in (2p+2k+4, (2p^2+kp+4p)q-2k-4)$, $1 \leqq k \leqq p-2$.

(7) $h_0 a_1 G \in (2p+2, (2p^2+3p+1)q-3)$,

$h_0 k_{1,l} a_2^2 \in (2p+l+3, (2p^2+lp+4p+l+2)q-3)$, $0 \leqq l \leqq p-4$,

$b_{01} h_0 k_{1,p-4} a_2^2 \in (3p+1, (3p^2+2p-2)q-5)$.

(8) $h_0 b_{01}^k k_{1,l} b_{02} \in (2k+l+5, (p^2+kp+lp+3p+l+2)q-2k-5)$,

$0 \leqq k \leqq p-2$ *if* $l=0$, $0 \leqq k \leqq p-1$ *if* $1 \leqq l \leqq p-4$,

$0 \leqq k \leqq p$ *if* $l = p-3$.

(9) fc$_0$6fcι*ίιe(2fc + 5, $(2p^2+kp+1)q-2k-5)$, $0\leqq k\leqq p-1$.

(10) $b_{01}^k h_0 x\in(2k+p+$ ί, $(2p^2+kp+p)q-2k-3)$, $0\leqq k\leqq p-1$,

$b_{01}^k h_0 k_{1,0} x\in(2k+p+3, (2p^2+kp+3p+1)q-2k-5)$, $0\leqq k\leqq p-1$.

(11) $b_{01}^k h_2 b_{02}\in(2k+3, (2p^2+kp+p)q-2k-3)$, $0\leqq k\leqq p-1$,

$h_0 h_2 b_{02}\in(4, (2p^2+p+1)q-4)$,

$b_{01}^k k_{1,0} b_{11}^2\in(2k+6, (2p^2+kp+2p+1)q-2k-6)$, $0\leqq$ fc $\leqq p-2$.

(12) $a_0^{p-1}d\in(p+1, (2p^2+p)q-2)$,

$g_{1,l}h_2 a_2\in(p+l+2, (2p^2+p+l+1)q-2)$, $0\leqq$ / $\leqq p-2$.

(13) $g_{1,l}b_{11}a_2\in(p+l+3, (2p^2+p+l+1)q-3)$, $0\leqq l\leqq p-3$.

(14) fe$_0$ι/ioί'Iι«2e(P+5, $(2p^2+2p+1)q-5)$,

$b_{01}^k b_{11} a_2\in(p+2k+2, (2p^2+kp+p)q-2k-2)$, $1\leqq$ fc $\leqq p$.

(15) fe$_{IIC}$+Vrfe(4,(2$_P^2$+2p)g-4).

(16) $a_0^i b_{01} h_2 a_2\in(p+\iota+3, (2p^2+2p)q-3)$, $i=p-3, p-2$.

(17) $b_{01}^k v_0\in(2k+3, (2p^2+kp+p+2)q-2k-3)$, ί $\leqq k\leqq p$.

(18) $b_{01}^k g_{2,p-3}b_{11}a_2\in(2p+k+1, (2p^2+kp+3p-1)q-2k-4)$, $0\leqq k\leqq p-1$.

(19) $b_{11}k_{2,0}\in(5, (2p^2+3p+1)q-5)$,

$k_{1,1}b_{11}^2\in(7, (2p^2+3p+2)q-6)$.

(20) fc$_{1>i}$b$_u$α$_2$e(p $+l+4, (2p^2+lp+3p+l+1)q-4)$, $0\leqq l\leqq p-4$,

$b_{01}k_{1,p-4}b_{11}a_2\in(2p+2, (3p^2+p-3)q-6)$.

REMARK 3.2. All indecomposable elements of $E_\infty$ listed above except for (1) and (2) are of total degree less than $3p^2q$. We can also obtain several partial informations in the range $t-s\geqq 3p^2q$. For example, we obtain the following survivors.

(21) $h_0 b_{01}^k b_{11}^3\in(2k+7, (3p^2+kp+1)q-2k-7)$, $k=0, 1$.

(22) $g_{1,l}b_{11}^2 h_2\in(l+6, (3p^2+l+1)q-6)$, $0\leqq l\leqq p-5$,

$g_{1,l}b_{11}^2 a_2\in(p+l+5, (3p^2+p+l+1)q-5)$, $0\leqq l\leqq p-4$,

$g_{1,l}b_{11}h_2 a_2\in(p+l+4, (3p^2+p+l+1)q-4)$, $0\leqq l\leqq p-4$.

## §4. Generators for $\pi_k(S;p)$

In this section we shall determine the group $\pi_k(S;p)$, $(2p^2+p)q-3 \leqq k \leqq (2p^2+4p+1)q-7$, and its generator. Partial results for $k>(2p^2+4p+1)q-7$ and several relations on compositions will be also obtained. We shall discuss them in order of (1),..., (20) in Theorem 3.1. All elements are of order $p$ and all summands and groups are $Z_p$, unless explicitly stated otherwise. For any survivor $a \in E_\infty^{s,t}$, we denote by $\{a\}$ the coset of $\pi_{t-s}(S;p)$ which is mapped to $a$. If $\{a\}$ consists of a single element $\alpha$, we write simply $\alpha=\{a\}$ instead of $\alpha \in \{a\}$.

