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On the Asymptotic Relationships between Two Systems of Differential Equations

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1. Introduction

In this paper we study the problem of asymptotic relationships between solutions of two systems of differential equations, one of which involves deviating arguments. We consider the systems

$$x'(t) = A(t)x(t)+f(t, x(g(t))), \tag{1}$$

$$y'(t) = A(t)y(t), \tag{2}$$

where $A(t)$ is a continuous $n\times n$ matrix function on $R_+=[0, \infty)$, $f(t, z)$ is a continuous n-vector function on $R_+\times R^n$, $g(t)$ is a continuous n-vector function on R_+ such that each component $g_i(t)$ is positive and satisfies $\lim_{t\to\infty} g_i(t)=\infty$, and

$$x(g(t)) = (x_1(g_1(t)),\dots, x_n(g_n(t))).$$

An important special case of (1) is the ordinary differential equation

$$x'(t) = A(t)x(t)+f(t, x(t)). \tag{3}$$

The problem of asymptotic relationships and/or asymptotic equivalence has been studied in many papers; see e.g. Brauer [1], Brauer and Wong [2], Cooke [3], Kato [5], Kitamura [6], Ráb [7], Švec [8], and the references cited in these papers. Recently, Ráb [7] and Kitamura [6] have presented conditions that lead to an equivalence between certain components of the solutions of (3) and the corresponding components of the solutions of (2).

The main purpose of this paper is to extend results of [6] to the systems (1) and (2) with general deviating argument $g(t)$ and to establish conditions that ensure the asymptotic equivalence of (1) and (2) when the deviating argument $g(t)$ is retarded.

In what follows we assume that the components $f_j(t, z)$ of $f(t, z)$ depend essentially on t and the q components $z_1, z_2,\dots, z_q$ $(1\leqq q\leqq n)$ of z in the sense that

$$|f_j(t, z_1,\dots, z_n)| \leqq \omega_j(t, |z_1|,\dots, |z_q|) \tag{4}$$

for $(t, z)\in R_+\times R^n$ and $j=1,\dots, n$, where each $\omega_j(t, r_1,\dots, r_q)$ is continuous on

$R_+ \times R_+^q$ and nondecreasing in $(r_1,\dots, r_q)$ for fixed $t \in R_+$.

We are particularly interested in some asymptotic relationships between the p components $x_1(t),\dots, x_p(t)$ $(q \leqq p \leqq n)$ of the solutions $x(t)$ of (1) and the p components $y_1(t),\dots, y_p(t)$ of the solutions $y(t)$ of (2).

2. Results

Let P_1, P_2 be $n \times n$ matrices such that

$$P_1 + P_2 = I \quad \text{(identity matrix)} \tag{5}$$

and let $Y(t) = (y_{ij}(t))$ be a fundamental matrix of the linear system (2). Then we define

$$Y(t)P_1 Y^{-1}(s) = (y_{ij}(t, s;\, P_1)),$$

$$Y(t)P_2 Y^{-1}(s) = (y_{ij}(t, s;\, P_2)).$$

THEOREM 1. *Assume that the condition* (4) *holds. Let* $\mu_i(t)$, $m_i(t)$ $(i = 1,\dots, p)$ *be positive continuous functions defined on* R_+ *which satisfy*

$$m_i(t) \geqq \mu_i(t) \qquad \text{for} \quad t \in R_+, \quad i = 1,\dots, p. \tag{6}$$

Suppose that there exist matrices P_1, P_2 *satisfying* (5) *and a constant* $T \geqq 0$ *such that for any* $\kappa > 0$

$$\int_t^\infty |y_{ij}(t, s;\, P_1)|\omega_j(s, \kappa m(g(s)))ds < \infty \tag{7}$$

for $t \geqq T$, $i, j = 1,\dots, n$, *and*

$$\int_t^\infty |y_{ij}(t, s;\, P_1)|\omega_j(s, \kappa m(g(s)))ds = o(\mu_i(t)), \tag{8}$$

$$\int_T^t |y_{ij}(t, s;\, P_2)|\omega_j(s, \kappa m(g(s)))ds = o(\mu_i(t)) \tag{9}$$

as $t \to \infty$ *for* $i = 1,\dots, p$, $j = 1,\dots, n$, *where*

$$\kappa m(g(s)) = (\kappa m_1(g_1(s)),\dots, \kappa m_q(g_q(s))).$$

Then, to any solution $y(t) = (y_1(t),\dots, y_n(t))$ *of* (2) *such that* $y_i(t) = O(m_i(t))$ *as* $t \to \infty$ $(i = 1,\dots, p)$, *there exists a solution* $x(t) = (x_1(t),\dots, x_n(t))$ *of* (1) *such that*

$$x_i(t) = y_i(t) + o(\mu_i(t)) \quad \text{as} \quad t \longrightarrow \infty \ (i = 1,\dots, p). \tag{10}$$

Conversely, to any solution $x(t) = (x_1(t),\dots, x_n(t))$ *of* (1) *such that* $x_i(t) = O(m_i(t))$ *as* $t \to \infty$ $(i = 1,\dots, p)$, *there exists a solution* $y(t) = (y_1(t),\dots, y_n(t))$ *of* (2) *such that* (10) *holds.*

