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(Z, λ)-Nuclear Mappings and Z-Nuclear Spaces

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Introduction

Persson and Pietsch [5] introduced the concepts of p-nuclear and p-quasi-nuclear mappings in Banach spaces. These concepts were recently extended in Miyazaki [4] to (p, q)-nuclear and (p, q)-quasi-nuclear mappings by using the sequence spaces $l_{p,q}$. On the other hand, these were extended in Ceĭtlin [1] to (Z, p)-nuclear and (Z, p)-quasi-nuclear mappings. The object of this paper is to extend these two kinds of concepts to (Z, λ)-nuclear and (Z, λ)-quasi-nuclear mappings in Banach spaces by making use of abstract sequence spaces λ. In case $1 \leq p \leq \infty$, $1 \leq q < \infty$, if $\lambda = l_{p,q}$ and Z is one-dimensional, a (Z, λ)-nuclear mapping coincides with a (p, q)-nuclear mapping, and if $\lambda = l_p$, a (Z, λ)-nuclear mapping coincides with a (Z, p)-nuclear mapping. We shall also extend the notion of nuclear spaces to Z-nuclear spaces by using (Z, l_1)-nuclear mappings introduced by Ceĭtlin [1]. We see that the tensor product of a nuclear space and a Banach space Z is Z-nuclear, and thus the space $\mathfrak{S}(R^n, Z)$ of rapidly decreasing functions defined in R^n and valued in Z is a Z-nuclear space.

In Section 1, we define the sequence space λ of type Λ and of type Λ_0 in such a way that $l_{p,q}$ is a space of type Λ_0 for $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and is a space of type Λ for $q \neq \infty$. In Section 2, we introduce the space $\lambda(Z)$ and consider the dual space of $\lambda(Z)$. Section 3 is devoted to studying (Z, λ)-nuclear mappings and Section 4 to studying (Z, λ)-quasi-nuclear mappings. We investigate Z-nuclear spaces in Section 5.

§1. Notations and Definitions

Let E and F be Banach spaces. We shall denote by $L(E, F)$ the space of continuous linear mappings T from E to F with the usual mapping norm

$$\|T\| = \sup_{\|u\| \leq 1} \|Tu\|.$$

We denote by $K(E, F)$ the space of compact mappings and by $L_0^Z(E, F)$ the space of mappings of Z-finite rank. Here $T \in L_0^Z(E, F)$ means that it can be written in the form

$$Tu = \sum_{i=1}^{n} B_i A_i u \qquad \text{for each} \quad u \in E$$

with $\{A_i\} \subset L(E, Z)$ and $\{B_i\} \subset L(Z, F)$, where Z is a Banach space.

Now we start with the sequence space C_0 of all scalar sequences converging to zero in which an extended quasi-norm p is given. We shall then define the sequence space $\lambda \subset C_0$ to be the space consisting of all $x \in C_0$ such that $p(x) < \infty$. We shall denote the extended quasi-norm p by $\|\cdot\|_\lambda$. We assume that λ is a non-zero space satisfying the following conditions:

(a) *If for any* $u = (u_1, \ldots, u_n, \ldots) \in \lambda$ *we set* $u^i = (u_1, \ldots, u_i, 0, \ldots)$ *for* $i = 1, 2, \ldots$, *then* $\|u - u^i\|_\lambda \to 0$ *as* $i \to \infty$.

(b) $\|\cdot\|_\lambda$ *is absolutely monotone, i.e.,* $|u_i| \leq |v_i|$ *for all i imply* $\|u\|_\lambda \leq \|v\|_\lambda$.

(c) λ *is a K-symmetric space. That is, if* u_π *is the sequence which is obtained as a rearrangement of the sequence u corresponding to a permutation* π *of the positive integers, then* $\|u\|_\lambda = \|u_\pi\|_\lambda$ *for each* $u \in \lambda$ *and each* π.

(d) *For any* $u = (u_1, \ldots, u_n, \ldots) \in \lambda$, *let v be its subsequence* $(u_{i_1}, \ldots, u_{i_n}, \ldots)$ *such that* $u_j = 0$ *for any* $j \neq i_n$ $(n = 1, 2, \ldots)$. *Then* $\|v\|_\lambda = \|u\|_\lambda$.

We say the above λ to be a space of type Λ, and if λ satisfies only the conditions (b), (c), (d), then we say λ to be a space of type Λ_0. We remark that $l_{p,q}$ is a space of type Λ_0 for $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and is a space of type Λ for $q \neq \infty$ (Proposition 3 in [3]).

We denote by λ' the topological dual of λ. λ' is a Banach space. If λ is of type Λ, then λ' is realized as a sequence space.

§2. The dual space of $\lambda(Z)$

We begin with the following

Definition 1. *Let* λ *be of type* Λ_0 *and let Z be a Banach space. Then* $\lambda(Z)$ *is the space of zero sequences* (u_i) *with values in the Banach space Z such that*

$$\|(u_i)\|_{\lambda(Z)} = \|(\|u_i\|)\|_\lambda$$

is finite. Then $\|(u_i)\|_{\lambda(Z)}$ *is a quasi-norm in* $\lambda(Z)$.

Definition 2. *Let* λ *be of type* Λ *and let Z be a Banach space. Then* $\lambda'(Z')$ *is the space of sequences* (u'_i) *with values in the Banach space* Z' *such that*

$$\|(u'_i)\|_{\lambda'(Z')} = \|(\|u'_i\|)\|_{\lambda'}$$

is finite. Then $\|(u'_i)\|_{\lambda'(Z')}$ *is a norm in* $\lambda'(Z')$.

Theorem 1. *Let* λ *be of type* Λ_0 *and complete and let Z be a Banach space. Then* $\lambda(Z)$ *is complete.*

PROOF. Let $(u_i^{(\nu)}) \in \lambda(Z)$ and

$$\lim_{\mu, \nu \to \infty} \|(u_i^{(\mu)} - u_i^{(\nu)})\|_{\lambda(Z)} = 0.$$

Then for each i $u_i^{(\nu)}$ is a Cauchy sequence in Z. Hence there exist $u_i \in Z$ $(i \in N)$ such that

$$u_i = \lim_{\nu \to \infty} u_i^{(\nu)} \qquad \text{for each} \quad i.$$

Since $\{(\|u_i^{(\nu)}\|)\}$ is a Cauchy sequence in λ and since λ is complete, $\{(\|u_i^{(\nu)}\|)\}$ converges to $(a_i) \in \lambda$, and $(a_i) = (\|u_i\|)$. Hence $(u_i) \in \lambda(Z)$. If we put

$$v_i^{(\nu)} = \|u_i^{(\nu)} - u_i\|,$$

$\{(v_i^{(\nu)})\}$ is a Cauchy sequence in λ. Since λ is complete, $\{(v_i^{(\nu)})\}$ converges to $(0, 0, \ldots)$. Therefore $\{(u_i^{(\nu)})\}$ converges to (u_i) in $\lambda(Z)$. The proof is complete.

