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Stability of Difference Schemes for Nonsymmetric Linear Hyperbolic Systems with Variable Coefficients

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1. Introduction

Let us consider the Cauchy problem for a hyperbolic system

$$\frac{\partial u}{\partial t}(x, t) = \sum_{j=1}^{n} A_j(x) \frac{\partial u}{\partial x_j}(x, t) \qquad (0 \leqq t \leqq T,\ -\infty < x_j < \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad u_0(x) \in L_2, \tag{1.2}$$

where $u(x, t)$ and $u_0(x)$ are N-vectors and $A_j(x)$ $(j=1, 2,\dots, n)$ are $N \times N$ matrices, and assume that this problem is well posed. For the numerical solution of this problem we consider the difference scheme

$$v(x, t+k) = S_h(x, h)v(x, t) \qquad (0 \leqq t \leqq T,\ -\infty < x_j < \infty), \tag{1.3}$$

$$v(x, 0) = u_0(x), \qquad k = \lambda h, \tag{1.4}$$

and study the stability of the scheme in the sense of Lax-Richtmyer, where $S_h(x, h)$ is a difference operator and h is a space mesh width.

The stability of schemes for symmetric hyperbolic systems was studied by Lax [7], Lax and Wendroff [8, 9], Kreiss [5] and Parlett [12] in the case

$$S_h(x, h) = \sum_\alpha c_\alpha(x, h) T_h^\alpha, \tag{1.5}$$

where α is a multi-index, c_α is an $N \times N$ matrix and T_h is the translation operator.

The stability for nonsymmetric hyperbolic systems was treated first by Yamaguti and Nogi [20]. They defined a family of bounded linear operators in L_2 associated with an $N \times N$ matrix $k(x, \omega)$ which is homogeneous of degree zero in ω, is independent of x for $|x| \geqq R$ $(R > 0)$ and belongs to $C^\infty(R_x^n \times (R_\omega^n - \{0\}))$. They studied the properties of the algebra of such families and applied the results to the investigation of the stability of Friedrichs' scheme under the assumption: The system (1.1) is regularly hyperbolic and $A_j(x)$ $(j=1, 2,\dots, n)$ are independent of x for $|x| \geqq R$ and belong to C^∞. Under the same assumption, Vaillancourt [16, 17] obtained an improved stability condition for Friedrichs' scheme and a condition for the modified Lax-Wendroff scheme; Kametaka [4] treated the regularly hyperbolic systems with nearly constant coefficients.

In this paper we are concerned with the nonsymmetric hyperbolic systems that satisfy the conditions: Eigenvalues of $A(x, \xi)=\sum_{j=1}^{n} A_j(x)\xi_j/|\xi|$ $(\xi \neq 0)$ are all real and their multiplicities are independent of x and ξ; elementary divisors of $A(x, \xi)$ are all linear; there exists a constant $\delta>0$ such that

$$|\lambda_i(x, \xi)-\lambda_j(x, \xi)| \geqq \delta \qquad (i \neq j;\ i, j = 1, 2,\dots, n),$$

where $\lambda_i(x, \xi)$ $(i=1, 2,\dots, s)$ are all the distinct eigenvalues of $A(x, \xi)$.

We consider the case where $S_h(x, h)$ is a sum of products of operators of the form (1.5). Our proof of stability is based on the following result: If $S_h(x, h)$ and $S_h(x, 0)$ are the families of bounded linear operators in L_2 and if there exist positive constants c_0 and c_1 and a norm $|||\cdot|||$ equivalent to the L_2-norm $\|\cdot\|$ such that

$$|||S_h(x, 0)u||| \leqq (1+c_0 h)\, |||u|||\,, \tag{1.6}$$

$$\|(S_h(x, h)-S_h(x, 0))u\| \leqq c_1 h\|u\| \qquad \text{for all} \quad u \in L_2, \quad h > 0, \tag{1.7}$$

then the scheme (1.3) is stable.

To construct such a norm $|||\cdot|||$, after Friedrichs [3] and Kumano-go [6] we introduce a family of bounded linear operators in L_2 associated with an $N \times N$ matrix $p(x, \omega)$ such that

$$p(x, \omega) = p_0(x, \omega)+p_\infty(\omega), \qquad \lim_{|x|\to\infty} p_0(x, \omega) = 0 \qquad \text{for each} \quad \omega \in R^n$$

and the Fourier transform of $p_0(x, \omega)$ with respect to x satisfies some conditions. We construct an algebra $\mathscr{K}_h$ of such families and show an analogue of Lax-Nirenberg Theorem [10] for elements of $\mathscr{K}_h$ in order to obtain sufficient conditions under which (1.6) holds.

Taking the properties of $\mathscr{K}_h$ into consideration, in Section 5 we construct an algebra of difference operators $S_h(x, h)$ for which (1.7) holds and in Section 6 the stability of the schemes with elements of this algebra is studied. For instance Vaillancourt's result is valid under the assumption:

$$A_j(x) = A_{j0}(x)+A_{j\infty}, \quad \lim_{|x|\to\infty} A_{j0}(x) = 0 \qquad (j = 1, 2,\dots, n)$$

and $(\partial^m/\partial x_k^m) A_{j0}(x)$ $(j, k=1, 2,\dots, n;\ m=0, 1,\dots, n+3)$ are bounded, continuous and integrable.

In Section 7 some examples of the schemes are given. Lemmas and theorems stated without proof are proved in the last section.

2. Notations and preliminaries

2.1. Notations

Let $\boldsymbol{C}$ be the field of complex numbers. Let $\bar{c}$ and c^* stand for the conjugate and the conjugate transpose of a matrix c respectively. We denote by $|a|$, $|z|$ and $\rho(a)$ the spectral norm of an $N \times N$ matrix a, the Euclidean norm of an N-vector z and the spectral radius of a respectively. For any hermitian matrices a and b we use the notation $a \geqq b$ if $a-b$ is positive semi-definite.

We denote by R^n the real n-space and write it as R^n_x, R^n_ω, etc. to specify its space variables. Unless otherwise stated, we denote by $u(x)$, $\varphi(x)$, etc. the N-vector functions defined on R^n.

The space L_p ($p \geqq 1$) consists of all measurable functions $u(x)$ in R^n such that $|u(x)|^p$ is integrable, i.e. $\int |u(x)|^p dx < \infty$, where two functions are identified if they coincide almost everywhere. The scalar product and the norm in L_2 are denoted by $(\ ,\ )$ and $\|\cdot\|$ respectively.

Let $\mathscr{S}$ be the space of all C^∞ functions on R^n which, together with all their derivatives, decrease faster than any negative power of $|x|$ as $|x| \to \infty$. We denote by $\hat{u}(\xi)$ ($\xi \in R^n$) the Fourier transform of $u(x)$. For each $\varphi(x)$ in $\mathscr{S}$, $\hat{\varphi}(\xi)$ can be written as follows:

$$\hat{\varphi}(\xi) = \kappa \int e^{-ix\cdot\xi} \varphi(x) dx \qquad \text{for all} \quad \varphi \in \mathscr{S},$$

where

$$\kappa = (2\pi)^{-n/2}, \qquad x \cdot \xi = \sum_{j=1}^n x_j \xi_j. \tag{2.1}$$

We denote by $\hat{p}(\xi, \omega)$ the Fourier transform of $p(x, \omega)$ with respect to x and by $a*b(x)$ the convolution $\int a(x-t)b(t)dt$ of two measurable functions $a(x)$ and $b(x)$.

For simplicity we make use of the notations

$$D_l = \frac{\partial}{\partial x_l}, \ \partial_j = \frac{\partial}{\partial \omega_j}.$$

We denote by $\sup_{\omega \neq 0} u(x, \omega)$ and $\sup_{\omega \neq z} u(x, \omega)$ the supremum of $u(x, \omega)$ on $R^n_\omega - \{0\}$ and that on $R^n_\omega - Z$ for each fixed x in R^n respectively.

Let S^{n-1} be the unit spherical surface in R^n_ω, and let $\omega' = (\omega'_1, \omega'_2, \dots, \omega'_n)$ denote a point on S^{n-1}. Then we have $|\omega'| = 1$.

We say that $l(\chi, \omega)$ is absolutely continuous with respect to ω_k, if it is so on any finite closed interval for each fixed χ and ω_j ($j = 1, 2, \dots, n$; $j \neq k$). We say that a scalar function $c(x, \omega)$ satisfies conditions imposed on matrix functions, if $c(x, \omega)I$ does.

2.2. The difference approximations

We consider a mesh imposed on (x, t)-space with a spacing of h in each x_j-direction $(j=1, 2,\dots, n)$ and a spacing of k in the t-direction. The ratio $\lambda=k/h$ is to be kept constant as h varies. We approximate (1.1) and (1.2) by the difference scheme of the form:

$$v(x, t+k) = S_h(x, h)v(x, t) \qquad (0 \leqq t \leqq T), \tag{2.2}$$

$$v(x, 0) = u_0(x), \tag{2.3}$$

where

$$S_h(x, h) = \textstyle\sum_m \prod_{j=1}^{\nu} C_{m_j}(x, h, T), \quad m = (m_1, m_2,\dots, m_\nu), \tag{2.4}$$

$$C_{m_j}(x, h, T) = \textstyle\sum_\alpha c_{\alpha m_j}(x, h)T_h^\alpha, \quad \alpha = (\alpha_1, \alpha_2,\dots, \alpha_n), \tag{2.5}$$

$$T_h^\alpha = T_{1h}^{\alpha_1} T_{2h}^{\alpha_2} \cdots T_{nh}^{\alpha_n}, \; T_{jh}u(x) = u(x_1,\dots, x_{j-1}, x_j+h, x_{j+1},\dots, x_n), \tag{2.6}$$

m_j ($m_j \geqq 0$; $j=0, 1,\dots, \nu$) and α_j ($j=1, 2,\dots, n$) are integers and $c_{\alpha m_j}(x, h)$'s are $N \times N$ matrices.

We approximate the partial differential operator hD_j ($1 \leqq j \leqq n$) by the difference operator Δ_{jh} of the form

$$\Delta_{jh} = \textstyle\sum_l b_l(T_{jh}^l - T_{jh}^{-l})/2, \tag{2.7}$$

where the summation is over a finite set of l ($l \geqq 0$) and b_l's are real constants. We put

$$s_j(\omega) = \textstyle\sum_l b_l \sin l\omega_j \qquad (j = 1, 2,\dots, n), \tag{2.8}$$

$$s(\omega) = (s_1(\omega), s_2(\omega),\dots, s_n(\omega)),$$

and assume that for some positive integer r $s_j(\omega)$ can be written as follows:

$$s_j(\omega) = \omega_j + O(|\omega_j|^{r+1}) \qquad (|\omega_j| \leqq \pi). \tag{2.9}$$

From (2.9) it follows that for all $u \in \mathscr{S}$

$$\Delta_{jh}u(x) = hD_j u(x) + O(h^{r+1}) \qquad \text{as} \quad h \to 0 \quad (j = 1, 2,\dots, n).$$

For example the following difference operators are well known:

$$F_h(x) = C_h + \lambda P_h, \tag{2.10}$$

$$M_h(x) = I + \lambda P_h(C_h + \lambda P_h/2), \tag{2.11}$$

where

$$P_h = \sum_{j=1}^{n} A_j(x)\Delta_{jh}, \quad C_h = (1/n)\sum_{j=1}^{n}(T_{jh}+T_{jh}^{-1})/2, \tag{2.12}$$
$$\Delta_{jh} = (T_{jh}-T_{jh}^{-1})/2 \qquad (j = 1, 2,\dots, n).$$

The schemes (2.2) with operators (2.10) and (2.11) are called Friedrichs' scheme and the modified Lax-Wendroff scheme respectively.

We say that the difference scheme (2.2) approximates (1.1) with accuracy of order p [13, 15] if all smooth solutions u of (2.1) satisfy

$$|u(x, t+k)-S_h(x, h)u(x, t)| = O(h^{p+1}) \qquad (h \to 0). \tag{2.13}$$

In the sequel we consider only the schemes with $p\geqq 1$.

The difference scheme is said to be stable in the sense of Lax-Richtmyer if for any $T>0$ there exists a constant $M(T)$ such that

$$\|S_h^\nu u\| \leqq M(T)\,\|u\| \tag{2.14}$$

for all $u\in L_2$ and for all $h>0$ and integers $\nu\geqq 0$ satisfying $0\leqq \nu k\leqq T$, where $M(T)$ is a function of T but is independent of h. Since S_h is a family of bounded linear operators in L_2 depending on h, we have to investigate the boundedness of powers of such families of operators.

Let $\mathscr{H}_h$ be the set of all families of bounded linear operators H_h that maps L_2 into itself and depends on a parameter $h>0$ and such that

$$\|H_h u\| \leqq c(h)\,\|u\| \qquad \text{for all} \quad u\in L_2, \quad h>0, \tag{2.15}$$

where $c(\mu)$ is a continuous function on $[0, \infty)$.

For two families K_h and L_h of $\mathscr{H}_h$ we use the notation $K_h\equiv L_h$ if there exists a constant c such that

$$\|(K_h-L_h)u\| \leqq ch\|u\| \qquad \text{for all} \quad u\in L_2, \quad h>0. \tag{2.16}$$

Then we have the following

THEOREM 2.1. *Let $L_h\in\mathscr{H}_h$ and suppose there exist a constant c_0 and a norm $|||\cdot|||$ equivalent to the L_2-norm such that*

$$|||L_h u||| \leqq (1+c_0 h)\,|||u||| \qquad \textit{for all} \quad u\in L_2, \quad h>0. \tag{2.17}$$

Then for any $T>0$ there exists a constant $M(T)$ such that

$$\|L_h^\nu u\| \leqq M(T)\,\|u\| \qquad \textit{for all} \quad u\in L_2, \quad 0\leqq \nu k\leqq T. \tag{2.18}$$

PROOF. By the assumption there exist positive constants c_1 and c_2 such that

$$(2.19) \qquad c_1\|u\| \leqq |||u||| \leqq c_2\|u\| \qquad \text{for all} \quad u \in L_2.$$

From (2.17) it follows that

$$|||L_h^\nu u||| \leqq (1+c_0h)^\nu |||u||| \qquad \text{for all} \quad u \in L_2, \quad h>0,$$

so that by (2.19) we have

$$c_1\|L_h^\nu u\| \leqq |||L_h^\nu u||| \leqq c_3|||u||| \leqq c_2c_3\|u\|,$$

where $c_3 = \exp(c_0T/\lambda)$. From this (2.18) follows with $M = c_2c_3/c_1$.

COROLLARY 2.1. *For any $S_h \in \mathscr{H}_h$ let $L_h \in \mathscr{H}_h$ be a family such that $L_h \equiv S_h$ and which satisfies the assumption of the theorem. Then for any $T>0$ there exists a constant $M(T)$ such that*

$$(2.20) \qquad \|S_h^\nu u\| \leqq M(T)\|u\| \qquad \textit{for all} \quad u \in L_2, \quad 0 \leqq \nu k \leqq T.$$

PROOF. Since for some constant c_4

$$\|(L_h - S_h)u\| \leqq c_4h\|u\| \qquad \text{for all} \quad u \in L_2, \quad h>0,$$

by (2.17) and (2.19) we have

$$\begin{aligned} |||S_hu||| &\leqq |||L_hu||| + |||(S_h - L_h)u||| \\ &\leqq |||L_hu||| + c_2c_4h\|u\| \\ &\leqq (1+c_5h)\,|||u|||, \end{aligned}$$

where $c_5 = c_0 + c_2c_4/c_1$. Hence (2.17) is satisfied and (2.20) follows from the theorem.

By Theorem 2.1 and its corollary, in proving the stability of the scheme (2.2), the problem is to find a norm $|||\cdot|||$ and a family $L_h \in \mathscr{H}_h$ such that $L_h \equiv S_h(x, h)$ in order to establish (2.17).

Now we study the algebraic structure of $\mathscr{H}_h$. For $A_h, B_h \in \mathscr{H}_h$ and $\alpha \in \boldsymbol{C}$ let $A_h + B_h$, A_hB_h and αA_h be defined by

$$(A_h+B_h)u = A_hu + B_hu, \quad (A_hB_h)u = A_h(B_hu), \quad (\alpha A_h)u = \alpha(A_hu).$$

Then $\mathscr{H}_h$ is an algebra over $\boldsymbol{C}$ with unit element I_h. Since the adjoint A_h^* of a family A_h also belongs to $\mathscr{H}_h$, the operation * is an involution in $\mathscr{H}_h$ and $\mathscr{H}_h$ is an algebra with involution [2].

3. One-parameter families of operators

3.1. Definitions

We introduce the set $\mathscr{K}$ consisting of all $N\times N$ matrix functions $p(x, \omega)$ defined on $R_x^n\times R_\omega^n$ with the properties:

1) $p(x, \omega)$ can be written as

$$p(x, \omega) = p_0(x, \omega)+p_\infty(\omega),$$

where $p_0(x, \omega)$ and $p_\infty(\omega)$ are bounded and measurable on $R_x^n\times R_\omega^n$ and on R_ω^n respectively, and $\lim_{|x|\to\infty} p_0(x, \omega)=0$ for each $\omega\in R^n$;

2) $p_0(x, \omega)$ is integrable as a function of x for each $\omega\in R^n$;

3) The Fourier transform $\hat{p}_0(\chi, \omega)$ of $p_0(x, \omega)$ is integrable as a function of χ for each $\omega\in R^n$ and $\operatorname{ess.sup}_\omega|\hat{p}_0(\chi, \omega)|$ is integrable.

(Two elements of $\mathscr{K}$ are identified if they coincide almost everywhere.)

The element $p(x, \omega)$ of $\mathscr{K}$ has the Fourier transform $\hat{p}(\chi, \omega)$ in the sense of distributions, which can be written as follows:

$$\hat{p}(\chi, \omega) = \hat{p}_0(\chi, \omega)+\delta(\chi)p_\infty(\omega), \tag{3.1}$$

where $\delta(\chi)$ is the delta function. We define $\|\hat{p}\|_F$ by

$$\|\hat{p}\|_F = \int \operatorname{ess.sup}_\omega |\hat{p}_0(\chi, \omega)|d\chi+\operatorname{ess.sup}_\omega |p_\infty(\omega)|. \tag{3.2}$$

In the following for simplicity we often omit x, ω and χ from $p(x, \omega)$, $\hat{p}(\chi, \omega)$, $u(x)$, $u(\omega)$, etc., when no confusion can arise.

We introduce into $\mathscr{K}$ matrix addition, matrix multiplication, scalar multiplication and conjugate transposition. Then we have

LEMMA 3.1. *If $p, q\in\mathscr{K}$ and $\alpha\in\boldsymbol{C}$, then $p+q$, pq, αp, $p^*\in\mathscr{K}$ and*

$$\|\widehat{p+q}\|_F \leqq \|\hat{p}\|_F+\|\hat{q}\|_F, \quad \|\widehat{\alpha p}\|_F = |\alpha|\,\|\hat{p}\|_F, \quad \|\widehat{p^*}\|_F = \|\hat{p}\|_F, \tag{3.3}$$

$$\|\widehat{pq}\|_F \leqq \|\hat{p}\|_F\|\hat{q}\|_F. \tag{3.4}$$

PROOF. It suffices to show that $pq\in\mathscr{K}$ and (3.4) holds. Put $d=pq$. Then d can be written as $d=d_0+d_\infty$, where

$$d_0 = p_0q_0+p_0q_\infty+p_\infty q_0, \quad d_\infty = p_\infty q_\infty.$$

By definition d satisfies conditions 1) and 2) of $\mathscr{K}$, and $\hat{d}_0(\chi, \omega)$ can be written as

$$\hat{d}_0(\chi, \omega) = \hat{p}_0*\hat{q}_0+\hat{p}_0q_\infty+p_\infty\hat{q}_0. \tag{3.5}$$

Since

$$|\hat{d}_0(\chi, \omega)| \leqq |\hat{p}_0 * \hat{q}_0| + |\hat{p}_0|\,|q_\infty| + |p_\infty|\,|\hat{q}_0|, \tag{3.6}$$

integrating (3.6) with respect to χ and applying Young's Theorem, we have

$$\int|\hat{d}_0(\chi, \omega)|d\chi \leqq \int|\hat{p}_0(\chi, \omega)|d\chi\int|\hat{q}_0(\chi, \omega)|d\chi$$

$$+|q_\infty(\omega)|\int|\hat{p}_0(\chi, \omega)|d\chi + |p_\infty(\omega)|\int|\hat{q}_0(\chi, \omega)|d\chi.$$

Hence $\hat{d}_0(\chi, \omega)$ is integrable as a function of χ for each ω.

Taking the essential suprema of both sides of (3.6) over R^n_ω and integrating them with respect to χ, we have

$$\|\hat{d}_0\|_F \leqq \|\hat{p}_0\|_F\|\hat{q}_0\|_F + (\operatorname{ess\,sup}_\omega |q_\infty|)\,\|\hat{p}_0\|_F + (\operatorname{ess\,sup}_\omega |p_\infty|)\,\|\hat{q}_0\|_F.$$

Therefore d_0 satisfies condition 3) of $\mathscr{K}$ and the proof is complete.

By this lemma $\mathscr{K}$ forms an algebra with involution over $\boldsymbol{C}$.

To define a family of operators associated with $p \in \mathscr{K}$, we show the following

LEMMA 3.2. *Let $p \in \mathscr{K}$ and $u \in \mathscr{S}$. Then*

$$\left\|\int\hat{p}(\xi-\xi', h\xi')\hat{u}(\xi')d\xi'\right\| \leqq \|\hat{p}\|_F\|\hat{u}\| \qquad \textit{for} \quad h>0, \tag{3.7}$$

and for almost all x

$$\text{l.i.m.}\,\kappa^{-1}\int e^{ix\cdot\xi}\int\hat{p}(\xi-\xi', h\xi')\hat{u}(\xi')d\xi'd\xi \tag{3.8}$$

$$= \kappa^{-1}\int e^{ix\cdot\xi}p(x, h\xi)\hat{u}(\xi)d\xi \qquad \textit{for} \quad h>0.$$

PROOF. For simplicity put

$$r_0(\chi) = \operatorname{ess\,sup}_\omega |\hat{p}_0(\chi, \omega)|, \quad r_\infty = \operatorname{ess\,sup}_\omega |p_\infty(\omega)|,$$

$$v(\xi, h) = \int\hat{p}(\xi-\xi', h\xi')\hat{u}(\xi')\,d\xi'.$$

Then for almost all ξ

$$|v(\xi, h)| \leqq r_\infty|\hat{u}(\xi)| + \int r_0(\xi-\xi')\,|\hat{u}(\xi')|d\xi'. \tag{3.9}$$

Integrating (3.9) with respect to ξ and changing the order of integration, we have

$$\int |v(\xi, h)| d\xi \leqq \|\hat{p}\|_F \int |\hat{u}(\xi)| d\xi \quad \text{for} \quad h > 0. \tag{3.10}$$

Since by Young's Theorem

$$\left\| \int r_0(\xi - \xi') \, |\hat{u}(\xi')| \, d\xi' \right\| \leqq \int r_0(\chi) d\chi \|\hat{u}\|,$$

from (3.9) it follows that

$$\|v\| \leqq r_\infty \|\hat{u}\| + \int r_0(\chi) d\chi \|\hat{u}\| = \|\hat{p}\|_F \|\hat{u}\|,$$

which shows (3.7).

By (3.7) and (3.10) $v(\xi, h)$ belongs to L_1 and to L_2 as a function of ξ for each fixed $h>0$. Therefore the inverse Fourier transform of $v(\xi, h)$ in L_1 and that in L_2 coincide almost everywhere on R^n_x and

$$\text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} v(\xi, h) d\xi = \kappa^{-1} \int e^{ix\cdot\xi} v(\xi, h) d\xi$$

for almost all x. By the change of order of integration we have for almost all x

$$\kappa^{-1} \int e^{ix\cdot\xi} v(\xi, h) d\xi = \kappa^{-1} \int e^{ix\cdot\xi} p(x, h\xi) \hat{u}(\xi) d\xi.$$

Thus (3.8) holds and the proof is complete.

With each $p \in \mathscr{K}$ we associate a one-parameter family of operators P_h by the formula:

$$P_h u(x) = \text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} \int \hat{p}(\xi - \xi', h\xi') \hat{u}(\xi') d\xi' d\xi \tag{3.11}$$

$$\text{for all} \quad u \in \mathscr{S}, \quad h > 0.$$

Then by (3.7) P_h is a family of bounded linear operators from $\mathscr{S}$ into L_2. Hence it can be extended to the closure $\bar{\mathscr{S}} = L_2$ with preservation of norm and the extension is unique. Denoting this extension of P_h again by P_h, we call P_h the family (of operators) associated with p and denote this mapping by ϕ i.e. $P_h = \phi(p)$. Unless otherwise stated, we denote by Q_h, $\tilde{L}_h$, $\overline{W}_h$, etc. the families associated with q, $\tilde{l}$, w^{-1}, etc. respectively.

We note that by (3.8) $P_h u$ $(u \in \mathscr{S})$ can be rewritten as follows:

$$P_h u(x) = \kappa^{-1} \int e^{ix\cdot\xi} p(x, h\xi) \hat{u}(\xi) d\xi \quad \text{for all} \quad u \in \mathscr{S}, \quad h > 0. \tag{3.12}$$

Let $\mathscr{K}_h = \phi(\mathscr{K})$. Then we have

LEMMA 3.3. *The mapping ϕ is one-to-one.*

PROOF. Suppose for some $p \in \mathscr{K}$

$$P_h v = 0 \qquad \text{for all} \quad v \in \mathscr{S}.$$

Then by (3.12) for almost all x

$$\int e^{ix\cdot\xi} p(x, h\xi)\hat{v}(\xi)d\xi = 0 \qquad \text{for all} \quad v \in \mathscr{S}, \quad h > 0.$$

Since for each $w(\xi) \in \mathscr{S}$ the inverse Fourier transform of $w(\xi)$ belongs to $\mathscr{S}$, it follows that for almost all x

$$\int e^{ix\cdot\xi} p(x, h\xi)w(\xi)d\xi = 0 \qquad \text{for all} \quad w \in \mathscr{S}, \quad h > 0.$$

Put $r(\xi) = \prod_{j=1}^{n}(1+\xi_j^2)^{-1}$. Then for almost all x

$$\int e^{ix\cdot\xi} p(x, h\xi)r(\xi)u(\xi)d\xi = 0 \qquad \text{for all} \quad u \in \mathscr{S},$$

because $r(\xi)u(\xi) \in \mathscr{S}$. Since $p(x, \omega)$ is bounded, $p(x, h\xi)r(\xi)$ belongs to L_1 as a function of ξ for almost all x. Hence for almost all (x, ξ)

$$p(x, h\xi) = 0 \qquad \text{for} \quad h > 0,$$

so that $p(x, \omega) = 0$ a.e., which completes the proof.

For $\phi(p), \phi(q) \in \mathscr{K}_h$ and $\alpha \in \boldsymbol{C}$ let

$$\phi(p) + \phi(q) = \phi(p+q), \quad \phi(p)\circ\phi(q) = \phi(pq),$$
$$\phi(p)^{\#} = \phi(p^*), \quad \alpha\phi(p) = \phi(\alpha p).$$

Then $\mathscr{K}_h$ forms a unitary algebra over $\boldsymbol{C}$ with respect to the operations $+$ and $\circ$, and the operation $^{\#}$ is an involution in $\mathscr{K}_h$. It is readily seen that $\mathscr{K}_h$ is an algebra with involution and the mappings ϕ and ϕ^{-1} are morphisms [1].