(1) $rq-1$ stem $(r \not\equiv 0 \bmod p, r \geqq 1)$ contains a summand $\operatorname{Im} J$, generated by the element $\alpha_r=\{g_{1,l}a_1^j\}$ for $r=jp+l+1$, $0 \leqq l \leqq p-2$ ([2], [13; §4]).

(2) $rpq-1$ stem $(r \not\equiv 0 \bmod p,\ r \geqq 1)$ contains a summand $\operatorname{Im} J=Z_{p^2}$, generated by the element $\alpha'_{rp}$ ([2], [13; §4]), which corresponds to $a_0^{p-1}a_1^{r-2}u$ for $r \geqq 2$.

(3) $rp^2q-l$ stem $(r \not\equiv 0 \bmod p,\ r \geqq 1)$ contains a summand $\operatorname{Im} J=Z_{p^3}$, generated by $\alpha''_{rp^2}$([2], [13; §4]), and $\alpha''_{3p^2}$ 6 $\{a_0^{p^2+p-2}a_1^{2p-3}ua_2\}$.

(4) $pq-2$ stem is generated by the element $\beta_1=\{b_{01}\}$ [21-IV]. For $k \leqq 3p+2$, the element $\beta_1^k=\{b_{01}^k\}$ generates a summand in $kpq-2k$ stem, since $\beta_1^{k+1} \neq 0$.

(5) $(rp+r-1)q-2$ stem contains the element $\beta_r$ of L. Smith [18] and H. Toda [24]. For $2 \leqq r \leqq 3p-1$, this stem is $Z_p$ by Theorem 3.1, and hence we have

$$\beta_r=\{k_{1,r-2}\} \text{ for } 2 \leqq r \leqq p-1,\quad \beta_p=\{g_{1,p-2}h_2\},\quad \beta_{p+1}=\{a_0^{p-1}c\},$$

$$\beta_{p+r}=\{k_{1,r-2}a_2\} \text{ for } 2 \leqq r \leqq p-1,\quad \beta_{2p}=\{g_{1,p-2}h_2a_2\},$$

$$\beta_{2p+1}=\{a_1G-2a_0^{p-1}ca_2\},\quad \beta_{2p+r}=\{k_{1,r-2}a_2^2\} \text{ for } 2 \leqq r \leqq p-1.$$

Let $k$ and $r$ satisfy $0 \leqq k<p$, $p+2 \leqq r \leqq 3p-1$, $r \neq 2p$, $2p+1$ and $k+r \leqq 3p-1$. Then the element $\beta_1^k\beta_r$ $(=\{b_{01}^k k_{1,r-p-2}a_2\}$ fo$p+2 \leqq r \leqq 2p-1$, $=\{b_{01}^k k_{1,r-2p-2}a_2^2\}$ for $2p+2 \leqq r \leqq 3p-1)$ generates a summand in $((k+r)p+r-1)q-2k-2$ stem.

(6) By [24; Th. 5.3], there is a relation $\beta_1^2\beta_2\beta_{2p+1}=\beta_1^3\beta_{2p+2}$, which is nonzero. Hence $\beta_1\beta_{2p+1}$ and $\beta_1^2\beta_{2p+1}$ are nonzero. Since $(2p^2+4p)q-4$ and $(2p^2+5p)q-6$ stems are $Z_p$ by Theorem 3.1, we see that these stems are generated by $\beta_1\beta_{2p+1}=\{a_0^{p+1}wb_{02}\}$ and $\beta_1^2\beta_{2p+1}=\{b_{01}(Z)\}$, respectively. The element $\beta_1^k\beta_{2p+1}=\{b_{01}^{k-1}(Z)\}$ $2 \leqq k<p$, is nonzero, and generates a summand for $k<p-1$ since $\beta_1^{k+1}\beta_{2p+1} \neq 0$. There are relations $\beta_1^p\beta_r=0$ for $r \geqq 2$ and $\beta_1\beta_{rp}=0$ [24; Th. 5.3, Th. 5.8].

(7) $(2p^2+3p+1)q-3$ stem is generated by $\alpha_1\beta_{2p+1}=\{h_1a_1G\}$. For $2p+2$

$\leqq r\leqq 3p-2$, $\alpha_1\beta_r=\{h_0k_{1,r-2p-2}a_2^2\}$generates a summand in $(rp+r)q-3$ stem. The element $\alpha_1\beta_1\beta_{3p-2}\in\{b_{01}h_0k_{1,p-4}a_2^2\}$ is nonzero. There are relations $\alpha_1\beta_1\beta_r=0$ for $r\geqq p$, $r\not\equiv -2 \bmod p$, $\alpha_1\beta_1^2\beta_r=0$ for $r\geqq p$ [15] and $\alpha_1\beta_{2p-1}=0$ [12–III].

(8) $(p^2+(r+2)p+r+1)q-5$ stem ($1\leqq r\leqq p-3$) is generated by the element $\kappa_r=\{h_0k_{1,r-1}b_{02}\}$ [12–III]. The element $\beta_1^k\kappa_r=\{b_{01}^k h_0k_{1,r-1}b_{02}\}$ ($k<p-1$ if $r=1$, $k\leqq p-1$ if $2\leqq r\leqq p-3$) is nonzero and, except for $\beta_1^{p-2}\kappa_1$, generates a summand. By Proposition 5.6 in the next section, the element $\beta_1^{p-2}\kappa_1$ also generates a summand. $(2p^2+p-1)q-5$ stem is $Z_{p^2}$ generated by the element $\mu\in\{h_0k_{1,p-3}b_{02}\}$ [12–III]. For $1\leqq k\leqq p$, the element $\beta_1^k\mu\in\{b_{01}^k h_0k_{1,p-3}b_{02}\}$ is nonzero, and if $k<p$ this generates a summand in $(2p^2+kp+p-1)q-2k-5$ stem. There are relations $\beta_1^{p-1}\kappa_1=0$, $\beta_1^p\kappa_r=0$ $(2\leqq r\leqq p-3)$ and $\beta_1^{p+1}\mu=0$.