PROOF. The first half of the theorem will be proved with use of the Schauder-Tychonoff fixed point theorem as formulated in Coppel [4, p. 9]. Let $y(t) = (y_1(t), \dots, y_n(t))$ be a solution of (2) satisfying $y_i(t) = O(m_i(t))$ as $t \to \infty (i = 1, \dots, p)$. There exist positive numbers $t_0 \geqq T$ and M such that

$$(11) \qquad |y_i(t)| \leqq Mm_i(t) \qquad \text{for} \quad t \geqq t_0 \ (i = 1, \dots, p).$$

In view of (8) and (9), we can choose, for a constant $\kappa > \max\{M, 1\}$, a number $t_1 \geqq t_0$ so large that the following inequalities hold:

$$(12) \qquad \int_t^\infty |y_{ij}(t, s; P_1)| \omega_j(s, \kappa m(g(s)))ds \leqq \frac{\kappa - M}{2n} \mu_i(t),$$

$$(13) \qquad \int_{t_1}^t |y_{ij}(t, s; P_2)| \omega_j(s, \kappa m(g(s)))ds \leqq \frac{\kappa - M}{2n} \mu_i(t)$$

for $t \geqq t_1$, $i = 1, \dots, p$ and $j = 1, \dots, n$. Put $\tau = \min\{\inf_{t \geqq t_1} g_i(t): i = 1, \dots, n\}$, and define the functions $m_i(t)$ $(i = p+1, \dots, n)$ on $[\tau, \infty)$ by

$$(14) \qquad m_i(t) = \begin{cases} |y_i(t)| + \sum_{j=1}^n \int_t^\infty |y_{ij}(t, s; P_1)| \omega_j(s, \kappa m(g(s)))ds \\ \qquad + \sum_{j=1}^n \int_{t_1}^t |y_{ij}(t, s; P_2)| \omega_j(s, \kappa m(g(s)))ds, & t \geqq t_1, \\ m_i(t_1), \qquad \tau \leqq t < t_1. \end{cases}$$

We denote by F the set of all vector functions $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ that are continuous on $[\tau, \infty)$ and satisfy $|\xi_i(t)| \leqq \kappa m_i(t)$ for $t \in [\tau, \infty)$, $i = 1, \dots, n$. We now define the operator Φ by

$$(15) \qquad (\Phi x)_i(t) = \begin{cases} y_i(t) - \sum_{j=1}^n \int_t^\infty y_{ij}(t, s; P_1) f_j(s, x(g(s)))ds \\ \qquad + \sum_{j=1}^n \int_{t_1}^t y_{ij}(t, s; P_2) f_j(s, x(g(s)))ds, & t \geqq t_1, \\ \dfrac{(\Phi x)_i(t_1)}{m_i(t_1)} m_i(t), \qquad \tau \leqq t < t_1, & (i = 1, \dots, n). \end{cases}$$

Clearly, Φ is well-defined on F by the condition (7). We shall show that Φ is continuous and maps F into a compact subset of F.

(i) *Φ maps F into F.* If $x \in F$ and $i = 1, \dots, p$, then by (4), (11), (12), (13) and (6) we have

$$\begin{aligned}|(\Phi x)_i(t)| &\leqq |y_i(t)| + \sum_{j=1}^{n} \int_t^{\infty} |y_{ij}(t, s; P_1)|\,|f_j(s, x(g(s)))|ds \\ &\quad + \sum_{j=1}^{n} \int_{t_1}^{t} |y_{ij}(t, s; P_2)|\,|f_j(s, x(g(s)))|ds \\ &\leqq Mm_i(t) + \sum_{j=1}^{n} \int_t^{\infty} |y_{ij}(t, s; P_1)|\omega_j(s, \kappa m(g(s)))ds \\ &\quad + \sum_{j=1}^{n} \int_{t_1}^{t} |y_{ij}(t, s; P_2)|\omega_j(s, \kappa m(g(s)))ds \\ &\leqq Mm_i(t) + \sum_{j=1}^{n} \frac{\kappa - M}{2n}\mu_i(t) + \sum_{j=1}^{n} \frac{\kappa - M}{2n}\mu_i(t) \leqq \kappa m_i(t)\end{aligned}$$

for $t \geqq t_1$, and

$$|(\Phi x)_i(t)| = \frac{|(\Phi x)_i(t_1)|}{m_i(t_1)} m_i(t) \leqq \frac{\kappa m_i(t_1)}{m_i(t_1)} m_i(t) = \kappa m_i(t)$$

for $\tau \leqq t < t_1$. If $x \in F$ and $i = p+1,\dots, n$, then by virtue of (4) and (14) we easily see that $|(\Phi x)_i(t)| \leqq m_i(t) \leqq \kappa m_i(t)$ for $t \geqq \tau$. Therefore Φ maps F into itself.

(ii) *Φ is continuous.* Let x_k $(k=1, 2,\dots)$ and x be functions in F such that $x_k(t) \to x(t)$ uniformly on every compact subinterval of $[\tau, \infty)$. First, consider any interval of the form $[t_1, T]$. Given an $\varepsilon > 0$, there is $t_2 \geqq T$ such that

$$\int_{t_2}^{\infty} \max_{t \in [t_1, T]} |y_{ij}(t, s; P_1)|\omega_j(s, \kappa m(g(s)))ds < \varepsilon/6n \tag{16}$$