3.2. Products and adjoints

To study the relations between the products $P_h Q_h$ and $P_h \circ Q_h$ we introduce the following two conditions.

CONDITION I. 1) $p \in \mathscr{K}$;

2) $\hat{p}_0(\chi, \omega)$ and $p_\infty(\omega)$ are absolutely continuous with respect to ω_j $(j=1, 2,\dots, n)$ and $\partial_j \hat{p}_0(\chi, \omega)$ and $\partial_j p_\infty(\omega)$ are measurable in $R_\chi^n \times R_\omega^n$ and in R_ω^n respec-

tively;

3) $\operatorname{ess\,sup}_{\omega} |\partial_j \hat{p}_0(\chi, \omega)|$ $(j=1, 2,\dots, n)$ are integrable and $\operatorname{ess\,sup}_{\omega} |\partial_j p_\infty(\omega)|$ $(j=1, 2,\dots, n)$ are finite.

CONDITION II. $q \in \mathscr{K}$ and $\operatorname{ess\,sup}_{\omega} (|\chi|\,|\hat{q}_0(\chi, \omega)|)$ is integrable.

We have

THEOREM 3.1. *Let p satisfy Condition* I *and q satisfy Condition* II. *Then*

$$P_h Q_h \equiv P_h \circ Q_h. \tag{3.13}$$

PROOF. By continuity of the L_2-norm it suffices to prove the theorem in the case $u \in \mathscr{S}$. From the definition of $P_h Q_h$ it follows that

$$\begin{aligned}\widehat{P_h Q_h u}(\xi) &= \widehat{P_h(Q_h u)}(\xi) \\ &= \iint \hat{p}_0(\xi-\eta,\, h\eta)\hat{q}_0(\eta-\xi',\, h\xi')\hat{u}(\xi')d\xi' d\eta \\ &\quad + \int p_\infty(h\xi)\hat{q}_0(\xi-\xi',\, h\xi')\hat{u}(\xi')d\xi' + w(\xi),\end{aligned}$$

where

$$w(\xi) = \int \hat{p}_0(\xi-\xi',\, h\xi')q_\infty(h\xi')\hat{u}(\xi')d\xi' + p_\infty(h\xi)q_\infty(h\xi)\hat{u}(\xi).$$

Changing the order of integration and setting $t=\eta-\xi'$, we have

$$\begin{aligned}\widehat{P_h Q_h u}(\xi) &= \iint \hat{p}_0(\chi-t,\, \omega+ht)\hat{q}_0(t,\, \omega)\hat{u}(\xi')dtd\xi' \\ &\quad + \int p_\infty(\omega+h\chi)\hat{q}_0(\chi,\, \omega)\hat{u}(\xi')d\xi' + w(\xi),\end{aligned} \tag{3.14}$$

where $\chi=\xi-\xi'$, $\omega=h\xi'$.

Since $P_h \circ Q_h$ is a family associated with pq,

$$\begin{aligned}\widehat{P_h \circ Q_h u}(\xi) &= \iint \hat{p}_0(\chi-t,\, \omega)\hat{q}_0(t,\, \omega)\hat{u}(\xi')dtd\xi' \\ &\quad + \int p_\infty(\omega)\hat{q}_0(\chi,\, \omega)\hat{u}(\xi')d\xi' + w(\xi),\end{aligned} \tag{3.15}$$

where $\chi=\xi-\xi'$, $\omega=h\xi'$. Comparison of (3.14) and (3.15) shows that the proof is complete by the first part of Lemma 3.2, if

$$\int \operatorname{ess\,sup}_{\omega} \Big| \int \{\hat{p}_0(\chi-t,\, \omega+ht) - \hat{p}_0(\chi-t,\, \omega)\}\hat{q}_0(t,\, \omega)dt \Big| d\chi = O(h), \tag{3.16}$$

$$\int \operatorname{ess\cdot sup}_{\omega} |\{p_\infty(\omega+h\chi)-p_\infty(\omega)\}\hat{q}_0(\chi,\omega)|d\chi = O(h). \tag{3.17}$$

Since $p_0(\chi, \omega)$ is absolutely continuous with respect to ω_j, we have

$$\begin{aligned}
&|\{\hat{p}_0(\chi-t,\omega+ht)-\hat{p}_0(\chi-t,\omega)\}\hat{q}_0(t,\omega)| \\
&\quad = |\textstyle\sum_{j=1}^n\{\hat{p}_0(\chi-t,\omega_1,\dots,\omega_{j-1},\omega_j+\theta_j,\omega_{j+1}+\theta_{j+1},\dots,\omega_n+\theta_n) \\
&\qquad -\hat{p}_0(\chi-t,\omega_1,\dots,\omega_j,\omega_{j+1}+\theta_{j+1},\dots,\omega_n+\theta_n)\}\hat{q}_0(t,\omega)| \\
&\quad = |\textstyle\sum_{j=1}^n\int_0^{\theta_j}\partial_j\hat{p}_0(\chi-t,\omega_1,\dots,\omega_{j-1},\omega_j+\zeta_j,\omega_{j+1}+\theta_{j+1},\dots, \\
&\qquad\qquad \omega_n+\theta_n)d\zeta_j\hat{q}_0(t,\omega)|,
\end{aligned}$$

where $\theta_j = ht_j$. Taking the essential suprema of both sides over R^n_ω and integrating them with respect to χ, we have

$$\begin{aligned}
&\iint \operatorname{ess\cdot sup}_{\omega}|\{\hat{p}_0(\chi-t,\omega+ht)-\hat{p}_0(\chi-t,\omega)\}\hat{q}_0(t,\omega)|d\chi dt \\
&\quad \leqq \iint \textstyle\sum_{j=1}^n \operatorname{ess\cdot sup}_{\omega}(|\partial_j\hat{p}_0(\chi-t,\omega)|)h|t_j| \operatorname{ess\cdot sup}_{\omega}(|\hat{q}_0(t,\omega)|)d\chi dt.
\end{aligned}$$

Hence (3.16) follows by I–3) and II.[1] Similarly we have (3.17).

From the proof of this theorem we have

COROLLARY 3.1. *If $a(x)$, $b(\omega)$, $p(x,\omega)\in\mathscr{K}$, then*

$$A_hP_h = A_h\circ P_h, \tag{3.18}$$

$$P_hB_h = P_h\circ B_h. \tag{3.19}$$

To study the relations between the adjoint P_h^* of P_h and the family $P_h^\sharp$ we introduce

CONDITION III. 1) $p\in\mathscr{K}$;

2) $\hat{p}_0(\chi,\omega)$ is absolutely continuous with respect to ω_j $(j=1,2,\dots,n)$ and $\partial_j\hat{p}_0(\chi,\omega)$ $(j=1,2,\dots,n)$ are measurable in $R^n_\chi\times R^n_\omega$;

3) $\operatorname{ess\cdot sup}_{\omega}(|\chi_j|\,|\partial_j\hat{p}_0(\chi,\omega)|)$ $(j=1,2,\dots,n)$ are integrable.

THEOREM 3.2. *Let $p\in\mathscr{K}$. Then*

$$P_h^\sharp u(x) = \text{l.i.m.}\,\kappa^{-1}\int e^{ix\cdot\xi}\int\widehat{p^*}(\xi-\xi', h\xi)\hat{u}(\xi')d\xi'd\xi \tag{3.20}$$

for all $u\in\mathscr{S}$, $h>0$.

1) The term Condition is often omitted when no confusion can arise.

If p satisfies Condition III, *then*

$$P_h^* \equiv P_h^\sharp. \tag{3.21}$$

PROOF. Since $\widehat{p^*}(\xi-\xi', h\xi)=\widehat{p^*}(\xi-\xi', h\xi'+h(\xi-\xi'))$, by the same argument as in the proof of Lemma 3.2 we have for $w \in \mathscr{S}$

$$\left\|\int \widehat{p^*}(\xi-\xi', h\xi)\hat{w}(\xi')d\xi'\right\| \leqq \|\widehat{p^*}\|_F \|\hat{w}\|. \tag{3.22}$$

For u, $w \in \mathscr{S}$

$$\begin{aligned}(u, P_h^* w) &= (P_h u, w) = (\widehat{P_h u}, \hat{w}) \\ &= \int\left\{\int \hat{p}(\xi-\xi', h\xi')\hat{u}(\xi')d\xi'\right\}^* \hat{w}(\xi)d\xi \\ &= \iint \hat{u}^*(\xi')\hat{p}^*(\xi-\xi', h\xi')\hat{w}(\xi)d\xi' d\xi \\ &= \iint \hat{u}^*(\xi')\widehat{p^*}(\xi'-\xi, h\xi')\hat{w}(\xi)d\xi d\xi'.\end{aligned}$$

From this (3.20) follows by (3.22).

It suffices to prove (3.21) in the case $u \in \mathscr{S}$. From (3.20) and the definition of $P_h^\sharp$ it follows that

$$\widehat{P_h^* u}(\xi)-\widehat{P_h^\sharp u}(\xi) = \int\{\widehat{p_0^*}(\chi, \omega+h\chi)-\widehat{p_0^*}(\chi, \omega)\}\hat{u}(\xi')d\xi', \tag{3.23}$$

where $\chi=\xi-\xi'$ and $\omega=h\xi'$. By III–2) we have

$$\begin{aligned}&|\widehat{p_0^*}(\chi, \omega+h\chi)-\widehat{p_0^*}(\chi, \omega)| \\ &\quad = |\textstyle\sum_{j=1}^n \displaystyle\int_0^{\theta_j} \partial_j\widehat{p_0^*}(\chi, \omega_1,\dots, \omega_{j-1}, \omega_j+\zeta_j, \omega_{j+1}+\theta_{j+1},\dots, \omega_n+\theta_n)d\zeta_j|,\end{aligned}$$

where $\theta_j=h\chi_j$. Taking the essential suprema of both sides over R_ω^n and integrating them with respect to χ, we find

$$\int \operatorname*{ess.sup}_{\omega} |\widehat{p_0^*}(\chi, \omega+h\chi)-\widehat{p_0^*}(\chi, \omega)|d\chi \leqq h\textstyle\sum_{j=1}^n \displaystyle\int \operatorname*{ess.sup}_{\omega}(|\chi_j|\,|\partial_j\widehat{p_0^*}(\chi, \omega)|)d\chi.$$

Hence (3.21) holds by III–3) and Lemma 3.2.

From (3.23) we have

COROLLARY 3.2. *If $k(\omega) \in \mathscr{K}$, then*

$$K_h^* = K_h^\sharp. \tag{3.24}$$

3.3. Construction of a new norm

We construct a norm which is equivalent to the L_2-norm and is useful for establishing (2.17).

Let ε and R $(R \geqq \varepsilon)$ be positive numbers and let $S(R, \varepsilon)=\{x \mid |x|<R+\varepsilon\}$. Let $x^{(i)}$ $(i=1, 2,\dots, s)$ be all the lattice-points $(\varepsilon\eta_1, \varepsilon\eta_2,\dots, \varepsilon\eta_n)$ contained in $S(R, \varepsilon)$ $(\eta_j=m_j/\sqrt{n};\ m_j=0, \pm 1, \pm 2,\dots;\ j=1, 2,\dots, n)$ and let

$$V_0 = \{x \mid |x| > R\}, \quad V_i = \{x \mid |x-x^{(i)}| < \varepsilon\} \quad (i = 1, 2,\dots, s).$$

Then we can construct a partition of unity $\{\alpha_i^2(x)\}_{i=0,1,\dots,s}$ with the properties:

1) $\alpha_i(x) \geqq 0$, $\alpha_i(x) \in C^\infty$, $\operatorname{supp}\alpha_i(x) \subset V_i$ $(i = 0, 1,\dots, s)$;

2) $\sum_{i=0}^{s} \alpha_i^2(x)=1$;

3) $\alpha_0(x)$ and all its partial derivatives are bounded uniformly with respect to R for each ε.

We introduce the following

CONDITION N. 1) $g \in \mathscr{K}$ and $D_j g(x, \omega)$ $(j=1, 2,\dots, n)$ are bounded on $R_x^n \times R_\omega^n$ and continuous on R_x^n for each ω; $D_j g(x, \omega)$ $(j=1, 2,\dots, n)$ are integrable as functions of x for each ω; $\widehat{D_j g}(\chi, \omega)$ $(j=1, 2,\dots, n)$ are integrable as functions of χ for each ω and $\operatorname{ess\,sup}_{\omega} |\widehat{D_j g}(\chi, \omega)|$ $(j=1, 2,\dots, n)$ are integrable;

2) $\lim_{R\to\infty} \|\widehat{\alpha_0 g_0}\|_F = 0.$

We have

THEOREM 3.3. *Suppose*
1) *$g(x, \omega)$ satisfies Condition* N;
2) *$g(x, \omega) \geqq eI$ for some constant $e > 0$.*
Then for sufficiently large R and small ε there exist positive constants d_1 and d_2 such that

$$d_1^2\|u\|^2 \leqq \sum_{i=0}^{s} \operatorname{Re}(G_h\alpha_i u, \alpha_i u) \leqq d_2^2\|u\|^2 \tag{3.25}$$

for all $u \in L_2$, $h > 0$,

where d_j $(j=1, 2)$ are independent of u and h.

This theorem enables us to introduce the norm

$$\|u\|_{G_h} = \{\sum_{i=0}^{s} \operatorname{Re}(G_h\alpha_i u, \alpha_i u)\}^{1/2} \qquad \text{for all} \quad u \in L_2, \tag{3.26}$$

and by (3.25) we have

$$d_1\|u\| \leqq \|u\|_{G_h} \leqq d_2\|u\|. \tag{3.27}$$

LEMMA 3.4. *If p and q satisfy Condition* N, *so also do $p+q$, pq and p^*.*

PROOF. It suffices to prove the lemma in the case of pq. Put $d=pq$. Then d satisfies Condition N–1). Since

$$d_0 = p_0q_0 + p_0q_\infty + p_\infty q_0,$$

it follows that

$$\widehat{\alpha_0 d_0}(\chi, \omega) = \int \widehat{\alpha_0 p_0}(\chi - t, \omega)\hat{q}_0(t, \omega)dt$$

$$+\widehat{\alpha_0 p_0}(\chi, \omega)q_\infty(\omega) + p_\infty(\omega)\widehat{\alpha_0 q_0}(\chi, \omega).$$

Taking the essential suprema of both sides over R^n_ω and integrating them with respect to χ, we have by Young's Theorem

$$\|\widehat{\alpha_0 d_0}\|_F \leqq \|\widehat{\alpha_0 p_0}\|_F\|\hat{q}_0\|_F + \|\widehat{\alpha_0 p_0}\|_F\|q_\infty\|_F + \|p_\infty\|_F\|\widehat{\alpha_0 q_0}\|_F,$$

the right side of which tends to zero as $R\to\infty$ by N–2). Hence $\|\widehat{\alpha_0 d_0}\|_F\to 0$ as $R\to\infty$ and pq satisfies Condition N–2).

3.4. Lax-Nirenberg Theorem

We have the following analogue of Lax-Nirenberg Theorem [10] which plays an important role in establishing (2.17).

THEOREM 3.4. *Suppose $p\in\mathscr{K}$ satisfies the conditions:*

1) *$\partial_j\hat{p}_0(\chi, \omega)$ and $\partial_j p_\infty(\omega)$ $(j=1, 2,\dots, n)$ are continuous on R^n_ω for each χ and absolutely continuous with respect to ω_k $(k=1, 2,\dots, n)$;*

2) *$\partial_k\partial_j\hat{p}_0(\chi, \omega)$ and $\partial_k\partial_j p_\infty(\omega)$ $(j, k=1, 2,\dots, n)$ are measurable in $R^n_\chi \times R^n_\omega$ and in R^n_ω respectively; $\operatorname{ess.sup}_\omega(|\partial_k\partial_j\hat{p}_0(\chi, \omega)|)$ $(j, k=1, 2,\dots, n)$ are integrable and $\operatorname{ess.sup}_\omega(|\partial_k\partial_j p_\infty(\omega)|)$ $(j, k=1, 2,\dots, n)$ are finite;*

3) *$\operatorname{ess.sup}_\omega(|\chi|^2|\hat{p}_0(\chi, \omega)|)$ is integrable;*

4) *$p(x, \omega)\geqq 0$.*

Then there exists a positive constant c independent of u and h such that

$$\operatorname{Re}(P_h u, u) \geqq -ch\|u\|^2 \qquad \textit{for all} \quad u\in L_2, \quad h>0. \tag{3.28}$$

4. Powers of families of operators

4.1. The family of operators Λ_h

In this section $s(\omega)$ denotes a real-valued vector function with the properties:

1) $s_l(\omega)$, $\partial_j s_l(\omega)$ and $\partial_k \partial_j s_l(\omega)$ ($j, k, l=1, 2,\dots, n$) are bounded and continuous on R^n;

2) Zeros of $|s(\omega)|$ are isolated points.

(The function $s(\omega)$ given in 2.2 has these properties.)

Let $Z=\{\omega \mid |s(\omega)|=0\}$. Then $R^n_\omega - Z$ is an open set by continuity of $|s(\omega)|$ and by properties 1) and 2) $|s(\omega)|$ satisfies Condition I. Let Λ_h be the family associated with $|s(\omega)|I$. Then by Corollary 3.2 we have

$$\Lambda_h = \Lambda_h^{\#} = \Lambda_h^{*}.$$

Let $p(x, \omega)$ be an element of $\mathscr{K}$ such that $p(x, \omega)/|s(\omega)|$ is bounded on $R^n_x \times (R^n_\omega - Z)$. Then we seek sufficient conditions under which P_h can be written as $P_h = Q_h \circ \Lambda_h$ for some $Q_h \in \mathscr{K}_h$. For any constant α let

$$q_\alpha(x, \omega) = \begin{cases} p(x, \omega)/|s(\omega)| & \text{for} \quad \omega \in R^n - Z, \\ \alpha I & \text{for} \quad \omega \in Z, \end{cases}$$

and suppose $q_\alpha(x, \omega) \in \mathscr{K}$. Then

$$\begin{aligned} |\widehat{Q_{\alpha h} u}(\xi) - \widehat{Q_{\beta h} u}(\xi)| &= \left| \int \{\hat{q}_\alpha(\xi-\xi', h\xi') - \hat{q}_\beta(\xi-\xi', h\xi')\} \hat{u}(\xi') d\xi' \right| \\ &\leqq |q_{\alpha\infty}(h\xi) - q_{\beta\infty}(h\xi)|\, |\hat{u}(\xi)| \qquad \text{for all} \quad u \in \mathscr{S}, \end{aligned}$$

where $Q_{\alpha h}$ and $Q_{\beta h}$ are the families associated with q_α and q_β ($\beta \neq \alpha$) respectively. Since Z is a set of measure zero, for all $u \in \mathscr{S}$ we have for almost all ξ

$$|q_{\alpha\infty}(h\xi) - q_{\beta\infty}(h\xi)|\, |\hat{u}(\xi)| = 0.$$

Hence $Q_{\alpha h}$ and $Q_{\beta h}$ can be identified. In the following we identify $q_\alpha(x, \omega)$ and $q_\beta(x, \omega)$ and denote them by $p(x, \omega)/|s(\omega)|$. Then we have $P_h = P_{1h} \circ \Lambda_h$, where P_{1h} is the family associated with $p/|s|$.

When $e(\omega)$ is a scalar function with isolated zeros such that $e(\omega) I \in \mathscr{K}$, we can define $p(x, \omega)/e(\omega)$ similarly by replacing $|s(\omega)|$ by $e(\omega)$.

In particular let $r(\omega)$ be a scalar function such that $r(\omega) I \in \mathscr{K}$ and for some constant c_0

$$|r(\omega)| \leqq c_0 |s(\omega)| \qquad \text{for all} \quad \omega \in R^n.$$

Then $r(\omega)/|s(\omega)| \in \mathscr{K}$ and $R_h = R_{1h} \circ \Lambda_h$, where R_h and R_{1h} are the families associated with rI and $(r/|s|)I$ respectively.

To study the relation between $P_hQ_h\Lambda_h$ and $P_h\circ Q_h\circ\Lambda_h$ and that between $(P_h\Lambda_h)^*$ and $P_h^\#\circ\Lambda_h$, we introduce the following conditions:

CONDITION I′. 1) $p\in\mathscr{K}$;

2) $\hat{p}_0(\chi,\omega)$ is bounded on $R_\chi^n\times(R_\omega^n-Z)$;

3) $\partial_j l_0(\chi,\omega)$ and $\partial_j l_\infty(\omega)$ $(j=1,2,\dots,n)$ are bounded on $R_\chi^n\times(R_\omega^n-Z)$ and continuous on R_ω^n-Z for each χ, where $l_0(\chi,\omega)=\hat{p}_0|s|$, $l_\infty(\omega)=p_\infty|s|$;

4) $\operatorname{ess.sup}_\omega|\partial_j l_0|$ $(j=1,2,\dots,n)$ are integrable.

CONDITION III′. 1), 2) the same as I′−1), I′−2) respectively;

3) $\partial_j l_0(\chi,\omega)$ $(j=1,2,\dots,n)$ are bounded on $R_\chi^n\times(R_\omega^n-Z)$ and continuous on R_ω^n-Z for each χ;

4) $\operatorname{ess.sup}_\omega(|\chi_j|\,|\partial_j l_0(\chi,\omega)|)$ $(j=1,2,\dots,n)$ are integrable.

We have

LEMMA 4.1. (i) *If p satisfies Condition* I′, *then $p|s|$ satisfies Condition* I.

(ii) *If p satisfies Condition* III′, *then $p|s|$ satisfies Condition* III.

Next we prove

LEMMA 4.2. (i) *If p satisfies Condition* I′ *and q satisfies Condition* II, *then*

$$P_hQ_h\Lambda_h\equiv P_h\circ Q_h\circ\Lambda_h. \tag{4.1}$$

(ii) *If p satisfies Condition* III′, *then*

$$(P_h\Lambda_h)^*\equiv P_h^\#\circ\Lambda_h. \tag{4.2}$$

PROOF. The assertion (ii) follows from Lemma 4.1 and Theorem 3.2. By Theorem 3.1 and its corollary

$$\Lambda_h\circ Q_h\equiv\Lambda_hQ_h,\quad Q_h\Lambda_h=Q_h\circ\Lambda_h,\quad P_h\Lambda_h=P_h\circ\Lambda_h.$$

As $\Lambda_h\circ Q_h=Q_h\circ\Lambda_h$, we have $Q_h\Lambda_h\equiv\Lambda_hQ_h$, so that

$$P_hQ_h\Lambda_h\equiv P_h\Lambda_hQ_h=(P_h\circ\Lambda_h)Q_h.$$

Since $p|s|$ satisfies Condition I by Lemma 4.1, by Theorem 3.1 we have

$$(P_h\circ\Lambda_h)Q_h\equiv(P_h\circ\Lambda_h)\circ Q_h.$$

Hence

$$P_hQ_h\Lambda_h\equiv P_h\circ\Lambda_h\circ Q_h=P_h\circ Q_h\circ\Lambda_h$$

and the proof is complete.

Now we introduce the following conditions:

CONDITION IV. $p \in \mathscr{K}$ and $\operatorname{ess\,sup}_{\omega}(|\chi|^2 |\hat{p}_0(\chi, \omega)|)$ is integrable.

CONDITION V. 1) p satisfies Condition I′;

2) $\partial_k m_{j0}(\chi, \omega)$ and $\partial_k m_{j\infty}(\omega)$ $(j, k=1, 2,\dots, n)$ are bounded on $R^n_\chi \times (R^n_\omega - Z)$ and continuous on $R^n_\omega - Z$ for each χ, where $m_{j0}(\chi, \omega)=(\partial_j l_0)|s|$, $m_{j\infty}(\omega)=(\partial_j l_\infty)|s|$, $l_0=\hat{p}_0|s|$ and $l_\infty=p_\infty|s|$;

3) $\operatorname{ess\,sup}_{\omega}(|\partial_k m_{j0}(\chi, \omega)|)$ $(j, k=1, 2,\dots, n)$ are integrable.

Condition IV implies Condition II and we have

LEMMA 4.3. *If p satisfies Conditions* IV *and* V, *then* $p(x, \omega)|s(\omega)|^2$ *satisfies conditions* 1), 2) *and* 3) *of Theorem* 3.4.

4.2. Subalgebras $\mathscr{M}$ and $\mathscr{L}$ of $\mathscr{K}$

Let $\mathscr{M}$ be the set of all elements of $\mathscr{K}$ that satisfy Conditions I′, II and III′ and let the set $\mathscr{L}$ consist of all elements of $\mathscr{M}$ that satisfy Conditions IV and V. ($\mathscr{M}$ and $\mathscr{L}$ depend on $s(\omega)$.) For instance $|s(\omega)|I$ and $(s_j(\omega)/|s(\omega)|)I$ $(j=1, 2,\dots, n)$ belong to $\mathscr{M}$ and $\mathscr{L}$.

LEMMA 4.4. (i) *If p and q satisfy Condition* II, *so also do $p+q$, pq and p^*.*

(ii) *If $p, q \in \mathscr{M}$, then $p+q$, pq, $p^* \in \mathscr{M}$.*

(iii) *If $p, q \in \mathscr{L}$, then $p+q$, pq, $p^* \in \mathscr{L}$.*

We show

LEMMA 4.5. *Let $g(x, \omega)$ satisfy Conditions* I′ *and* II, *and let*

$$l(x, \omega) = c(\omega)I + q(x, \omega)|s(\omega)|, \tag{4.3}$$

where $q(x, \omega) \in \mathscr{M}$ and $c(\omega)$ is a scalar function satisfying Condition I. *Then*

$$L_h^* G_h L_h \equiv L_h^\sharp \circ G_h \circ L_h. \tag{4.4}$$

PROOF. L_h can be written as $L_h = C_h + Q_h \circ \Lambda_h$, where $C_h = \phi(cI)$. By Corollary 3.2 and Lemma 4.2 we have

$$C_h^* = C_h^\sharp, \qquad (Q_h \circ \Lambda_h)^* \equiv Q_h^\sharp \circ \Lambda_h.$$

Therefore $L_h^* \equiv L_h^\sharp$, and

$$L_h^* G_h L_h \equiv L_h^\sharp G_h L_h. \tag{4.5}$$

By Corollary 3.1 and Lemma 4.2 we have

$$G_hC_h = G_h \circ C_h, \qquad G_hQ_h\Lambda_h \equiv G_h \circ Q_h \circ \Lambda_h.$$

Hence $G_hL_h \equiv G_h \circ L_h$ and by (4.5)

$$L_h^* G_h L_h \equiv L_h^\sharp (G_h \circ L_h). \tag{4.6}$$

Since gl satisfies Condition II by Lemma 4.4 and l^* satisfies Condition I, by Theorem 3.1, we have

$$L_h^\sharp (G_h \circ L_h) \equiv L_h^\sharp \circ (G_h \circ L_h). \tag{4.7}$$

Hence (4.4) follows from (4.6) and (4.7).