(9) $(2p^2+1)q-5$ stem is generated by $\lambda'=\{h_0b_{11}^2\}$ [12–III]. For $1\leqq k<p$, the element $\beta_1^k\lambda'\in\{b_{01}^k h_0b_{11}^2\}$ generates a summand, and there is a relation $\beta_1^p\lambda'=0$.

(10) $(2p^2+p)q-3$ stem contains an element $\nu=\{h_0x\}$. Since $\beta_1\nu=\{b_{01}h_0x\}\neq 0, \nu$ generates a summand. By Proposition 5.1 (i) in the next section, $\nu$ is equal to the Toda bracket $\langle\beta_{2p-1}, \alpha_1, \alpha_1\rangle$. For $1\leqq k<p$, the element $\beta_1^k\nu\in\{b_1^k h_0x\}$ generates a summand in $(2p^2+kp+p)q-2k-3$ stem. For $0\leqq k<p-1$, the element $\beta_1^k\beta_2\nu=\{b_{01}^k h_0k_{1,0}x\}$ generates a summand, and $\beta_1^{p-1}\beta_2\nu$ is nonzero. There is a relation $\beta_1^p\nu=0$.

(11) $(2p^2+p)q-3$ stem is $Z_p+Z_p$; one factor is generated by $\nu$ and other is generated by an element $\gamma$ in $\{h_2b_{02}\}$ This element is not unique, and Thomas-Zahler's element $\gamma_2$ [20] may possibly represent $\{h_2b_{02}\}$. For $1\leqq k\leqq p-1$, the element $\beta_1^k\gamma\in\{b_{01}^k h_2b_{02}\}$ generates a summand. The element $\alpha_1\gamma=\{h_0h_2b_{02}\}$ generates $(2p^2+p+1)q-4$ stem. By the relations 5 of [10; Prop. 4.3] and (iii) of Theorem 1.1, we have $h_0\cdot h_2b_{01}b_{02}=h_0\cdot h_1b_{11}b_{02}=-k_{1,0}b_{11}^2$ in $H^{**}(A)$. Therefore $\alpha_1\beta_1\gamma=\{k_{1,0}b_{11}^2\}$ (up to sign), which generates $(2p^2+2p+1)q-6$ stem. For $2\leqq k\leqq p-1$, the element $\alpha_1\beta_1^k\gamma=\{b_{01}^{k-1}k_{1,0}b_{11}^2\}$ generates a summand.

(12) $(2p^2+p)q-2$ stem is generated by an element, which we call $\rho_0$, corresponding to $a_0^{p-1}d$. For $1\leqq i\leqq p-1$, $(2p^2+p+i)q-2$ stem contains the element $\rho_{2,i}$ [14]. We simply write $\rho_i$ instead of $\rho_{2,i}$. Since this stem is $Z_p$, the element $\rho_i=\{g_{1,i-1}h_2a_2\}$ generates this stem. By Proposition 5.1 (ii) in the next section and [14; Th. A], $\rho_i=\langle\rho_{i-1}, p\iota, \alpha_1\rangle$ for $1\leqq i\leqq p-1$ and $\rho_{p-1}=\beta_{2p}$. There is a relation $\beta_1\rho_i=0$ for $1\leqq i\leqq p-1$.

(13) $(2p^2+p+1)q-3$ stem is $Z_p+Z_p$ by Proposition 5.6 in the next section; one factor is generated by $\beta_1^{p-2}\kappa_1$ and other by $\rho'_1\in\{h_0b_{11}a_2\}$. $(2p^2+p+i)q-3$ stem $(2\leqq i\leqq p-2)$ is generated by $\rho'_i=\{g_{1,i-1}b_{11}a_2\}$. By Proposition 5.1 (iii), the last element $\rho'_{p-2}$ is equal to the Toda bracket $\langle\beta_1, p\iota, \beta_{2p-1}\rangle$. There are relations $\beta_1^2\rho'_1=0$ and $\beta_1\rho'_i=0$ for $2\leqq i\leqq p-2$.

(14) $(2p^2+2p)q-4$ stem contains a summand generated by an element $\rho''$

$=\{b_{01}b_{11}a_2\}$, since $\beta_1\rho''=\{b_{01}^2b_{11}a_2\}\neq 0$. For $1\leqq k\leqq p-1$, the element $\beta_1^k\rho''=\{b_{01}^{k+1}b_{11}a_2\}$ generates a summand. $(2p^2+2p+1)q-5$ stem is generated by $\alpha_1\rho''=\beta_1\rho'_1=\{b_{01}h_0b_{11}a_2\}$. There is a relation $\alpha_1\beta_1\rho''=\beta_1^2\rho'_1=0$.

(15) $(2p^2+2p)q-4$ stem is $Z_p+Z_p$ and generated by $\rho''$ and an element in $\{b_{11}c+b_{01}d\}$.