for $i, j = 1,\dots, n$. Then,

$$\begin{aligned}&|(\Phi x_k)_i(t) - (\Phi x)_i(t)| \\ &\leqq \sum_{j=1}^{n} \int_t^{t_2} |y_{ij}(t, s; P_1)|\,|f_j(s, x_k(g(s))) - f_j(s, x(g(s)))|ds \\ &\quad + \sum_{j=1}^{n} \int_{t_2}^{\infty} |y_{ij}(t, s; P_1)|\,|f_j(s, x_k(g(s))) - f_j(s, x(g(s)))|ds \\ &\quad + \sum_{j=1}^{n} \int_{t_1}^{t} |y_{ij}(t, s; P_2)|\,|f_j(s, x_k(g(s))) - f_j(s, x(g(s)))|ds \qquad (17) \\ &\leqq \sum_{j=1}^{n} \int_{t_1}^{t_2} \max_{t \in [t_1, T]} |y_{ij}(t, s; P_1)|ds \cdot \max_{s \in [t_1, t_2]} |f_j(s, x_k(g(s))) - f_j(s, x(g(s)))| \\ &\quad + 2\sum_{j=1}^{n} \int_{t_2}^{\infty} \max_{t \in [t_1, T]} |y_{ij}(t, s; P_1)|\omega_j(s, \kappa m(g(s)))ds \\ &\quad + \sum_{j=1}^{n} \int_{t_1}^{T} \max_{t \in [t_1, T]} |y_{ij}(t, s; P_2)|ds \cdot \max_{s \in [t_1, T]} |f_j(s, x_k(g(s))) - f_j(s, x(g(s)))|\end{aligned}$$

for $t \in [t_1, T]$ and $i=1,\dots, n$. Since $f_j(s, z)$ $(j=1,\dots, n)$ are continuous and the sequence $\{x_k(t)\}$ converges uniformly to $x(t)$ on any compact subinterval of $[\tau, \infty)$, there exists a positive integer k_0 such that if $k \geqq k_0$, then

$$(18) \qquad \max_{s \in [t_1, t_2]} |f_j(s, x_k(g(s))) - f_j(s, x(g(s)))| < \varepsilon/3nN$$

for $j=1,\dots, n$, where

$$N = \max\left\{\int_{t_1}^{t_2} \max_{t \in [t_1, T]} |y_{ij}(t, s;\, P_l)| ds \colon l = 1, 2;\ i, j = 1,\dots, n\right\}.$$

Using (16), (17) and (18), we conclude that if $k \geqq k_0$, then

$$(19) \qquad |(\Phi x_k)_i(t) - (\Phi x)_i(t)| < \sum_{j=1}^{n} N \cdot \varepsilon/3nN + 2\sum_{j=1}^{n} \varepsilon/6n + \sum_{j=1}^{n} N \cdot \varepsilon/3nN = \varepsilon$$

for any $t \in [t_1, T]$, $i=1,\dots, n$. Next, consider the interval $[\tau, t_1]$. The inequality (19) implies that for a given $\varepsilon > 0$ there is a positive integer k_0 such that if $k \geqq k_0$, then

$$|(\Phi x_k)_i(t_1) - (\Phi x)_i(t_1)| < \varepsilon(\max_{t \in [\tau, t_1]} m_i(t)/m_i(t_1))^{-1}$$

for $i=1,\dots, n$. Therefore, if $k \geqq k_0$, we have

$$(20) \qquad |(\Phi x_k)_i(t) - (\Phi x)_i(t)| = \{m_i(t)/m_i(t_1)\} |(\Phi x_k)_i(t_1) - (\Phi x)_i(t_1)| < \varepsilon$$

for any $t \in [\tau, t_1]$, $i=1,\dots, n$. The inequalities (19) and (20) show that, for each i, $(\Phi x_k)_i(t)$ converges uniformly to $(\Phi x)_i(t)$ on any compact subinterval of $[\tau, \infty)$. This implies that Φ is continuous.

(iii) *ΦF is uniformly bounded and equicontinuous at every point of* $[\tau, \infty)$. The uniform boundedness of ΦF is obvious. Differentiating (15), we obtain

$$\begin{aligned}(\Phi x)_i'(t) = y_i'(t) &- \sum_{j=1}^{n} \int_t^\infty \frac{\partial}{\partial t} y_{ij}(t, s;\, P_1) f_j(s, x(g(s))) ds \\ &+ \sum_{j=1}^{n} \int_{t_1}^{t} \frac{\partial}{\partial t} y_{ij}(t, s;\, P_2) f_j(s, x(g(s))) ds \\ &+ \sum_{j=1}^{n} [y_{ij}(t, t;\, P_1) + y_{ij}(t, t;\, P_2)] f_j(t, x(g(t))),\end{aligned}$$

from which, noting that $|x_i(t)| \leqq \kappa m_i(t)$ for $t \geqq \tau$, we see that

$$\begin{aligned}|(\Phi x)_i'(t)| \leqq |y_i'(t)| &+ \sum_{j=1}^{n} \int_t^\infty \left|\frac{\partial}{\partial t} y_{ij}(t, s;\, P_1)\right| \omega_j(s, \kappa m(g(s))) ds \\ &+ \sum_{j=1}^{n} \int_{t_1}^{t} \left|\frac{\partial}{\partial t} y_{ij}(t, s;\, P_2)\right| \omega_j(s, \kappa m(g(s))) ds\end{aligned}$$

$$+\sum_{j=1}^{n}|y_{ij}(t, t; P_1)+y_{ij}(t, t; P_2)|\omega_j(t, \kappa m(g(t)))$$

for $t \geqq t_1$. This implies that, on any finite subinterval of $[t_1, \infty)$, the functions $(\Phi x)'_i(t)$ $(i=1,\dots, n)$ are bounded by a constant independent of $x \in F$. Hence, ΦF is equicontinuous on every finite subinterval of $[t_1, \infty)$. The equicontinuity of $(\Phi x)_i(t)$, $x \in F$, on $[\tau, t_1)$ is obvious, since $(\Phi x)_i(t_1)$ $(i=1,\dots, n)$ are bounded independently of $x \in F$. Thus we conclude that ΦF is equicontinuous on every finite subinterval of $[\tau, \infty)$.