COROLLARY 4.1. *Under the assumption of Lemma* 4.5 *let*

$$g(x, \omega) = w^*(x, \omega) w(x, \omega),$$

where w, $w^{-1} \in \mathscr{K}$. *Then*

$$G_h - L_h^* G_h L_h \equiv G_h - L_h^\sharp \circ G_h \circ L_h = W_h^\sharp \circ (I_h - \tilde{L}_h^\sharp \circ \tilde{L}_h) \circ W_h, \tag{4.8}$$

$$g - l^* g l = w^*(I - \tilde{l}^* \tilde{l}) w, \qquad \tilde{l} = w l w^{-1}. \tag{4.9}$$

PROOF. Since

$$\overline{W}_h \circ W_h = W_h^\sharp \circ \overline{W}_h^\sharp = I_h, \qquad G_h = W_h^\sharp \circ W_h,$$

we have from (4.4)

$$\begin{aligned} L_h^* G_h L_h \equiv L_h^\sharp \circ G_h \circ L_h &= W_h^\sharp \circ \overline{W}_h^\sharp \circ L_h^\sharp \circ W_h^\sharp \circ W_h \circ L_h \circ \overline{W}_h \circ W_h \\ &= W_h^\sharp \circ \tilde{L}_h^\sharp \circ \tilde{L}_h \circ W_h. \end{aligned}$$

Hence (4.8) holds and we have (4.9) by matrix calculation.

4.3. Integrability of Fourier transforms

Our next step is to obtain sufficient conditions under which an $N \times N$ matrix function $p(x, \omega)$ belongs to $\mathscr{K}$, $\mathscr{M}$ or $\mathscr{L}$. To this end we introduce

CONDITION VI. 1) $p(x, \omega)$ can be written as

$$p(x, \omega) = p_0(x, \omega) + p_\infty(\omega),$$

where $p_0(x, \omega)$ and $p_\infty(\omega)$ are bounded and measurable on $R_x^n \times R_\omega^n$ and on R_ω^n respectively, and $\lim_{|x| \to \infty} p_0(x, \omega) = 0$ for each $\omega \in R^n$;

2) $D_l^m p_0(x, \omega)$ $(l = 1, 2, \ldots, n;\ m = 0, 1, \ldots, n+3)$ are continuous on $R_x^n \times (R_\omega^n - Z)$ and continuous on R_x^n for each $\omega \in Z$; $\sup_\omega (|D_l^m p_0(x, \omega)|)$ $(l = 1, 2, \ldots,$

n; $m=0, 1,\dots, n+3$) are bounded and integrable;

3) $D_l^q \partial_j p_0(x, \omega)$ and $\partial_j p_\infty(\omega)$ ($j, l=1, 2,\dots, n$; $q=0, 1,\dots, n+2$) are continuous on $R_x^n \times (R_\omega^n - Z)$;

4) $\sup_{\omega \notin Z}(|D_l^q \partial_j p_0(x, \omega)|\,|s(\omega)|)$ ($j, l=1, 2,\dots, n$; $q=0, 1,\dots, n+2$) are bounded and integrable; $\sup_{\omega \notin Z}(|\partial_j p_\infty(\omega)|\,|s(\omega)|)$ ($j=1, 2,\dots, n$) are finite;

5) $D_l^r \partial_k \partial_j p_0(x, \omega)$ and $\partial_k \partial_j p_\infty(\omega)$ ($j, k, l=1, 2,\dots, n$; $r=0, 1,\dots, n+1$) are continuous on $R_x^n \times (R_\omega^n - Z)$;

6) $\sup_{\omega \notin Z}(|D_l^r \partial_k \partial_j p_0(x, \omega)|\,|s(\omega)|^2)$ ($j, k, l=1, 2,\dots, n$; $r=0, 1,\dots, n+1$) are bounded and integrable; $\sup_{\omega \notin Z}(|\partial_k \partial_j p_\infty(\omega)|\,|s(\omega)|^2)$ ($j, k=1, 2,\dots, n$) are finite.

We have the following results.

LEMMA 4.6. (i) *If p satisfies Conditions* VI–1) *and* VI–2), *then p satisfies Conditions* II *and* IV.

(ii) *If p satisfies Conditions* VI–1)–VI–4), *then $p \in \mathscr{M}$.*

(iii) *If p satisfies Condition* VI, *then $p \in \mathscr{L}$.*

COROLLARY 4.2. *Let $a(x)$ be an $N \times N$ matrix such that*

$$a(x) = a_0(x) + a_\infty,$$

where $\lim_{|x| \to \infty} a_0(x) = 0$. Suppose $D_l^m a_0(x)$ ($l=1, 2,\dots, n$; $m=0, 1,\dots, n+1+p$; $p=0, 1, 2$) are bounded and continuous on R^n and are integrable. Then $|\chi|^p |\hat{a}_0(\chi)|$ ($p=0, 1, 2$) are integrable.

LEMMA 4.7. *If $g(x, \omega)$ satisfies Conditions* VI–1) *and* VI–2), *then it satisfies Condition* N.

4.4. Powers of families of operators

To prove the boundedness of L_h^ν ($0 \leqq \nu k \leqq T$), in view of Theorem 2.1, it suffices to show that L_h satisfies (2.17). We show first the following

THEOREM 4.1. *Let $g(x, \omega) \in \mathscr{L}$ satisfy conditions of Theorem* 3.3 *and let*

$$l(x, \omega) = c(\omega)I + q(x, \omega)\,|s(\omega)| + r(x, \omega)\,|s(\omega)|^2, \tag{4.10}$$

where $q, r \in \mathscr{L}$ and $c(\omega)$ is a real-valued scalar function which is bounded and continuous together with the first and second partial derivatives. Suppose

1) $q^* g + g q = 0$ *for all* $\omega \in R^n - Z$;

2) $1 - c^2(\omega) = |s(\omega)|^2 a(\omega) + b(\omega)$;

3) $g - l^* g l \geqq b g$;

4) $b(\omega) = \sum_{j=1}^{m} b_j^2(\omega)$,

where $a(\omega)$ *and* $b_j(\omega)$ $(j=1, 2,\ldots, m)$ *are real-valued scalar functions such that* $b_j(\omega)$ $(j=1, 2,\ldots, m)$ *satisfy Condition* I *and* $a(\omega)I \in \mathscr{L}$. *Then for some* $c_0 \geqq 0$

$$\|L_h u\|_{G_h}^2 \leqq (1+c_0 h)\|u\|_{G_h}^2 \qquad \textit{for all} \quad u \in L_2, \quad h>0, \tag{4.11}$$

where $\|\cdot\|_{G_h}$ *is the norm given by* (3.26).

Proof. By Lemma 4.5 we have

$$L_h^* G_h L_h \equiv L_h^\# \circ G_h \circ L_h. \tag{4.12}$$

By conditions 1) and 2)

$$g - l^* g l = (ag-p)|s|^2 + bg, \tag{4.13}$$

where

$$p = (q^*gq + r^*gc + cgr) + (q^*gr + r^*gq)|s| + r^*gr|s|^2.$$

From condition 3) it follows that

$$(ag-p)|s|^2 \geqq 0. \tag{4.14}$$

Since $ag-p \in \mathscr{L}$, by Lemma 4.3 and Theorem 3.4 we have for some $c_1 \geqq 0$

$$\operatorname{Re}((A_h \circ G_h - P_h) \circ \Lambda_h^2 u, u) \geqq -c_1 h\|u\|^2 \qquad \text{for all} \quad u \in L_2, \quad h>0, \tag{4.15}$$

where $A_h = \phi(aI)$.

Let $\{\alpha_i^2(x)\}_{i=0,1,\ldots,s}$ be the partition of unity given in 3.3 and let $\Omega = \{x \mid |x| > R+\varepsilon\}$. Then $\alpha_0(x)=1$ on Ω, so that $\beta_0(x)=\alpha_0(x)-1=0$ on Ω. Since $\beta_0(x)$ and $\alpha_j(x)$ $(j=1, 2,\ldots, s)$ are smooth functions with compact supports, $|\chi|^k|\hat\beta_0(\chi)|$ and $|\chi|^k|\hat\alpha_j(\chi)|$ $(k=0, 1; j=1, 2,\ldots, s)$ are integrable. Hence $\alpha_i(x)$ $(i=0, 1,\ldots, s)$ satisfy Condition II.

Let $B_h = \phi(bI)$, $B_{jh} = \phi(b_j I)$ $(j=1, 2,\ldots, m)$ and $\alpha_i = \phi(\alpha_i I)$ $(i=0, 1,\ldots, s)$. Then by Theorem 3.1 $\alpha_i B_{jh} \equiv B_{jh}\alpha_i$ and $G_h B_{jh} \equiv B_{jh} G_h$. Since $B_{jh}^* = B_{jh}$ by Corollary 3.2, for some $c_2, c_3 \geqq 0$ we have

$$\begin{aligned}\operatorname{Re}((G_h \circ B_h)\alpha_i u, \alpha_i u) &\geqq \operatorname{Re}\textstyle\sum_{j=1}^{m}(B_{jh}G_h B_{jh}\alpha_i u, \alpha_i u) - c_2 h\|u\|^2 \\ &= \operatorname{Re}\textstyle\sum_{j=1}^{m}(G_h B_{jh}\alpha_i u, B_{jh}\alpha_i u) - c_2 h\|u\|^2 \\ &\geqq \operatorname{Re}\textstyle\sum_{j=1}^{m}(G_h \alpha_i B_{jh} u, \alpha_i B_{jh} u) - c_3 h\|u\|^2.\end{aligned}$$

Hence

$$\textstyle\sum_{i=0}^{s} \operatorname{Re}((G_h \circ B_h)\alpha_i u, \alpha_i u) \tag{4.16}$$

$$\geqq \sum_{j=1}^{m} \sum_{i=0}^{s} \operatorname{Re}(G_h \alpha_i B_{jh} u, \alpha_i B_{jh} u) - c_4 h \|u\|^2$$
$$\geqq \sum_{j=1}^{m} d_1^2 \|B_{jh} u\|^2 - c_4 h \|u\|^2 \geqq -c_4 h \|u\|^2,$$

where d_1 is given by (3.25) and $c_4 = (s+1)c_3$.

Since $L_h \alpha_i \equiv \alpha_i L_h$ by Theorem 3.1, we have for some $c_5 \geqq 0$

$$(4.17) \quad |(G_h \alpha_i L_h u, \alpha_i L_h u) - (G_h L_h \alpha_i u, L_h \alpha_i u)|$$
$$\leqq |(G_h(\alpha_i L_h - L_h \alpha_i) u, \alpha_i L_h u)| + |(G_h L_h \alpha_i u, (\alpha_i L_h - L_h \alpha_i) u)| \leqq c_5 h \|u\|^2.$$

From (4.12) for some $c_6 \geqq 0$

$$(4.18) \qquad |(G_h L_h u, L_h u) - (L_h^{\#} \circ G_h \circ L_h u, u)| \leqq c_6 h \|u\|^2.$$

Since by definition

$$\|L_h u\|_{G_h}^2 = \sum_{i=0}^{s} \operatorname{Re}(G_h \alpha_i L_h u, \alpha_i L_h u),$$

by (4.17) and (4.18) there is a constant $c_7 \geqq 0$ such that

$$(4.19) \qquad \|L_h u\|_{G_h}^2 \leqq \sum_{i=0}^{s} \operatorname{Re}((L_h^{\#} \circ G_h \circ L_h) \alpha_i u, \alpha_i u) + c_7 h \|u\|^2.$$

By (4.13) we have

$$(4.20) \qquad (G_h - L_h^{\#} \circ G_h \circ L_h) u = (A_h \circ G_h - P_h) \circ \Lambda_h^2 u + B_h \circ G_h u.$$

Hence by (4.15), (4.16), (4.19) and (4.20)

$$(4.21) \quad \|u\|_{G_h}^2 - \|L_h u\|_{G_h}^2 \geqq \sum_{i=0}^{s} \operatorname{Re}((G_h - L_h^{\#} \circ G_h \circ L_h) \alpha_i u, \alpha_i u) - c_7 h \|u\|^2$$
$$\geqq -c_8 h \|u\|^2,$$

where $c_8 = c_1 + c_4 + c_7$. By (3.27) we have (4.11) with $c_0 = c_8 / d_1^2$ and the proof is complete.

We note that the theorem remains valid even if condition 4) is replaced by the condition

$$\sum_{i=0}^{s} \operatorname{Re}((G_h \circ B_h) \alpha_i u, \alpha_i u) \geqq -ch\|u\|^2 \qquad \text{for all} \quad u \in L_2, \quad h > 0,$$

where c is a non-negative constant.

THEOREM 4.2. *Let $g(x, \omega) \in \mathscr{M}$ satisfy conditions of Theorem 3.3 and let*

$$(4.22) \qquad l(x, \omega) = c(\omega) I + q(x, \omega) |s(\omega)|,$$

$$(4.23) \qquad g(x, \omega) - l^*(x, \omega) g(x, \omega) l(x, \omega) = |e(\omega)|^2 r(x, \omega),$$

where $q \in \mathscr{M}$ and $c(\omega)$ and $e(\omega)$ are scalar functions satisfying Condition I.

Suppose

1) $r(x, \omega)$ *satisfies Conditions* II *and* N;
2) $r(x, \omega) \geqq \beta I$ *for some* $\beta > 0$.

Then for some $c_0 \geqq 0$

$$\|L_h u\|_{G_h}^2 \leqq (1+c_0 h)\,\|u\|_{G_h}^2 \qquad \text{for all} \quad u \in L_2, \quad h>0. \tag{4.24}$$

PROOF. By Theorem 3.3 there exist positive constants d_j, ε_j $(j=1, 2)$, ε and R such that

$$d_1^2\|u\|^2 \leqq \textstyle\sum_{i=0}^s \operatorname{Re}(G_h\alpha_i u, \alpha_i u) \leqq d_2^2\|u\|^2, \tag{4.25}$$

$$\varepsilon_1^2\|u\|^2 \leqq \textstyle\sum_{i=0}^s \operatorname{Re}(R_h\alpha_i u, \alpha_i u) \leqq \varepsilon_2^2\|u\|^2. \tag{4.26}$$

By Lemma 4.5 we have

$$L_h^* G_h L_h \equiv L_h^\# \circ G_h \circ L_h. \tag{4.27}$$

By the same argument as in the proof of Theorem 4.1 there is a constant $c_1 \geqq 0$ such that

$$\|L_h u\|_{G_h}^2 \leqq \textstyle\sum_{i=0}^s \operatorname{Re}((L_h^\# \circ G_h \circ L_h)\alpha_i u, \alpha_i u) + c_1 h\|u\|^2. \tag{4.28}$$

By Corollary 3.2 for $E_h = \phi(eI)$ we have

$$E_h^* = E_h^\# \tag{4.29}$$

and by Theorem 3.1 and its corollary

$$E_h^\# \circ E_h \circ R_h = (E_h^\# \circ R_h) \circ E_h = (E_h^\# \circ R_h) E_h \equiv E_h^\# R_h E_h. \tag{4.30}$$

Since by (4.23)

$$G_h - L_h^\# \circ G_h \circ L_h = E_h^\# \circ E_h \circ R_h,$$

by (4.29) and (4.30) we have

$$G_h - L_h^\# \circ G_h \circ L_h \equiv E_h^* R_h E_h. \tag{4.31}$$

Hence by (4.28) and (4.31) for some $c_2 \geqq 0$

$$\begin{aligned}
\|u\|_{G_h}^2 - \|L_h u\|_{G_h}^2 &\geqq \textstyle\sum_{i=0}^s \operatorname{Re}((G_h - L_h^\# \circ G_h \circ L_h)\alpha_i u, \alpha_i u) - c_1 h\|u\|^2 \\
&\geqq \textstyle\sum_{i=0}^s \operatorname{Re}(E_h^* R_h E_h \alpha_i u, \alpha_i u) - c_2 h\|u\|^2 \\
&= \textstyle\sum_{i=0}^s \operatorname{Re}(R_h E_h \alpha_i u, E_h \alpha_i u) - c_2 h\|u\|^2.
\end{aligned}$$

Since $E_h\alpha_i \equiv \alpha_i E_h$, we have for some $c_3 \geqq 0$

$$\|u\|_{G_h}^2 - \|L_h u\|_{G_h}^2 \geqq \textstyle\sum_{i=0}^s \operatorname{Re}(R_h \alpha_i E_h u, \alpha_i E_h u) - c_3 h\|u\|^2 - c_2 h\|u\|^2,$$

so that by (4.26) with $c_4=c_2+c_3$

$$\|u\|_{G_h}^2-\|L_h u\|_{G_h}^2 \geqq \varepsilon_1^2\|E_h u\|^2-c_4h\|u\|^2 \geqq -c_4h\|u\|^2.$$

Thus (4.24) holds by (4.25) with $c_0=c_4/d_1^2$.

5. Two algebras of difference operators

5.1. Algebra $\mathscr{F}_h$

Let $\mathscr{A}_0$ be the set of all $N\times N$ matrix functions $a(x)$ defined on R^n with the properties:

1) $a(x)$ can be written as

$$a(x) = a_0(x)+a_\infty,$$

where $\lim_{|x|\to\infty} a_0(x)=0$;

2) $a_0(x)$ is bounded and integrable;

3) $|\chi|^p|\hat{a}_0(\chi)|$ $(p=0, 1, 2)$ are integrable.

(Two elements of $\mathscr{A}_0$ are identified if they coincide almost everywhere.)

We denote by α an n-tuple $(\alpha_1, \alpha_2,\dots, \alpha_n)$ of integers, i.e. $\alpha=(\alpha_1, \alpha_2,\dots, \alpha_n)$. Let $\mathscr{A}$ be the set of all matrices $a(x, \omega)$ such that $a(x, \omega)=\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega}$, where $a_\alpha\in\mathscr{A}_0$ and the summation is over a finite set of α. It is clear that $a(x, \omega)$ satisfies Conditions I, II and III. Let

$$a(x, \omega) = \textstyle\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega}, \quad b(x, \omega) = \sum_\beta b_\beta(x)e^{i\beta\cdot\omega}. \tag{5.1}$$

Then we have

$$a(x, \omega)+b(x, \omega) = \textstyle\sum_\gamma(a_\gamma(x)+b_\gamma(x))e^{i\gamma\cdot\omega}, \tag{5.2}$$

$$a(x, \omega)b(x, \omega) = \textstyle\sum_\gamma(\sum_{\alpha+\beta=\gamma}a_\alpha(x)b_\beta(x))e^{i\gamma\cdot\omega}, \tag{5.3}$$

$$a^*(x, \omega) = \textstyle\sum_\alpha a_\alpha^*(x)e^{-i\alpha\cdot\omega}. \tag{5.4}$$

Hence $\mathscr{A}$ is a subalgebra of $\mathscr{K}$ with involution.

By (2.6) T_h^α is a family of bounded linear operators mapping L_2 into itself. Since for $a(x)\in\mathscr{A}_0$

$$\|a(x)T_h^\alpha u(x)\| \leqq (\operatorname{ess\,sup}_x|a(x)|)\,\|u\|,$$

the family $a(x)T_h^\alpha$ belongs to $\mathscr{H}_h$. We define a mapping ψ from $\mathscr{A}$ into $\mathscr{H}_h$ by

$$\psi(\textstyle\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega}) = \sum_\alpha a_\alpha(x)T_h^\alpha,$$

and let $\mathscr{A}_h=\psi(\mathscr{A})$.

For $\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega}\in\mathscr{A}$ let $A_h=\phi(\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega})$. Then for $u\in\mathscr{S}$

$$\kappa\int e^{ix\cdot\xi}\sum_\alpha a_\alpha(x)T_h^\alpha u(x)dx$$

$$=\int\sum_\alpha\widehat{a_{\alpha 0}}(\xi-\xi')e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi'+\sum_\alpha a_{\alpha\infty}e^{i\alpha\cdot h\xi}\hat{u}(\xi)$$

$$=\int\sum_\alpha\widehat{a_\alpha}(\xi-\xi')e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi' = \widehat{A_h u}(\xi)\quad\text{a.e.,}$$

so that for $u\in\mathscr{S}$ we have in L_2

$$\sum_\alpha a_\alpha(x)T_h^\alpha u(x) = A_h u(x)\,. \tag{5.5}$$

By the uniqueness of the extension of operators (5.5) holds for all $u\in L_2$, so that $\sum_\alpha a_\alpha(x)T_h^\alpha$ and A_h can be identified. Hence ψ is the restriction of ϕ to $\mathscr{A}$ and is a one-to-one mapping from $\mathscr{A}$ onto $\mathscr{A}_h$. We call $\sum_\alpha a_\alpha(x)e^{i\alpha\cdot\omega}$ the symbol of $\sum_\alpha a_\alpha(x)T_h^\alpha$. Let A_h, $B_h\in\mathscr{A}_h$ and let

$$A_h = \sum_\alpha a_\alpha(x)T_h^\alpha,\qquad B_h = \sum_\beta b_\beta(x)T_h^\beta.$$

Then their symbols $a(x,\omega)$ and $b(x,\omega)$ are given by (5.1). Since $\mathscr{A}_h\subset\mathscr{K}_h$, A_h+B_h, $A_h\circ B_h$ and $A_h^\sharp$ can be defined in $\mathscr{K}_h$ and they are the families associated with $a+b$, ab and a^* respectively. By (5.2)–(5.4) we have

$$A_h+B_h = \sum_\gamma(a_\gamma(x)+b_\gamma(x))T_h^\gamma, \tag{5.6}$$

$$A_h\circ B_h = \sum_\gamma(\sum_{\alpha+\beta=\gamma}a_\alpha(x)b_\beta(x))\,T_h^\gamma, \tag{5.7}$$

$$A_h^\sharp = \sum_\alpha a_\alpha^*(x)T_h^{-\alpha}\,. \tag{5.8}$$

Hence $\mathscr{A}_h$ is a subalgebra of $\mathscr{K}_h$ with involution and it follows that ψ and ψ^{-1} are morphisms.

LEMMA 5.1. *Let $F_{jh}\in\mathscr{A}_h$ $(j=1, 2,\dots, r)$ and let*

$$F_h = F_{1h}F_{2h}\cdots F_{rh},\qquad L_h = F_{1h}\circ F_{2h}\circ\cdots\circ F_{rh}.$$

Then $F_h\equiv L_h$.

PROOF. We have

$$F_h-L_h = \sum_{j=1}^{r-1}(F_{0h}\cdots F_{j-1h})\{F_{jh}(F_{j+1h}\circ\cdots\circ F_{rh})$$
$$-F_{jh}\circ(F_{j+1h}\circ\cdots\circ F_{rh})\}\qquad(F_{0h}=I_h)\,.$$

The symbol $f_j(x,\omega)$ of F_{jh} satisfies Conditions I and II, because $f_j\in\mathscr{A}$. By Lemma 4.4 $f_{j+1}(x,\omega)f_{j+2}(x,\omega)\cdots f_r(x,\omega)$ $(j=1, 2,\dots, r-1)$ satisfy Condition II.

Hence by Theorem 3.1

$$F_{jh}(F_{j+1h}\circ\cdots\circ F_{rh}) \equiv F_{jh}\circ(F_{j+1h}\circ\cdots\circ F_{rh}) \qquad (1 \leqq j < r)$$

and so we have $F_h \equiv L_h$, which completes the proof.

Let $\mathscr{F}_h$ be the subalgebra of $\mathscr{H}_h$ generated by $\mathscr{A}_h$. Then $F_h \in \mathscr{F}_h$ can be expressed as

$$F_h = \sum_r F_{1h}^{(r)} F_{2h}^{(r)} \cdots F_{kh}^{(r)} \qquad (F_{jh}^{(r)} \in \mathscr{A}_h).$$

Corresponding to this we put

$$L_h = \sum_r F_{1h}^{(r)} \circ F_{2h}^{(r)} \circ \cdots \circ F_{kh}^{(r)},$$

$$l(x, \omega) = \sum_r f_1^{(r)} f_2^{(r)} \cdots f_k^{(r)},$$

where $f_j^{(r)}(x, \omega)$ is the symbol of $F_{jh}^{(r)}$. Then $L_h \in \mathscr{A}_h$, $F_h \equiv L_h$ and $l(x, \omega)$ is the symbol of L_h. In the following we call $l(x, \omega)$ a symbol belonging to F_h.

5.2. Algebra $\mathscr{G}_h$

We consider the case where coefficient matrices of T_h^α depend not only on x but also on h.

Let $\mathscr{B}_0$ be the set of all $N \times N$ matrix functions $b(x, \mu)$ defined on $R_x^n \times [0, \infty)$ with the properties:

1) $b(x, 0) \in \mathscr{A}_0$;
2) $b(x, \mu)$ can be written as

$$b(x, \mu) = b_0(x, \mu) + b_\infty(\mu),$$

where $\lim_{|x| \to \infty} b_0(x, \mu) = 0$ for each μ;

3) For each μ $b_0(x, \mu)$ is bounded on R_x^n and integrable;
4) $\hat{b}_0(\chi, \mu)$ is integrable for each μ;
5) For some $c \geqq 0$

$$\int |\hat{b}_0(\chi, \mu) - \hat{b}_0(\chi, 0)| d\chi \leqq c\mu,$$

$$|b_\infty(\mu) - b_\infty(0)| \leqq c\mu \qquad \text{for all} \quad \mu \geqq 0.$$

For instance $\Delta_{j\mu} a(x)$ $(j = 1, 2, \ldots, n)$ belong to $\mathscr{B}_0$ for $a(x) \in \mathscr{A}_0$.

LEMMA 5.2. *Let $b(x, \mu) \in \mathscr{B}_0$ and let B_h be the family associated with $b(x, 0)e^{i\alpha\cdot\omega}$. Then $b(x, h)T_h^\alpha \in \mathscr{H}_h$ and*

$$b(x, h)T_h^\alpha \equiv B_h. \tag{5.9}$$

PROOF. Let $u(x)\in\mathscr{S}$. Then since

$$\|b(x, h)T_h^\alpha u\|^2 \leqq (\operatorname{ess.sup}_x |b(x, h)|)^2 \|u\|^2,$$

$b(x, h)T_h^\alpha u(x)$ belongs to L_2 for each fixed h and its Fourier transform can be written as follows:

$$\text{l.i.m.}\,\kappa\int e^{-ix\cdot\xi}b(x, h)T_h^\alpha u(x)dx$$

$$= \int \hat{b}_0(\xi-\xi', h)e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi' + b_\infty(h)e^{i\alpha\cdot h\xi}\hat{u}(\xi) \quad \text{a.e.}.$$

Hence

$$\|b(x, h)T_h^\alpha u - B_h u\| \leqq \left\|\int\{\hat{b}_0(\xi-\xi', h)-\hat{b}_0(\xi-\xi', 0)\}e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi'\right\|$$

$$+|b_\infty(h)-b_\infty(0)|\,\|\hat{u}\|.$$

By Young's Theorem and condition 5) we have

$$\|b(x, h)T_h^\alpha u - B_h u\| \leqq 2ch\|u\|,$$

which implies (5.9) if $b(x, h)T_h^\alpha\in\mathscr{H}_h$. Since

$$\|b(x, h)T_h^\alpha u\| \leqq \|B_h u\| + 2ch\|u\|,$$

$b(x, h)T_h^\alpha$ belongs to $\mathscr{H}_h$ and the proof is complete.

Let $\mathscr{B}_h$ be the set of all finite sums of families of the form $\sum_\alpha b_\alpha(x, h)T_h^\alpha$ ($b_\alpha(x, \mu)\in\mathscr{B}_0$) and let $\mathscr{G}_h$ be the subalgebra of $\mathscr{H}_h$ generated by $\mathscr{B}_h$. It is clear that $\mathscr{A}_0\subset\mathscr{B}_0$ and $\mathscr{F}_h\subset\mathscr{G}_h$.