DEFINITION 3. *Let λ be of type Λ_0 and complete and let Z be a Banach space. Then $\lambda'_0(Z')$ is the space of sequences (u'_i) with values in the Banach space Z' such that for every $(u_i) \in \lambda(Z)$ the series $\sum_{i=1}^{\infty} u'_i(u_i)$ converges. The norm $\|\cdot\|^0_{\lambda'}$ in $\lambda'_0(Z')$ is given by*

$$\|(u'_i)\|^0_{\lambda'} = \sup_{\|(u_i)\|_{\lambda(Z)} \leq 1} |\textstyle\sum_{i=1}^{\infty} u'_i(u_i)|.$$

We show that $\|(u'_i)\|^0_{\lambda'} < \infty$ for all $(u'_i) \in \lambda'_0(Z')$ and $\|\cdot\|^0_{\lambda'}$ is a norm. In fact, if $(u'_i) \in \lambda'_0(Z')$, then (u'_i) can be considered as the linear form f on $\lambda(Z)$ defined by $f((u_i)) = \sum_{i=1}^{\infty} u'_i(u_i)$. Define a sequence $\{f_n\}$ of linear forms on $\lambda(Z)$ by $f_n((u_i)) = \sum_{i=1}^{n} u'_i(u_i)$. It is easy to see that each f_n is continuous. Furthermore $\{f_n\}$ converges to f at each point of $\lambda(Z)$. Since $\lambda(Z)$ is a complete quasi-normed space by Theorem 1, from the Banach-Steinhaus Theorem it follows that f is continuous and $\|(u'_i)\|^0_{\lambda'} = \|f\|$. Hence $\|\cdot\|^0_{\lambda'}$ is a norm.

PROPOSITION 1. *Let λ be of type Λ and complete and let Z be a Banach space. Then the dual space of $\lambda(Z)$ is norm isomorphic with $\lambda'_0(Z')$, where a sequence (u'_i) in $\lambda'_0(Z')$ is identified with the linear form f defined by*

$$(1) \qquad f((u_i)) = \textstyle\sum_{i=1}^{\infty} u'_i(u_i) \qquad \text{for each} \quad (u_i) \in \lambda(Z).$$

PROOF. Let $(u'_i) \in \lambda'_0(Z')$. Then the linear form f defined by (1) is continuous and $\|f\| = \|(u'_i)\|^0_{\lambda'}$, which we have already shown in the paragraph after Definition 3. Conversely, let $f \in \lambda(Z)'$. If for each $i \in N$ we define u'_i by

$$u'_i(u) = f((0, \ldots, 0, \overset{i}{u}, 0, \ldots)) \qquad \text{for each} \quad u \in Z,$$

then $u'_i \in Z'$. If for any $(u_i) \in \lambda(Z)$, we put $u^n = (u_1, \ldots, u_n, 0, \ldots)$, then

$$u^n \longrightarrow (u_i) \qquad (n \longrightarrow \infty) \quad \text{in} \quad \lambda(Z)$$

by the condition (a). Hence we have

$$\begin{aligned} f((u_i)) &= f(\lim_{n\to\infty} u^n) \\ &= \lim_{n\to\infty} f(u^n) \\ &= \lim_{n\to\infty} \textstyle\sum_{i=1}^{n} u_i'(u_i) \\ &= \textstyle\sum_{i=1}^{\infty} u_i'(u_i). \end{aligned}$$

Consequently we have

$$(u_i') \in \lambda_0'(Z')$$

and

$$\|f\| = \|(u_i')\|_{\lambda'}^{0}.$$

The proof is complete.

THEOREM 2. *Let λ be of type Λ and complete and let Z be a Banach space. Then the dual space of $\lambda(Z)$ is norm isomorphic with $\lambda'(Z')$, where a sequence (u_i') in $\lambda'(Z')$ is identified with the linear form f defined by*

$$f((u_i)) = \textstyle\sum_{i=1}^{\infty} u_i'(u_i) \qquad \textit{for each} \quad (u_i) \in \lambda(Z).$$

PROOF. Since $\lambda(Z)'$ is norm isomorphic with $\lambda_0'(Z')$ by Proposition 1, we have only to prove that $\lambda'(Z')$ and $\lambda_0'(Z')$ are norm isomorphic. Let $(u_i') \in \lambda'(Z')$. Then, for any $(u_i) \in \lambda(Z)$

$$\textstyle\sum_{i=1}^{\infty} |u_i'(u_i)| \leq \|(u_i)\|_{\lambda(Z)} \|(u_i')\|_{\lambda'(Z')} < \infty,$$

from which it follows that $(u_i') \in \lambda_0'(Z')$ and $\|(u_i')\|_{\lambda'}^{0} \leq \|(u_i')\|_{\lambda'(Z')}$. Thus we have

$$\lambda'(Z') \subset \lambda_0'(Z') \quad \text{and} \quad \|\cdot\|_{\lambda'}^{0} \leq \|\cdot\|_{\lambda'(Z')}.$$

On the other hand, let $(u_i') \in \lambda_0'(Z')$. Put

$$e_i' = \begin{cases} u_i'/\|u_i'\| & \text{for} \quad u_i' \neq 0 \\ 0 & \text{for} \quad u_i' = 0 \end{cases}$$

and $\alpha_i = \|u_i'\|$. Then, $u_i' = \alpha_i e_i'$ for each $i \in N$. For any $\varepsilon > 0$, if $e_i' \neq 0$, there exists an $e_i \in Z$ such that $\|e_i\| = 1$ and $e_i'(e_i) > 1 - \varepsilon$. If $e_i' = 0$, we put $e_i = 0$. Then, for any $(\xi_i) \in \lambda$ with $\|(\xi_i)\|_\lambda \leq 1$ we have

$$\begin{aligned}\sum_{i=1}^{\infty} |\xi_i\alpha_i| &\leq 1/(1-\varepsilon) \sum_{i=1}^{\infty} |\alpha_i e_i'(\xi_i e_i)| \\ &\leq 1/(1-\varepsilon) \sup_{\|(u_i)\|_{\lambda(Z)}\leq 1} \sum_{i=1}^{\infty} |u_i'(u_i)| \\ &= 1/(1-\varepsilon)\, \|(u_i')\|_{\lambda'}^{0}.\end{aligned}$$

Thus we have

$$\lambda_0'(Z') \subset \lambda'(Z') \quad \text{and} \quad \|\cdot\|_{\lambda'(Z')} \leq \|\cdot\|_{\lambda'}^{0}.$$

This completes the proof.

§3. (Z, λ)-nuclear mappings

We shall define (Z, λ)-nuclear mappings as follows.