From the preceding considerations we are able to apply the Schauder-Tychonoff fixed point theorem to the operator Φ. Let $x(t)=(x_1(t),\dots, x_n(t)) \in F$ be a fixed point of Φ. It is easy to see that $x(t)$ is a solution of (1) for $t \geqq t_1$. Using (15) we have

$$|x_i(t)-y_i(t)| \leqq \sum_{j=1}^{n}\int_t^{\infty}|y_{ij}(t, s; P_1)|\omega_j(s, \kappa m(g(s)))ds$$
$$+\sum_{j=1}^{n}\int_{t_1}^{t}|y_{ij}(t, s; P_2)|\omega_j(s, \kappa m(g(s)))ds$$

for $t \geqq t_1$, $i=1,\dots, p$. This inequality together with (8) and (9) shows that the solution $x(t)$ has the required asymptotic property (10).

To prove the second assertion of the theorem, let $x(t)=(x_1(t),\dots, x_n(t))$ be a solution of (1) such that $x_i(t)=O(m_i(t))$ as $t \to \infty$ $(i=1,\dots, p)$. Define $y(t)=(y_1(t),\dots, y_n(t))$ by

$$y_i(t) = x_i(t)+\sum_{j=1}^{n}\int_t^{\infty}y_{ij}(t, s; P_1)f_j(s, x(g(s)))ds$$
$$-\sum_{j=1}^{n}\int_{t_1}^{t}y_{ij}(t, s; P_2)f_j(s, x(g(s)))ds,$$

where t_1 is sufficiently large. Since there is a constant $\kappa>0$ such that $|x_i(g(t))| \leqq \kappa m_i(g(t))$ for $t \geqq t_1$, $i=1,\dots, p$, the function $y(t)$ is well-defined by the condition (7). It is easy to verify that $y(t)$ is a solution of (2) for $t \geqq t_1$. The required asymptotic relationship (10) follows readily from (8) and (9). This completes the proof of Theorem 1.

Remark 1. Theorem 1 is an extension of a result of Kitamura [6, Theorem 1].

Theorem 2. *Assume that the condition* (4) *holds. Let* $\mu_i(t)$, $m_i(t)$ $(i=1,\dots, p)$ *be positive continuous functions which satisfy* (6). *Suppose that there exist matrices* P_1, P_2 *satisfying* (5) *and constants* $T \geqq 0$ *and* $L>0$ *such that* *for any* $\kappa>0$

$$\int_t^\infty |y_{ij}(t,\ s;\ P_1)|\omega_j(s,\ \kappa m(g(s)))ds < \infty, \tag{7}$$

$$\int_t^\infty \omega_j(s,\ \kappa m(g(s)))ds < \infty \tag{21}$$

for $t \geqq T$, $i, j=1,\dots, n$, *and*

$$|y_{ij}(t,\ s;\ P_1)| \leqq L\mu_i(t), \qquad T \leqq t \leqq s, \tag{22}$$

$$|y_{ij}(t,\ s;\ P_2)| \leqq L\,\mu_i(t), \qquad T \leqq s \leqq t, \tag{23}$$

$$y_{ij}(t;\ P_2)/\mu_i(t) \longrightarrow 0 \quad as \quad t \longrightarrow \infty \tag{24}$$

for $i=1,\dots, p$, $j=1,\dots, n$, *where* $y_{ij}(t;\ P_2)$ *is the* (i, j)*-element of the matrix* $Y(t)P_2$.

Then, the conclusions of Theorem 1 *hold.*

PROOF. It suffices to show that the conditions (21)–(24) imply the conditions (8) and (9) of Theorem 1. The condition (8) is obvious from (21) and (22). To see that (9) is valid, let $\varepsilon>0$ be given arbitrarily. We choose $t_0 \geqq T$ so large that

$$\int_{t_0}^\infty \omega_j(s,\ \kappa m(g(s)))ds < \varepsilon/2L, \tag{25}$$

and take $t_1 \geqq t_0$ so that

$$|y_{ik}(t;\ P_2)|/\mu_i(t) < \varepsilon/2nM \qquad (k = 1,\dots, n) \tag{26}$$

for $t \geqq t_1$, where

$$M = \max\left\{\int_T^{t_0} |(k,\ j)\text{-element of } Y^{-1}(s)|\omega_j(s,\ \kappa m(g(s)))ds\colon k = 1,\dots, n\right\}.$$