Let $E_h\in\mathscr{G}_h$. Then E_h can be written as

$$E_h = \sum_r E_{1h}^{(r)}E_{2h}^{(r)}\cdots E_{kh}^{(r)} \qquad (E_{jh}^{(r)}\in\mathscr{B}_h),$$

where

$$E_{jh}^{(r)} = \sum_\alpha e_{j\alpha}^{(r)}(x, h)T_h^\alpha \qquad (e_{j\alpha}^{(r)}(x, \mu)\in\mathscr{B}_0).$$

Corresponding to these we put

$$F_{jh}^{(r)} = \sum_\alpha e_{j\alpha}^{(r)}(x, 0)T_h^\alpha, \qquad F_h = \sum_r F_{1h}^{(r)}F_{2h}^{(r)}\cdots F_{kh}^{(r)}.$$

Then $F_{jh}^{(r)}\in\mathscr{A}_h$ by the definition of $\mathscr{B}_0$ and $E_{jh}^{(r)}\equiv F_{jh}^{(r)}$ by Lemma 5.2. Hence $F_h\in\mathscr{F}_h$ and $E_h\equiv F_h$. Thus we have

THEOREM 5.1. *Let $S_h(x, h)$ be the difference operator* (2.4) *with*

(5.10) $$c_{\alpha m_j}(x, \mu) \in \mathscr{B}_0 \qquad (j = 1, 2,\dots, \nu).$$

Then

$$S_h(x, h) \in \mathscr{G}_h, \qquad S_h(x, 0) \in \mathscr{F}_h.$$

Let L_h be the family associated with a symbol belonging to $S_h(x, 0)$. Then

$$L_h \in \mathscr{A}_h, \qquad S_h(x, h) \equiv S_h(x, 0) \equiv L_h.$$

By this theorem and Corollary 2.1, in proving the stability of the scheme (2.2) under the condition (5.10) the problem is to establish (2.17) for L_h.

Let

$$s(x, \omega) = \sum_m \prod_{j=1}^{\nu} c_{m_j}(x, \omega),$$

where

$$c_{m_j}(x, \omega) = \sum_\alpha c_{\alpha m_j}(x, 0) e^{i\alpha\cdot\omega}, \qquad c_{\alpha m_j}(x, \mu) \in \mathscr{B}_0.$$

Then $s(x, \omega)$ is a symbol belonging to $S_h(x, 0)$. For instance let

(5.11) $$f(x, \omega; \lambda) = c(\omega)I + i\lambda p(x, \omega),$$

(5.12) $$m(x, \omega; \lambda) = I + i\lambda p(x, \omega)[c(\omega)I + i\lambda p(x, \omega)/2],$$

where

(5.13) $$p(x, \omega) = \sum_{j=1}^{n} A_j(x) s_j(\omega), \quad c(\omega) = \left(\sum_{j=1}^{n} \cos\omega_j\right)/n,$$

(5.14) $$s_j(\omega) = \sin\omega_j, \qquad A_j(x) \in \mathscr{A}_0 \quad (j = 1, 2,\dots, n).$$

Then $f(x, \omega; \lambda)$ and $m(x, \omega; \lambda)$ are symbols belonging to F_h and M_h given by (2.10) and (2.11) respectively.

6. Stability of difference schemes

6.1. Assumptions and lemmas

In this section we study the stability of the scheme (2.2). Let

(6.1) $$A(x, \omega) = \sum_{j=1}^{n} A_j(x)\omega_j$$

and let Δ_{jh} $(j=1, 2,\dots, n)$ be the difference operators such that $s_j(\omega)$ $(j=1, 2,\dots, n)$ satisfy (2.9). Suppose the following conditions are satisfied:

CONDITION A. $A_j(x)$ $(j=1, 2,\dots, n)$ are bounded and continuous on R_x^n and can be written as

$$A_j(x) = A_{j0}(x) + A_{j\infty} \qquad (j = 1, 2, \ldots, n),$$

where

$$\lim_{|x|\to\infty} A_{j0}(x) = 0 \qquad (j = 1, 2, \ldots, n).$$

Condition B. $D_l^m A_{j0}(x)$ $(l=1, 2, \ldots, n;\ m=0, 1, \ldots, n+3)$ are bounded, continuous and integrable on R_x^n.

Condition C. 1) Eigenvalues of $A(x, \omega')$ are all real and their multiplicities are independent of x and ω';

2) There exists a constant $\delta > 0$ independent of x and ω' such that

$$|\lambda_i(x, \omega') - \lambda_j(x, \omega')| \geqq \delta \qquad (i \neq j;\ i, j = 1, 2, \ldots, s),$$

where $\lambda_i(x, \omega')$ $(i=1, 2, \ldots, s)$ are all the distinct eigenvalues of $A(x, \omega')$;

3) Elementary divisors of $A(x, \omega')$ are all linear.

By Corollary 4.2 $A_j(x)$ $(j=1, 2, \ldots, n)$ belong to $\mathscr{A}_0$.

Let

$$P_h = \textstyle\sum_{j=1}^n A_j(x) \Delta_{jh}, \tag{6.2}$$

$$p(x, \omega) = \textstyle\sum_{j=1}^n A_j(x) s_j(\omega), \tag{6.3}$$

$$p_z(x, \omega) = \textstyle\sum_{j=1}^n A_j(x) s_j(\omega)/|s(\omega)|. \tag{6.4}$$

Then $P_h \in \mathscr{A}_h$ and $ip(x, \omega)$ is the symbol of P_h. By Lemmas 4.6 and 4.7 $p_z(x, \omega)$ belongs to $\mathscr{L}$ and satisfies Condition N. We have the following two lemmas.

Lemma 6.1. *There exists an element $g(x, \omega)$ of $\mathscr{L}$ satisfying the conditions of Theorem 3.3 such that*

$$\{g(x, \omega) p_z(x, \omega)\}^* = g(x, \omega) p_z(x, \omega) \qquad \text{for} \quad \omega \in R^n - Z. \tag{6.5}$$

Lemma 6.2. *There exist elements $w(x, \omega)$ and $w^{-1}(x, \omega)$ of $\mathscr{L}$ satisfying Condition N such that*

$$g(x, \omega) = w^*(x, \omega) w(x, \omega). \tag{6.6}$$

For $a \in \mathscr{K}$ we denote waw^{-1} by $\tilde{a}$. By these lemmas $\tilde{p}_z$ and $\tilde{p}$ are hermitian matrices on $R_x^n \times (R_\omega^n - Z)$ and on $R_x^n \times R_\omega^n$ respectively. By Lemma 3.4 $\tilde{p}_z$ satisfies Condition N and by Lemma 4.4 it belongs to $\mathscr{L}$.

In the following we assume that $S_h(x, h) \in \mathscr{G}_h$. Then $S_h(x, 0) \in \mathscr{F}_h$ and a symbol belonging to $S_h(x, 0)$ is an element of $\mathscr{A}$.

From the results obtained in Sections 2, 4 and 5 we can conclude that if a symbol belonging to $S_h(x, 0)$ satisfies conditions of Theorem 4.1 or 4.2, then the scheme (2.2) is stable by Theorem 2.1 and its corollary.

Let $P[\lambda; \mathscr{L}]$ be the set of all polynomials in λ of the form

$$a(x, \omega; \lambda) = \textstyle\sum_{j=0}^{m} \lambda^j a_j(x, \omega), \quad a_j(x, \omega) \in \mathscr{L} \qquad (j = 0, 1, \dots, m),$$

and denote by $P[\lambda; p]$ the set of all polynomials in λ and $p(x, \omega)$. The set $P[\lambda; \mathscr{M}]$ is defined similarly. For a scalar function $t(\omega)$ we use the notation

$$a(x, \omega; \lambda)/t(\omega) = \textstyle\sum_{j=0}^{m} \lambda^j a_j/t \in \mathscr{K} \qquad (\text{or } \mathscr{L}, \mathscr{M})$$

if $a_j(x, \omega)/t(\omega) \in \mathscr{K}$ (or $\mathscr{L}$, $\mathscr{M}$) $(j=0, 1, \dots, m)$.

6.2. Special schemes

We have the following [17]

THEOREM 6.1. *Friedrichs' scheme is stable, if* $\lambda\rho(p_z(x, \omega)) \leqq 1/\sqrt{n}$. *The modified Lax-Wendroff scheme is stable if* $\lambda\rho(p_z(x, \omega)) \leqq 2/\sqrt{n}$.

PROOF. For Friedrichs' scheme by (5.11) $f(x, \omega; \lambda)$ can be rewritten in $\mathscr{K}$ as

$$f(x, \omega; \lambda) = c(\omega)I + i\lambda p_z(x, \omega)\,|s(\omega)|,$$

which is of the form (4.10). By the fact $p_z \in \mathscr{L}$ and by Lemma 6.1 the first part of the assumptions and condition 1) of Theorem 4.1 are satisfied.

From (5.13) and (5.14) it follows that

$$1 - c^2(\omega) = n^{-1}|s(\omega)|^2 + b(\omega), \quad b(\omega) = \textstyle\sum_{j>k} b_{jk}^2(\omega),$$

$$b_{jk}(\omega) = (\cos\omega_j - \cos\omega_k)/n.$$

Hence conditions 2) and 4) of Theorem 4.1 are satisfied.

By Corollary 4.1 we have

$$g - f^*gf = w^*(n^{-1}I - \lambda^2\tilde{p}_z^2)\,|s|^2 w + bg.$$

Since $\lambda\rho(\tilde{p}_z) \leqq 1/\sqrt{n}$, we have $g - f^*gf \geqq bg$ and condition 3) of Theorem 4.1 is satisfied. Hence Friedrichs' scheme is stable.

By (5.12) $m(x, \omega; \lambda)$ can be rewritten in $\mathscr{K}$ as

$$m(x, \omega; \lambda) = I + i\lambda p_z c|s| - \lambda^2 p_z^2|s|^2/2.$$

Since $p_z^2 \in \mathscr{L}$ by Lemma 4.4, the assumptions of Theorem 4.1 are satisfied except condition 3).

By Corollary 4.1 we have

$$g - m^*gm = w^*(\lambda\tilde{p})^2\,[(n^{-1}I - \lambda^2\tilde{p}_z^2/4)\,|s|^2 + b]w.$$

Since $\lambda\rho(\tilde{p}_z) \leqq 2/\sqrt{n}$, we have $g - m^*gm \geqq 0$. Hence the modified Lax-Wendroff scheme is stable.

6.3. Stability theorems

We consider the schemes (2.2) with accuracy of order $r \geqq 1$ and state stability conditions in terms of a symbol $l(x, \omega; \lambda)$ belonging to $S_h(x, 0)$. Suppose $s(\omega)$ satisfies (2.9) and let

$$d = r+k,\ k = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even,} \end{cases} \qquad y(x, \omega; \lambda) = \textstyle\sum_{j=2}^{r}(i\lambda p_z)^j |s|^{j-2}/j!.$$

Then since p_z, $|s|I \in \mathscr{L}$, by Lemma 4.4 $y \in \mathscr{L}$.

We denote by λ_0, c_1 and c_2 positive constants and by $t(\omega)$ a scalar function such that $t(\omega)I \in \mathscr{K}$.

THEOREM 6.2. *Let*

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{r}(i\lambda p)^j/j!, \tag{6.7}$$

where $r = 4m-1$ *or* $4m$ $(m \geqq 1)$. *Then the scheme* (2.2) *is stable for sufficiently small* λ.

PROOF. l can be rewritten in $\mathscr{K}$ as

$$l(x, \omega; \lambda) = I + i\lambda p_z|s| + y|s|^2,$$

and the assumptions of Theorem 4.1 are satisfied except condition 3).

We have

$$g - l^*gl = c_2 w^*(\lambda\tilde{p})^d(I - (\lambda\tilde{p})^2\tilde{q})w,$$

where $c_2 = 2/(r!d)$ and $q \in P[\lambda; p]$. Hence there exists λ_0 such that $g - l^*gl \geqq 0$ for $\lambda \leqq \lambda_0$. Thus the scheme (2.2) is stable for $\lambda \leqq \lambda_0$.

THEOREM 6.3. *Let*

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{r}(i\lambda p)^j/j! - (\lambda p)^m v(\lambda p)^m, \tag{6.8}$$

where $r \geqq 2m$ $(m \geqq 1)$ *and* $v(x, \omega; \lambda) \in P[\lambda; \mathscr{L}]$. *Suppose*

1) $|s(\omega)|^\sigma \leqq c_1 t(\omega)$;

2) $v_1(x, \omega; \lambda) = v/t \in \mathscr{K}$;

3) $u(x, \omega; \lambda) \geqq c_2 t(\omega)I$ *for* $\lambda \leqq \lambda_0$,

where $\sigma=d-2m$ and $u=\tilde{v}^*+\tilde{v}-\tilde{v}^*(\lambda\tilde{p})^{2m}\tilde{v}$. Then the scheme (2.2) *is stable for sufficiently small λ.*

PROOF. l can be rewritten in $\mathscr{K}$ as

$$l(x, \omega; \lambda) = I+f_1|s|+f_2|s|^2, \tag{6.9}$$

where

$$f_1 = i\lambda p_z, \quad f_2 = y-\lambda^{2m}p_z^m v p_z^m |s|^{2m-2}.$$

By Lemma 4.4 $f_1, f_2 \in \mathscr{L}$.

It suffices to show that condition 3) of Theorem 4.1 is satisfied. We have

$$g-l^*gl = w^*(\lambda\tilde{p})^m[u+\lambda q_2+(\lambda\tilde{p})^\sigma\tilde{q}_3](\lambda\tilde{p})^m w,$$

where $q_3 \in P[\lambda; p]$,

$$q_2 = \tilde{v}^*\tilde{q}_1+\tilde{q}_1^*\tilde{v}, \quad q_1 = \textstyle\sum_{j=1}^r (ip)^j\lambda^{j-1}/j!. \tag{6.10}$$

By condition 1) we can define $e(\omega)=|s(\omega)|^\sigma/t(\omega)$ as in 4.1 and it follows that $e(\omega)I \in \mathscr{K}$ and

$$\begin{aligned} g-l^*gl &= w^*(\lambda\tilde{p})^m t[c_2I+\lambda q_{21}+(\lambda\tilde{p}_z)^\sigma\tilde{q}_3 e](\lambda\tilde{p})^m w \\ &\quad + w^*(\lambda\tilde{p})^m(u-c_2tI)(\lambda\tilde{p})^m w, \end{aligned}$$

where

$$q_{21} = \tilde{v}_1^*\tilde{q}_1+\tilde{q}_1^*\tilde{v}_1, \qquad \sigma \geqq 2.$$

Hence by condition 3) there exists λ_1 $(0<\lambda_1\leqq\lambda_0)$ such that $g-l^*gl\geqq 0$ for $\lambda\leqq\lambda_1$. Thus the scheme is stable for $\lambda\leqq\lambda_1$.

THEOREM 6.4. *Let*

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^r (i\lambda p)^j/j! - (i\lambda p)^{2m+1}a - (\lambda p)^{m+1}v(\lambda p)^{m+1}, \tag{6.11}$$

where $r\geqq 2m+2$ $(m\geqq 0)$, $v(x, \omega; \lambda) \in P[\lambda; \mathscr{L}]$ and $a(\omega)$ is a real-valued scalar function such that $a(\omega)I \in \mathscr{L}$ and $(a(\omega)/t(\omega))I \in \mathscr{K}$. Suppose conditions 1), 2) *and* 3) *of Theorem* 6.3 *are satisfied, where $\sigma=d-2m-2$,*

$$u = \tilde{v}+\tilde{v}^*+(-1)^m 2aI-\tilde{b}^*(\lambda\tilde{p})^{2m}\tilde{b}, \quad b = (-1)^m(ia)+\lambda pv.$$

Then the scheme (2.2) *is stable for sufficiently small λ.*

PROOF. l can be rewritten in $\mathscr{K}$ as (6.9), where

$$f_1 = i\lambda p_z(1-a), \quad f_2 = y-(\lambda p_z)v(\lambda p_z) \quad \text{if} \quad m = 0,$$

$$f_1 = i\lambda p_z, \quad f_2 = y-(\lambda p_z)^m b(\lambda p_z)^{m+1}|s|^{2m-1} \quad \text{if} \quad m \geqq 1.$$

By Lemma 4.4 $f_1, f_2 \in \mathscr{L}$. We have

$$g - l^* g l = w^*(\lambda \tilde{p})^{m+1}[u + i\lambda q_3 + (\lambda \tilde{p})^\sigma \tilde{q}_4](\lambda \tilde{p})^{m+1} w,$$

where $\sigma \geqq 2$, $q_4 \in P[\lambda; p]$,

$$q_3 = q_2^* \tilde{p} - \tilde{p} q_2, \quad q_2 = \tilde{v} - i\tilde{q}_1^* \tilde{b}, \quad q_1 = \textstyle\sum_{j=0}^{r-2}(i\lambda p)^j/(j+2)!.$$

By condition 1) we can define $e(\omega) = |s(\omega)|^\sigma / t(\omega)$ and we have $e(\omega) I \in \mathscr{K}$,

$$g - l^* g l = w^*(\lambda \tilde{p})^{m+1} t[c_2 I + i\lambda q_{31} + (\lambda \tilde{p}_z)^\sigma \tilde{q}_4 e](\lambda \tilde{p})^{m+1} w$$
$$+ w^*(\lambda \tilde{p})^{m+1}(u - c_2 t I)(\lambda \tilde{p})^{m+1} w,$$

where

$$q_{31} = q_{21}^* \tilde{p} - \tilde{p} q_{21}, \quad q_{21} = \tilde{v}_1 - i\tilde{q}_1^* \tilde{b}_1,$$
$$b_1 = (-1)^m (i a_1) + \lambda p v_1, \quad a_1 = a/t.$$

Hence by condition 3) there exists λ_1 $(0 < \lambda_1 \leqq \lambda_0)$ such that $g - l^* g l \geqq 0$ for $\lambda \leqq \lambda_1$. Thus by Theorem 4.1 the scheme is stable for $\lambda \leqq \lambda_1$.

COROLLARY 6.1. *Let*

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{r}(i\lambda p)^j/j! - (i\lambda p)^{r-1} e, \tag{6.12}$$

where $r = 4m+1$ *or* $4m+2$ $(m \geqq 1)$, $e(\omega)$ *is a scalar function such that* $|s(\omega)|^2 \leqq c_1 e(\omega)$ *for some* $c_1 > 0$ *and* $e(\omega)$, $\partial_j e(\omega)$ *and* $\partial_k \partial_j e(\omega)$ $(j, k = 1, 2, \dots, n)$ *are bounded and continuous on* R^n_ω. *Then the scheme* (2.2) *is stable for sufficiently small* λ.

THEOREM 6.5. *Let*

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{r}(i\lambda p)^j/j! - \lambda^{2m} v, \tag{6.13}$$

where $r \geqq 2m$ $(m \geqq 0,\ r \geqq 1)$,

$$v(x, \omega; \lambda) = a + \lambda^\alpha b \qquad (\alpha \geqq 0),$$
$$a(x, \omega; \lambda) \in P[\lambda; \mathscr{L}], \quad b(x, \omega; \lambda) \in P[\lambda; \mathscr{L}],$$
$$a_1(x, \omega; \lambda) = a/|s|^2 \in \mathscr{L}, \quad b_1(x, \omega; \lambda) = b/|s| \in \mathscr{L}.$$

Suppose

1) $\tilde{b}^* + \tilde{b} = 0$;

2) $|s(\omega)|^{d-2} \leqq c_1 t(\omega)$;

3) $a_2(x, \omega; \lambda) = a_1/t \in \mathscr{K}, \quad b_2(x, \omega; \lambda) = b_1/t \in \mathscr{K}$;

4) $u(x, \omega; \lambda) \geqq c_2 t|s|^2 I \quad$ *for* $\lambda \leqq \lambda_0$,

where $u = \tilde{a}^* + \tilde{a} - \lambda^{2m}\tilde{v}^*\tilde{v}$. *Then the scheme* (2.2) *is stable for sufficiently small* λ.

PROOF. l can be rewritten in $\mathscr{K}$ as (6.9), where

$$f_1 = i\lambda p_z - \lambda^\beta b_1, \quad f_2 = y - \lambda^{2m} a_1, \quad \beta = 2m + \alpha.$$

By Lemma 4.4 $f_1, f_2 \in \mathscr{L}$. By (6.5) and condition 1) we have

$$f_1^* g + g f_1 = 0.$$

Hence the assumptions of Theorem 4.1 are satisfied except condition 3).

We have

$$g - l^* g l = \lambda^{2m} w^*(u + \lambda q_2 + \lambda^\sigma \tilde{p}^d \tilde{q}_3) w,$$

where $\sigma = d - 2m \geqq 2$, $q_3 \in P[\lambda; p]$ and q_2 is given by (6.10). By condition 2) we can define $e(\omega) = |s(\omega)|^{d-2}/t(\omega)$ and $e(\omega) I \in \mathscr{K}$. Put

$$q_{21} = q_2/(t|s|^2), \quad q_{11} = q_1/|s|, \quad q_4 = \tilde{a}_2^* \tilde{q}_1 + \lambda^\alpha \tilde{b}_2^* \tilde{q}_{11}.$$

Then

$$q_{21}(x, \omega; \lambda) = q_4 + q_4^* \in \mathscr{K},$$

$$g - l^* g l = \lambda^{2m} w^* t|s|^2 (c_2 I + \lambda q_{21} + \lambda^\sigma \tilde{p}_z^d e \tilde{q}_3) w$$
$$+ \lambda^{2m} w^*(u - c_2 t|s|^2 I) w$$

and by condition 4) there exists λ_1 $(0 < \lambda_1 \leqq \lambda_0)$ such that $g - l^* g l \geqq 0$ for $\lambda \leqq \lambda_1$. Thus the scheme is stable for $\lambda \leqq \lambda_1$.

THEOREM 6.6. *Let*

(6.14) $$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{r} (i\lambda p)^j / j! - \lambda^\alpha v,$$

where

$$v(x, \omega; \lambda) = mI + \lambda^\beta a + \lambda^\gamma b \qquad (\beta, \gamma \geqq 0),$$

$$m(\omega; \lambda) = \textstyle\sum_{j=0}^{\mu} \lambda^j m_j(\omega) I, \quad \gamma \geqq \alpha \geqq 0,$$

$$a(x, \omega; \lambda) \in P[\lambda; \mathscr{M}], \quad b(x, \omega; \lambda) \in P[\lambda; \mathscr{M}],$$

$$a_1(x, \omega; \lambda) = a/|s| \in \mathscr{M}, \quad b_1(x, \omega; \lambda) = b/|s| \in \mathscr{M},$$

$m_j(\omega)$ $(j=0, 1, \ldots, \mu)$ *are scalar functions satisfying Condition* I. *Suppose*

1) $\tilde{b}^* + \tilde{b} = 0$;

2) $t(\omega)$ *satisfies Condition* I;

3) $|s(\omega)|^d \leqq c_1 t^2(\omega)$, $|m_j(\omega)| \leqq c_1 t^2(\omega)$ $\quad (j = 0, 1, \ldots, \mu)$;

4) $a_2(x, \omega; \lambda) = a/t^2 \in \mathscr{K}$, $b_2(x, \omega; \lambda) = b|s|/t^2 \in \mathscr{K}$ *and* a_2, b_1 *and* b_2 *satisfy Conditions* N *and* II;

5) $u(x, \omega; \lambda) \geqq c_2 t^2 I \quad$ *for* $\lambda \leqq \lambda_0$,

where $u = (m^* + m)I + \lambda^\beta(\tilde{a}^* + \tilde{a}) - \lambda^\alpha \tilde{v}^* \tilde{v}$. *Then the scheme* (2.2) *is stable for sufficiently small* λ.

PROOF. l can be rewritten in $\mathscr{K}$ as

$$l(x, \omega; \lambda) = c(\omega; \lambda)I + f|s|,$$

where

$$c(\omega; \lambda) = I - \lambda^\alpha m, \quad f = i\lambda p_z + y|s| - \lambda^\alpha(\lambda^\beta a_1 + \lambda^\gamma b_1).$$

By Lemma 4.4 $f \in \mathscr{M}$ and $c(\omega; \lambda)$ satisfies Condition I. By (6.5) and condition 1) we have

$$g - l^* g l = \lambda^\alpha w^*(u + \lambda q_2 + \lambda^\sigma \tilde{p}^d \tilde{q}_3) w,$$

where $\sigma = d - \alpha \geqq 1$, $q_3 \in P[\lambda; p]$ and q_2 is given by (6.10).

By condition 3) we can define

$$e_1(\omega) = |s(\omega)|^d / t^2(\omega), \quad e_2(\omega; \lambda) = \textstyle\sum_{j=0}^{\mu} \lambda^j m_j(\omega)/t^2(\omega)$$

and $e_j I \in \mathscr{K}$ $(j=1, 2)$. Put

$$q_{21} = q_2/t^2, \quad v_1 = e_2 I + \lambda^\beta a_2, \quad q_{11} = q_1/|s|,$$

$$q_4 = \tilde{v}_1^* \tilde{q}_1 + \lambda^\gamma \tilde{b}_2^* \tilde{q}_{11}.$$

Then $q_{21}(x, \omega; \lambda) = q_4 + q_4^* \in \mathscr{K}$ and we have

$$g - l^* g l = \lambda^\alpha t^2(\omega) r(x, \omega; \lambda),$$

where

$$r(x, \omega; \lambda) = w^*(u_1 - c_2 I) w + w^*(c_2 I + \lambda q_{21} + \lambda^\sigma \tilde{p}_z^d \tilde{q}_3 e_1) w,$$

$$u_1(x, \omega; \lambda) = \tilde{v}_1^* + \tilde{v}_1 - \lambda^\alpha(\tilde{v}_1^* \tilde{v} + \lambda^\gamma \tilde{b}^* \tilde{v}_1 + \lambda^{2\gamma} \tilde{b}_1^* \tilde{b}_2).$$

By condition 4) v_1 and v satisfy Conditions N and II, so that r satisfies the same conditions. Since by condition 5)

$$u_1(x, \omega; \lambda) \geqq c_2 I \quad \text{for} \quad \lambda \leqq \lambda_0,$$

there exist $c_3>0$ and λ_1 $(0<\lambda_1 \leqq \lambda_0)$ such that

$$r(x, \omega) \geqq c_3 w^* w \geqq c_3 e I \quad \text{for} \quad \lambda \leqq \lambda_1.$$

Hence conditions 1) and 2) of Theorem 4.2 are satisfied and the scheme is stable for $\lambda \leqq \lambda_1$.

6.4. Case of a regularly hyperbolic system

In this section we assume that $A_j(x)$ $(j=1, 2,\dots, n)$ are real matrices and that (1.1) is a regularly hyperbolic system, that is, eigenvalues of $A(x, \omega')$ are all real and distinct ($s=N$ in Condition C) [19].

THEOREM 6.7. *For a regularly hyperbolic system with real coefficients let*

$$l(x, \omega; \lambda) = I + i\lambda p(x, \omega) + \lambda^2 q(x, \omega; \lambda)|s(\omega)|^2, \tag{6.15}$$

where q is a polynomial in λ with coefficients satisfying Condition VI. *Suppose*

$$\rho(l(x, \omega; \lambda)) \leqq 1 \quad \textit{for} \quad \lambda \leqq \lambda_0. \tag{6.16}$$

Then the scheme (2.2) *is stable for sufficiently small λ.*

To prove the theorem we need the following

LEMMA 6.3. *Under the assumptions of the theorem there exist λ_1 $(0<\lambda_1 \leqq \lambda_0)$ and a nonsingular matrix $u(x, \omega; \lambda)$ such that*

i) *u and u^{-1} belong to $\mathscr{L}$ for each λ $(0<\lambda \leqq \lambda_1)$;*
ii) *$g(x, \omega; \lambda)=u^*u$ satisfies Condition* N *for each λ $(0<\lambda \leqq \lambda_1)$;*
iii) *For some $e_1>0$*

$$g(x, \omega; \lambda) \geqq e_1 I \quad \textit{for} \quad \lambda \leqq \lambda_1;$$

iv) $u(p_z - i\lambda q|s|)u^{-1} = d + \lambda|s|f \quad \textit{for} \quad \omega \in R^n - Z, \quad \lambda \leqq \lambda_1,$

where $d(x, \omega; \lambda)$ and $f(x, \omega; \lambda)$ are diagonal matrices belonging to $\mathscr{L}$ and d is a real matrix.