DEFINITION 4. *Let λ be of type Λ and let E, F and Z be Banach spaces. $T \in L(E, F)$ is said to be a left (Z, λ)-nuclear or simply (Z, λ)-nuclear (resp. right (Z, λ)-nuclear) mapping, if T can be written in the form*

$$(2) \qquad Tu = \sum_{i=1}^{\infty} B_i A_i u \qquad \text{for each} \quad u \in E$$

with $\{A_i\} \subset L(E, Z)$ and $\{B_i\} \subset L(Z, F)$ such that

$$\|(\|A_i\|)\|_\lambda < \infty$$

and

$$\sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'} < \infty$$

(resp.

$$\sup_{\|u\|\leq 1} \|(\|A_i u\|)\|_{\lambda'} < \infty$$

and

$$\|(\|B_i\|)\|_\lambda < \infty),$$

where B_i' is the adjoint mapping of B_i. We denote by $N_{Z,\lambda}(E, F)$ (resp. $N^{Z,\lambda}(E, F)$) the collection of (Z, λ)-nuclear (resp. right (Z, λ)-nuclear) mappings. The quasi-norm (as proved later) is defined by

$$\nu_{Z,\lambda}(T) = \inf(\|(\|A_i\|)\|_\lambda \cdot \sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'})$$

(resp.

$$\nu^{Z,\lambda}(T) = \inf(\sup_{\|u\|\leq 1} \|(\|A_i u\|)\|_{\lambda'} \cdot \|(\|B_i\|)\|_\lambda)),$$

where infimum is taken over all representations (2) of T.

REMARK. *In case $1 \leq p \leq \infty$, $1 \leq q < \infty$, if $\lambda = l_{p,q}$ and Z is one-dimensional, a (Z, λ)-nuclear (resp. right (Z, λ)-nuclear) mapping coincides with a (p, q)-nuclear (resp. right (p, q)-nuclear) mapping introduced in Miyazaki [4], and* *if* $\lambda=l_p$, *a* (Z, λ)*-nuclear mapping coincides with a* (Z, p)*-nuclear mapping introduced in Ceĭtlin* [1].

For $T\in N_{Z,\lambda}(E, F)$ and for each $u\in E$, the series (2) is convergent. In fact, for any finite set J of positive integers and for each $u\in E$ we have

$$\|\sum_{i\in J} B_iA_iu\| \leq \sup_{\|v'\|\leq 1} \sum_{i\in J} \|A_iu\|\cdot\|B_i'v'\|$$
$$\leq \|u\|\cdot\|(\alpha_i)\|_\lambda\cdot \sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'},$$

where

$$\alpha_i = \begin{cases} \|A_i\| & \text{for} \quad i\in J \\ 0 & \text{for} \quad i \notin J. \end{cases}$$

Let ε be any positive number. Since $\|(\|A_i\|)\|_\lambda<\infty$, by the condition (a) of λ there exists an integer $p>0$ such that

$$\|(0,\dots, 0, \|A_p\|, \|A_{p+1}\|,\dots)\|_\lambda < \varepsilon.$$

For $J=\{p, p+1,\dots, q\}$, by the condition (b) of λ we have

$$\|(0,\dots, 0, \alpha_p,\dots, \alpha_q, 0,\dots)\|_\lambda < \varepsilon.$$

It follows that

$$\|\sum_{i\in J} B_iA_iu\| < C\cdot\varepsilon \quad \text{with a constant} \quad C.$$

Hence the series (2) is convergent. A similar fact is valid for $T\in N^{Z,\lambda}(E, F)$.

PROPOSITION 2. *Let* λ *be of type* Λ, *let* E, F *and* Z *be Banach spaces and let* $T\in N_{Z,\lambda}(E, F)$ (*resp.* $T\in N^{Z,\lambda}(E, F)$). *Then*

$$\|T\| \leq \nu_{Z,\lambda}(T) \qquad (\textit{resp.} \quad \|T\| \leq \nu^{Z,\lambda}(T)).$$

PROOF. If $T\in N_{Z,\lambda}(E, F)$, we have

$$\|Tu\| \leq \|u\|\cdot\|(\|A_i\|)\|_\lambda\cdot \sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'}.$$

Therefore we have

$$\|T\| \leq \inf(\|(\|A_i\|)\|_\lambda\cdot \sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'}),$$

where infimum is taken over all representations (2) of T. The proof is complete.

PROPOSITION 3. *Let* λ *be of type* Λ *and let* E, F *and* Z *be Banach spaces.*

Then if $T \in N_{Z,\lambda}(E, F)$, its adjoint T' belongs to $N^{Z',\lambda}(F', E')$ and it satisfies

$$\nu^{Z',\lambda}(T') \leq \nu_{Z,\lambda}(T).$$

Furthermore, assume E, F and Z are reflexive. Then if $T' \in N^{Z',\lambda}(F', E')$, we have

$$T \in N_{Z,\lambda}(E, F)$$

and

$$\nu^{Z',\lambda}(T') = \nu_{Z,\lambda}(T).$$

PROOF. If $T \in N_{Z,\lambda}(E, F)$, then for any $\varepsilon > 0$ it can be written as

$$Tu = \sum_{i=1}^{\infty} B_i A_i u \qquad \text{for each} \quad u \in E$$

with

$$\|(\|A_i\|)\|_{\lambda} \cdot \sup_{\|v'\| \leq 1} \|(\|B'_i v'\|)\|_{\lambda'} \leq \nu_{Z,\lambda}(T) + \varepsilon.$$

Hence we have

$$T'v' = \sum_{i=1}^{\infty} A'_i B'_i v' \qquad \text{for each} \quad v' \in F'$$

and we have

$$\nu^{Z',\lambda}(T') \leq \|(\|A'_i\|)\|_{\lambda} \cdot \sup_{\|v'\| \leq 1} \|(\|B'_i v'\|)\|_{\lambda'} \leq \nu_{Z,\lambda}(T) + \varepsilon.$$

Therefore we have

$$\nu^{Z',\lambda}(T') \leq \nu_{Z,\lambda}(T).$$

When E, F and Z are reflexive, in the same way $T' \in N^{Z',\lambda}(F', E')$ implies

$$T \in N_{Z,\lambda}(E, F)$$

and

$$\nu_{Z,\lambda}(T) \leq \nu^{Z',\lambda}(T').$$

Thus

$$\nu_{Z,\lambda}(T) = \nu^{Z',\lambda}(T').$$

This completes the proof.

THEOREM 3. *Let λ be of type Λ, let E, F and Z be Banach spaces and let $T_k \in N_{Z,\lambda}(E, F)$ for $k = 1, 2, \ldots, M$, M being a positive integer. Then $\sum_{k=1}^{M} T_k$*

$\in N_{Z,\lambda}(E, F)$ *and*

$$\nu_{Z,\lambda}(\sum_{k=1}^{M} T_k) \leq C^{M-1} \cdot M \cdot (\sum_{k=1}^{M} \nu_{Z,\lambda}(T_k)),$$

where C is a constant.