These are possible by (21) and (24). Then, using (23), (26) and (25), we compute as follows:

$$\begin{aligned} &\frac{1}{\mu_i(t)}\int_T^t |y_{ij}(t,\ s;\ P_2)|\omega_j(s,\ \kappa m(g(s)))ds \\ &\quad= \frac{1}{\mu_i(t)}\int_T^{t_0} |y_{ij}(t,\ s;\ P_2)|\omega_j(s,\ \kappa m(g(s)))ds \\ &\qquad+ \frac{1}{\mu_i(t)}\int_{t_0}^t |y_{ij}(t,\ s;\ P_2)|\omega_j(s,\ \kappa m(g(s)))ds \end{aligned}$$

$$\leqq \sum_{k=1}^{n} \frac{|y_{ik}(t; P_2)|}{\mu_i(t)} \int_T^{t_0} |(k, j)-\text{element of } Y^{-1}(s)| \omega_j(s, \kappa m(g(s)))ds$$

$$+\int_{t_0}^{t} \frac{|y_{ij}(t, s; P_2)|}{\mu_i(t)} \omega_j(s, \kappa m(g(s)))ds$$

$$\leqq \sum_{k=1}^{n} \frac{|y_{ik}(t; P_2)|}{\mu_i(t)} M + L\int_{t_0}^{\infty} \omega_j(s, \kappa m(g(s)))ds$$

$$< \sum_{k=1}^{n} \{\varepsilon/2nM\}M + L\{\varepsilon/2L\} = \varepsilon, \qquad t \geqq t_1.$$

This shows that (9) holds, and the proof of Theorem 2 is complete.

REMARK 2. Theorem 2 is a generalization of a result of Brauer and Wong [2, Theorem 1].

LEMMA. *Let $F_i(t, z_1,\dots, z_q)$ $(i=1,\dots, q)$ be nonnegative continuous functions on $[T, \infty)\times R_+^q$ which are nondecreasing in $(z_1,\dots, z_q)$ and satisfy*

$$\lim_{t\to\infty} (1/c) \int_t^{\infty} F_i(s, c,\dots, c)ds = 0, \qquad i = 1,\dots, q,$$

uniformly with respect to $c\in[1, \infty)$. Let $z_i(t)$ $(i=1,\dots, q)$ be nonnegative continuous functions on $[T, \infty)$ satisfying

$$z_i(t) \leqq K_i+\int_T^t F_i(s, z_1(s),\dots, z_q(s))ds, \qquad i = 1,\dots, q, \tag{27}$$

for $t\geqq T^*$ $(\geqq T)$, where K_i are constants.

Then $z_i(t)$ $(i=1,\dots, q)$ are bounded functions of t.

PROOF. We choose $t_0\geqq T^*$ sufficiently large that

$$(1/c)\int_{t_0}^{\infty} F_i(s, c,\dots, c)ds \leqq \tfrac{1}{2}, \qquad i=1,\dots, q, \tag{28}$$

for any $c\in[1, \infty)$. We put

$$I_i(t) = K_i+\int_T^t F_i(s, z_1(s),\dots, z_q(s))ds, \qquad i = 1,\dots, q.$$

To prove that $z_i(t)$ $(i=1,\dots, q)$ are bounded functions of t, it is sufficient to show that $I_i(t)$ are bounded. We may assume without loss of generality that $K_i\geqq 1$ $(i=1,\dots, q)$. So we have $I(t)\equiv\max\{I_i(t)\colon i=1,\dots, q\}\geqq 1$ for $t\geqq t_0$. From (28) it follows that

$$\int_{t_0}^{\infty} F_i(s, I(t),\dots, I(t))ds \leqq \tfrac{1}{2}I(t) \tag{29}$$

for $t \geqq t_0$, $i=1,\dots, q$. For any $t \geqq t_0$ there is an index $j \in \{1,\dots, q\}$ such that

$$I(t) = I_j(t) \leqq K + \int_{t_0}^{t} F_j(s, z_1(s),\dots, z_q(s))ds, \tag{30}$$

where

$$K = \max\left\{K_j + \int_T^{t_0} F_j(s, z_1(s),\dots, z_q(s))ds : j = 1,\dots, q\right\},$$

which does not depend on t. Then, using (30), (27) and (29), we have

$$\begin{aligned} I(t) &\leqq K + \int_{t_0}^{t} F_j(s, I_1(s),\dots, I_q(s))ds \\ &\leqq K + \int_{t_0}^{t} F_j(s, I(t),\dots, I(t))ds \leqq K + \tfrac{1}{2}I(t), \qquad t \geqq t_0, \end{aligned}$$

from which it follows that $I(t) \leqq 2K$ for $t \geqq t_0$. Since K is independent of t, the function $I(t)$ is bounded. Therefore $I_i(t)$ $(i=1,\dots, q)$ are bounded functions of t. This completes the proof of the lemma.

In the remaining part of this paper we assume that the deviating argument $g(t)=(g_1(t),\dots, g_n(t))$ is retarded in the sense that

$$g_i(t) \leqq t \qquad \text{for} \quad t \in R_+, \quad i = 1,\dots, q. \tag{31}$$

With the help of the lemma we prove a theorem which enables us to estimate the growth (or decay) of solutions of (1) in terms of a fundamental matrix of (2).

THEOREM 3. *Assume that the conditions* (4) *and* (31) *hold. Let* $Y(t) = (y_{ij}(t))$ *be a fundamental matrix of* (2) *and let* $Y(t)Y^{-1}(s) = (y_{ij}(t, s))$. *Suppose that there exist positive continuous functions* $\sigma_i(t)$, $M_{ij}(s)$ $(i=1,\dots, p, j=1,\dots, n)$ *defined on* R_+ *such that*

$$\sigma_i(t) \geqq \max\{|y_{ij}(t)| : j = 1,\dots, n\}, \qquad t \in R_+, \tag{32}$$

$$M_{ij}(s) \geqq |y_{ij}(t, s)|/\sigma_i(t), \qquad 0 \leqq s \leqq t, \tag{33}$$

$$\int_0^\infty M_{ij}(s)\omega_j(s, \kappa\sigma(g(s)))ds < \infty \qquad (i = 1,\dots, p, j = 1,\dots, n) \tag{34}$$

for any $\kappa > 0$ *and*

$$\lim_{t\to\infty} (1/c) \int_t^\infty M_{ij}(s)\omega_j(s, c\sigma(g(s)))ds = 0 \qquad (i = 1,\dots, q, j = 1,\dots, n) \tag{35}$$

uniformly with respect to $c \in [1, \infty)$.