PROOF OF THEOREM 6.7. By Lemma 4.5 and its corollary,

$$\begin{aligned} G_h - L_h^* G_h L_h &\equiv G_h - L_h^\sharp \circ G_h \circ L_h \\ &= U_h^\sharp \circ (I_h - \tilde{L}_h^\sharp \circ \tilde{L}_h) \circ U_h, \end{aligned}$$

where $\tilde{l}(x, \omega; \lambda) = ulu^{-1}$. We have in $\mathscr{K}$

$$I - \tilde{l}^*\tilde{l} = \lambda^2|s|^2[i(f^* - f) - (d + \lambda|s|f^*)(d + \lambda|s|f)],$$

which satisfies conditions 1), 2) and 3) of Theorem 3.4 for $\lambda \leqq \lambda_1$ by Lemma 4.3. Since $\tilde{l}$ is a diagonal matrix by Lemma 6.3, from (6.16) it follows that

$$I - \tilde{l}^*\tilde{l} \geqq (1 - \rho(l))I \geqq 0 \qquad \text{for} \quad \lambda \leqq \lambda_1.$$

Hence $u^*(I - \tilde{l}^*\tilde{l})u$ satisfies all conditions of Theorem 3.4 and we have for some $c_1 \geqq 0$

$$\begin{aligned} &\mathrm{Re}((G_h - L_h^\# \circ G_h \circ L_h)\alpha_i v, \alpha_i v) \\ &\quad = \mathrm{Re}((U_h^\# \circ (I_h - \tilde{L}_h^\# \circ \tilde{L}_h) \circ U_h)\alpha_i v, \alpha_i v) \geqq -c_1 h \|\alpha_i v\|^2 \end{aligned}$$

$$\text{for all} \quad v \in L_2, \quad h > 0.$$

By the same argument as in the proof of Theorem 4.1 we have for some $c_2 \geqq 0$

$$\|v\|_{G_h}^2 - \|L_h v\|_{G_h}^2 \geqq \sum_{i=0}^s \mathrm{Re}((G_h - L_h^\# \circ G_h \circ L_h)\alpha_i v, \alpha_i v) - c_2 h \|v\|^2,$$

so that

$$\|v\|_{G_h}^2 - \|L_h v\|_{G_h}^2 \geqq -(c_1 + c_2)h\|v\|^2.$$

Hence for some $c_0 \geqq 0$

$$\|L_h v\|_{G_h}^2 \leqq (1 + c_0 h)\|v\|_{G_h}^2,$$

and by Corollary 2.1 the scheme is stable for $\lambda \leqq \lambda_1$.

7. Examples of schemes

In this section Conditions A, B and C are assumed. To construct difference schemes with accuracy of order r, we assume that $A_j(x)$ ($j = 1, 2, \dots, n$) are bounded and continuous together with their partial derivatives up to the r-th order, where $r = 3$ in examples 2 and 3 and $r = 4$ in examples 4 and 5.

We introduce the following difference operators:

$$\Delta_{1j} = (T_j - T_j^{-1})/2, \quad \Delta_{2j} = [8(T_j - T_j^{-1}) - (T_j^2 - T_j^{-2})]/12,$$

$$\delta_j = (T_j + T_j^{-1} - 2I)/4 \qquad (j = 1, 2, \dots, n),$$

$$P_{mh}(x) = \sum_{j=1}^n A_j(x)\Delta_{mj} \qquad (m = 1, 2),$$

$$F_{mh}(x, h) = \sum_{j\neq k} A_j \Delta_{mj}(A_k \Delta_{mk}) + \sum_{j=1}^{n} A_j(\Delta_{mj} A_j)\Delta_{mj},$$

$$K_{1h}(x, h) = F_{1h} + 4\sum_{j=1}^{n} A_j^2 \delta_j,$$

$$K_{2h}(x, h) = F_{2h} + 4\sum_{j=1}^{n} A_j^2 \delta_j(1-\delta_j/3),$$

$$Q_h(x, h) = F_{2h} + \sum_{j=1}^{n} A_j^2 \Delta_{1j}^2(1-4\delta_j/3).$$

Since by Corollary 4.2 $A_j(x)\in\mathscr{A}_0$ and $\Delta_{mj}A_j(x)\in\mathscr{B}_0$ ($j=1, 2,\dots, n$; $m=1, 2$), $P_{mh}(x)$ ($m=1, 2$) belong to $\mathscr{A}_h$ and $F_{mh}(x, h)$, $K_{mh}(x, h)$ ($m=1, 2$) and $Q_h(x, h)$ belong to $\mathscr{G}_h$.

In connection with these operators we define the following functions:

$$\alpha_j(\omega) = \sin\omega_j, \quad \beta_j(\omega) = \sin^2(\omega_j/2),$$

$$s_j(\omega) = \alpha_j(1+2\beta_j/3) \qquad (j = 1, 2,\dots, n),$$

$$p_1(x, \omega) = \sum_{j=1}^{n} A_j\alpha_j, \quad p_2(x, \omega) = \sum_{j=1}^{n} A_j s_j, \tag{7.1}$$

$$n_1(x, \omega) = 4\sum_{j=1}^{n} A_j^2\beta_j^2, \quad n_2(x, \omega) = (16/9)\sum_{j=1}^{n} A_j^2(2+\beta_j)\beta_j^3, \tag{7.2}$$

$$f(x, \omega) = (4/9)\sum_{j=1}^{n} A_j^2\alpha_j^2\beta_j^2, \tag{7.3}$$

$$k_m(x, \omega) = -p_m^2 - n_m \quad (m = 1, 2), \quad q(x, \omega) = -p_2^2 + f, \tag{7.4}$$

$$r_1(x, \omega) = (2/3)\sum_{j=1}^{n} A_j\alpha_j\beta_j, \quad r_{j+1}(x, \omega) = p_2 r_j + r_1 p_1^j \qquad (j = 1, 2). \tag{7.5}$$

Matrices $ip_m(x, \omega)$, $k_m(x, \omega)$ ($m=1, 2$) and $q(x, \omega)$ are symbols belonging to $P_{mh}(x)$, $K_{mh}(x, 0)$ ($m=1, 2$) and $Q_h(x, 0)$ respectively. By Lemmas 4.6 and 4.7 p_m, n_m, k_m ($m=1, 2$), r_j ($j=1, 2, 3$), f and q belong to $\mathscr{L}$ and satisfy Condition N.

Put

$$|\alpha| = (\textstyle\sum_{j=1}^{n}\alpha_j^2)^{1/2}, \quad |\beta| = (\sum_{j=1}^{n}\beta_j^2)^{1/2},$$

$$\sigma(\omega) = (\textstyle\sum_{j=1}^{n}\beta_j^3)^{1/2}, \quad \tau(\omega) = \sum_{j=1}^{n}\beta_j.$$

Then we have

$$|\alpha| \leqq |s| \leqq 5|\alpha|/3, \tag{7.6}$$

$$|\alpha|^2 \leqq 4\sqrt{n}|\beta|, \quad |\beta| \leqq \tau, \quad |\beta|^3 \leqq \sqrt{n}\sigma^2, \quad 9|s|^2/100 \leqq \sqrt{n}|\beta|.$$

From these it follows that

$$(\alpha_j/|s|)I \quad (j = 1, 2,\dots, n), \qquad (|\alpha|/|s|)I \in \mathscr{L}, \tag{7.7}$$

$$(\alpha_j/|\alpha|)I,\ (\beta_j/|\beta|)I\ (j = 1, 2,\dots, n), \quad (|s|/|\alpha|)I,\ (|\alpha|^2/|\beta|)I, \tag{7.8}$$

$$(|\beta|/\tau)I,\ (|\beta|^3/\sigma^2)I,\ (|s|^2/|\beta|)I,\ (|s|^2/\tau)I \in \mathscr{K}.$$

Hence by (7.1)–(7.8)

$$p_m/|s|\ (m=1, 2),\quad r_j/|s|^j\ (j=1, 2, 3),\quad f/|s|^2 \in \mathscr{L}, \tag{7.9}$$

$$n_m/|\beta|^{m+1}\ (m=1, 2),\quad r_j/(|\alpha|^j|\beta|)\ (j=1, 2, 3),\quad f/(|\alpha|^2|\beta|^2) \in \mathscr{K}, \tag{7.10}$$

and they satisfy Conditions N and II. It is clear that $|\beta(\omega)|$ and $\sigma(\omega)$ satisfy Condition I and

$$r_j(x, \omega) = p_2^j - p_1^j \qquad (j=1, 2, 3).$$

For simplicity we put $\mu = 1/n$. For a difference operator $S_h(x, h)$ let $l(x, \omega; \lambda)$ be a symbol belonging to $S_h(x, 0)$ and let $M(x, \omega; \lambda)$ denote a hermitian element of $\mathscr{K}$.

EXAMPLE 1. Let

$$S_h(x) = \textstyle\sum_{j=0}^{r} (\lambda P_{2h})^j/j!, \tag{7.11}$$

where $r=3$ or 4. Then $l(x, \omega; \lambda)$ can be written as (6.7). By Theorem 6.2 the scheme (2.2) with the operator (7.11) is stable if $\lambda\rho(p_z) \leqq \sqrt{3d}/\sqrt{n}$ in the case $r=3$ and is so if $\lambda\rho(p_z) \leqq 2\sqrt{2d}/\sqrt{n}$ in the case $r=4$, where $p_z = p_2/|s|$, $d = (2/25)\sqrt{40\sqrt{6}-15}$.

EXAMPLE 2. Let

$$S_h(x) = I - E_h + \lambda P_{2h} + \lambda^2 P_{2h} P_{1h}/2 + \lambda^3 P_{1h}^3/6, \tag{7.12}$$

where $E_h = \mu^2 \sum_{j=1}^{n} \Delta_{1j}^2 \sum_{k=1}^{n} \delta_k$. Then $l(x, \omega; \lambda)$ can be written in $\mathscr{K}$ as

$$l(x, \omega; \lambda) = \textstyle\sum_{j=0}^{3} (i\lambda p_2)^j/j! - v, \tag{7.13}$$

where

$$v(x, \omega; \lambda) = eI - \lambda^2 p_2 r_1/2 - i\lambda^3 r_3/6,$$

$$e(\omega) = \mu^2|\alpha|^2 t, \qquad t = \tau.$$

By (7.7)–(7.10) $v/|s|^2 \in \mathscr{L}$ and $v/(t|s|^2) \in \mathscr{K}$. Since $\mu^2|\alpha|^2 t \leqq 1$, by (7.6) we have for some λ_0 and M

$$\begin{aligned} u &= \tilde{v}^* + \tilde{v} - \tilde{v}^*\tilde{v} \\ &= t|s|^2[\mu^2(2-\mu^2|\alpha|^2 t)(|\alpha|/|s|)^2 I - \lambda^2 M] \geqq 0 \qquad \text{for } \lambda \leqq \lambda_0. \end{aligned}$$

Application of Theorem 6.5 with $a(x, \omega; \lambda) = v$, $b(x, \omega; \lambda) = 0$, $r=3$ and $m=($

shows that the scheme (2.2) with the operator (7.12) is stable for sufficiently small λ.

EXAMPLE 3. Let

$$S_h(x, h) = I - C_h + \lambda P_{2h} + \lambda^2 P_{1h}^2/2 + \lambda^3 K_{1h} P_{1h}/6, \tag{7.14}$$

where $C_h = \mu \sum_{j=1}^n \delta_j^2$. Then we have (7.13), where

$$v(x, \omega; \lambda) = cI + \lambda^2 a, \quad c(\omega) = \mu \sum_{j=1}^n \beta_j^2,$$

$$a(x, \omega; \lambda) = -r_2/2 + i\lambda(n_1 p_1 - r_3)/6.$$

Put $t = |\beta|$. Then by (7.7)–(7.10) $a/|s| \in \mathcal{M}$ and a/t^2 satisfies Conditions N and II. Hence for some λ_0 and M we have

$$\begin{aligned} u &= 2cI + \lambda^2(\tilde{a}^* + \tilde{a}) - \tilde{v}^* \tilde{v} \\ &= t^2[\mu(2 - \mu t^2)I - \lambda^2 M] \geqq 0 \qquad \text{for} \quad \lambda \leqq \lambda_0. \end{aligned}$$

Application of Theorem 6.6 with $m(\omega; \lambda) = c$, $b(x, \omega; \lambda) = 0$ and $r = 3$ yields the stability of the scheme (2.2) with the operator (7.14) for sufficiently small λ.

EXAMPLE 4. Let

$$S_h(x, h) = I + E_h + \lambda(I + \lambda P_{2h}/2 + \lambda^2 Q_h/6 + \lambda^3 P_{1h}^3/24) P_{2h}, \tag{7.15}$$

where $E_h = \mu^2 \sum_{j=1}^n \Delta_{1j}^2 \sum_{k=1}^n \delta_k^2$. Then we have in $\mathscr{K}$

$$l(x, \omega; \lambda) = \sum_{j=0}^4 (i\lambda p_2)^j / j! - v, \tag{7.16}$$

where

$$v(x, \omega; \lambda) = eI - i\lambda^3 f p_2/6 + \lambda^4 r_3 p_2/24, \qquad e = \mu^2 |\alpha|^2 |\beta|^2.$$

Put $t = |\beta|^2$. Then by (7.7)–(7.10) $v/|s|^2 \in \mathscr{L}$, and $v/(t|s|^2) \in \mathscr{K}$. Hence by (7.6) we have for some λ_0 and M

$$\begin{aligned} u &= \tilde{v}^* + \tilde{v} - \tilde{v}^* \tilde{v} \\ &= t|s|^2[\mu^2(2 - \mu^2 |\alpha|^2 t)(|\alpha|/|s|)^2 I - \lambda^2 M] \geqq 0 \qquad \text{for} \quad \lambda \leqq \lambda_0. \end{aligned}$$

Thus the scheme (2.2) with the operator (7.15) is stable for sufficiently small λ by applying Theorem 6.5 with $r = 4$ and $m = 0$.

EXAMPLE 5. Let

$$S_h(x, h) = I + E_h + \lambda(I + \lambda P_{2h}/2 + \lambda^2 K_{2h}/6 + \lambda^3 K_{1h} P_{1h}/24) P_{2h}, \tag{7.17}$$

where $E_h = \mu \sum_{j=1}^n \delta_j^3$. Then we have (7.16), where

$$v(x, \omega; \lambda) = eI+\lambda^3 a, \qquad e = \mu\sigma^2,$$

$$a(x, \omega; \lambda) = [in_2+\lambda(r_3-n_1p_1)/4]p_2/6.$$

Put $t=\sigma$. Then by (7.7)-(7.10) $a/|s|$ belongs to $\mathscr{M}$ and a/t^2 satisfies Conditions N and II. Hence for some λ_0 and M we have

$$\begin{aligned} u &= 2eI+\lambda^3(\tilde{a}^*+\tilde{a})-\tilde{v}^*\tilde{v} \\ &= t^2[\mu(2-\mu t^2)I-\lambda^2 M] \geqq 0 \qquad \text{for} \quad \lambda \leqq \lambda_0. \end{aligned}$$

By Theorem 6.6 the scheme (2.2) with the operator (7.17) is stable for sufficiently small λ.

8. Proofs

In 8.1–8.5 we denote $\operatorname{ess.sup}_{\omega}$ by $\sup_{\omega}$ for short.

8.1. Proof of Theorem 3.3

Let α_i $(0\leqq i\leqq s)$ be the family associated with $\alpha_i(x)I$. Then $\alpha_i(x)u(x)=(\alpha_i u)(x)$ $(0\leqq i\leqq s)$. Since

$$|\textstyle\sum_{i=0}^{s} \operatorname{Re}(G_h\alpha_i u, \alpha_i u)| \leqq \sum_{i=0}^{s}\|\hat{g}\|_F\|\alpha_i u\|^2 = \|\hat{g}\|_F\|u\|^2,$$

we have the second inequality of (3.25).

By continuity of the L_2-norm it suffices to prove the first inequality in the case $u\in\mathscr{S}$. We consider first the case $1\leqq i\leqq s$. From (3.12) it follows that

$$(G_h\alpha_i u, \alpha_i u) = (\alpha_i G_h\alpha_i u, u),$$

$$\alpha_i G_h\alpha_i u = \alpha_i(x)\kappa^{-1}\int e^{ix\cdot\xi}g(x, h\xi)\widehat{\alpha_i u}(\xi)d\xi.$$

Without loss of generality we may assume that $x^{(i)}$ is the origin. By the mean value theorem we have

$$g(x, h\xi) = g(0, h\xi)+\textstyle\sum_j x_j\displaystyle\int_0^1 g_j(\theta x, h\xi)d\theta,$$

where $g_j(x, \omega)=D_j g(x, \omega)$. Since $g(0, h\xi)\geqq eI$ by condition 2), it follows that

$$\operatorname{Re}(G_h\alpha_i u, \alpha_i u) \geqq e\|\alpha_i u\|^2-\textstyle\sum_j|(G'_{jh}\alpha_i u, x_j\alpha_i u)|, \tag{8.1}$$

where

$$G'_{jh}\alpha_i u(x) = \kappa^{-1}\int e^{ix\cdot\xi}\int_0^1 g_j(\theta x, h\xi)d\theta\widehat{\alpha_i u}(\xi)d\xi.$$

Let $\{\varepsilon_k\}$ be any sequence such that $\varepsilon_k>0$ and $\varepsilon_k\to 0$ as $k\to\infty$. Then by the boundedness of g_j we have

$$(8.2)\qquad (G'_{jh}\alpha_i u,\ x_j\alpha_i u) = \lim_{k\to\infty}(w_{jk},\ x_j\alpha_i u),$$

where

$$w_{jk}(x) = \kappa^{-1}\int e^{ix\cdot\xi}g_{jk}(x,\ h\xi)\widehat{\alpha_i u}(\xi)d\xi,$$

$$g_{jk}(x,\ \omega) = \int_{\varepsilon_k}^{1} g_j(\theta x,\ \omega)d\theta.$$

Since $\operatorname{supp}(x_j\alpha_i u)\subset V_i$, we have

$$\|x_j\alpha_i u\| \leqq \varepsilon\|\alpha_i u\|.$$

Combining this with the estimate (to be shown later)

$$(8.3)\qquad \|w_{jk}\| \leqq c_j\|\alpha_i u\|, \qquad c_j = \int \sup_{\omega}|\hat{g}_j(\chi,\ \omega)|d\chi,$$

we obtain

$$|(w_{jk},\ x_j\alpha_i u)| \leqq \varepsilon c_j\|\alpha_i u\|^2,$$

which yields by (8.2)

$$|(G'_{jh}\alpha_i u,\ x_j\alpha_i u)| \leqq \varepsilon c_j\|\alpha_i u\|^2.$$

From this and (8.1) with $c=\sum_{j=1}^{n}c_j$ we have

$$\operatorname{Re}(G_h\alpha_i u,\ \alpha_i u) \geqq e\|\alpha_i u\|^2 - c\varepsilon\|\alpha_i u\|^2,$$

so that

$$(8.4)\qquad \textstyle\sum_{i=1}^{s}\operatorname{Re}(G_h\alpha_i u,\ \alpha_i u) \geqq e\sum_{i=1}^{s}\|\alpha_i u\|^2 - c\varepsilon(\sum_{i=1}^{s}\|\alpha_i u\|^2).$$

Next we consider the case $i=0$. Let $G_{\infty h}$ and G_{0h} be the families associated with $g_\infty(\omega)$ and $g_0(x,\ \omega)$ respectively. Then

$$\operatorname{Re}(G_h\alpha_0 u,\ \alpha_0 u) = \operatorname{Re}(G_{\infty h}\alpha_0 u,\ \alpha_0 u)+\operatorname{Re}(\alpha_0 G_{0h}\alpha_0 u,\ u),$$

$$(G_{\infty h}\alpha_0 u,\ \alpha_0 u) \geqq e\|\alpha_0 u\|^2,$$

because $g_\infty(\omega)\geqq eI$. Since by definition

$$\alpha_0 G_{0h}\alpha_0 u = \alpha_0(G_{0h}(\alpha_0 u)) = (\alpha_0 G_{0h})(\alpha_0 u)$$

and $\alpha_0 G_{0h}=\alpha_0\circ G_{0h}$ by Corollary 3.1, we have

$$\alpha_0 G_{0h} \alpha_0 u = (\alpha_0 \circ G_{0h})(\alpha_0 u).$$

Hence it follows that

$$\mathrm{Re}(G_h \alpha_0 u, \alpha_0 u) \geqq e\|\alpha_0 u\|^2 - \|\widehat{\alpha_0 g_0}\|_F \|\alpha_0 u\| \|u\|.$$

From this and (8.4) we have

$$\textstyle\sum_{i=0}^{s} \mathrm{Re}(G_h \alpha_i u, \alpha_i u) \geqq e\|u\|^2 - c\varepsilon\|u\|^2 - \|\widehat{\alpha_0 g_0}\|_F \|u\|^2.$$

Now we choose ε small so that $c\varepsilon \leqq e/4$, and then choose R large so that $\|\widehat{\alpha_0 g_0}\|_F \leqq e/4$. This choice of R is possible by N–2). For such ε and R we have

$$\textstyle\sum_{i=0}^{s} \mathrm{Re}(G_h \alpha_i u, \alpha_i u) \geqq (e/2)\|u\|^2, \tag{8.5}$$

which is the first inequality of (3.25).

It remains to show (8.3). Since $g_j(x, \omega)$ is continuous and integrable with respect to x for each ω, by the change of order of integration we have

$$\int |g_{jk}(x, \omega)| dx \leqq \int_{\varepsilon_k}^{1} \int |g_j(\theta x, \omega)| dx d\theta = \int |g_j(x, \omega)| dx \int_{\varepsilon_k}^{1} 1/|\theta|^n d\theta.$$

Hence $g_{jk}(x, \omega)$ is integrable for each ω, and

$$\begin{aligned} \hat{g}_{jk}(\chi, \omega) &= \kappa \int_{\varepsilon_k}^{1} \int e^{-ix\cdot\chi} g_j(\theta x, \omega) dx d\theta \\ &= \int_{\varepsilon_k}^{1} \hat{g}_j(\chi/\theta, \omega)/|\theta|^n d\theta. \end{aligned} \tag{8.6}$$

Since $\hat{g}_j(\chi, \omega)$ is integrable for each ω, it follows that

$$\begin{aligned} \int |\hat{g}_{jk}(\chi, \omega)| d\chi &\leqq \int\int_{\varepsilon_k}^{1} |\hat{g}_j(\chi/\theta, \omega)|/|\theta|^n d\theta d\chi \\ &= \int_{\varepsilon_k}^{1} \int |\hat{g}_j(\chi/\theta, \omega)|/|\theta|^n d\chi d\theta \\ &\leqq \int |\hat{g}_j(\chi, \omega)| d\chi. \end{aligned}$$

Hence $\hat{g}_{jk}(\chi, \omega)$ is integrable for each ω and by N–1) we have from (8.6)

$$\int \sup_{\omega} |\hat{g}_{jk}(\chi, \omega)| d\chi \leqq c_j \qquad (j = 1, 2, \ldots, n).$$

Put

$$v_{jk}(\xi) = \int \hat{g}_{jk}(\xi - \xi', h\xi') \widehat{\alpha_i u}(\xi') d\xi'.$$

Then by the same argument as in the proof of Lemma 3.2 we have

$$\int |v_{jk}(\xi)|d\xi \leqq c_j \int |\widehat{\alpha_i u}(\xi)|d\xi,$$

$$\|v_{jk}\| \leqq c_j \|\alpha_i u\| . \tag{8.7}$$

Since $v_{jk} \in L_1 \cap L_2$,

$$\text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} v_{jk}(\xi)d\xi = w_{jk}(x) \quad \text{a.e.}.$$

Thus $\|v_{jk}\| = \|w_{jk}\|$ and (8.3) holds by (8.7).

8.2. Proof of Theorem 3.4

By continuity of the L_2-norm it suffices to prove the theorem in the case $u \in \mathscr{S}$. Let σ be a space variable in R^n, $B_0 = \{\sigma \mid |\sigma| \leqq 1\}$ and $q(\sigma)$ be a C^∞ even function such that

i) $q(\sigma) \geqq 0$, $\operatorname{supp} q(\sigma) \subset B_0$;

ii) $\int q^2(\sigma)d\sigma = 1$.

After Vaillancourt [16] we introduce the functions

$$a(x, \omega) = c^{-n} \int p(x, \zeta) e^2(\omega, \zeta)d\zeta,$$

$$b(\tilde{\omega}, x, \omega) = c^{-n} \int e(\tilde{\omega}, \zeta) p(x, \zeta) e(\omega, \zeta)d\zeta,$$

where

$$c = h^{1/2}, \quad \zeta = \omega - c\sigma, \quad e(\omega, \zeta) = q(c^{-1}[\omega - \zeta]).$$

As will be shown in the proof of Lemma A, the families of operators A_h and B_h can be defined by

$$A_h u(x) = \text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} \int \hat{a}(\xi - \xi', h\xi') \hat{u}(\xi')d\xi' d\xi, \tag{8.8}$$

$$B_h u(x) = \text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} \int \hat{b}(h\xi, \xi - \xi', h\xi') \hat{u}(\xi')d\xi' d\xi \tag{8.9}$$

for all $u \in \mathscr{S}$,

where $\hat{b}(\tilde{\omega}, \chi, \omega)$ is the Fourier transform of $b(\tilde{\omega}, x, \omega)$ with respect to x.

LEMMA A. *A_h and B_h are families of bounded linear operators mapping*

$\mathscr{S}$ *into* L_2 *and*

(8.10) $$(B_h u, u) \geqq 0 \qquad \textit{for all} \quad u \in \mathscr{S},$$

(8.11) $$A_h \equiv P_h,$$

(8.12) $$A_h + A_h^* \equiv 2B_h.$$

By this lemma we have

$$\begin{aligned}\operatorname{Re}(P_h u, u) &\geqq \operatorname{Re}(P_h u, u) - (B_h u, u)\\ &\geqq \operatorname{Re}((P_h - A_h)u, u) + ((A_h + A_h^* - 2B_h)u, u)/2\\ &\geqq -\|P_h - A_h\|\,\|u\|^2 - \|A_h + A_h^* - 2B_h\|\,\|u\|^2/2.\end{aligned}$$

Hence (3.28) holds by (8.11) and (8.12).

Proof of Lemma A. Let

$$w(\xi) = \int \hat{b}(h\xi, \xi - \xi', h\xi')\hat{u}(\xi')d\xi'.$$

Then

$$w(\xi) = \int r_0(\xi - \xi', h\xi')\hat{u}(\xi')d\xi' + r_\infty(h\xi)\hat{u}(\xi),$$

where

$$r_0(\chi, \omega) = c^{-n}\int e(h\chi + \omega, \zeta)\hat{p}_0(\chi, \zeta)e(\omega, \zeta)d\zeta,$$

$$r_\infty(\omega) = c^{-n}\int e(\omega, \zeta)p_\infty(\zeta)e(\omega, \zeta)d\zeta.$$

By condition i) we have

$$\int \sup_\omega |r_0(\chi, \omega)|d\chi \leqq L\int \sup_\omega |\hat{p}_0(\chi, \omega)|d\chi,$$

$$\sup_\omega |r_\infty(\omega)| \leqq L \sup_\omega |p_\infty(\omega)|,$$

where $L = \max_\eta q^2(\eta)\int_{|\sigma|\leqq 1} 1\, d\sigma$.