(16) $(2p^2+2p)q-3$ stem is $Z_{p^2}$ generated by an element $\varphi_2\in\{a_0^{p-3}b_{01}h_2a_2\}$. This is the third element of order $p^2$ *in* Coker $J=$ Ker $e$ [2]; the first and the second ones are $\varphi$ [12–I] and $\mu$ [12–III] in $(p^2+p)q-3$ and $(2p^2+p-1)q-5$ stems, respectively. By Proposition 5.1 (iv) in the next section, $\varphi_2\in\langle\rho_{p-2},\alpha_1,\alpha_1\rangle$. Since $h_0\,g_{1,p-2}h_2a_2=-a_0^{p-2}h_{01}h_2a_2$ in $H^{**}(A)$, there is a relation $p\varphi_2=-\alpha_1\rho_{p-1}$. Also, $\beta_1\varphi_2=0$ holds.

(17) $(2p^2+2p+2)q-5$ stem is generated by $\{b_{01}v_0\}$. For $1\leqq k\leqq p-2$, $\beta_1^k\{b_{01}v_0\}\in\{b_{01}^{k+1}v_0\}$ generates a summand, and $\beta_1^{p-1}\{b_{01}v_0\}\neq 0$. By Proposition 5.2, the element $\{b_{01}v_0\}$ is given by the Toda bracket $\langle\gamma,\alpha_2,\beta_1\rangle$.

(18) Since $g_{2,p-3}b_{11}a_2=-$ fe$_{1>0}$ fc$_{1\}j,-3}\alpha_2$ by the relation 11. a) of [10; Th. 3.3], $(2p^2+3p-1)q-4$ stem is generated by $\beta_2\beta_{2p-1}=\{g_{2,p-3}b_{11}a_2\}$. For $1\leqq k\leqq p-2$, $\beta_1^k\beta_2\beta_{2p-1}=\{b_1^{p-1}g_{2,p-3}b_{11}a_2\}$ generates a summand. The element $\beta_1^{p-1}\beta_2\beta_{2p-1}=\{b_{01}^{p-1}g_{2,p-3}b_{11}a_2\}$ is nonzero.

(19) $(2p^2+3p+1)q-5$ stem is $Z_p+Z_p$, generated by $\beta_2\nu$ and $\{b_{11}k_{2,0}\}$. The element $\beta_2\gamma$ may possibly represent $\{b_{11}k_{2,0}\}$ $(2p^2+3p+2)q-6$ stem is $Z_p$ generated by $\{k_{1,0}b_{11}^2\}$. In this stem, $\alpha_1\{b_{11}k_{2,0}\}$ may possibly be nonzero.

(20) $(2p^2+rp+2p+r)q-4$ stem $(1\leqq r\leqq p-3)$ contains a summand generated by $\{k_{1,r-1}b_{11}a_2\}$. The following problem seems very difficult.

PROBLEM. *Is* $\beta_1\{k_{1,0}b_{11}a_2\}$ *trivial*?

For the composition $\beta_1\{k_{1,l}b_{11}a_2\}, 1\leqq l\leqq p-5$, the same problem can be considered. But we see that the element $\beta_1\{k_{1,p-4}b_{11}a_2\}$ is nontrivial.

(21) $(3p^2+1)q-7$ stem contains an element $\{h_0b_{11}^3\}$, which is equal to $\langle\lambda',\beta_1^p,\alpha_1\rangle$ by Proposition 5.2. $\beta_1\langle\lambda',\beta_1^p,\alpha_1\rangle$ is also nonzero.

(22) $(3p^2+i)q-6$ stem $(1\leqq i\leqq p-4)$ contains a nonzero element similar to the elements $\varepsilon_i$ and $\lambda_i$. $(3p^2+p+i)q-5$ and $(3p^2+p+i)q-4$ stems $(1\leqq i\leqq p-3)$ contain nonzero elements similar to $\rho'_i$ and $\rho_i$, respectively.

From the above discussions, we have obtained the following results.

THEOREM 4.1. *The group* $\pi_k(S;p)$, $(2p^2+p)q-3\leqq k\leqq(2p^2+4p+1)q-7$, $p=5$, $q=2(p-1)$, is *the direct sum of the cyclic groups generated by the following elements of degree* $k$:

$\alpha_r$ $(2p^2+p+1\leqq r\leqq 2p^2+4p-1, r\not\equiv 0 \bmod p)$,

$\alpha'_{rp}$ $(2p+1\leqq r\leqq 2p+4)$, $\beta_1^k$ $(2p+2\leqq k\leqq 2p+4)$,

$\beta_1^k\beta_r$ ($p+2 \leqq r \leqq 2p+1, r \neq 2p, 0 \leqq k \leq p-1, 2p \leqq k+r \leqq 2p+2$),

$\alpha_1\beta_{2p+1}$, $\beta_2\beta_{2p-1}$, $\beta_1\beta_2\beta_{2p-1}$,

$\beta_1^k\kappa_r$ ($1 \leq r \leq p-3, 0 \leqq k \leqq p-1 \leqq k+r \leqq p+1, (k, r) \neq (p-1, 1)$),

$\beta_1^k\lambda'$ ($1 \leqq k \leqq 4$), $\beta_1^k\mu$ ($1 \leqq k \leqq 3$), $\beta_1^k\nu$ ($0 \leqq k \leqq 3$),

$\beta_1^k\gamma$ ($0 \leqq k \leqq 3$), $\alpha_1\beta_1^k\gamma$ ($0 \leqq k \leq 3$), $\beta_2\nu$, $\beta_1\beta_2\nu$,

$\rho_i$ ($0 \leqq i \leqq p-1$), $\rho'_i$ ($1 \leqq i \leqq p-2$), $\beta_1\rho'_1$ ($=\alpha_1\rho''$),

$\beta_1^k\rho''$ ($0 \leqq k \leqq 2$), $\beta_1^k\langle\gamma, \alpha_2, \beta_1\rangle$ ($0 \leqq k \leqq 2$), $\varphi_2$,

$\{b_{11}c+b_{01}d\}$, $\{b_{11}k_{2,0}\}$, $\{k_{1,0}b_{11}a_2\}$, $\{k_{1,1}b_{11}^2\}$.