Then every solution $x(t)=(x_1(t),\dots, x_n(t))$ *of* (1) *which exists on some interval* $[t_x, \infty)$ *satisfies* $x_i(t)=O(\sigma_i(t))$ *as* $t\to\infty$ $(i=1,\dots, p)$.

PROOF. Let $x(t)=(x_1(t),\dots, x_n(t))$ be a solution of (1) defined on $[t_x, \infty)$. By the variation of constants formula $x(t)$ admits the expression

$$x_i(t) = \sum_{j=1}^{n} y_{ij}(t)c_j+\int_T^t \sum_{j=1}^{n} y_{ij}(t, s)f_j(s, x(g(s)))ds,$$

where $T\geqq t_x$ is sufficiently large. Dividing the above equality by $\sigma_i(t)$ and using (32) and (33), we obtain

$$\frac{|x_i(t)|}{\sigma_i(t)} \leqq K+\int_T^t \sum_{j=1}^{n} M_{ij}(s)\omega_j(s, |x_1(g_1(s))|,\dots, |x_q(g_q(s))|)ds$$

for $t\geqq T$, $i=1,\dots, p$, where $K=\sum_{j=1}^n |c_j|$. Now, set

$$z_i(t) = |x_i(g_i(t))|/\sigma_i(g_i(t)),$$

$$F_i(s, z_1,\dots, z_q) = \sum_{j=1}^{n} M_{ij}(s)\omega_j(s, z_1\sigma_1(g_1(s)),\dots, z_q\sigma_q(g_q(s))).$$

Then, we have

$$\begin{aligned} (36) \qquad z_i(t) &\leqq K +\int_T^{g_i(t)} \sum_{j=1}^{n} M_{ij}(s)\omega_j(s, z_1(s)\sigma_1(g_1(s)),\dots, z_q(s)\sigma_q(g_q(s)))ds \\ &= K+\int_T^{g_i(t)} F_i(s, z_1(s),\dots, z_q(s))ds \end{aligned}$$

for $i=1,\dots, p$ and $t\geqq T^*$, where T^* is chosen so that $g_i(t)\geqq T$ for $t\geqq T^*$. If $i=1,\dots, q$, then by virtue of (31) we get

$$z_i(t) \leqq K+\int_T^t F_i(s, z_1(s),\dots, z_q(s))ds$$

for $i=1,\dots, q$ and $t\geqq T^*$. Since the hypothesis (35) yields

$$\lim_{t\to\infty} (1/c) \int_t^{\infty} F_i(s, c,\dots, c)ds = 0 \qquad (i = 1,\dots, q)$$

uniformly with respect to $c\in[1, \infty)$, we are able to apply the lemma to conclude that the functions $z_i(t)$ $(i=1,\dots, q)$ are bounded. It follows from (34) and (36) that $z_i(t)$ $(i=q+1,\dots, p)$ are bounded functions of t. As the functions $z_i(t) =|x_i(g_i(t))|/\sigma_i(g_i(t))$ $(i=1,\dots, p)$ are bounded, $|x_i(t)|/\sigma_i(t)$ are also bounded. Thus Theorem 3 is proved.

Combining Theorem 1 with Theorem 3, we have the following theorem which establishes the asymptotic equivalence of the systems (1) and (2) with retarded argument $g(t)$.

THEOREM 4. *Suppose that the hypotheses of Theorem* 1 *and* 3 *are satisfied. Suppose in addition that*

$$m_i(t) \geqq \sigma_i(t) \qquad \textit{for} \quad t \in R_+, \quad i = 1,\dots, p.$$

Then, to any solution $y(t)=(y_1(t),\dots, y_n(t))$ *of* (2), *there exists a solution* $x(t)=(x_1(t),\dots, x_n(t))$ *of* (1) *such that*

$$x_i(t) = y_i(t)+o(\mu_i(t)) \quad \textit{as} \quad t \longrightarrow \infty, \quad (i = 1,\dots, p). \tag{10}$$

Conversely, to any solution $x(t)=(x_1(t),\dots, x_n(t))$ *of* (1), *there exists a solution* $y(t)=(y_1(t),\dots, y_n(t))$ *of* (2) *such that* (10) *holds.*

PROOF. By Theorem 1, we have only to show that any solution $x(t)$ of (1) satisfies $x_i(t)=O(m_i(t))$ as $t\to\infty$ $(i=1,\dots, p)$ and that any solution $y(t)$ of (2) satisfies $y_i(t)=O(m_i(t))$ as $t\to\infty$ $(i=1,\dots, p)$. The former is a consequence of Theorem 3, and the latter is clear, since

$$|y_i(t)| = |\textstyle\sum_{j=1}^n y_{ij}(t)c_j| \leqq \sum_{j=1}^n \sigma_i(t)|c_j| \leqq (\sum_{j=1}^n |c_j|)m_i(t).$$

The next theorem follows immediately from Theorem 4.