By the same argument as in the proof of Lemma 3.2 we have $\|w\| \leqq L\|\hat{p}\|_F\|\hat{u}\|$. Hence $w \in L_2$, and the formula (8.9) defines a family of bounded linear operators B_h. The same reasoning applies also to A_h.

We show (8.10). Put

(8.13) $$\hat{v}(\xi, \zeta) = e(h\xi, \zeta)\hat{u}(\xi).$$

Then $|\hat{v}(\xi, \zeta)|^2$ is integrable for each fixed ζ. Hence there exists the Fourier inverse transform $v(x, \zeta)$ such that $|v(x, \zeta)|^2$ is integrable for each fixed ζ. Since $p(x, \zeta) \geqq 0$, it follows that

$$v^*(x, \zeta)p(x, \zeta)v(x, \zeta) \geqq 0.$$

Integration of this inequality with respect to x yields by Plancherel's formula

(8.14) $$\int v^*(x, \zeta)p(x, \zeta)v(x, \zeta)dx$$
$$= \iint \hat{v}^*(\xi, \zeta)\hat{p}(\xi-\xi', \zeta)\hat{v}(\xi', \zeta)d\xi' d\xi \geqq 0.$$

Substituting (8.13) into (8.14) and then integrating it with respect to ζ, by the change of order of integration we have $(\hat{u}, w) \geqq 0$, which shows (8.10), because $w = \widehat{B_h u}$ by (8.9).

Since

$$a(x, \omega) = \int p(x, \omega - c\sigma)q^2(\sigma)d\sigma,$$

from (8.8) it follows that

(8.15) $$\widehat{(P_h - A_h)u}(\xi) = \int \{\hat{p}(\chi, \omega) - \hat{a}(\chi, \omega)\}\hat{u}(\xi')d\xi'$$
$$= \iint \{\hat{p}(\chi, \omega) - \hat{p}(\chi, \omega - c\sigma)\}q^2(\sigma)d\sigma\hat{u}(\xi')d\xi',$$

where $\chi = \xi - \xi'$, $\omega = h\xi'$.

Owing to condition 1) we have by the mean value theorem

(8.16) $$\hat{p}_0(\chi, \omega) - \hat{a}_0(\chi, \omega) = c\int \sum_{j=1}^n \sigma_j \int_0^1 \partial_j \hat{p}_0(\chi, \omega - \theta c\sigma)q^2(\sigma)d\theta d\sigma.$$

Since $\partial_j \hat{p}_0(\chi, \omega)$ is absolutely continuous with respect to ω_k,

(8.17) $$\partial_j \hat{p}_0(\chi, \omega) - \partial_j \hat{p}_0(\chi, \omega - \rho)$$
$$= \sum_{k=1}^n \{\partial_j \hat{p}_0(\chi, \omega_1, \ldots, \omega_{k-1}, \omega_k, \eta_{k+1}, \ldots, \eta_n)$$
$$- \partial_j \hat{p}_0(\chi, \omega_1, \ldots, \omega_{k-1}, \eta_k, \eta_{k+1}, \ldots, \eta_n)\}$$
$$= \sum_{k=1}^n m_{kj}(\chi, \eta, \omega),$$

where $\rho = \theta c\sigma$, $\eta = \omega - \rho$,

$$m_{kj}(\chi, \eta, \omega) = -\int_0^{\rho_l} \partial_k\partial_j \hat{p}_0(\chi, \omega_1, \dots, \omega_{k-1}, \omega_k - t_k, \eta_{k+1}, \dots, \eta_n) dt_k.$$

Hence by (8.16) and (8.17)

(8.18) $$\hat{p}_0(\chi, \omega) - \hat{a}_0(\chi, \omega)$$
$$= c\int \sum_{j=1}^n \sigma_j \int_0^1 \partial_j \hat{p}_0(\chi, \omega) q^2(\sigma) d\theta d\sigma - ck(\chi, \omega),$$

where

$$k(\chi, \omega) = \int \sum_{j=1}^n \sigma_j \int_0^1 \sum_k m_{kj}(\chi, \eta, \omega) q^2(\sigma)\, d\theta d\sigma.$$

The first term on the right side of (8.18) vanishes, because $q^2(\sigma)$ is even. Since

$$|ck(\chi, \omega)| \leqq c\iint_0^1 \sum_{j,k} |\sigma_j|\, |\rho_l| \sup_\omega |\partial_k\partial_j \hat{p}_0(\chi, \omega)| q^2(\sigma) d\theta d\sigma$$
$$\leqq h \sum_{j,k} \sup_\omega |\partial_k\partial_j \hat{p}_0(\chi, \omega)| \quad \text{a.e.},$$

from (8.18) it follows that

$$|\hat{p}_0(\chi, \omega) - \hat{a}_0(\chi, \omega)| \leqq h \sum_{j,k} \sup_\omega |\partial_k\partial_j \hat{p}_0(\chi, \omega)| \quad \text{a.e.}.$$

Similarly we have

$$|p_\infty(\omega) - a_\infty(\omega)| \leqq h \sum_{j,k} \sup_\omega |\partial_k\partial_j p_\infty(\omega)| \quad \text{a.e.}.$$

The same argument as in the proof of Lemma 3.2 yields from (8.15)

$$\|\widehat{(P_h - A_h)u}\| \leqq Mh \|\hat{u}\|,$$

where

$$M = \sum_{j,k}\left(\int \sup_\omega |\partial_k\partial_j \hat{p}_0(\chi, \omega)| d\chi + \sup_\omega |\partial_k\partial_j p_\infty(\omega)|\right).$$

Hence (8.11) holds.

From (8.8) and (3.20) it follows that

(8.19) $$\widehat{(A_h + A_h^* - 2B_h)u}(\xi)$$
$$= c^{-n}\iint \hat{p}(\chi, \zeta)\{e(h\chi + \omega, \zeta) - e(\omega, \zeta)\}^2 \hat{u}(\xi') d\zeta d\xi',$$
$$= \iint \hat{p}_0(\chi, \zeta)\{q(\chi' + \sigma) - q(\sigma)\}^2 \hat{u}(\xi') d\sigma d\xi',$$

where $\chi'=c\chi$, $\chi=\xi-\xi'$, $\omega=h\xi'$, $\zeta=\omega-c\sigma$. By the mean value theorem we have

$$\begin{aligned}
&\left|\int \hat{p}_0(\chi,\zeta)\{q(\chi'+\sigma)-q(\sigma)\}^2 d\sigma\right| \\
&\leqq h\int\left|\hat{p}_0(\chi,\zeta)\left\{\sum_j \chi_j \frac{\partial q}{\partial\sigma_j}(\sigma+\theta\chi')\right\}^2\right| d\sigma \\
&\leqq hK_1 \sup_{\omega}(|\chi|^2|\hat{p}_0(\chi,\omega)|) \quad \text{a.e.},
\end{aligned}$$

where

$$K_1 = n\max_j\left\{\max_\eta\left(\left|\frac{\partial q}{\partial\eta_j}(\eta)\right|^2\right)\right\}\int_{|\sigma|\leqq 1} d\sigma.$$

From (8.19) it follows as in the proof of (8.11) that

$$\|\widehat{(A_h+A_h^*-2B_h)u}\| \leqq K_2 h\|\hat{u}\|,$$

where $K_2=\int\sup_\omega(|\chi|^2|\hat{p}_0(\chi,\omega)|)d\chi$. Hence (8.12) holds.

In the following for simplicity we put

$$S_\omega = R^n_\omega,\quad S_z = R^n_\omega - Z,\quad S_\chi = R^n_\chi,\quad S_x = R^n_x,\quad S_t = R^n_t,\quad S_0 = R^n_\omega-\{0\}$$

and let

$$S_{ab} = S_a\times S_b,\quad S_{abc} = S_a\times S_b\times S_c,$$

where a, b and c denote ω, z, χ, x, t or 0. We denote by $M[x,\chi,z]$ the set of all bounded and measurable $N\times N$ matrix functions on $S_{x\chi z}$ and denote by $C[\chi, z]$ the set of all bounded and continuous $N\times N$ matrix functions on $S_{\chi z}$. The sets $M[z]$, $M[\chi,z]$, $C[0]$, $C[\chi,0]$, etc. are also defined in the same manner.

8.3. Proof of Lemma 4.1

We show (i). Let $l(\chi,\omega)=\hat{p}|s|$. Then by I′–1) l belongs to $\mathscr{K}$ and satisfies I–1). Let c_j $(j=1, 2, 3)$ be constants such that

$$|\partial_j s_k(\omega)| \leqq c_1 \qquad \text{on} \quad S_\omega \quad (j,k=1,2,\dots,n),$$

$$|\partial_j l_0(\chi,\omega)| \leqq c_2,\quad |\hat{p}_0(\chi,\omega)| \leqq c_3 \qquad \text{on} \quad S_{\chi z} \quad (j=1,2,\dots,n).$$

Denote by $L(\tilde{\omega},\omega)$ the line segment joining the points $\tilde{\omega}$ and ω, where

$$\tilde{\omega} = (\omega_1,\dots,\omega_{j-1},\tilde{\omega}_j,\omega_{j+1},\dots,\omega_n),\quad \omega=(\omega_1,\omega_2,\dots,\omega_n).$$

When there lies no point of Z on $L(\tilde{\omega}, \omega)$, by I′–3) we have

$$(8.20) \qquad l_0(\chi, \tilde{\omega}) - l_0(\chi, \omega) = (\tilde{\omega}_j - \omega_j)\partial_j l_0(\chi, \eta),$$

where η is some point on $L(\tilde{\omega}, \omega)$.

When a point $\hat{\omega}$ of Z lies on $L(\tilde{\omega}, \omega)$, we have $|s(\hat{\omega})| = 0$ and

$$l_0(\chi, \tilde{\omega}) - l_0(\chi, \omega) = \hat{p}_0(\chi, \tilde{\omega})(|s(\tilde{\omega})| - |s(\hat{\omega})|) + \hat{p}_0(\chi, \omega)(|s(\hat{\omega})| - |s(\omega)|),$$

where the first (or second) term on the right side vanishes if $\tilde{\omega} \in Z$ (or $\omega \in Z$). Hence it follows that

$$(8.21) \qquad \begin{aligned} |l_0(\chi, \tilde{\omega}) - l_0(\chi, \omega)| &\leqq c_3(|s(\tilde{\omega}) - s(\hat{\omega})| + |s(\hat{\omega}) - s(\omega)|) \\ &\leqq \sqrt{n c_1} c_3(|\tilde{\omega}_j - \hat{\omega}_j| + |\hat{\omega}_j - \omega_j|) \\ &= \sqrt{n c_1} c_3 |\tilde{\omega}_j - \omega_j|. \end{aligned}$$

From (8.20) and (8.21) we have

$$|l_0(\chi, \tilde{\omega}) - l_0(\chi, \omega)| \leqq c_4 |\tilde{\omega}_j - \omega_j| \qquad \text{for} \quad \tilde{\omega}, \omega \in R^n,$$

where $c_4 = \max(c_2, \sqrt{n c_1} c_3)$. Thus $l_0(\chi, \omega)$ is absolutely continuous with respect to ω_j. Hence l_0 satisfies I–2), because $\partial_j l_0 \in M[\chi, z]$. Similarly l_∞ satisfies I–2).

From I′–3) and I′–4) it follows that l satisfies I–3).

The assertion (ii) can be shown similarly.

8.4. Proof of Lemma 4.3

Since $\sup_\omega (|\chi|^2 |\hat{p}_0 |s|^2|)$ is integrable by IV, it suffices to show that conditions 1) and 2) of Theorem 3.4 are satisfied. By I′–1)–I′–3) and V–2) $\partial_j l_0(\chi, \omega)$, $\partial_k m_{j0}(\chi, \omega) \in M[\chi, z]$; $\sup_\omega |\partial_j l_0(\chi, \omega)|$, $\sup_\omega |\partial_k m_{j0}(\chi, \omega)| \in M[\chi]$ and $\partial_j l_\infty(\omega)$, $\partial_k m_{j\infty}(\omega) \in M[z]$.

Let $r_0 = \hat{p}_0 |s|^2$ and $r_\infty = p_\infty |s|^2$. Then by I′–1) and I′–2) $r_0 \in M[\chi, \omega]$ and $r_\infty(\omega) \in M[\omega]$. By I′–3) we have for $\omega \in S_z$

$$(8.22) \qquad \partial_j r_0 = m_{j0} + l_0(\partial_j |s|), \quad \partial_j r_\infty = m_{j\infty} + l_\infty(\partial_j |s|).$$

Since the terms on the right sides are continuous on S_z for each χ, so are $\partial_j r_0$ and $\partial_j r_\infty$.

Let $\omega^{(0)}$ be any point of Z. Then $\partial_j r_0(\chi, \omega^{(0)})$ and $\partial_j r_\infty(\omega^{(0)})$ are calculated to be zero. By I′–2) and I′–3) $\hat{p}_0$ and $\partial_j l_0$ are bounded on $S_{\chi z}$; p_∞, $\partial_j l_\infty$ and $\partial_j |s|$ are bounded on S_z. Hence the terms on the right sides of (8.22) tend to zero as $\omega\to\omega^{(0)}$. Therefore $\partial_j r_0$ and $\partial_j r_\infty$ are continuous on S_ω for each χ.

By the same argument as in the proof of Lemma 4.1 m_{j0}, $l_0(\partial_j|s|)$, $m_{j\infty}$ and $l_\infty(\partial_j|s|)$ are absolutely continuous with respect to ω_k. Hence by (8.22) $\partial_j r_0$ and $\partial_j r_\infty$ have the same property and condition 1) is satisfied.

By I′–3) and V–2) we have from (8.22) for $\omega\in S_z$

$$\partial_k\partial_j r_0 = \partial_k m_{j0}+(\partial_k l_0)(\partial_j|s|)+\hat{p}_0|s|(\partial_k\partial_j|s|),$$

$$\partial_k\partial_j r_\infty = \partial_k m_{j\infty}+(\partial_k l_\infty)(\partial_j|s|)+p_\infty|s|(\partial_k\partial_j|s|),$$

and $\partial_k\partial_j r_0\in M[\chi, z]$, $\partial_k\partial_j r_\infty\in M[z]$, $\sup_\omega|\partial_k\partial_j r_0|\in M[\chi]$. By the conditions $\sup_\omega|\partial_k\partial_j r_0|$ is integrable and $\sup_\omega|\partial_k\partial_j r_\infty|$ is finite, so that condition 2) is satisfied.

8.5. Proof of Lemma 4.4

We prove that *if p and q satisfy* (a) II (*or* IV) (b) I′ (c) I′, II *and* III′ *or* (d) V, *then $p+q$, pq and p^* satisfy the corresponding conditions.* For properties (i) and (ii) of the lemma follow from (a) and (c) respectively; property (iii) follows from (a), (c) and (d). It suffices to show these assertions only for pq.

Put $d=pq$. Then by Lemma 3.1 $d\in\mathscr{K}$, $d_\infty\in M[\omega]$ and $\sup_\omega|\hat{d}_0(\chi, \omega)|$ is integrable.

We prove (a). Since

$$\hat{d}_0(\chi, \omega) = \hat{p}_0*\hat{q}_0+\hat{p}_0q_\infty+p_\infty\hat{q}_0, \qquad d_\infty = p_\infty q_\infty, \tag{8.23}$$

we have

$$|\chi|\,|\hat{d}_0| \leqq \int|\chi-t|\,|\hat{p}_0(\chi-t, \omega)|\,|\hat{q}_0(t, \omega)|dt+\int|\hat{p}_0(\chi-t, \omega)|\,|t|\,|\hat{q}_0(t, \omega)|dt + |\chi|\,|\hat{p}_0|\,|q_\infty|+|p_\infty|\,|\chi|\,|\hat{q}_0|, \tag{8.24}$$

$$|\chi|^2|\hat{d}_0| \leqq 2\Big\{\int|\chi-t|^2|\hat{p}_0(\chi-t, \omega)|\,|\hat{q}_0(t, \omega)|dt + \int|\hat{p}_0(\chi-t, \omega)|\,|t|^2|\hat{q}_0(t, \omega)|dt\Big\}+|\chi|^2|\hat{p}_0|\,|q_\infty|+|p_\infty|\,|\chi|^2|\hat{q}_0|. \tag{8.25}$$

Taking the essential suprema of both sides of (8.24) and (8.25) over S_ω and integrating them with respect to χ, we find that $\sup_\omega(|\chi|^k|\hat{d}_0(\chi, \omega)|)$ is integrable in the case $k=1$ (or $k=2$) if p and q satisfy II (or IV).

We prove (b). Let

$$v_0(\chi, \omega) = \hat{q}_0|s|,\ v_\infty(\omega) = q_\infty|s|,\ e_0(\chi, \omega) = \hat{d}_0|s|,\ e_\infty(\omega) = d_\infty|s|.$$

Then $\partial_j l_0(\chi, \omega)$, $\partial_j v_0(\chi, \omega) \in M[\chi, z]$ and $\partial_j l_\infty(\omega)$, $\partial_j v_\infty \in C[z]$; $\partial_j l_0(\chi, \omega)$ and $\partial_j v_0(\chi, \omega)$ are measurable on S_χ for each $\omega \in S_z$.

It can be shown that if $f(\chi, \omega)$ is measurable on $S_{\chi z}$ and is continuous on S_z for each χ, then $\sup_\omega |f(\chi, \omega)|$ is measurable on S_χ and

$$(8.26) \qquad |f(\chi, \omega)| \leqq \sup_\omega |f(\chi, \omega)| \qquad \text{on} \quad S_{\chi z}.$$

Hence by I′–1)–I′–3) $\sup_\omega |\hat{p}_0(\chi, \omega)|$, $\sup_\omega |\hat{q}_0(\chi, \omega)|$, $\sup_\omega |\partial_j l_0(\chi, \omega)|$ and $\sup_\omega |\partial_j v_0(\chi, \omega)|$ belong to $M[\chi]$.

Let c_k $(k=1, 2, 3, 4)$ be constants such that

$$|s(\omega)| \leqq c_1 \qquad \text{on} \quad S_\omega,$$

$$(8.27) \qquad |\partial_j |s(\omega)|| \leqq c_2 \quad (j = 1, 2, \ldots, n) \qquad \text{on} \quad S_z,$$

$$|\hat{p}_0(\chi, \omega)| \leqq c_3, \quad |\partial_j l_0(\chi, \omega)| \leqq c_4 \quad (j = 1, 2, \ldots, n) \qquad \text{on} \quad S_{\chi z}.$$

Then by (8.26)

$$|\hat{p}_0(\chi-t, \omega)\hat{q}_0(t, \omega)| \leqq c_3 \sup_\omega |\hat{q}_0(t, \omega)| \qquad \text{for} \quad (t, \chi, \omega) \in S_{t\chi z}.$$

Integration of both sides with respect to t shows that $\hat{p}_0 * \hat{q}_0$ is bounded on $S_{\chi z}$. By I′–1) and I′–2) $p_\infty \hat{q}_0$ and $\hat{p}_0 q_\infty$ are bounded on $S_{\chi z}$. Hence I′–2) is satisfied by (8.23).

By (8.23) we have

$$(8.28) \qquad e_0 = l_0 * \hat{q}_0 + l_0 q_\infty + l_\infty \hat{q}_0, \quad e_\infty = l_\infty q_\infty.$$

By I′–1) and I′–2) $l_0(\chi-t, \omega)\hat{q}_0(t, \omega)$ belong to $M[t, \chi, z]$ and is integrable with respect to t for each $(\chi, \omega) \in S_{\chi z}$. By I′–3) we have for $\omega \in S_z$

$$(8.29) \qquad \partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\} = (\partial_j l_0(\chi-t, \omega))\hat{q}_0(t, \omega)$$
$$+ \hat{p}_0(\chi-t, \omega)\partial_j v_0(t, \omega) - \hat{p}_0(\chi-t, \omega)\hat{q}_0(t, \omega)(\partial_j |s|),$$

so that by (8.26)

$$|\partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\}| \leqq \varphi(t),$$

where

$$\varphi(t) = (c_2 c_3 + c_4) \sup_\omega |\hat{q}_0(t, \omega)| + c_3 \sup_\omega |\partial_j v_0(t, \omega)|,$$

which is integrable by I′–1) and I′–4). Hence

$$(8.30) \qquad \partial_j(l_0 * \hat{q}_0) = \int \partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\} dt \qquad \text{for} \quad (\chi, \omega) \in S_{\chi z},$$

$\partial_j(l_0 * \hat{q}_0) \in M[\chi, z]$ and $\sup_\omega |\partial_j(l_0 * \hat{q}_0)| \in M[\chi]$.
By I′–3) and (8.29) $\partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\}$ is continuous on S_z for each (χ, t) and is dominated by $\varphi(t)$, so that $\partial_j(l_0 * \hat{q}_0)$ is continuous on S_z for each χ.

By I′–1)–I′–3) $\partial_j(l_\infty \hat{q}_0)$, $\partial_j(l_0 q_\infty) \in M[\chi, z]$ and $\partial_j(l_\infty q_\infty) \in M[z]$; they are continuous on S_z for each χ. Hence by (8.28) d satisfies I′–3).

Since d satisfies I′–1) and I′–3), $\sup_\omega |\partial_j e_0| \in M[\chi]$. From (8.29) it follows that

$$(8.31) \qquad \sup_\omega |\partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\}| \leqq \sup_\omega |\partial_j l_0(\chi-t, \omega)| \sup_\omega |\hat{q}_0(t, \omega)|$$
$$+ \sup_\omega |\hat{p}_0(\chi-t, \omega)| (\sup_\omega |\partial_j v_0(t, \omega)| + c_2 \sup_\omega |\hat{q}_0(t, \omega)|).$$

By I′–1) and I′–4) the terms on the right side are integrable with respect to χ and t. Hence from (8.30) and (8.31) we have

$$\int \sup_\omega |\partial_j(l_0 * \hat{q}_0)| d\chi \leqq \int \sup_\omega |\partial_j l_0(\chi, \omega)| d\chi \| \hat{q}_0 \|_F$$
$$+ \| \hat{p}_0 \|_F \int \sup_\omega |\partial_j v_0(\chi, \omega)| d\chi + c_2 \| \hat{p}_0 \|_F \| \hat{q}_0 \|_F$$

and $\sup_\omega |\partial_j(l_0 * \hat{q}_0)|$ is integrable.

Since

$$\sup_\omega |\partial_j(l_\infty \hat{q}_0)| \leqq \sup_\omega |\partial_j l_\infty| \sup_\omega |\hat{q}_0| + \sup_\omega |p_\infty| \sup_\omega |\partial_j v_0|$$
$$+ c_2 \sup_\omega |p_\infty| \sup_\omega |\hat{q}_0|,$$

by I′–1), I′–3) and I′–4) $\sup_\omega |\partial_j(l_\infty \hat{q}_0)|$ is integrable. Similarly $\sup_\omega |\partial_j(l_0 q_\infty)|$ is integrable. Hence by (8.28) $\sup_\omega |\partial_j e_0|$ is integrable and I′–4) is satisfied.

We prove (c). By (a) and (b) it suffices to show that d satisfies III′–4). From (8.29) it follows that

$$(8.32) \qquad |\chi_j| |\partial_j\{l_0(\chi-t, \omega)\hat{q}_0(t, \omega)\}| \leqq |\chi_j - t_j| |\partial_j l_0(\chi-t, \omega)| |\hat{q}_0(t, \omega)|$$
$$+ |\partial_j l_0(\chi-t, \omega)| |t_j| |\hat{q}_0(t, \omega)| + |\chi_j - t_j| |\hat{p}_0(\chi-t, \omega)| |\partial_j v_0(t, \omega)|$$
$$+ |\hat{p}_0(\chi-t, \omega)| |t_j| |\partial_j v_0(t, \omega)| + |\chi_j - t_j| |\hat{p}_0(\chi-t, \omega)| |\hat{q}_0(t, \omega)| |\partial_j |s||$$
$$+ |\hat{p}_0(\chi-t, \omega)| |t_j| |\hat{q}_0(t, \omega)| |\partial_j |s||.$$

Each term of (8.32) is measurable on $S_{t\chi z}$ and its essential supremum over S_ω is measurable on $S_{t\chi}$, so that the integrability of $\sup_\omega (|\chi_j| |\partial_j(l_0 * \hat{q}_0)|)$ follows from the conditions.

By I′, II and III′ it can be shown that $\sup_{\omega}(|\chi_j|\,|\partial_j(l_\infty \hat{q}_0)|)$ and $\sup_{\omega}(|\chi_j|\cdot|\partial_j(l_0 q_\infty)|)$ are also integrable. Hence by (8.28) $\sup_{\omega}(|\chi_j|\,|\partial_j e_0|)$ is integrable and III′–4) is satisfied.

We prove (d). By (b) it suffices to show that d satisfies V–2) and V–3). Let $w_{j0}(\chi, \omega)=(\partial_j v_0)|s|$. Then by V–1) and V–2) $\partial_k m_{j0}(\chi, \omega)$ and $\partial_k w_{j0}(\chi, \omega)$ belong to $M[\chi, z]$ and are measurable on S_χ for each $\omega \in S_z$; $\sup_{\omega}|\partial_k m_{j0}(\chi, \omega)|$, $\sup_{\omega}|\partial_k w_{j0}(\chi, \omega)| \in M[\chi]$.

Multiplying both sides of (8.30) by $|s(\omega)|$, we have by (8.29)

$$\{\partial_j(l_0 * \hat{q}_0)\}\,|s| = m_{j0} * \hat{q}_0 + \hat{p}_0 * w_{j0} - (l_0 * \hat{q}_0)(\partial_j |s|)\,. \tag{8.33}$$

By the same argument as in the proof of (b) $\partial_k(m_{j0} * \hat{q}_0)$ belongs to $M[\chi, z]$ and is continuous on S_z for each χ; $\sup_{\omega}|\partial_k(m_{j0} * \hat{q}_0)|$ belongs to $M[\chi]$ and is integrable. Similarly for $\partial_k(\hat{p}_0 * w_{j0})$ and $\partial_k\{(l_0 * \hat{q}_0)(\partial_j |s|)\}$ we have the same results. Therefore by (8.33) $p_0 q_0$ satisfies V–2) and V–3).

It is readily verified that $p_\infty q_0$, $p_0 q_\infty$ and $p_\infty q_\infty$ satisfy the same conditions. Hence by (8.28) d satisfies V–2) and V–3).

In the following $\sup_{\omega}$ does not stand for $\operatorname{ess.sup}_{\omega}$.

8.6. Proof of Lemma 4.6

We prove (i). By VI–1) and VI–2) p satisfies conditions 1) and 2) of $\mathscr{K}$. Since

$$|\hat{p}_0(\chi, \omega)| \leqq \kappa \int \sup_{\omega} |p_0(x, \omega)| dx, \tag{8.34}$$

by VI–2) $\hat{p}_0(\chi, \omega)$ belongs to $M[\chi, \omega]$; it belongs to $M[\omega]$ for each χ and is continuous on S_χ for each ω. Hence $\operatorname{ess.sup}_{\omega} |\hat{p}_0(\chi, \omega)|$, $\sup_{\omega} |\hat{p}_0(\chi, \omega)| \in M[\chi]$.