*Here the elements $\alpha'_{rp}$ and $\varphi_2$ are of order $p^2$, and the others are of order $p$. The group $\pi_{(2p^2+4p+1)q-6}(S; p)$ is $Z_p$ or 0, and generated by $\beta_1\{k_{1,0}b_{11}a_2\}$.*

PROPOSITION 4.2. (i) *For $\xi=\alpha_r$ ($r \geqq 2$), $\alpha'_{rp}$ ($r \geqq 1$), $\alpha''_{rp^2}$ ($r \geqq 1$), $\alpha_1\beta_r$ ($p+1 \leqq r \leqq 2p-3$ or $2p+1 \leqq r \leqq 3p-3$), $\varepsilon_i$ ($1 \leqq i \leqq p-1$), $\lambda_i$ ($1 \leqq i \leqq p-2$), $\rho_i$ ($1 \leqq i \leqq p-1$), $\rho'_i$ ($2 \leqq i \leqq p-2$) and $\varphi_2$, the composition $\beta_1\xi$ is trivial.*

(ii) *For $\xi=\alpha_1\beta_{2p-2}, \rho'_1$ and $\alpha_1\beta_{3p-2}$, $\beta_1\xi \neq 0$ and $\beta_1^2\xi=0$.*

(iii) $\beta_1^{p-2}\kappa_1 \neq 0$ *and* $\beta_1^{p-1}\kappa_1=0$.

(iv) *For $\xi=\alpha_1$, $\beta_r$ ($2 \leqq r \leqq 2p+1$, $r \not\equiv 0 \bmod p$), $\alpha_1\beta_r$ ($2 \leqq r \leqq p-1$), $\beta_2\beta_{p-1}$, $\alpha_1\beta_2\beta_{p-1}$, $\beta_2\beta_{2p-1}$, $\varepsilon'$, $\lambda'$, $\kappa_r$ ($2 \leqq r \leqq p-3$), $\nu$ and $\beta_2\nu$, the element $\beta_1^{p-1}\xi$ is non-trivial but $\beta_1^p\xi$ is trivial.*

(v) *For $\xi=\varphi$ and $\mu$, $\beta_1^p\xi \neq 0$ and $\beta_1^{p+1}\xi=0$.*

## §5. Toda brackets and group extensions in $\pi_k(S; p)$

In this section we shall represent some generators of $\pi_*(S; p)$ by making use of Toda brackets, and prove that the group extension in $\pi_{(2p^2+p+1)q-3}(S; p)$ is trivial.

We recall the elements $\nu$, $\rho_0$, $\rho_1$, $\rho'_{p-2}$, $\rho_{p-2}$ and $\varphi_2$ of $\pi_*(S; p)$, which correspond to the survivors $h_0x$, $a_0^{p-1}d$, $h_0h_2a_2$, $g_{1,p-3}b_{11}a_2$, $g_{1,p-3}h_2a_2$ and $a_0^{p-3}b_{01}h_2a_2$, respectively.

PROPOSITION 5.1. *The following Toda bracket formulas hold, up to non-zero coefficients.*

(i) $\nu = \langle\beta_{2p-1}, \alpha_1, \alpha_1\rangle$.

(ii) $\rho_1 = \langle \rho_0, p\iota, \alpha_1 \rangle$.

(iii) $\rho'_{p-2} = \langle \beta_1, p\iota, \beta_{2p-1} \rangle$.

(iv) $\varphi_2 \in \langle \rho_{p-2}, \alpha_1, \alpha_1 \rangle$

**Proof.** (i) This is the mod $p$ version of [11; Lemma 3.11]. Since $d_p(x) = h_0 k_{1,p-3} a_2$ [12; Prop. 21.4] and $h_0 h_0 = 0$ in the $E_p$ term of the Adams spectral sequence, the Massey product $\langle k_{1,p-3} a_2, h_0, h_0 \rangle$ is defined and equal to $h_0 x$ in the $E_{p+1}$ term. The element $k_{1,p-3}a_2$ and $h_0$ converge to $\beta_{2p-1}$ and $\alpha_1$, respectively, and there are relations $\alpha_1 \beta_{2p-1} = 0$ [12; Cor. 21.5] and $\alpha_1 \alpha_1 = 0$. Hence the Toda bracket in (i) is defined. By Moss' convergence theorem, we obtain the desired result.

(ii) and (iii) These are proved in the same way as Lemmas 3.14 and 3.13 of [11] respectively*).