A similar statement holds for elements of $N^{Z,\lambda}(E, F)$.

PROOF. For any $\varepsilon > 0$ T_k can be written in the form

$$T_k u = \sum_{i=1}^{\infty} B_{k,i} A_{k,i} u, \qquad k = 1, 2, \ldots, M$$

with

$$\{A_{k,i}\} \subset L(E, Z) \quad \text{and} \quad \{B_{k,i}\} \subset L(Z, F)$$

such that

$$\|(\|A_{k,i}\|)_i\|_\lambda \leq 1$$

and

$$\sup_{\|v'\| \leq 1} \|(\|B'_{k,i} v'\|)\|_{\lambda'} \leq \nu_{Z,\lambda}(T_k) + \varepsilon/2^k, \qquad k = 1, 2, \ldots, M.$$

Hence we have

$$\begin{aligned} \|(\|A_{k,i}\|)_{i,k}\|_\lambda &\leq C^{M-1} \cdot \sum_{k=1}^{M} \|(\|A_{k,i}\|)_i\|_\lambda \\ &\leq C^{M-1} \cdot M, \end{aligned}$$

where C is a constant. On the other hand, we have

$$\begin{aligned} \sup_{\|v'\| \leq 1} \|(\|B'_{k,i} v'\|)_{i,k}\|_{\lambda'} &\leq \sum_{k=1}^{M} \sup_{\|v'\| \leq 1} \|(\|B'_{k,i} v'\|)_i\|_{\lambda'} \\ &\leq \sum_{k=1}^{M} \nu_{Z,\lambda}(T_k) + \varepsilon. \end{aligned}$$

Thus we have

$$\nu_{Z,\lambda}(\sum_{k=1}^{M} T_k) \leq C^{M-1} \cdot M \cdot (\sum_{k=1}^{M} \nu_{Z,\lambda}(T_k) + \varepsilon).$$

Since ε is arbitrary, this completes the proof.

PROPOSITION 4. *Let λ be of type Λ and let E, F, G and Z be Banach spaces. If $T \in N_{Z,\lambda}(E, F)$ and $S \in L(F, G)$, then $ST \in N_{Z,\lambda}(E, G)$ and*

$$\nu_{Z,\lambda}(ST) \leq \|S\| \cdot \nu_{Z,\lambda}(T).$$

If $T \in L(E, F)$ and $S \in N_{Z,\lambda}(F, G)$, then $ST \in N_{Z,\lambda}(E, G)$

and

$$\nu_{Z,\lambda}(ST) \leq \nu_{Z,\lambda}(S) \cdot \|T\| .$$

The analogues for the mappings of $N^{Z,\lambda}$ are valid.

PROOF. First let $T \in N_{Z,\lambda}(E, F)$ and $S \in L(F, G)$. If $S=0$, the assertion is trivial. So we assume $S \neq 0$. Then we have

$$STu = \sum_{i=1}^{\infty} SB_i A_i u \qquad \text{for each} \quad u \in E,$$

with

$$\{A_i\} \subset L(E, Z) \quad \text{and} \quad \{B_i\} \subset L(Z, F)$$

such that

$$\|(\|A_i\|)\|_\lambda < \infty$$

and

$$\begin{aligned} &\sup_{\|w'\|_{G'} \leq 1} \|(\|(SB_i)'w'\|)\|_{\lambda'} \\ &\leq \|S\| \cdot \sup_{\|w'\|_{G'} \leq 1} \|(\|(B_i' \|S\|^{-1} S')w'\|)\|_{\lambda'} \\ &\leq \|S\| \cdot \sup_{\|u'\|_{F'} \leq 1} \|(\|B_i' u'\|)\|_{\lambda'} < \infty. \end{aligned}$$

This implies $ST \in N_{Z,\lambda}(E, G)$ and

$$\nu_{Z,\lambda}(ST) \leq \|S\| \cdot \nu_{Z,\lambda}(T) .$$

Secondly, let $T \in L(E, F)$ and $S \in N_{Z,\lambda}(F, G)$. Then we have

$$STu = \sum_{i=1}^{\infty} B_i A_i Tu \qquad \text{for each} \quad u \in E,$$

with

$$\|(\|A_i T\|)\|_\lambda \leq \|T\| \cdot \|(\|A_i\|)\|_\lambda < \infty$$

and

$$\sup_{\|w'\|_{G'} \leq 1} \|(\|B_i' w'\|)\|_{\lambda'} < \infty.$$

Hence

$$ST \in N_{Z,\lambda}(E, G)$$

and

$$\nu_{Z,\lambda}(ST) \leq \nu_{Z,\lambda}(S) \cdot \|T\| .$$

This completes the proof.

PROPOSITION 5. *Let λ be of type Λ and let E, F and Z be Banach spaces. Then $L_0^Z(E, F)$ is dense in $N_{Z,\lambda}(E, F)$ and $N^{Z,\lambda}(E, F)$.*

PROOF. Let $T \in N_{Z,\lambda}(E, F)$. Then $Tu = \sum_{i=1}^{\infty} B_i A_i u$ for each $u \in E$, with

$$\|(\|A_i\|)\|_\lambda < \infty$$

and

$$\sup_{\|v'\| \leq 1} \|(\|B_i' v'\|)\|_{\lambda'} < \infty.$$

If we set

$$T_k u = \sum_{i=1}^{k} B_i A_i u,$$

we obtain

$$T_k \in L_0^Z(E, F)$$

and

$$(T - T_k)u = \sum_{i=1}^{\infty} B_{k+i} A_{k+i} u \qquad \text{for each} \quad u \in E.$$

Consequently we have

$$\nu_{Z,\lambda}(T - T_k) \leq \|(\|A_{k+i}\|)\|_\lambda \cdot \sup_{\|v'\| \leq 1} \|(\|B_{k+i}' v'\|)\|_{\lambda'}.$$

Owing to (a), this converges to 0 as $k \to \infty$. Hence $L_0^Z(E, F)$ is dense in $N_{Z,\lambda}(E, F)$.
In the same way we can show that $L_0^Z(E, F)$ is dense in $N^{Z,\lambda}(E, F)$.