By integration by parts we have for each ω

$$\widehat{D_l^{n+3} p_0}(\chi, \omega) = (i\chi_l)^{n+3} \hat{p}_0(\chi, \omega)\,,$$

so that

$$\textstyle\sum_{l=1}^n |\widehat{D_l^{n+3} p_0}(\chi, \omega)| = \sum_{l=1}^n |\chi_l|^{n+3} |\hat{p}_0(\chi, \omega)|\,.$$

Let d be a positive constant such that $\sum_{l=1}^n |\chi_l|^{n+3} \geqq d|\chi|^{n+3}$. Then since

$$d|\chi|^{n+3}|\hat{p}_0| \leqq \textstyle\sum_{l=1}^n |\chi_l|^{n+3} |\hat{p}_0| \leqq \kappa \sum_{l=1}^n \int |D_l^{n+3} p_0| dx,$$

we have for any fixed $A>0$

$$\int_{|\chi|\geqq A}\sup_{\omega}(|\chi|^k|\hat{p}_0(\chi,\omega)|)d\chi \leqq c\int_{|\chi|\geqq A} 1/|\chi|^{n+3-k}d\chi \qquad (k=0,1,2),$$

where

$$c=(\kappa/d)\sum_{l=1}^{n}\int\sup_{\omega}|D_l^{n+3}p_0(x,\omega)|dx.$$

Hence $\sup_{\omega}(|\chi|^k|\hat{p}_0(\chi,\omega)|)$ ($k=0, 1, 2$) are integrable, because by (8.34)

$$\int_{|\chi|\leqq A}\sup_{\omega}(|\chi|^k|\hat{p}_0(\chi,\omega)|)d\chi<\infty.$$

Thus p satisfies condition 3) of $\mathscr{K}$, II and IV.

We prove (ii). Since p belongs to $\mathscr{K}$ and $\hat{p}_0(\chi,\omega)$ is bounded on $S_{\chi z}$ by (i), p satisfies I′–1), I′–2), III′–1) and III′–2). By VI–3) and VI–4) $\operatorname{ess.sup}_{\omega}(\widehat{|\partial_j p_0|}\,|s|)$, $\sup_{\omega\notin Z}(\widehat{|\partial_j p_0|}\,|s|)\in M[\chi]$ ($j=1, 2,\dots, n$).

By VI–2) $e^{-ix\cdot\chi}p_0(x,\omega)|s(\omega)|$ is measurable on $S_{x\chi z}$ and is integrable with respect to x for each $(\chi,\omega)\in S_{\chi z}$. By VI–3) we have for $\omega\in S_z$

$$\partial_j(e^{-ix\cdot\chi}p_0|s|)=e^{-ix\cdot\chi}(\partial_j p_0)\,|s|+e^{-ix\cdot\chi}p_0\partial_j|s|,$$

so that

$$|\partial_j(e^{-ix\cdot\chi}p_0|s|)|\leqq\varphi(x)\qquad\text{for}\quad\omega\in S_z,$$

where

$$\varphi(x)=\sup_{\omega\notin Z}(|\partial_j p_0|\,|s|)+c_2\sup_{\omega\notin Z}|p_0|$$

and c_2 is given by (8.27). By VI–2) and VI–4) $\varphi(x)$ is integrable. Hence

$$\partial_j(\hat{p}_0|s|)=\widehat{\partial_j(p_0|s|)}\qquad\text{for}\quad(\chi,\omega)\in S_{\chi z},\tag{8.35}$$

$\partial_j(\hat{p}_0|s|)\in M[\chi, z]$ and

$$\partial_j(\hat{p}_0|s|)=\widehat{\partial_j p_0}|s|+\hat{p}_0\partial_j|s|\qquad\text{for}\quad(\chi,\omega)\in S_{\chi z}.\tag{8.36}$$

By VI–2) and VI–3) $\partial_j(e^{-ix\cdot\chi}p_0|s|)$ is continuous on $S_{\chi z}$ and is dominated by $\varphi(x)$, so that $\partial_j(\hat{p}_0|s|)$ is continuous on $S_{\chi z}$ and p_0 satisfies I′–3) and III′–3). Since by VI–3)

$$\partial_j(p_\infty|s|)=(\partial_j p_\infty)\,|s|+p_\infty\partial_j|s|\qquad\text{for}\quad\omega\in S_z,\tag{8.37}$$

by VI–1), VI–3) and VI–4) p_∞, $(\partial_j p_\infty)\,|s|\in C[z]$ and p_∞ satisfies I′–3). Thus

I′–3) and III′–3) are satisfied.

By integration by parts we have

$$\widehat{D_l^{n+2}\partial_j p_0}(\chi, \omega)\,|s(\omega)| = (i\chi_l)^{n+2}\widehat{\partial_j p_0}(\chi, \omega)\,|s(\omega)| \qquad \text{for} \quad \omega \in S_z,$$

and $\sup_{\omega\notin Z}(|\chi|^k|\widehat{\partial_j p_0}|\,|s|)$ $(k=0, 1)$ are integrable by the same argument as for $\sup_{\omega}(|\chi|^k|\hat{p}_0(\chi, \omega)|)$. Hence by (i) and (8.36) $\sup_{\omega\notin Z}(|\chi|^k|\partial_j(\hat{p}_0|s|)|)$ $(k=0, 1)$ are integrable and p satisfies I′–4) and III′–4). Therefore by (i) $p \in \mathscr{M}$.

We prove (iii). By (ii) it suffices to show that V–2) and V–3) are satisfied. By VI–5) and VI–6) $\operatorname{ess.sup}_{\omega}(|\widehat{\partial_k\partial_j p_0}|\,|s|^2)$, $\sup_{\omega\notin Z}(|\widehat{\partial_k\partial_j p_0}|\,|s|^2) \in M[\chi]$ $(j, k=1, 2,\dots, n)$.

Multiplying both sides of (8.36) by $|s(\omega)|$, we have

$$\text{(8.38)} \qquad \{\partial_j(\hat{p}_0|s|)\}\,|s| = \widehat{\partial_j p_0}|s|^2 + \hat{p}_0|s|\partial_j|s| \qquad \text{for} \quad \omega \in S_z.$$

By the same argument as in the proof of (8.35)

$$\partial_k(\widehat{\partial_j p_0}|s|^2) = \widehat{\partial_k\{(\partial_j p_0)\,|s|^2\}} \qquad \text{for} \quad \omega \in S_z$$

and $\partial_k(\widehat{\partial_j p_0}|s|^2) \in C[\chi, z]$.

Since p satisfies V–1), we have for $\omega \in S_z$

$$\partial_k(\hat{p}_0|s|\partial_j|s|) = \{\partial_k(\hat{p}_0|s|)\}\partial_j|s| + \hat{p}_0|s|\partial_k\partial_j|s|\,,$$

which belongs to $C[\chi, z]$. Hence by (8.38) $\partial_k[\{\partial_j(\hat{p}_0|s|)\}\,|s|] \in C[\chi, z]$ and p_0 satisfies V–2).

Multiplying both sides of (8.37) by $|s(\omega)|$, we have

$$\text{(8.39)} \qquad \{\partial_j(p_\infty|s|)\}\,|s| = (\partial_j p_\infty)\,|s|^2 + p_\infty|s|\partial_j|s|\,.$$

Calculating the partial derivatives of (8.39) with respect to ω_k, by VI–3)–VI–6) we find $\partial_k[\{\partial_j(p_\infty|s|)\}\,|s|] \in C[z]$. Hence p_∞ satisfies V–2).

From (8.38) it follows for $(\chi, \omega) \in S_{\chi z}$ that

$$\text{(8.40)} \qquad \begin{aligned}\partial_k[\{\partial_j(\hat{p}_0|s|)\}\,|s|] &= \widehat{\partial_k\partial_j p_0}|s|^2 + 2\widehat{\partial_j p_0}|s|\partial_k|s| \\ &\quad + \{\partial_k(\hat{p}_0|s|)\}\partial_j|s| + \hat{p}_0|s|\partial_k\partial_j|s|\,.\end{aligned}$$

By the same argument as for $\sup_{\omega\notin Z}(|\widehat{\partial_j p_0}|\,|s|)$ we have the integrability of $\sup_{\omega\notin Z}(|\widehat{\partial_k\partial_j p_0}|\cdot|s|^2)$. Since $\sup_{\omega\notin Z}|\partial_k(\hat{p}_0|s|)|$ is integrable by (ii), so is $\sup_{\omega\notin Z}|\partial_k[\{\partial_j(\hat{p}_0|s|)\}\,|s|]|$ by (8.40) and V–3) is satisfied.

8.7. Proof of Lemma 4.7

By VI–1) and VI–2) $D_l^m g_0(x, \omega) \in M[x, \omega]$ and $\sup_\omega D_l^m g_0(x, \omega) \in M[x]$ ($l=1, 2, \dots, n$; $m=0, 1, \dots, n+3$). Hence $\hat{g}_0(\chi, \omega)$, $\widehat{D_l g_0}(\chi, \omega) \in M[\chi, \omega]$; $\operatorname{ess.sup}_\omega |\hat{g}_0(\chi, \omega)|$, $\operatorname{ess.sup}_\omega |\widehat{D_l g_0}(\chi, \omega)|$, $\sup_\omega |\hat{g}_0(\chi, \omega)|$ and $\sup_\omega |\widehat{D_l g_0}(\chi, \omega)|$ belong to $M[\chi]$.

By Lemma 4.6 $g \in \mathscr{K}$. Since $D_l g = D_l g_0$, by VI–2) $D_l g(x, \omega)$ is bounded on $S_{x\omega}$, and is continuous and integrable with respect to x for each ω.

From VI–2) it follows as in the proof of Lemma 4.6 that $\widehat{D_l g}(\chi, \omega)$ ($l=1, 2, \dots, n$) are integrable with respect to χ and that $\operatorname{ess.sup}_\omega |\widehat{D_l g}(\chi, \omega)|$ ($l=1, 2, \dots, n$) are also integrable. Thus g satisfies N–1).

By the same argument as in the proof of Lemma 4.6 we have for any fixed $A>0$

$$\int_{|\chi| \geqq A} \sup_\omega |\widehat{\alpha_0 g_0}(\chi, \omega)| d\chi \leqq c_1(R) \int_{|\chi| \geqq A} |\chi|^{-n-1} d\chi,$$

$$\int_{|\chi| \leqq A} \sup_\omega |\widehat{\alpha_0 g_0}(\chi, \omega)| d\chi \leqq c_0(R) \int_{|\chi| \leqq A} 1\, d\chi,$$

where

$$c_1(R) = (\kappa/d') \textstyle\sum_{l=1}^n \displaystyle\int \sup_\omega |D_l^{n+1}(\alpha_0(x) g_0(x, \omega))| dx,$$

$$c_0(R) = \kappa \int \sup_\omega |\alpha_0(x) g_0(x, \omega)| dx$$

and d' is a positive constant such that $\sum_{l=1}^n |\chi_l|^{n+1} \geqq d' |\chi|^{n+1}$.

Since the supports of $\sup_\omega |\alpha_0(x) g_0(x, \omega)|$ and $\sup_\omega |D_l^{n+1} \alpha_0(x) g_0(x, \omega)|$ ($l=1, 2, \dots, n$) are contained in V_0 and $D_l^m \alpha_0(x)$ ($m=0, 1, \dots, n+1$) are bounded uniformly with respect to R, by the integrability of $\sup_\omega |D_l^m g_0(x, \omega)|$ we have

$$\lim_{R \to \infty} c_j(R) = 0 \qquad (j = 0, 1).$$

Hence

$$\lim_{R \to \infty} \int \sup_\omega |\widehat{\alpha_0 g_0}(\chi, \omega)| d\chi = 0,$$

and Condition N–2) is satisfied.

8.8. Proof of Lemma 6.1

8.8.1. Preliminary results and proof

Assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ and let p_i $(1 \leqq i \leqq s)$ be the multiplicity of λ_i. We denote by $\sup_{\omega'} u(x, \omega')$ the supremum of $u(x, \omega')$ over S^{n-1}. Unless otherwise stated, in this section we denote by j, k, l, m, q and r the integers such that $1 \leqq j, k, l \leqq n$, $0 \leqq m \leqq n+3$, $0 \leqq q \leqq n+2$ and $0 \leqq r \leqq n+1$. To prove Lemma 6.1 we need the following three lemmas.

LEMMA B. *Under Conditions* A *and* C *there exists a hermitian matrix* $S(x, \omega')$ *such that*

$$S(x, \omega') = S_0(x, \omega') + S_\infty(\omega'), \tag{8.41}$$

$$S(x, \omega') \geqq eI, \tag{8.42}$$

$$\{S(x, \omega')A(x, \omega')\}^* = S(x, \omega')A(x, \omega'), \tag{8.43}$$

where $S_0(x, \omega') \to 0$ *uniformly with respect to* ω' *as* $|x| \to \infty$ *and* e *is a positive constant which does not depend on* x *and* ω'.

Let $a(x, \omega)$ be a scalar function defined on S_{x0}. Then we introduce the following

Property D. 1) $a(x, \omega)$ can be written as

$$a(x, \omega) = a_0(x, \omega) + a_\infty(\omega),$$

where $\lim_{|x| \to \infty} a_0(x, \omega) = 0$ for $\omega \in S_0$;

2) $D_l^m a_0(x, \omega)$, $D_l^q \partial_j a_0(x, \omega)$ and $D_l^r \partial_k \partial_j a_0(x, \omega)$ are continuous on S_{x0}; $\partial_j a_\infty(\omega)$ and $\partial_k \partial_j a_\infty(\omega)$ are continuous on S_0;

3) $\sup_{\omega \neq 0}(|D_l^m a_0(x, \omega)|)$, $\sup_{\omega \neq 0}(|D_l^q \partial_j a_0(x, \omega)|\,|\omega|)$ and $\sup_{\omega \neq 0}(|D_l^r \partial_k \partial_j a_0(x, \omega)|\,|\omega|^2)$ are bounded and integrable; $\sup_{\omega \neq 0}(|a_\infty(\omega)|)$, $\sup_{\omega \neq 0}(|\partial_j a_\infty(\omega)|\,|\omega|)$ and $\sup_{\omega \neq 0}(|\partial_k \partial_j a_\infty(\omega)| \cdot |\omega|^2)$ are finite.

LEMMA C. *Let* $a(x, \omega)$ *and* $b(x, \omega)$ *be scalar functions with property* D. *Then*

(i) $a+b$, ab *and* $\bar{a}$ *have property* D;

(ii) *If* $|b| \geqq \alpha$ *for some* $\alpha > 0$, *then* a/b *has property* D;

(iii) *If* $a \geqq \beta$ *for some* $\beta > 0$, *then* $\sqrt{a}$ *has property* D.

LEMMA D. *Under Conditions* A, B *and* C *the eigenvalues* $\lambda_i(x, \omega/|\omega|)$ $(i = 1, 2, \ldots, s)$ *of* $A(x, \omega/|\omega|)$ $(|\omega| \neq 0)$ *and the entries of* $S(x, \omega/|\omega|)$ *have property* D.

PROOF OF LEMMA 6.1. Let

$$g(x, \omega) = \begin{cases} S(x, s(\omega)/|s(\omega)|) & \text{if } \omega \in S_z, \\ eI & \text{if } \omega \in Z. \end{cases} \tag{8.44}$$

We show that $g(x, \omega)$ satisfies VI. Since by Lemma D the entries of $S(x, \omega/|\omega|)$ have property D, by D–1) we have

$$S(x, \omega/|\omega|) = S_0(x, \omega/|\omega|) + S_\infty(\omega/|\omega|),$$

where $\lim_{|x|\to\infty} S_0(x, \omega/|\omega|) = 0$. Let

$$g_\infty(\omega) = \begin{cases} S_\infty(s(\omega)/|s(\omega)|) & \text{if } \omega \in S_z, \\ eI & \text{if } \omega \in Z, \end{cases} \tag{8.45}$$

and put $g_0(x, \omega) = g(x, \omega) - g_\infty(\omega)$. Then

$$\lim_{|x|\to\infty} g_0(x, \omega) = 0 \qquad \text{for} \quad \omega \in R^n,$$

$$g_0(x, \omega) = 0 \qquad \text{for} \quad \omega \in Z. \tag{8.46}$$

By D–2) and D–3) $g_0(x, \omega) \in C[x, z]$ and $g_\infty(\omega) \in C[z]$. Hence by (8.45) and (8.46) $g_0(x, \omega) \in M[x, \omega]$ and $g_\infty(\omega) \in M[\omega]$. Thus g satisfies VI–1).

Since $\sup_{\omega \neq 0} |D_l^m S_0(x, \omega/|\omega|)|$ belongs to $M[x]$ and is integrable by D–2) and D–3), $\sup_{\omega \notin Z} |D_l^m g_0(x, \omega)|$ is bounded and integrable. Hence g satisfies VI–2).

For $\omega \in S_z$ we have

$$D_l^q \partial_j g_0(x, \omega) = \textstyle\sum_{k=1}^n \{\partial_j s_k(\omega)\} \, [D_l^q \partial_k S_0(x, \omega/|\omega|)]_{\omega = s(\omega)}, \tag{8.47}$$

$$\partial_j g_\infty(\omega) = \textstyle\sum_{k=1}^n \{\partial_j s_k(\omega)\} \, [\partial_k S_\infty(\omega/|\omega|)]_{\omega = s(\omega)}, \tag{8.48}$$

so that by D–2) $D_l^q \partial_j g_0(x, \omega)$ and $\partial_j g_\infty(\omega)$ are continuous on S_{xz} and on S_z respectively. Thus g satisfies VI–3).

From (8.47) and (8.48) it follows that for $(x, \omega) \in S_{xz}$

$$|D_l^q \partial_j g_0| \, |s| \leqq c \textstyle\sum_{k=1}^n \sup_{\omega \neq 0} (|D_l^q \partial_k S_0(x, \omega/|\omega|)| \, |\omega|),$$

$$|\partial_j g_\infty| \, |s| \leqq c \textstyle\sum_{k=1}^n \sup_{\omega \neq 0} (|\partial_k S_\infty(\omega/|\omega|)| \, |\omega|),$$

where c is a constant such that $|\partial_j s_k(\omega)| \leqq c$. Hence by D–3) $\sup_{\omega \notin Z} (|D_l^q \partial_j g_0| \, |s|)$ is bounded and integrable and $\sup_{\omega \notin Z} (|\partial_j g_\infty| \, |s|)$ is finite. Thus g satisfies VI–4). Similarly it can be shown that g fulfills VI–5) and VI–6).

By Lemma 4.6 $g \in \mathscr{L}$. Since by (8.42) and (8.44)

$$g(x, \omega) \geqq eI \qquad (e > 0),$$

by Lemma 4.7 g satisfies the conditions of Theorem 3.3. Finally (6.5) follows from (8.43).

8.8.2. Proof of Lemma B

Let

(8.49) $$d(\lambda;\, x,\, \omega') = \det(\lambda I - A) = \prod_{j=1}^{s}(\lambda - \lambda_j)^{p_j},$$

$$d_\lambda(\lambda;\, x,\, \omega') = D_\lambda d(\lambda;\, x,\, \omega') \qquad (D_\lambda = \partial/\partial\lambda),$$

$$A_\infty(\omega') = \sum_{j=1}^{n} A_{j\infty}\omega'_j, \quad d_\infty(\lambda;\, \omega') = \det(\lambda I - A_\infty(\omega')),$$

$$d_{\lambda\infty}(\lambda;\, \omega') = D_\lambda d_\infty(\lambda;\, \omega').$$

As λ_j $(j=1, 2,\ldots, s)$ are real, we have

(8.50) $$d_\lambda(\lambda;\, x,\, \omega') = N\prod_{j=1}^{s}(\lambda - \lambda_j)^{p_j - 1}\prod_{k=1}^{s-1}(\lambda - \mu_k),$$

where $\mu_k(x, \omega')$ $(k=1, 2,\ldots, s-1)$ are real and $\lambda_k < \mu_k < \lambda_{k+1}$.

By Condition A $A(x, \omega') \to A_\infty(\omega')$ uniformly with respect to ω' as $|x| \to \infty$. Hence by continuity of eigenvalues of matrices we have the following results:

1) Eigenvalues of $A_\infty(\omega')$ are all real and their multiplicities are independent of ω';

2) $|\lambda_{i\infty}(\omega') - \lambda_{j\infty}(\omega')| \geqq \delta \quad (i \neq j;\ i, j = 1, 2,\ldots, s)$,

(8.51) $$\lambda_j(x,\, \omega') \longrightarrow \lambda_{j\infty}(\omega') \qquad (j = 1, 2,\ldots, s)$$

uniformly with respect to ω' as $|x| \to \infty$, where $\lambda_{j\infty}(\omega')$ $(j=1, 2,\ldots, s)$ are all the distinct eigenvalues of $A_\infty(\omega')$ and $\lambda_{1\infty} < \lambda_{2\infty} < \cdots < \lambda_{s\infty}$;

3) $\mu_k(x, \omega') \to \mu_{k\infty}(\omega')$ $(k=1, 2,\ldots, s-1)$ uniformly with respect to ω' as $|x| \to \infty$, where $\mu_{k\infty}(\omega')$ $(k=1, 2,\ldots, s-1)$ are zeros of $d_{\lambda\infty}(\lambda, \omega')$ such that $\lambda_{k\infty} < \mu_{k\infty} < \lambda_{k+1\,\infty}$;

4) There exists a constant $\rho > 0$ independent of x and ω' such that

$$|\lambda_j(x,\, \omega') - \mu_k(x,\, \omega')| \geqq 2\rho \quad (j = 1, 2,\ldots, s;\ k = 1, 2,\ldots, s-1).$$

Put $\lambda_{j0}(x, \omega') = \lambda_j - \lambda_{j\infty}$ $(j=1, 2,\ldots, s)$. Then from (8.51) it follows that

(8.52) $$\lambda_j(x,\, \omega') = \lambda_{j0}(x,\, \omega') + \lambda_{j\infty}(\omega'), \quad \lim_{|x|\to\infty} \lambda_{j0}(x,\, \omega') = 0.$$

Let $D_j(\rho)$ and $D_{j\infty}(\rho)$ $(j=1, 2,\ldots, s)$ be the open disks on the complex λ-plane with radius ρ and centers at λ_j and $\lambda_{j\infty}$ respectively. Let $E(\lambda;\, x, \omega')$ and $E_\infty(\lambda;\, \omega')$ be the matrices whose (i, j) entries are (j, i) cofactors of $\lambda I - A(x, \omega')$ and λI

$-A_\infty(\omega')$ respectively. Then $E(\lambda; x, \omega') \to E_\infty(\lambda; \omega')$ uniformly with respect to ω' for each fixed λ as $|x| \to \infty$.

By C-3) $(\lambda I - A(x, \omega'))^{-1}$ has a simple pole at $\lambda = \lambda_j(x, \omega')$ $(1 \leqq j \leqq s)$. Let $C_j(x, \omega')$ be the residue of $(\lambda I - A(x, \omega'))^{-1}$ at $\lambda = \lambda_j$ and let

$$r_j(\lambda; x, \omega') = \prod_{i=1, i\neq j}^{s} (\lambda - \lambda_i)^{p_i}, \quad r_{j\infty}(\lambda; \omega') = \prod_{i=1, i\neq j}^{s} (\lambda - \lambda_{i\infty})^{p_i}.$$

Then

$$r_j(\lambda_j; x, \omega') \longrightarrow r_{j\infty}(\lambda_{j\infty}; \omega') \qquad \text{as} \quad |x| \to \infty$$

and we have

$$|r_j(\lambda_j; x, \omega')| \geqq \delta^{N-p_j}, \quad |r_{j\infty}(\lambda_{j\infty}; \omega')| \geqq \delta^{N-p_j}. \tag{8.53}$$

Since

$$(\lambda I - A(x, \omega'))^{-1} = E(\lambda; x, \omega')/d(\lambda; x, \omega'),$$

$E(\lambda; x, \omega')$ can be written on $D_j(\rho)$ as

$$E(\lambda; x, \omega') = (\lambda - \lambda_j(x, \omega'))^{p_j - 1} B_j(\lambda; x, \omega'), \tag{8.54}$$

where the entries of $B_j(\lambda; x, \omega')$ are sums of products of λ, $\lambda_j(x, \omega')$ and entries of $A(x, \omega')$. Hence $B_j(\lambda; x, \omega')$ converges to a matrix, say $B_{j\infty}(\lambda; \omega')$, uniformly with respect to ω' as $|x| \to \infty$ for each fixed λ. It follows that

$$C_j(x, \omega') = B_j(\lambda_j; x, \omega')/r_j(\lambda_j; x, \omega'), \tag{8.55}$$

$$B_{j\infty}(\lambda_{j\infty}; \omega') = \lim_{|x| \to \infty} B_j(\lambda_j; x, \omega'), \tag{8.56}$$

and by (8.54) we have on $D_{j\infty}(\rho)$

$$E_\infty(\lambda; \omega') = (\lambda - \lambda_{j\infty}(\omega'))^{p_j - 1} B_{j\infty}(\lambda; \omega'). \tag{8.57}$$

Let

$$C_{j\infty}(\omega') = B_{j\infty}(\lambda_{j\infty}; \omega')/r_{j\infty}(\lambda_{j\infty}; \omega'). \tag{8.58}$$

Then by (8.53) and (8.56) $C_j(x, \omega') \to C_{j\infty}(\omega')$ uniformly with respect to ω' as $|x| \to \infty$. Since

$$(\lambda I - A_\infty(\omega'))^{-1} = E_\infty(\lambda; \omega')/d_\infty(\lambda; \omega'),$$

by (8.57) and (8.58) we have

$$\lim_{\lambda \to \lambda_{j\infty}} (\lambda I - A_\infty(\omega'))^{-1}(\lambda - \lambda_{j\infty}) = C_{j\infty}(\omega').$$

Hence $(\lambda I - A_\infty(\omega'))^{-1}$ has simple poles at $\lambda = \lambda_{j\infty}$ $(j = 1, 2, \ldots, s)$.

We prove (8.41)–(8.43). After Friedrichs [3] we define $S(x, \omega')$ by

$$S(x, \omega') = \sum_{j=1}^{s} \frac{1}{2\pi i}\int_{\Gamma_j}(\lambda I - A^*(x, \omega'))^{-1}(\lambda I - A(x, \omega'))^{-1} \times d_\lambda^{-1}(\lambda; x, \omega')d(\lambda; x, \omega')d\lambda,$$

where Γ_j $(1 \leqq j \leqq s)$ is the positively oriented path running along the circumference of $D_j(\rho)$. Then it follows that

$$(8.59)\qquad \begin{aligned} S(x, \omega') &= \sum_{j=1}^{s} \lim_{\lambda\to\lambda_j}\{(\lambda I - A^*)^{-1}(\lambda I - A)^{-1}(\lambda-\lambda_j)^2 d_\lambda^{-1}d/(\lambda-\lambda_j)\} \\ &= \sum_{j=1}^{s} p_j^{-1}C_j^*(x, \omega')C_j(x, \omega'). \end{aligned}$$

Hence

$$(8.60)\qquad S(x, \omega') \longrightarrow S_\infty(\omega') \equiv \sum_{j=1}^{s} p_j^{-1}C_{j\infty}^*(\omega')C_{j\infty}(\omega')$$

uniformly with respect to ω' as $|x|\to\infty$. Put $S_0(x, \omega')=S(x, \omega')-S_\infty(\omega')$. Then (8.41) holds.