THEOREM 5. *Assume that the conditions* (4) *and* (31) *hold. Suppose that a fundamental matrix* $Y(t)$ *of* (2) *and its inverse matrix* $Y^{-1}(t)$ *are bounded. Suppose that*

$$\int_0^\infty \omega_j(s, \kappa,\dots, \kappa)ds < \infty \qquad (j = 1,\dots, n)$$

for any $\kappa>0$ *and*

$$\lim_{t\to\infty} (1/c)\int_t^\infty \omega_j(s, c,\dots, c)ds = 0 \qquad (j = 1,\dots, n)$$

uniformly with respect to $c \in [1, \infty)$.

Then, to any solution $y(t)=(y_1(t),\dots, y_n(t))$ *of* (2), *there exists a solution* $x(t) =(x_1(t),\dots, x_n(t))$ *of* (1) *such that*

$$\lim_{t\to\infty} |x_i(t)-y_i(t)| = 0 \qquad (i = 1,\dots, n). \tag{37}$$

Conversely, to any solution $x(t)=(x_1(t),\dots, x_n(t))$ *of* (1), *there exists a solution* $y(t)=(y_1(t),\dots, y_n(t))$ *of* (2) *such that* (37) *holds.*

PROOF. Take P_1 to be the identity matrix and P_2 to be the zero matrix. Set $M_i=\sup_{t\geqq 0}\{\max\{|y_{ij}(t)|: j=1,\dots, n\}\}$ and $M_{ij}=\sup_{t,s\geqq 0} |y_{ij}(t, s)|$ $(i, j=1,\dots, n)$. Since $Y(t)=(y_{ij}(t))$ and $Y(t)Y^{-1}(s)=(y_{ij}(t, s))$ are bounded, M_i and M_{ij} are determined as positive constants. Taking $\mu_i(t)\equiv\sigma_i(t)\equiv m_i(t)\equiv M_i$ and $M_{ij}(s)\equiv M_{ij}/M_i$, we can easily verify that all the conditions of Theorem 4 are satisfied. Hence the conclusion follows from Theorem 4. This completes the proof.

REMARK 3. Theorem 5 is an extension of a result of Brauer [1, Theorem 3].

Finally, we examine the case that $A(t)=A$ is a constant $n\times n$ matrix. The differential equations are the following:

$$x'(t) = Ax(t)+f(t, x(g(t))), \tag{1'}$$

$$y'(t) = Ay(t). \tag{2'}$$

We assume that A has the Jordan canonical form:

$$A = \operatorname{diag}[J_1,\dots, J_k, J_{k+1},\dots, J_l],$$

where J_h $(1\leqq h\leqq l)$ are square matrices of order n_h with λ_h on the diagonal, 1 on the subdiagonal, and 0 elsewhere. Without loss of generality we will suppose that $\operatorname{Re}\lambda_h$ $(1\leqq h\leqq k)$ are negative and $\operatorname{Re}\lambda_h$ $(k+1\leqq h\leqq l)$ are nonnegative. Let λ be the largest real part of the eigenvalues λ_h $(1\leqq h\leqq l)$ of A, and let m and r be the maximum orders of the Jordan blocks which correspond to eigenvalues of A with real part equal to λ and 0 respectively. (If no real parts equal zero, then put $r=1$.)

THEOREM 6. *Assume that the conditions* (4) *and* (31) *hold.*

(i) *If $\lambda<0$, then suppose that*

$$\int_0^\infty e^{-\lambda s}\omega_j(s, \kappa[g(s)]^{m-1}e^{\lambda g(s)})ds < \infty \qquad (j = 1,\dots, n)$$

for any $\kappa>0$ and

$$\lim_{t\to\infty}(1/c)\int_t^\infty e^{-\lambda s}\omega_j(s, c[g(s)]^{m-1}e^{\lambda g(s)})ds = 0 \qquad (j=1,\dots,n) \tag{38}$$

uniformly with respect to $c\in[1, \infty)$, where

$$c[g(s)]^{m-1}e^{\lambda g(s)} = (c[g_1(s)]^{m-1}e^{\lambda g_1(s)},\dots, c[g_q(s)]^{m-1}e^{\lambda g_q(s)}).$$

(ii) *If $\lambda\geqq 0$, then suppose that*

$$\int_0^\infty s^{r-1}\omega_j(s, \kappa[g(s)]^{m-1}e^{\lambda g(s)})ds < \infty \qquad (j = 1,\dots, n)$$

for any $\kappa>0$. Suppose in addition that the condition (38) *is satisfied.*

Then, to any solution $y(t)=(y_1(t),\dots, y_n(t))$ of (2'), *there exists a solution*

$x(t)=(x_1(t),\dots, x_n(t))$ of (1′) such that

$$\lim_{t\to\infty}|x_i(t)-y_i(t)| = 0 \qquad (i = 1,\dots, n). \tag{37}$$

Conversely, to any solution $x(t)=(x_1(t),\dots, x_n(t))$ of (1′), there exists a solution $y(t)=(y_1(t),\dots, y_n(t))$ of (2′) such that (37) holds.