COROLLARY. *Let λ be of type Λ, let E and F be Banach spaces and let Z be a finite dimensional Banach space; then $N_{Z,\lambda}(E, F) \subset K(E, F)$ and $N^{Z,\lambda}(E, F) \subset K(E, F)$.*

LEMMA 1. *Let λ be of type Λ and a Banach space, let Z be a Banach space, let $(\delta_i) \in \lambda$ and let D_1 be the mapping from $l_\infty(Z)$ into $\lambda(Z)$ defined by*

$$D_1((a_i)) = (\delta_i a_i) \qquad \textit{for each} \quad (a_i) \in l_\infty(Z).$$

Then

$$D_1 \in N_{Z,\lambda}(l_\infty(Z), \lambda(Z))$$

and

$$\nu_{Z,\lambda}(D_1) = \|(\delta_i)\|_\lambda.$$

PROOF. Let $I_i(z)=(0,\dots, 0, \overset{i}{z}, 0,\dots)$. Then $I_i(z)\in\lambda(Z)$ for $z\in Z$, since $(0,\dots, 0, \overset{i}{1}, 0,\dots)\in\lambda$. Hence I_i is a mapping of Z into $\lambda(Z)$. Define $A_i\colon l_\infty(Z)\to Z$ by $A_i u=\delta_i a_i$ for each $u=(a_i)\in l_\infty(Z)$. Then

$$\{I_i\}\subset L(Z, \lambda(Z)), \qquad \{A_i\}\subset L(l_\infty(Z), Z),$$

$$D_1 u = \sum_{i=1}^{\infty} I_i A_i u \qquad \text{for each} \quad u\in l_\infty(Z),$$

$$\|(\|A_i\|)\|_\lambda = \|(\delta_i)\|_\lambda,$$

and by Theorem 2

$$\sup_{\|v'\|_{\lambda(Z)'}\le 1}\|(\|I_i'v'\|)\|_{\lambda'} = \sup_{\|v'\|_{\lambda'(Z')}\le 1}\|(\|I_i'v'\|)\|_{\lambda'} = 1.$$

Hence

$$D_1\in N_{Z,\lambda}(l_\infty(Z), \lambda(Z)) \quad\text{and}\quad \nu_{Z,\lambda}(D_1)\le\|(\delta_i)\|_\lambda.$$

On the other hand, let I be $(z, z,\dots)$ with $\|z\|=1$. Then we have

$$\|(\delta_i)\|_\lambda = \|D_1 I\|_{\lambda(Z)}\le\|D_1\|\le\nu_{Z,\lambda}(D_1),$$

where the last inequality follows from Proposition 2. Hence

$$\nu_{Z,\lambda}(D_1) = \|(\delta_i)\|_\lambda.$$

The proof is complete.

THEOREM 4. *Let λ be of type A and a Banach space and let E, F and Z be Banach spaces. Then $T\in L(E, F)$ is (Z, λ)-nuclear if and only if T can be factorized in the form $T=Q_1D_1P_1$:*

$$E\xrightarrow{P_1} l_\infty(Z)\xrightarrow{D_1}\lambda(Z)\xrightarrow{Q_1} F,$$

where $P_1\in L(E, l_\infty(Z))$ with $\|P_1\|\le 1$, $Q_1\in L(\lambda(Z), F)$ with $\|Q_1\|\le 1$ and D_1 is a mapping of the type given in Lemma 1.

PROOF. The sufficiency is evident by Proposition 4 and Lemma 1. The necessity is proved by virtue of the definition of $T\in N_{Z,\lambda}(E, F)$ and the following decomposition of T. Since, for any $\varepsilon>0$,

$$Tu = \sum_{i=1}^{\infty} B_i A_i u$$

with

$$\|(\|A_i\|)\|_\lambda\le\nu_{Z,\lambda}(T)+\varepsilon,$$

and

$$\sup_{\|v'\|\leq 1} \|(\|B_i'v'\|)\|_{\lambda'} \leq 1,$$

we can get the decomposition of T:

$$E \xrightarrow{P_1} l_\infty(Z) \xrightarrow{D_1} \lambda(Z) \xrightarrow{Q_1} F,$$

where for each $u \in E$ $P_1 u = (v_i)$ with

$$v_i = \begin{cases} A_i u / \|A_i\| & \text{for} \quad A_i \neq 0 \\ 0 & \text{for} \quad A_i = 0, \end{cases}$$

and

$$D_1((a_i)) = (\|A_i\| a_i) \in \lambda(Z) \qquad \text{for each} \quad (a_i) \in l_\infty(Z),$$

$$Q_1((b_i)) = \textstyle\sum_{i=1}^\infty B_i b_i \in F \qquad \text{for each} \quad (b_i) \in \lambda(Z).$$

It is easy to verify that $\|P_1\| \leq 1$. Furthermore we have $\|Q_1\| \leq 1$. In fact,

$$\begin{aligned} \|\textstyle\sum_{i=1}^\infty B_i b_i\|_F &= \sup_{\|v'\|\leq 1} |\textstyle\sum_{i=1}^\infty v'(B_i b_i)| \\ &\leq \sup_{\|v'\|\leq 1} \textstyle\sum_{i=1}^\infty |B_i' v'(b_i)| \\ &\leq \|(b_i)\|_{\lambda(Z)} \cdot \sup_{\|v'\|\leq 1} \|(B_i' v')\|_{\lambda'(Z')} \\ &\leq \|(b_i)\|_{\lambda(Z)} . \end{aligned}$$

Thus the proof is complete.

LEMMA 2. *Let λ be of type A and a Banach space, let Z be a Banach space, let $(\delta_i) \in \lambda$ and let D_2: $\lambda'(Z) \to l_1(Z)$ be defined by*

$$D_2((b_i)) = (\delta_i b_i) \qquad \textit{for each} \quad (b_i) \in \lambda'(Z).$$

Then

$$D_2 \in N^{Z,\lambda}(\lambda'(Z), l_1(Z))$$

and

$$\nu^{Z,\lambda}(D_2) = \|(\delta_i)\|_\lambda .$$

PROOF. If we define A_i: $\lambda'(Z) \to Z$ by $A_i u = \delta_i b_i$ for $u = (b_i) \in \lambda'(Z)$, we have

$$D_2 u = \textstyle\sum_{i=1}^\infty I_i A_i u = \textstyle\sum_{i=1}^\infty \delta_i I_i \delta_i' A_i u$$

with

$$\delta_i' = \begin{cases} 1/\delta_i & \text{for} \quad \delta_i \neq 0 \\ 0 & \text{for} \quad \delta_i = 0, \end{cases}$$

where

$$\|(\|\delta_i I_i\|)\|_\lambda = \|(\delta_i)\|_\lambda < \infty$$

and

$$\sup_{\|u\|_{\lambda'(Z)} \le 1} \|(\|\delta_i' A_i u\|)\|_{\lambda'} \le 1.$$

Hence

$$D_2 \in N^{Z,\lambda}(\lambda'(Z), l_1(Z))$$

and

$$\nu^{Z,\lambda}(D_2) \le \|(\delta_i)\|_\lambda.$$

On the other hand, we have

$$\|(\delta_i)\|_\lambda = \sup_{\|(b_i)\|_{\lambda'(Z)}=1} \|(\delta_i b_i)\|_{l_1(Z)}.$$

Hence for any $\varepsilon > 0$, there exists (b_i) with $\|(b_i)\|_{\lambda'(Z)} = 1$ such that

$$\|(\delta_i)\|_\lambda - \varepsilon \le \|D_2((b_i))\|_{l_1(Z)}.$$

Since

$$\|D_2((b_i))\|_{l_1(Z)} \le \|D_2\|,$$

by Proposition 2 we have

$$\|(\delta_i)\|_\lambda - \varepsilon \le \nu^{Z,\lambda}(D_2).$$

Thus we obtain

$$\|(\delta_i)\|_\lambda \le \nu^{Z,\lambda}(D_2).$$

Consequently

$$\nu^{Z,\lambda}(D_2) = \|(\delta_i)\|_\lambda.$$

The proof is complete.