(iv) In the same way as Lemma 2.3 (ii), we have $g_{1,p-3} h_2 a_2 = \langle g_{1,p-3}, a_0^p, d \rangle$ in $H^{**}(A)$. Then $g_{1,1}\, g_{1,p-3} h_2 a_2 \in \langle g_{1,1}\, g_{1,p-3}, a_0^p, d \rangle = 1/2 \langle a_0^{p-2} b_{01}, a_0^p, d \rangle \ni 1/2 a_0^{p-2} b_{01} h_2 a_2$ by Lemma 2.3 (ii). Since these Massey products have trivial indeterminancy, we obtain

$$(*) \qquad g_{1,1} \cdot g_{1,p-3} h_2 a_2 = -1/2 a_0^{p-2} b_{01} h_2 a_2 \quad \text{in} \quad H^{**}(A).$$

Consider the Massey product $\langle g_{1,p-3} h_2 a_2, h_0, h_0 \rangle$ in $H^{**}(A)$. By dimensional consideration, we can put $\langle g_{1,p-3} h_2 a_2, h_0, h_0 \rangle = \alpha a_0^{p-3} b_{01} h_2 a_2$ for some $\alpha \in Z_p$. Then $\alpha a_0^{p-2} b_{01} h_2 a_2 = -g_{1,p-3} h_2 a_2 \langle h_0, h_0, a_0 \rangle = g_{1,p-3} h_2 a_2 \cdot g_{1,1} = -1/2 a_0^{p-2} b_{01} h_2 a_2$ by (*). Hence $\alpha = -1/2$, and we get

$$\langle g_{1,p-3} h_2 a_2, h_0, h_0 \rangle = -1/2 a_0^{p-3} b_{01} h_2 a_2 \quad \text{in} \quad H^{**}(A).$$

Applying Moss' convergence theorem to this Massey product, we obtain the desired result. *q. e. d.*

PROPOSITION 5.2. *The elements $b_{01} v_0$ and $h_0 b_{11}^3$ converge, up to nonzero coefficients, to the Toda brackets $\langle \gamma, \alpha_2, \beta_1 \rangle$ and $\langle \lambda', \beta_1^p, \alpha_1 \rangle$, respectively. Here $\gamma$ and $\lambda'$ are the generators corresponding to the survivors $h_2 b_{02}$ and $h_0 b_{11}^2$.*

PROOF. Let $\alpha \in Z_p$ be the nonzero coefficient in the equality $d_2(h_1) = \alpha a_0 b_{01}$. Then we have $d_2(v_0) = \alpha g_{1,1} h_2 b_{02}$ by (2.3) and $d_2(g_{2,0}) = -\alpha g_{1,1} b_{01}$. The Massey product $\langle h_2 b_{02}, g_{1,1}, b_{01} \rangle$ is defined and equal to $-(1/\alpha) b_{01} v_0 - (1/\alpha) h_2 b_{02} \cdot g_{2,0} = -(2/\alpha) b_{01} v_0$ in the $E_3$ term. By Moss' convergence theorem, $-(2/\alpha) b_{01} v_0$ converges to $\langle \gamma, \alpha_2, \beta_1 \rangle$. Similarly we have $h_0 b_{11}^3 = \langle h_0 b_{11}^2, b_{01}^p, h_0 \rangle$ in $E_{p+1}$, which converges to $\langle \lambda', \beta_1^p, \alpha_1 \rangle$. *q. e. d.*

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*) In the statement of [11 Lemma 3.13], there is a misprint: $\alpha_1$ should be read $\beta_1$.

Now we shall consider the ring $\mathscr{A}_*(M)$ studied in [13]. Put $M^n = S^{n-1} \cup_{p\iota} e^n$ and let

$$S^{n-1} \xrightarrow{i} M^n \xrightarrow{\pi} S^n$$

be the cofibering for $M^n$. Define $\mathscr{A}_k(M)$ by the limit group $\lim [M^{n+k}, M^n]$, where $[X, Y]$ denotes the set of homotopy classes of maps from $X$ to $Y$, and the limit is taken over the suspension. The direct sum $\mathscr{A}_*(M) = \sum_k \mathscr{A}_k(M)$ forms a (graded) algebra over $Z_p$. We introduced in [13] a linear map

$$D: \mathscr{A}_k(M) \longrightarrow \mathscr{A}_{k+1}(M)$$

having the following properties:

(5.1) [13; (1.7)] $D(\xi\eta) = D(\xi)\eta + (-1)^{\deg \xi}\xi D(\eta)$.

(5.2) [13; (1.8)] $D^2(\xi) = 0$.

(5.3) [13; Lemma 3.2] *For any element $\gamma \in \pi_k(S; p)$ of order $p$, there exists an element $[\gamma] \in \mathscr{A}_{k+1}(M)$ such that $D[\gamma] = 0$ and $\pi[\gamma]i = \gamma$.*

(5.4) [13; (1.9)] *Let $\delta = i\pi \in \mathscr{A}_{-1}(M)$. Then $D(\delta) = 1_M$, the identity class of $M^n$.*

(5.5) [13; (1.11)] *The subalgebra* $\operatorname{Ker} D$ *of $\mathscr{A}_*(M)$ is commutative, i.e., $\xi\eta = (-1)^{\deg \xi \deg \eta}\eta\xi$ for $\xi, \eta \in \operatorname{Ker} D$.*