We show (8.42). From (8.59) we have $S(x, \omega')\geqq 0$. Suppose $S(x, \omega') > 0$ does not hold. Then there exist a point $(\tilde{x}, \tilde{\omega}')$ and a vector u $(u\neq 0)$ such that $S(\tilde{x}, \tilde{\omega}')u=0$, and (8.59) yields

$$C_j(\tilde{x}, \tilde{\omega}')u = 0 \qquad (j = 1, 2,\ldots, s).$$

Since in general

$$u = \frac{1}{2\pi i}\sum_{j=1}^{s}\int_{\Gamma_j}(\lambda I - A(x, \omega'))^{-1}d\lambda u,$$

it follows that $u=\sum_{j=1}^{s} C_j(x, \omega')u$, and so we have $u=0$, which is a contradiction. Hence

$$S(x, \omega') > 0 \qquad \text{for all} \quad x\in R^n, \quad \omega'\in S^{n-1}.$$

By the same argument it follows from continuity of $S_\infty(\omega')$ that $S_\infty(\omega')\geqq \alpha I$ for some $\alpha>0$.

By (8.60) there is $R_0>0$ such that

$$S(x, \omega') \geqq (\alpha/2)I \qquad \text{for} \quad |x| \geqq R_0.$$

By continuity of $S(x, \omega')$ there exists $\beta>0$ such that

$$S(x, \omega') \geqq \beta I \qquad \text{for} \quad |x| \leqq R_0 \quad \text{and} \quad \omega'\in S^{n-1}.$$

Hence (8.42) holds with $e=\min(\alpha/2, \beta)$.

Finally we have

$$\begin{aligned}\{S(x, \omega')A(x, \omega')\}^* &= A^*(x, \omega')S(x, \omega')\\ &= \sum_{j=1}^{s} \frac{1}{2\pi i}\left\{\int_{\Gamma_j} \lambda(\lambda I - A^*)^{-1}(\lambda I - A)^{-1} d_\lambda^{-1} d\lambda\right.\\ &\qquad \left. - \int_{\Gamma_j} (\lambda I - A^*)(\lambda I - A^*)^{-1}(\lambda I - A)^{-1} d_\lambda^{-1} d\lambda\right\}\\ &= S(x, \omega')A(x, \omega'),\end{aligned}$$

because the second integral vanishes.

8.8.3. Proof of Lemma C

It is clear that $a+b$ and $\bar{a}$ have property D. Let $d=ab$. Then $d=d_0+d_\infty$, where

$$d_0 = a_0 b_0 + a_\infty b_0 + a_0 b_\infty, \quad d_\infty = a_\infty b_\infty.$$

From this it follows that d has property D.

In the case (ii) let $e(x, \omega)=a/b$. Then $e=e_0+e_\infty$, where

$$e_0(x, \omega) = u/v, \quad e_\infty = a_\infty/b_\infty,$$

$$u(x, \omega) = a_0 b_\infty - b_0 a_\infty, \quad v(x, \omega) = b b_\infty.$$

By (i) u and v have property D. Since

$$u = u_0, \quad u_\infty = 0, \quad |v| \geqq \alpha^2, \quad |b_\infty| \geqq \alpha,$$

it follows that e has property D.

In the case (iii) let $f(x, \omega)=\sqrt{a}$ and $\gamma=\sqrt{\beta}$. Then $f=f_0+f_\infty$, where

$$f_0(x, \omega) = \sqrt{a} - \sqrt{a_\infty}, \quad f_\infty(\omega) = \sqrt{a_\infty}.$$

Since

$$f_\infty \geqq \gamma, \quad f_\infty \partial_j f_\infty = (\partial_j a_\infty)/2,$$

$$f_\infty \partial_k \partial_j f_\infty + (\partial_k f_\infty)(\partial_j f_\infty) = (\partial_k \partial_j a_\infty)/2,$$

f_∞ has property D. As $f_0=a_0/(\sqrt{a}+\sqrt{a_\infty})$ and $f\geqq\gamma$, f_0 has property D.

8.8.4. Proof of Lemma D

Since by (8.52) $\lambda_i(x, \omega/|\omega|)$ $(1\leqq i\leqq s)$ has property D–1), we show first that it has property D–2). The coefficients of the polynomial $d(\lambda; x, \omega/|\omega|)$ are sums of products of entries of $A(x, \omega/|\omega|)$, which have property D by Lemma C. Hence $\lambda_i(x, \omega/|\omega|) \in C[x, 0]$. Similarly we have $\lambda_{i\infty}(\omega/|\omega|) \in C[0]$.

Put

$$(8.61)\qquad q(\lambda;\, x, \omega/|\omega|) = D_\lambda^{p_i-1} d(\lambda;\, x, \omega/|\omega|) \qquad (D_\lambda = \partial/\partial\lambda),$$

$$(8.62)\qquad q_\infty(\lambda;\, \omega/|\omega|) = D_\lambda^{p_i-1} d_\infty(\lambda;\, \omega/|\omega|), \qquad p = N - p_i.$$

Then $q(\lambda_i(x, \omega/|\omega|);\, x, \omega/|\omega|) = 0$, $q_\infty(\lambda_{i\infty}(\omega/|\omega|);\, \omega/|\omega|) = 0$ and by C–2) we have for $(x, \omega) \in S_{x0}$

$$(8.63)\qquad |D_\lambda q(\lambda_i(x, \omega/|\omega|);\, x, \omega/|\omega|)| = \textstyle\prod_{k=1, k\neq i}^{s} |\lambda_i - \lambda_k|^{p_k} p_i! \geqq p_i! \delta^p > 0,$$

$$(8.64)\qquad |D_\lambda q_\infty(\lambda_{i\infty}(\omega/|\omega|);\, \omega/|\omega|)| = \textstyle\prod_{k=1, k\neq i}^{s} |\lambda_{i\infty} - \lambda_{k\infty}|^{p_k} p_i! \geqq p_i! \delta^p > 0.$$

Hence by the implicit function theorem $\lambda_i(x, \omega/|\omega|)$ has partial derivatives $D_l \lambda_i$ and $\partial_j \lambda_i$ on S_{x0}, which can be written as

$$(8.65)\qquad D_l \lambda_i(x, \omega/|\omega|) = -[D_l q(\lambda;\, x, \omega/|\omega|)/D_\lambda q(\lambda;\, x, \omega/|\omega|)]_{\lambda=\lambda_i},$$

$$(8.66)\qquad \partial_j \lambda_i(x, \omega/|\omega|) = -[\partial_j q(\lambda;\, x, \omega/|\omega|)/D_\lambda q(\lambda;\, x, \omega/|\omega|)]_{\lambda=\lambda_i}.$$

Similarly $\lambda_{i\infty}(\omega/|\omega|)$ has a partial derivative $\partial_j \lambda_{i\infty}(\omega/|\omega|)$ on S_0, which can be written as

$$(8.67)\qquad \partial_j \lambda_{i\infty}(\omega/|\omega|) = -[\partial_j q_\infty(\lambda;\, \omega/|\omega|)/D_\lambda q_\infty(\lambda;\, \omega/|\omega|)]_{\lambda=\lambda_{i\infty}}.$$

On the other hand by (8.61) and (8.62) $q(\lambda;\, x, \omega/|\omega|)$ and $q_\infty(\lambda;\, \omega/|\omega|)$ can be written as follows:

$$(8.68)\qquad q(\lambda;\, x, \omega/|\omega|) = b\lambda^{p+1} + a_0(x, \omega/|\omega|)\lambda^p + \cdots + a_p(x, \omega/|\omega|),$$

$$(8.69)\qquad q_\infty(\lambda;\, \omega/|\omega|) = b\lambda^{p+1} + a_{0\infty}(\omega/|\omega|)\lambda^p + \cdots + a_{p\infty}(\omega/|\omega|),$$

where $b = N!/(p+1)!$, a_t $(t = 0, 1, \ldots, p)$ have property D and can be written as $a_t = a_{t0} + a_{t\infty}$. Hence by (8.63) and (8.65) $D_l \lambda_i(x, \omega/|\omega|) \in C[x, 0]$, because $\lambda_i(x, \omega/|\omega|) \in C[x, 0]$. By consideration of the successive derivatives of (8.65) with respect to x_l $D_l^m \lambda_{i0}(x, \omega/|\omega|)$ belongs to $C[x, 0]$.

Since by (8.66) and (8.67) $\partial_j \lambda_i(x, \omega/|\omega|)$ and $\partial_j \lambda_{i\infty}(\omega/|\omega|)$ are continuous on S_{x0}, so is $\partial_j \lambda_{i0}(x, \omega/|\omega|)$. Calculating the successive derivatives of (8.66) with respect to x_l, we see that $D_l^m \partial_j \lambda_{i0}(x, \omega/|\omega|)$ is continuous on S_{x0}.

By consideration of the derivatives of (8.66) and (8.67) with respect to ω_k $\partial_k \partial_j \lambda_i(x, \omega/|\omega|)$ and $\partial_k \partial_j \lambda_{i\infty}(\omega/|\omega|)$ are continuous on S_{x0} and on S_0 respectively. Hence $\partial_k \partial_j \lambda_{i0}(x, \omega/|\omega|)$ is continuous on S_{x0}. Similarly $D_l^r \partial_k \partial_j \lambda_{i0}(x, \omega/|\omega|)$ is continuous on S_{x0}. Thus $\lambda_i(x, \omega/|\omega|)$ has property D–2).

We prove that $\lambda_i(x, \omega/|\omega|)$ has property D–3). Put $q_i(x, \omega) = q(\lambda_{i\infty}(\omega/|\omega|);$

$x, \omega/|\omega|)$. Then from (8.61) and (8.49) we have

$$q_i(x, \omega) = \lambda_{i0}(x, \omega/|\omega|)e_i(x, \omega), \tag{8.70}$$

where

$$e_i(x, \omega) = -\prod_{j=1, j\neq i}^{s}(\lambda_{i\infty}-\lambda_j)^{p_j}p_i! + \lambda_{i0}\tilde{q}(x, \omega)$$

and $\tilde{q}(x, \omega)$ is a sum of products of $\lambda_{i\infty}$ and λ_t $(t=1, 2,\dots, s)$ which are bounded on S_{x0}. Hence there exists $K>0$ such that

$$|e_i(x, \omega)| \geqq (\delta/4)^p \qquad \text{for} \quad |x| \geqq K. \tag{8.71}$$

From (8.68) and (8.69) it follows that

$$q_i(x, \omega) = \sum_{t=0}^{p} a_{t0}\lambda_{i\infty}^{p-t}, \tag{8.72}$$

and from (8.70)–(8.72) we have for $|x|\geqq K$

$$|\lambda_{i0}(x, \omega/|\omega|)| \leqq \left(\sum_{t=0}^{p}|a_{t0}|\,|\lambda_{i\infty}|^{p-t}\right)/(\delta/4)^p. \tag{8.73}$$

Since $\lambda_{i0}(x, \omega/|\omega|)$ and $a_{t0}(x, \omega/|\omega|)$ $(t=0, 1,\dots, p)$ belong to $C[x, 0]$, $\sup_{\omega\neq 0}|\lambda_{i0}(x, \omega/|\omega|)|$ and $\sup_{\omega\neq 0}|a_{t0}(x, \omega/|\omega|)|$ $(t=0, 1,\dots, p)$ belong to $M[x]$. Put $c_i(x)=\sup_{\omega\neq 0}|\lambda_{i0}(x, \omega/|\omega|)|$. Then $\int_{|x|\leqq K} c_i(x)dx<\infty$, and by (8.73) $\int_{|x|\geqq K} c_i(x)dx <\infty$, because $\int\sup_{\omega\neq 0}|a_{t0}(x, \omega/|\omega|)|dx<\infty$ $(t=0, 1,\dots, p)$. Hence $c_i(x)$ is integrable.

Since $D_l\lambda_{i0}(x, \omega/|\omega|)\in C[x, 0]$, we have $\sup_{\omega\neq 0}|D_l\lambda_{i0}(x, \omega/|\omega|)|\in M[x]$. As $\lambda_i(x, \omega/|\omega|)$ is bounded on S_{x0}, by (8.65) and (8.63) $\sup_{\omega\neq 0}|D_l\lambda_{i0}(x, \omega/|\omega|)|$ is integrable. By calculating the successive derivatives of (8.65) with respect to x_l, it can be shown similarly that $\sup_{\omega\neq 0}|D_l^m\lambda_{i0}(x, \omega/|\omega|)|$ is bounded and integrable.

As $a_t(x, \omega/|\omega|)$ $(t=0, 1,\dots, p)$ have property D, $\{\partial_j a_t(x, \omega/|\omega|)\}|\omega|\in C[x, 0]$ $(t=0, 1,\dots, p)$ and by (8.66) and (8.63) $\{\partial_j\lambda_i(x, \omega/|\omega|)\}|\omega|\in C[x, 0]$. Similarly $\{\partial_j\lambda_{i\infty}(\omega/|\omega|)\}\,|\omega|\in C[0]$. Therefore $\sup_{\omega\neq 0}(|\partial_j\lambda_{i0}(x, \omega/|\omega|)|\,|\omega|)\in M[x]$ and $\sup_{\omega\neq 0}(|\partial_j\lambda_{i\infty}(\omega/|\omega|)|\,|\omega|)$ is finite.

From (8.70) we have

$$\partial_j q_i(x, \omega) = (\partial_j\lambda_{i0})e_i + \lambda_{i0}\partial_j e_i. \tag{8.74}$$

By D–3) $\sup_{\omega\neq 0}(|\partial_j a_{t0}(x, \omega/|\omega|)|\,|\omega|)$ and $\sup_{\omega\neq 0}|a_{t0}(x, \omega/|\omega|)|$ $(t=0, 1,\dots, p)$ are integrable. Hence from (8.72) it follows that $\sup_{\omega\neq 0}(|\partial_j q_i(x, \omega)|\,|\omega|)$ is integrable. By (8.73) and (8.74) we have for $|x|\geqq K$

$$|\partial_j\lambda_{i0}(x, \omega/|\omega|)|\,|\omega| \leqq \{|\partial_j q_i|\,|\omega| + |\lambda_{i0}|\,|\partial_j e_i|\,|\omega|\}/(\delta/4)^p,$$

so that $\sup_{\omega\neq 0}(|\partial_j\lambda_{i0}(x, \omega/|\omega|)|\,|\omega|)$ is integrable.

Calculating the successive derivatives of (8.74) with respect to x_l, we see that $\{D_l^q \partial_j \lambda_{i0}(x, \omega/|\omega|)\} |\omega| \in M[x, 0]$ and that $\sup_{\omega \neq 0}(|D_l^q \partial_j \lambda_{i0}(x, \omega/|\omega|)| |\omega|)$ is integrable. Similarly it can be shown that $\sup_{\omega \neq 0}(|D_l^r \partial_k \partial_j \lambda_{i0}(x, \omega/|\omega|)| |\omega|^2)$ is bounded and integrable and that $\sup_{\omega \neq 0}(|\partial_k \partial_j \lambda_{i\infty}(\omega/|\omega|)| |\omega|^2)$ is finite. Hence $\lambda_i(x, \omega/|\omega|)$ has property D–3).

By (8.55) the entries of $C_i(x, \omega/|\omega|)$ have property D by Lemma C, because the entries of $B_i(\lambda_i; x, \omega/|\omega|)$ and $r_i(\lambda_i; x, \omega/|\omega|)$ are sums of products of $\lambda_i(x, \omega/|\omega|)$ and entries of $A(x, \omega/|\omega|)$. Hence the entries of $S(x, \omega/|\omega|)$ have property D.

8.9. Proof of Lemma 6.2

Let $S(x, \omega/|\omega|) = (s_{ij}(x, \omega))$ and

$$q_k(x, \omega) = \det \begin{bmatrix} s_{11} \cdots s_{1k} \\ \vdots \qquad \vdots \\ s_{k1} \cdots s_{kk} \end{bmatrix} \qquad (k = 1, 2, \ldots, N).$$

Since $S(x, \omega/|\omega|)$ is positive definite, it can be written as $S(x, \omega/|\omega|) = W^* W$, where $W(x, \omega) = (w_{ij})$ is an upper triangular matrix and

$$w_{ii} = d_i = (q_i/q_{i-1})^{1/2} \qquad (i = 1, 2, \ldots, N;\ q_0 = 1),$$

$$w_{ij} = d_i u_{ij} \qquad (j > i;\ i = 1, 2, \ldots, N-1),$$

$$u_{ij} = (s_{ij} - \textstyle\sum_{k=1}^{i-1} d_k^2 \bar{u}_{ki} u_{kj})/d_i^2.$$

Put

$$w(x, \omega) = \begin{cases} W(x, s(\omega)) & \text{for} \quad \omega \in S_z, \\ \sqrt{e} I & \text{for} \quad \omega \in Z. \end{cases}$$

Then $g(x, \omega)$ can be written as (6.6).

As $S(x, \omega/|\omega|) \geqq eI$, there exist positive constants c_j $(j = 1, 2, 3)$ such that

$$c_1 \leqq q_k(x, \omega) \leqq c_2, \quad c_3 \leqq d_k(x, \omega) \qquad (k = 1, 2, \ldots, N).$$

Since s_{ij} $(i, j = 1, 2, \ldots, N)$ have property D by Lemma D, it follows that w_{ij} $(j \geqq i;\ i = 1, 2, \ldots, N)$ have property D and as in the proof of Lemma 6.1 $w(x, \omega)$ satisfies VI.

Since $\det w(x, \omega) \geqq \min(\sqrt{c_1}, \sqrt{e}) > 0$, $w^{-1}(x, \omega)$ exists and satisfies VI. Hence $w(x, \omega)$ and $w^{-1}(x, \omega)$ belong to $\mathscr{L}$ and fulfill Condition N by Lemmas 4.6 and 4.7.

8.10. Proof of Lemma 6.3

We construct first the matrix u which diagonalizes $p_z - i\lambda q|s|$ for $\omega \in S_z$. By regular hyperbolicity there exist a nonsingular matrix $w(x, \omega)$ and a real diagonal matrix $d(x, \omega)$ with the following

Property E. 1) w, w^{-1} and d satisfy Condition VI;

2) For some constant $e_0 > 0$

(8.75) $$w^*(x, \omega)w(x, \omega) \geqq e_0 I;$$

3) $d = wp_z w^{-1}$ for $\omega \in S_z^{1)}$.

Put

$$e(x, \omega; \lambda) = w(p_z - i\lambda q|s|)w^{-1}.$$

Then by E–3) we have

(8.76) $$e(x, \omega; \lambda) = d - \lambda|s|\tilde{q},$$

where $\tilde{q}(x, \omega; \lambda) = iwqw^{-1}$. Let $\tilde{q} = (\tilde{q}_{ij})$ and $d = \mathrm{diag}(d_1, d_2, \dots, d_N)$. By the condition of Theorem 6.7 and E–1) $\tilde{q}_{ij}$ $(i, j = 1, 2, \dots, N)$ are bounded on $S_{x\omega} \times (0, \lambda_0]$. Hence for some λ_2 $(0 < \lambda_2 \leqq \lambda_0)$

(8.77) $$\lambda|s|\textstyle\sum_{j=1}^{N}|\tilde{q}_{kj}| \leqq \delta/4 \quad (k = 1, 2, \dots, N) \qquad \text{for} \quad \lambda \leqq \lambda_2,$$

and by C–2)

(8.78) $$|d_i - d_j| \geqq \delta \qquad \text{for} \quad \omega \in S_z \quad (i \neq j;\ i, j = 1, 2, \dots, N).$$

By Gershgorin's Theorem the eigenvalues $\mu_i(x, \omega; \lambda)$ $(i = 1, 2, \dots, N)$ of $e(x, \omega; \lambda)$ can be numbered so that

$$|\mu_i - d_i| \leqq \delta/4 \quad (i = 1, 2, \dots, N) \qquad \text{for} \quad \omega \in S_z, \quad \lambda \leqq \lambda_2.$$

Therefore they are bounded on $S_{xz} \times (0, \lambda_2]$ and

(8.79) $$|\mu_i - \mu_j| \geqq \delta/2, \quad |\mu_i - d_j| \geqq 3\delta/4 \qquad \text{for} \quad \omega \in S_z, \quad \lambda \leqq \lambda_2 \qquad (i \neq j;\ i, j = 1, 2, \dots, N).$$

We construct an eigenvector of e corresponding to μ_i $(1 \leqq i \leqq N)$.

1) The construction of $w(x, \omega)$ is given in [11] and it follows as in the proof of Lemma 6.1 that $w(x, \omega)$ has property E.

From (8.76) we have

$$\prod_{j=1}^{N}(d_i-\mu_j) = \det\{(d_iI-d)+\lambda|s|\tilde{q}\} = \lambda|s|y_i, \tag{8.80}$$

where $y_i(x, \omega; \lambda)$ is a sum of products of d_k, $\tilde{q}_{kl}$ $(k, l=1, 2,\dots, N)$ and $\lambda|s|$. Let

$$\phi_i(x, \omega; \lambda) = \prod_{j=1, j\neq i}^{N}(d_i-\mu_j).$$

Since by (8.79) $|\phi_i|\geqq(3\delta/4)^{N-1}$ for $\lambda\leqq\lambda_2$, from (8.80) it follows that

$$d_i-\mu_i = \lambda|s|\varphi_i \qquad \text{for} \quad \lambda \leqq \lambda_2, \tag{8.81}$$

where $\varphi_i(x, \omega; \lambda)=y_i/\phi_i$.

Let $\Delta_{ij}(x, \omega; \lambda)$ $(j=1, 2,\dots, N)$ be the (i, j) cofactors of the matrix μ_iI-e. Since

$$\mu_iI-e = (\mu_i-d_i)I+(d_iI-d)+\lambda|s|\tilde{q},$$

by (8.81) we have

$$\Delta_{ii} = \varepsilon_i+\lambda|s|v_{ii}, \quad \varepsilon_i(x, \omega; \lambda) = \prod_{j=1, j\neq i}^{N}(d_i-d_j),$$

$$\Delta_{ij} = \lambda|s|v_{ij} \qquad (j \neq i;\, j = 1, 2,\dots, N),$$

where $v_{ij}(x, \omega; \lambda)$ $(j=1, 2,\dots, N)$ are sums of products of $\lambda|s|$, φ_i, d_k and $\tilde{q}_{kl}$ $(k, l=1, 2,\dots, N)$. Hence for some λ_3 $(0<\lambda_3\leqq\lambda_2)$

$$\lambda|s|\,|v_{ii}| \leqq \delta^{N-1}/2 \qquad \text{for} \quad \lambda \leqq \lambda_3. \tag{8.82}$$

Since by (8.78) $|\varepsilon_i|\geqq\delta^{N-1}$, it follows that

$$|\mathrm{Re}(\Delta_{ii})| \geqq \delta^{N-1}/2 \qquad \text{for} \quad \lambda \leqq \lambda_3. \tag{8.83}$$

Hence $(\Delta_{i1}, \Delta_{i2},\dots, \Delta_{iN})^T$ is an eigenvector of e corresponding to μ_i.

We normalize this eigenvector and find its expression. Since ε_i is of constant sign, we may assume that $\varepsilon_i>0$. Then $\varepsilon_i\geqq\delta^{N-1}$ and by (8.82) $\mathrm{Re}(\Delta_{ii})\geqq\delta^{N-1}/2$ for $\lambda\leqq\lambda_3$. Setting $\Delta_i=(\sum_{k=1}^{N}|\Delta_{ik}|^2)^{1/2}$, we have

$$\Delta_i \geqq \delta^{N-1}/2, \quad |\bar{\Delta}_{ii}+\Delta_i| \geqq \delta^{N-1} \qquad \text{for} \quad \lambda \leqq \lambda_3. \tag{8.84}$$

The vector $m_i=(m_{i1}, m_{i2},\dots, m_{iN})^T$ is defined as follows:

$$m_i(x, \omega; \lambda) = 0 \qquad \text{for} \quad \omega\in Z, \tag{8.85}$$

$$m_{ii}(x, \omega; \lambda) = a_i/b_i \qquad \text{for} \quad \omega\in S_z, \tag{8.86}$$

$$m_{ij}(x, \omega; \lambda) = v_{ij}/\Delta_i \quad (j \neq i) \qquad \text{for} \quad \omega\in S_z, \tag{8.87}$$

where

$$a_i(x, \omega; \lambda) = \Delta_i(v_{ii} - \bar{v}_{ii}) - \lambda|s|\eta_i, \quad \eta_i = \textstyle\sum_{k=1, k\neq i}^{N} |v_{ik}|^2,$$

$$b_i(x, \omega; \lambda) = \Delta_i(\bar{\Delta}_{ii} + \Delta_i).$$

Then

$$\Delta_{ii}/\Delta_i = 1 + \lambda|s|m_{ii} \quad \text{for} \quad \omega \in S_z, \tag{8.88}$$

$$\Delta_{ij}/\Delta_i = \lambda|s|m_{ij} \quad (j \neq i) \quad \text{for} \quad \omega \in S_z. \tag{8.89}$$

Hence $\sigma_i + \lambda|s|m_i$ is a normalized eigenvector of e corresponding to μ_i, where σ_i is the i-th column vector of I.

We define matrices $m(x, \omega; \lambda)$, $\Lambda(x, \omega; \lambda)$ and $t(x, \omega; \lambda)$ as follows:

$$m = (m_1, m_2, \dots, m_N), \quad \Lambda = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_N),$$

$$t = I + \lambda|s|m \quad \text{for} \quad \lambda \leqq \lambda_3. \tag{8.90}$$

Then

$$et = t\Lambda \quad \text{for} \quad \omega \in S_z, \quad \lambda \leqq \lambda_3. \tag{8.91}$$

Since by (8.84)–(8.87) $m(x, \omega; \lambda)$ is bounded on $S_{x\omega} \times (0, \lambda_3]$, we have for some λ_4 $(0 < \lambda_4 \leqq \lambda_3)$

$$|\det t| \geqq 1/2 \quad \text{for} \quad \lambda \leqq \lambda_4. \tag{8.92}$$

Hence t^{-1} exists for $\lambda \leqq \lambda_4$ and is bounded on $S_{x\omega} \times (0, \lambda_4]$. From (8.90) and (8.91) it follows that

$$\Lambda = t^{-1}et \quad \text{for} \quad \lambda \leqq \lambda_4, \tag{8.93}$$

$$t^{-1} = I - \lambda|s|t^{-1}m. \tag{8.94}$$

Therefore for some λ_1 $(0 < \lambda_1 \leqq \lambda_4)$

$$(t^{-1})^* t^{-1} \geqq (1/2)I \quad \text{for} \quad \lambda \leqq \lambda_1. \tag{8.95}$$

Let $u(x, \omega; \lambda) = t^{-1}w$. Then from (8.93)

$$\Lambda = u(p_z - i\lambda q|s|)u^{-1} \quad \text{for} \quad \omega \in S_z, \quad \lambda \leqq \lambda_1, \tag{8.96}$$

so that u transforms $p_z - i\lambda q|s|$ into a diagonal matrix.

We show that u has properties of Lemma 6.3. By (8.75) and (8.95) we have

$$u^*u \geqq (e_0/2)I \quad \text{for} \quad (x, \omega) \in S_{x\omega}, \quad \lambda \leqq \lambda_1,$$

and so u has property iii).

By the argument similar to that in 8.9 t and t^{-1} satisfy VI and belong to $\mathscr{L}$.

Hence by E–1) and Lemma 4.4 u and u^{-1} belong to $\mathscr{L}$ and by Lemmas 4.7 and 3.4 satisfy conditions of Theorem 3.3.

By (8.76), (8.90), (8.93) and (8.94) we have

$$\Lambda = t^{-1}et = d+\lambda|s|f, \tag{8.97}$$

where $f = dm - t^{-1}mdt - t^{-1}\tilde{q}t$. Since Λ and d are diagonal, so is f. It is clear that $f \in \mathscr{L}$. Thus by (8.96) and (8.97) u has property iv).

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