THEOREM 5. *Let λ be of type Λ and a Banach space and let E, F and Z be Banach spaces. Then $T \in L(E, F)$ is right (Z, λ)-nuclear if and only if T can be written in the form $T = Q_2 D_2 P_2$:*

$$E \xrightarrow{P_2} \lambda'(Z) \xrightarrow{D_2} l_1(Z) \xrightarrow{Q_2} F,$$

where $P_2 \in L(E, \lambda'(Z))$ with $\|P_2\| \leq 1$, $Q_2 \in L(l_1(Z), F)$ with $\|Q_2\| \leq 1$ and D_2 is a mapping of the type given in Lemma 2.

PROOF. The sufficiency is evident by Proposition 4 and Lemma 2. The necessity is proved by virtue of the definition of $T \in N^{Z,\lambda}(E, F)$ and the following decomposition of T. Since for any $\varepsilon > 0$ we have

$$Tu = \sum_{i=1}^{\infty} B_i A_i u$$

with

$$\|(\|B_i\|)\|_{\lambda} \leq \nu^{Z,\lambda}(T) + \varepsilon$$

and

$$\sup_{\|u\| \leq 1} \|(\|A_i u\|)\|_{\lambda'} \leq 1,$$

we can get the decomposition of T:

$$E \xrightarrow{P_2} \lambda'(Z) \xrightarrow{D_2} l_1(Z) \xrightarrow{Q_2} F,$$

where

$$P_2 u = (A_i u) \in \lambda'(Z) \qquad \text{for each} \quad u \in E,$$

$$D_2((a_i)) = (\|B_i\| a_i) \in l_1(Z) \qquad \text{for each} \quad (a_i) \in \lambda'(Z)$$

and for each $(b_i) \in l_1(Z)$

$$Q_2((b_i)) = \sum_{i=1}^{\infty} C_i b_i$$

with

$$C_i = \begin{cases} B_i / \|B_i\| & \text{for} \quad B_i \neq 0 \\ 0 & \text{for} \quad B_i = 0. \end{cases}$$

It is easy to verify that $\|P_2\| \leq 1$ and $\|Q_2\| \leq 1$. The proof is complete.

§4. (Z, λ)-quasi-nuclear mappings

In this section we shall introduce and investigate the (Z, λ)-quasi-nuclear mappings.

DEFINITION 5. *Let λ be of type Λ_0 and let E, F and Z be Banach spaces.*

$T \in L(E, F)$ is said to be a (Z, λ)-quasi-nuclear mapping if there exists a sequence $\{A_i\} \subset L(E, Z)$ such that

$$(\|A_i\|) \in \lambda$$

and

$$\|Tu\| \leq \|(\|A_i u\|)\|_\lambda \qquad \text{for each} \quad u \in E.$$

The $\inf \|(\|A_i\|)\|_\lambda$ *which is taken over all $\{A_i\}$ satisfying the above condition is denoted by $\nu^Q_{Z,\lambda}(T)$. The collection of all (Z, λ)-quasi-nuclear mappings is denoted by $N^Q_{Z,\lambda}(E, F)$.*

PROPOSITION 6. *Let λ be of type Λ_0 and let E, F and Z be Banach spaces. Then for any $T \in N^Q_{Z,\lambda}(E, F)$ we have*

$$\|T\| \leq \nu^Q_{Z,\lambda}(T).$$

PROOF. Let $T \in N^Q_{Z,\lambda}(E, F)$. Then

$$\|Tu\| \leq \|u\| \cdot \|(\|A_i\|)\|_\lambda.$$

Thus the proof is complete.

PROPOSITION 7. *Let λ be of type Λ and let E, F and Z be Banach spaces. Then we have*

$$N_{Z,\lambda}(E, F) \subset N^Q_{Z,\lambda}(E, F)$$

and

$$\nu^Q_{Z,\lambda}(T) \leq \nu_{Z,\lambda}(T)$$

for each $T \in N_{Z,\lambda}(E, F)$.

PROOF. $T \in N_{Z,\lambda}(E, F)$ can be expressed as follows. For any $\varepsilon > 0$ there exist sequences $\{A_i\} \subset L(E, Z)$ and $\{B_i\} \subset L(Z, F)$ such that

$$Tu = \textstyle\sum_{i=1}^{\infty} B_i A_i u \qquad \text{for each} \quad u \in E,$$

$$\|(\|A_i\|)\|_\lambda \leq \nu_{Z,\lambda}(T) + \varepsilon$$

and

$$\sup_{\|v'\| \leq 1} \|(\|B_i' v'\|)\|_{\lambda'} \leq 1.$$

Therefore we have

$$\|Tu\| \leq \sup_{\|v'\|\leq 1} \sum_{i=1}^{\infty} \|A_i u\| \cdot \|B_i' v'\|$$

$$\leq \|(\|A_i u\|)\|_\lambda \cdot \sup_{\|v'\|\leq 1} \|(\|B_i' v'\|)\|_{\lambda'}$$

$$\leq \|(\|A_i u\|)\|_\lambda ,$$

which shows $T\in N^Q_{Z,\lambda}(E, F)$ and $\nu^Q_{Z,\lambda}(T)\leq \nu_{Z,\lambda}(T)$. The proof is complete.

Proposition 8. *Let λ be of type Λ_0 and let E, F, G and Z be Banach spaces. If $T\in N^Q_{Z,\lambda}(E, F)$ and $S\in L(F, G)$, then $ST\in N^Q_{Z,\lambda}(E, G)$ and*

$$\nu^Q_{Z,\lambda}(ST) \leq \|S\| \cdot \nu^Q_{Z,\lambda}(T).$$

If $T\in L(E, F)$ and $S\in N^Q_{Z,\lambda}(F, G)$, then $ST\in N^Q_{Z,\lambda}(E, G)$ and

$$\nu^Q_{Z,\lambda}(ST) \leq \nu^Q_{Z,\lambda}(S) \cdot \|T\|.$$

Proof. If $T\in N^Q_{Z,\lambda}(E, F)$ and $S\in L(F, G)$, there exists a sequence $\{A_i\} \subset L(E, Z)$ such that

$$(\|A_i\|)\in\lambda \quad \text{and} \quad \|Tu\| \leq \|(\|A_i u\|)\|_\lambda \quad \text{for each} \quad u\in E.$$