We also introduced in [13] and [14; Th. B, (7.4)] the following elements in $\operatorname{Ker} D$:

$$\alpha \in \mathscr{A}_q(M) \quad \text{with} \quad \pi\alpha i = \alpha_1,$$

$$\beta_{(r)} \in \mathscr{A}_{(rp+r-1)q-1}(M),\ r \geqq 1, \quad \text{with} \quad \pi\beta_{(r)}i = \beta_r \quad \text{and} \quad \alpha\beta_{(r)} = \beta_{(r)}\alpha = 0,$$

$$\rho(t) \in \mathscr{A}_{(tp^2+(t-1)p+1)q-1}(M),\ t \geqq 1, \quad \text{with} \quad \alpha^{p-2}\rho(t) = \rho(t)\alpha^{p-2} = \beta_{(tp)},$$

$$\sigma(t) \in \mathscr{A}_{(tp^2+(t-1)p+1)q-2}(M),\ t \geqq 1, \quad \text{with} \quad \alpha^{p-3}\sigma(t) = \sigma(t)\alpha^{p-3} = \beta_{(1)}\beta_{(tp-1)}.$$

LEMMA 5.3. *Let $N = (2p^2+p)q-1$. Then there exists uniquely an element $\rho \in \mathscr{A}_N(M)$ such that $\pi\rho i = \rho_0$, $D(\rho) = 0$ and $\rho\alpha^{p-1} = \alpha^{p-1}\rho = x\beta_{(2p)}$ for some $x \not\equiv 0 \bmod p$. For $k = N$ and $N+q$, a $Z_p$-basis for $\mathscr{A}_k(M)$ is given by, respectively,*

$$\{\rho,\ \alpha^{2p^2+p}\delta,\ \alpha^{2p^2+p-1}\delta\alpha,\ \delta(\beta_{(1)}\delta)^{p-2}\beta_{(p+2)}\delta\}$$

*and*

$$\{\rho\alpha = \alpha\rho,\ \alpha^{2p^2+p+1}\delta,\ \alpha^{2p^2+p}\delta\alpha\}.$$

PROOF. By Theorem 4.1, $\pi_{N-1}(S; p) = Z_p$, $\pi_N(S; p) = Z_{p^2}$ and $\pi_{N+1}(S; p) = Z_p$ are generated by $\rho_0$, $\alpha'_{2p^2+p}$ and $\beta_1^{p-2}\beta_{p+2}$, respectively. Therefore, by [13; Th. 3.5, discussions in pp. 648-649 and (5.11)], we obtain the result on $\mathscr{A}_N(M)$ for an element $\mathrm{p}=[\rho_0]$ satisfying $\pi p i=\rho_0$ and $\mathrm{D(p)}=0$. Such $p$ is determined up to $\mathscr{A}_N(M) \cap \operatorname{Ker}\pi_* i^* \cap \operatorname{Ker}\mathrm{D}=Z_p$, generated by $\xi=\alpha^{2p^2+p}\delta-\alpha^{2p^2+p-1}\delta\alpha$.

By Theorem 4.1 and Proposition 5.1 (ii), $\pi_{N+q-1}(S;\mathrm{p})$ and $\pi_{N+q}(S;p)$ are $Z_p$ generated by $\rho_1=\langle\rho_0, p\iota, \alpha_1\rangle$ and $\alpha_{2p^2+p+1}$ respectively, and $\pi_{N+q+1}(S;p)=0$. Then we can take $[\rho_1]=\rho\alpha=\alpha\rho$ by [13; Prop. 3.9], and hence the result on $\mathscr{A}_{N+q}(M)$ follows similarly. In particular, $\mathscr{A}_{N+q}(M) \cap \operatorname{Ker} D$ is $Z_p+Z_p$, generated by $\rho\alpha$ and $\xi\alpha=\alpha\xi$.

Consider the element $\rho(2)\in\mathscr{A}_{N+q}(M)$. Since $\mathrm{D(p(2))}=0$, we can put $\mathrm{p}(2)=x\rho\alpha+y\xi\alpha$ for some $x, y\in Z_p$. Then $x\rho_1=\pi\rho(2)i$, which is equal to the nonzero element $\rho_{2,1}$ by [14; Th. A]. Hence $x\neq 0$. Replacing p by $p-(y/x)\xi$, the equality $\mathrm{p}(2)=x\rho\alpha$ holds for a unique p. By [14; Th. B], $\rho\alpha^{p-1}=\alpha^{p-1}\rho=(1/x)\alpha^{p-2}\rho(2)=(1/x)\beta_{(2p)}$ as desired. *q. e. d.*

We consider the group $\pi_{(2p^2+p+1)q-3}(S;p)$. By Theorem 3.1, this consists of $p^2$ elements and one of the following two cases occurs:

(I) $\pi_{(2p^2+p+1)q-3}(S;p)=Z_p+Z_p$, *generated by* $\rho'_1$ *and* $\beta_1^{p-2}\kappa_1$;

(II) $\pi_{(2p^2+p+1)q-3}(S;p)=Z_{p^2}$, *generated by* $\rho'_1$, *and* $p\rho'_1=\beta_1^{p-2}\kappa_1$.

Here $\rho'_1\in\{h_0b_{11}a_2\}$ up to a nonzero coefficient and $\beta_1^{p-2}\kappa_1\in\{b_{01}^{p-2}h_0k_{1,0}b_{02}\}$.