PROOF. A fundamental matrix $Y(t)$ of (2′) is given explicitly by

$$Y(t) = e^{tA} = \mathrm{diag}\,[e^{tJ_1},\dots, e^{tJ_k}, e^{tJ_{k+1}},\dots, e^{tJ_l}],$$

where

$$e^{tJ_h} = e^{\lambda_h t}\begin{bmatrix} 1 & t & t^2/2! & \cdots t^{n_h-1}/(n_h-1)! \\ 0 & 1 & t & \cdots t^{n_h-2}/(n_h-2)! \\ \cdots & \cdots & \cdots & \cdots\ \cdots\cdots\cdots \\ 0 & 0 & 0 & \cdots\quad 1 \end{bmatrix}, \; h=1,\dots, l.$$

Letting $Y(t)=(y_{ij}(t))$ and $Y(t)Y^{-1}(s)=(y_{ij}(t, s))$, we have $y_{ij}(t, s)=y_{ij}(t-s)$. Take $\sigma_i(t)=K(t+1)^{m-1}e^{\lambda t}$ and $M_{ij}(s)=Ke^{-\lambda s}$ $(i, j=1,\dots, n)$, where K is a suitable positive constant. Then, it is easy to see that all the conditions of Theorem 3 are satisfied.

(i) If $\lambda<0$, then we apply Theorem 3. It follows that every solution $x(t)=(x_1(t),\dots, x_n(t))$ of (1′) has the property $x_i(t)=O(t^{m-1}e^{\lambda t})$ as $t\to\infty$ $(i=1,\dots, n)$. Since every solution $x(t)$ of (1′) converges to zero as $t\to\infty$ and the same is true also for the solutions $y(t)$ of (2′), the asymptotic equivalence (37) is evident.

(ii) If $\lambda\geqq 0$, then we apply Theorem 4. Take $m_i(t)\equiv\sigma_i(t)=K(t+1)^{m-1}e^{\lambda t}$ and $\mu_i(t)\equiv K$ $(i=1,\dots, n)$. Let P_1, P_2 be the diagonal matrices such that $P_1=\mathrm{diag}\,[0_1,\dots, 0_k, I_{k+1},\dots, I_l]$ and $P_2=\mathrm{diag}\,[I_1,\dots, I_k, 0_{k+1},\dots, 0_l]$, where I_h and 0_h $(0\leqq h\leqq l)$ are the identity matrices and the zero matrices of order n_h respectively. We have

$$Y(t)P_1Y^{-1}(s) = (y_{ij}(t, s; P_1)) = \mathrm{diag}\,[0_1,\dots, 0_k, e^{(t-s)J_{k+1}},\dots, e^{(t-s)J_l}]$$

and

$$Y(t)P_2Y^{-1}(s) = (y_{ij}(t, s; P_2)) = \mathrm{diag}\,[e^{(t-s)J_1},\dots, e^{(t-s)J_k}, 0_{k+1},\dots, 0_l].$$

Noting that the real parts of the eigenvalues λ_h of J_h $(k+1\leqq h\leqq l)$ are positive or equal to zero and that the real parts of the eigenvalues λ_h of J_h $(1\leqq h\leqq k)$ are negative, we obtain

$$|y_{ij}(t, s; P_1)| \leqq K_1(s-t+1)^{r-1}, \qquad t\leqq s,$$

$$|y_{ij}(t, s; P_2)| \leqq K_2(t-s+1)^{m^*-1}e^{\alpha(t-s)}, \qquad s \leqq t,$$

for $i, j=1,\dots, n$, where K_1, K_2 are some constants and $\alpha<0$ is the largest real part of λ_h $(1\leqq h\leqq k)$ and m^* is the maximum order of those blocks in the Jordan form which correspond to eigenvalues with real part equal to α. Then we have

$$\begin{aligned}
&\frac{1}{\mu_i(t)}\int_t^\infty |y_{ij}(t, s; P_1)|\omega_j(s, \kappa m(g(s)))ds \\
&\leqq (K_1/K)\int_t^\infty (s-t+1)^{r-1}\omega_j(s, \kappa K[g(s)+1]^{m-1}e^{\lambda g(s)})ds \\
&\leqq (K_1/K)\int_t^\infty s^{r-1}\omega_j(s, 2\kappa K[g(s)]^{m-1}e^{\lambda g(s)})ds \longrightarrow 0
\end{aligned}$$

as $t\to\infty$, and we have

$$\begin{aligned}
&\frac{1}{\mu_i(t)}\int_T^t |y_{ij}(t, s; P_2)|\omega_j(s, \kappa m(g(s)))ds \\
&\leqq (K_2/K)\int_T^t (t-s+1)^{m^*-1}e^{\alpha(t-s)}\omega_j(s, \kappa K[g(s)+1]^{m-1}e^{\lambda g(s)})ds \\
&\leqq C\int_T^{t/2} e^{\alpha(t-s)/2}\omega_j(s, 2\kappa K[g(s)]^{m-1}e^{\lambda g(s)})ds \\
&\quad + C\int_{t/2}^t e^{\alpha(t-s)/2}\omega_j(s, 2\kappa K[g(s)]^{m-1}e^{\lambda g(s)})ds \\
&\leqq Ce^{\alpha t/4}\int_T^\infty \omega_j(s, 2\kappa K[g(s)]^{m-1}e^{\lambda g(s)})ds \\
&\quad + C\int_{t/2}^\infty \omega_j(s, 2\kappa K[g(s)]^{m-1}e^{\lambda g(s)})ds \longrightarrow 0
\end{aligned}$$

as $t\to\infty$, where $C=(K_2/K)\sup_{z\geqq 0}(z+1)^{m^*-1}e^{\alpha z/2}<\infty$. Hence all the conditions of Theorem 1 are satisfied.

The above observation enables us to apply Theorem 4, from which we conclude that the required asymptotic equivalence (37) is satisfied. This completes the proof.

REMARK 4. Theorem 6 generalizes a result of Švec [8, Theorem 6].

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