Then

$$\{\|S\| A_i\} \subset L(E, Z), \qquad (\|S\| \cdot \|A_i\|)\in\lambda$$

and

$$\|STu\| \leq \|S\| \cdot \|Tu\| \leq \|S\| \cdot \|(\|A_i u\|)\|_\lambda$$

$$= \|(\|S\| \cdot \|A_i u\|)\|_\lambda \qquad \text{for each} \quad u\in E.$$

Therefore we have

$$ST\in N^Q_{Z,\lambda}(E, G)$$

and

$$\nu^Q_{Z,\lambda}(ST) \leq \|S\| \cdot \nu^Q_{Z,\lambda}(T).$$

In the same way, if $T\in L(E, F)$ and $S\in N^Q_{Z,\lambda}(F, G)$, there exists a sequence $\{B_i\} \subset L(F, Z)$ such that

$$(\|B_i\|)\in\lambda \quad \text{and} \quad \|Sv\| \leq \|(\|B_i v\|)\|_\lambda \qquad \text{for each} \quad v\in F.$$

Therefore we have

$$\|STu\| \leq \|(\|B_i Tu\|)\|_\lambda \qquad \text{for each} \quad u\in E$$

and

$$\|(\|B_iT\|)\|_\lambda \leq \|T\| \cdot \|(\|B_i\|)\|_\lambda < \infty,$$

that is,

$$(\|B_iT\|) \in \lambda.$$

Consequently we have

$$ST \in N^Q_{Z,\lambda}(E, G)$$

and

$$\nu^Q_{Z,\lambda}(ST) \leq \nu^Q_{Z,\lambda}(S) \cdot \|T\|.$$

The proof is complete.

THEOREM 6. *Let λ be of type Λ_0, let E, F and Z be Banach spaces and $T_k \in N^Q_{Z,\lambda}(E, F)$, $k=1, 2,\ldots, M$. Then $\sum_{k=1}^M T_k \in N^Q_{Z,\lambda}(E, F)$ and*

$$\nu^Q_{Z,\lambda}(\textstyle\sum_{k=1}^M T_k) \leq M \cdot C^{M-1} \cdot (\sum_{k=1}^M \nu^Q_{Z,\lambda}(T_k)).$$

PROOF. For any $\varepsilon > 0$ there exist sequences

$$\{A_{k,i}\}_{1 \leq i < \infty} \subset L(E, Z), \qquad k = 1, 2,\ldots, M,$$

such that

$$\|(\|A_{k,i}\|)_i\|_\lambda < \nu^Q_{Z,\lambda}(T_k) + \varepsilon/2^k$$

and

$$\|T_k u\| \leq \|(\|A_{k,i}u\|)_i\|_\lambda \qquad \text{for each} \quad u \in E, \qquad k = 1, 2,\ldots, M.$$

Therefore we have

$$\begin{aligned}\|(\textstyle\sum_{k=1}^M T_k)u\| &\leq \textstyle\sum_{k=1}^M \|T_k u\| \\ &\leq \textstyle\sum_{k=1}^M \|(\|A_{k,i}u\|)_i\|_\lambda \\ &\leq M \cdot \|(\|A_{k,i}u\|)_{k,i}\|_\lambda.\end{aligned}$$

Hence we have

$$\begin{aligned}\nu^Q_{Z,\lambda}(\textstyle\sum_{k=1}^M T_k) &\leq M \cdot \|(\|A_{k,i}\|)_{k,i}\|_\lambda \\ &\leq M \cdot C^{M-1} \cdot \textstyle\sum_{k=1}^M \|(\|A_{k,i}\|)_i\|_\lambda \\ &\leq M \cdot C^{M-1} \cdot (\textstyle\sum_{k=1}^M \nu^Q_{Z,\lambda}(T_k) + \varepsilon).\end{aligned}$$

Consequently we have

$$\sum_{k=1}^{M} T_k \in N_{Z,\lambda}^{Q}(E, F)$$

and

$$\nu_{Z,\lambda}^{Q}(\sum_{k=1}^{M} T_k) \leq M \cdot C^{M-1} \cdot (\sum_{k=1}^{M} \nu_{Z,\lambda}^{Q}(T_k)).$$

The proof is complete.

DEFINITION 6. *A Banach space F is said to have the extension property if each mapping $T_0 \in L(E_0, F)$, E_0 being any linear subspace of an arbitrary quasi-normed space E, can be extended to a $T \in L(E, F)$ preserving its norm.*

PROPOSITION 9. *Let λ be of type Λ, let E, F and Z be Banach spaces, and let us assume that F has the extension property. Then any (Z, λ)-quasi-nuclear mapping T: $E \to F$ is (Z, λ)-nuclear, and*

$$\nu_{Z,\lambda}(T) = \nu_{Z,\lambda}^{Q}(T).$$

PROOF. From the definition, for any $\varepsilon > 0$ there exists a sequence $\{A_i\} \subset L(E, Z)$ such that

$$\|Tu\| \leq \|(\|A_i u\|)\|_\lambda \qquad \text{for each} \quad u \in E$$

and

$$\|(\|A_i\|)\|_\lambda \leq \nu_{Z,\lambda}^{Q}(T) + \varepsilon.$$

Let us denote by Q_0 the mapping from the subspace $\{(A_i u) | u \in E\}$ of $\lambda(Z)$ into F defined by

$$Q_0((A_i u)) = Tu.$$

Then we have $\|Q_0\| \leq 1$. Thus, by our assumption there exists an extension Q: $\lambda(Z) \to F$ of Q_0 with $\|Q\| \leq 1$. If we define I_i: $Z \to \lambda(Z)$ such that

$$I_i(z) = (0,\dots, 0, \overset{i}{\check{z}}, 0,\dots),$$

then $\sum_{i=1}^{\infty} I_i A_i u$ is convergent in $\lambda(Z)$ by condition (a) for λ and equal to $(A_i u)$. Hence we have

$$Tu = \sum_{i=1}^{\infty} Q I_i A_i u$$

and for each $v' \in F'$

$$\|(\|(QI_i)' v'\|)\|_{\lambda'} = \|(\|I_i' Q' v'\|)\|_{\lambda'}$$

$$= \|Q'v'\|_{\lambda'(Z')}.$$

By Theorem 2

$$\|(\|(QI_i)'v'\|)\|_{\lambda'} = \|Q'v'\|_{\lambda(Z)'} \leq \|v'\|.$$

Therefore we have

$$\sup_{\|v'\|\leq 1} \|(\|(QI_i)'v'\|)\|_{\lambda'} \leq 1$$

and

$$\nu_{Z,\lambda}(T) \leq \nu^{Q}_{Z,\lambda}(T)+\varepsilon.$$

Owing to Proposition 7 we now obtain the conclusion of the proposition.

§5. Z-nuclear spaces

We first recall the definition of Z-nuclear mappings ([1]).