LEMMA 5.4. *There exists uniquely an element* $\kappa_{(1)}$ *in* $\mathscr{A}_{(p^2+3p+2)q-4}(M)$ *such that* $\pi\kappa_{(1)}i=\kappa_1$ *and* $D(\kappa_{(1)})=0$. *This element satisfies* $\alpha\kappa_{(1)}=\kappa_{(1)}\alpha=0$ *and* $\beta_{(1)}\kappa_{(1)}=\kappa_{(1)}\beta_{(1)}=0$.

PROOF. By [13; Th. 3.5] and the results on $\pi_*(S;p)$, we see that $\mathscr{A}_{(p^2+3p+2)q-4}(M)\cap\operatorname{Ker}D=Z_p$ generated by $\kappa_{(1)}=[\kappa_1]$ and that $\mathscr{A}_{(p^2+3p+3)q-4}(M)\cap\operatorname{Ker}D$ and $\mathscr{A}_{(p^2+4p+2)q-5}(M)$ are trivial. Therefore the desired relations on $\kappa_{(1)}$ hold. *q. e. d.*

LEMMA 5.5. *Let* $N'=(2p^2+p+1)q-2$. *If the case* (I) *is valid, then there is an element* $\bar{\rho}\in\mathscr{A}_{N'}(M)$ *satisfying* $\pi\bar{\rho}i=\rho'_1$ *and* $D(\bar{\rho})=0$, *and then a* $Z_p$*-basis for* $\mathscr{A}_{N'}(M)\cap\operatorname{Ker}\mathrm{D}$ *is given by* $\{\bar{\rho}, (\beta_{(1)}\delta)^{p-2}\kappa_{(1)}\rho\alpha\delta+\delta\rho\alpha\}$. *The element* $p$ *is determined up to* $\rho\alpha\delta+\delta\rho\alpha$. *If the case* (II) *is valid, then a* $Z_p$*-basis for* $\mathscr{A}_{N'}(M)\cap\operatorname{Ker}D$ *is given by* $\{(\beta_{(1)}\delta)^{p-2}\kappa_{(1)}\rho\alpha\delta+\delta\rho\alpha\}$.

PROOF. From the results on $\pi_k(S;p)$, $k=N'-1, N', N'+1$, and Lemmas 5.3-5.4, $\mathscr{A}_{N'}(M)$ is easily computed by [13; Th. 3.5], and we have the results. *q. e. d.*

PROPOSITION 5.6. *The case* (II) *is not valid, that is, the group extension in* $\pi_{(2p^2+p+1)q-3}(S;p)$ *is trivial.*

PROOF. Consider the element $\sigma(2)\in\mathscr{A}_{(2p^2+p+1)q-2}(M)$ [14; (7.4)]. This satisfies $D(\sigma(2))=0$ and $\sigma(2)\alpha^{p-3}=\alpha^{p-3}\sigma(2)=\beta_{(1)}\beta_{(2p-1)}$. If the case (II) is valid, we can put $\sigma(2)=x(\beta_{(1)}\delta)^{p-2}\kappa_{(1)}+y(\rho\alpha\delta+\delta\rho\alpha)$ for some x, $y\in Z_p$, by Lemma 5.5. Since $\sigma(2)\alpha^{p-2}=\beta_{(1)}\beta_{(2p-1)}\alpha=0$, $\kappa_{(1)}\alpha=0$ and $\rho\alpha^p=\beta_{(2p)}\alpha=0$, we have $y\rho\alpha\delta\alpha^{p-1}=0$. Since $\alpha\delta\alpha^{p-1}=-\alpha^{p-1}\delta\alpha+2\alpha^p\delta$ [13; (4.4)], we have $\rho\alpha\delta\alpha^{p-1}=-\rho\alpha^{p-1}\delta\alpha=-\beta_{(2p)}\delta\alpha$ and so $y\alpha_1\beta_{2p}=-y\pi\rho\alpha\delta\alpha^{p-1}i=0$. The element $\alpha_1\beta_{2p}$ is nonzero, and hence $y=0$. Therefore $\sigma(2)=x(\beta_{(1)}\delta)^{p-2}\kappa_{(1)}$ and $\beta_{(1)}\beta_{(2p-1)}=\sigma(2)\alpha^{p-3}=0$. This implies $\langle\beta_1, p\iota, \beta_{2p-1}\rangle=\pi\beta_{(1)}\beta_{(2p-1)}i=0$, which contradicts Proposition 5.1 (iii). Thus, the case (II) is negative. *q.e.d.*

REMARK. From a similar discussion, we see that $\pi\sigma(2)i$ is nontrivial and not a multiple of $\beta_1^{p-2}\kappa_1$, i.e., $\pi_{(2p^2+p+1)q-3}(S; p)=Z_p+Z_p$ is generated by $\pi\sigma(2)i$ and $\beta_1^{p-2}\kappa_1$. Furthermore one of the authors has proved in Part II of [14] (this journal 331-342) the following result: *the element* $\pi\sigma(2)\alpha^{j-1}i$, $2\leqq j\leqq p-2$, *is nontrivial and generates* $\pi_{(2p^2+p+j)q-3}(Sp)$, *and the relations* $\rho_j\alpha_k=ka\rho'_{j+k}$ (mod $\beta_1^{p-2}\kappa_1$ *if* $j+k=1$), $j\geqq 0$, fc^1, *hold, where* $\rho'_j=\pi\sigma(2)\alpha^{j-1}i$ *for* $1\leqq j\leqq p-2$, $=0$ *for* $j\geqq p-1$ flttd *the coefficient* $a\in Z_p$ *is independent of* $j$ *and* $k$.

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