DEFINITION 7. *Let E, F and Z be Banach spaces. Then $T\in L(E, F)$ is said to be a Z-nuclear mapping, if T can be written in the form*

$$Tu = \sum_{i=1}^{\infty} B_i A_i u \qquad \textit{for each} \quad u \in E$$

with

$$\{A_i\} \subset L(E, Z) \quad \textit{and} \quad \{B_i\} \subset L(Z, F)$$

such that

$$\sum_{i=1}^{\infty} \|A_i\| \cdot \|B_i\| < \infty,$$

that is, T is a (Z, l_1)-nuclear mapping.

Let E be a locally convex space, let U be an arbitrary absolutely convex neighborhood of zero and p_U be the gauge of U. Then the quotient space $E/p_U^{-1}(0)$ is normable by the norm $\hat{u}\to\|\hat{u}\|=p_U(u)$, where $u\in\hat{u}$. We shall denote by E_U the normed space $(E/p_U^{-1}(0), \|\cdot\|)$ and by $\widetilde{E_U}$ its completion. Then the quotient mapping is continuous on E into $\widetilde{E_U}$. This mapping will be denoted by ϕ_U.

If E is a locally convex space and $B\neq\phi$ an absolutely convex bounded subset of E, then $E_1=\bigcup_{n=1}^{\infty} nB$ is a subspace of E. The gauge function p_B of B in E_1 is seen to be a norm on E_1. Then the normed space (E_1, p_B) will be denoted by E_B. It is immediate that the imbedding mapping ψ_B: $E_B\to E$ is continuous.

If U and V are absolutely convex neighborhoods of zero in E with respective gauge functions p_U and p_V and such that $U \subset V$, then $p_U^{-1}(0) \subset p_V^{-1}(0)$ and each equivalence class $\hat{u} \bmod p_U^{-1}(0)$ is contained in a unique equivalence class $\hat{v} \bmod p_V^{-1}(0)$ and $\hat{u} \to \hat{v}$ is a linear mapping $\phi_{V,U}$, which is called the canonical mapping of E_U onto E_V. It has a unique continuous extension of $\widetilde{E_U}$ into $\widetilde{E_V}$, which is again called canonical and also denoted by $\phi_{V,U}$.

We now generalize the notion of a Z-nuclear mapping to arbitrary locally convex spaces E, F as follows.

Definition 8. *A linear mapping T of a locally convex space E into another locally convex space F is said to be Z-nuclear if there exist an absolutely convex neighborhood of zero U in E and an absolutely convex bounded set B in F with F_B complete such that $T(U) \subset B$ with respect to which T can be then decomposed in the form $T = \psi_B \circ T_0 \circ \phi_U$ with $T_0 \in L(E_U, F_B)$ such that the mapping $\overline{T}_0$ of $\widetilde{E_U}$ into F_B induced by T_0 is Z-nuclear.*

Proposition 10. *Let E and F be Banach spaces. Then $T: E \to F$ is Z-nuclear with E and F considered as Banach spaces if and only if T is Z-nuclear with E and F considered as locally convex spaces.*

Proof. Let $T: E \to F$ be Z-nuclear with E and F considered as Banach spaces. If we put $U = \{x \mid \|x\| \leq 1\}$ in E and $B = \widetilde{T(U)}$, then $\widetilde{E_U} = E$ and F_B is complete. T can be decomposed in the form

$$T = \psi_B \circ T \circ \phi_U$$

with ϕ_U the identity mapping. Therefore T is Z-nuclear with E and F considered as locally convex spaces. Conversely, if T is Z-nuclear with E and F considered as locally convex spaces, T can be decomposed in the form

$$T = \psi_B \circ \overline{T}_0 \circ \phi_U,$$

where U and B are sets of E and F stated in the definition above and $\overline{T}_0$ is Z-nuclear of $\widetilde{E_U}$ into F_B. Therefore by Proposition 4 T is Z-nuclear when E and F are considered as Banach spaces. The proof is complete.

Theorem 7. *Let E and F be locally convex spaces and let Z be a Banach space. Then a linear mapping $T \in L(E, F)$ is Z-nuclear if and only if it can be written in the form*

$$Tu = \textstyle\sum_{i=1}^{\infty} \lambda_i B_i A_i u \qquad \text{for each} \quad u \in E,$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, $\{A_i\}$ is an equicontinuous sequence in $L(E, Z)$ and $\{B_i\}$ is a sequence in $L(Z, F)$ for which there exist a neighborhood of zero W in Z and an absolutely convex bounded set B with F_B complete such that $B_i(W) \subset B$

for any $i=1, 2, \ldots$.

PROOF. The condition is necessary. For, if T is Z-nuclear, then

$$T = \psi_B \circ \overline{T}_0 \circ \phi_U,$$

where $\overline{T}_0$ is Z-nuclear in $L(\widetilde{E_U}, F_B)$, U being a suitable neighborhood of zero and B being a suitable bounded subset of F for which F_B is complete. Therefore $\overline{T}_0$ can be written in the form

$$\overline{T}_0 \hat{u} = \sum_{i=1}^{\infty} B_i A_i \hat{u} \qquad \text{for each} \quad \hat{u} \in \widetilde{E_U}$$

with

$$\{A_i\} \subset L(\widetilde{E_U}, Z) \quad \text{and} \quad \{B_i\} \subset L(Z, F_B)$$

such that

$$\sum_{i=1}^{\infty} \|A_i\| \cdot \|B_i\| < \infty.$$

That is, $\overline{T}_0 = \tau(S)$ with $S = \sum_{i=1}^{\infty} B_i \otimes A_i \in L(\widetilde{E_U}, Z) \widetilde{\otimes} L(Z, F_B)$, where τ is the canonical mapping of $L(\widetilde{E_U}, Z) \widetilde{\otimes} L(Z, F_B)$ into $L(\widetilde{E_U}, F_B)$. By virtue of Theorem 6.4 of [6], S can be written in the form

$$S = \sum_{i=1}^{\infty} \lambda_i \widetilde{B_i} \otimes \widetilde{A_i}$$

with

$$\sum_{i=1}^{\infty} |\lambda_i| < \infty,$$

where $\{\widetilde{A_i}\}$ and $\{\widetilde{B_i}\}$ are null sequences in $L(\widetilde{E_U}, Z)$ and $L(Z, F_B)$ respectively. Then $\{\widetilde{A_i} \circ \phi_U\}$ is equicontinuous in $L(E, Z)$. Since $\{\widetilde{B_i}\}$ is a null sequence in $L(Z, F_B)$, there exists a neighborhood of zero W in Z such that $\psi_B \circ \widetilde{B_i}(W) \subset B$ for any $i=1, 2, \ldots$. It is clear that the mapping $T = \sum_{i=1}^{\infty} \lambda_i \psi_B \circ \widetilde{B_i} \circ \widetilde{A_i} \circ \phi_U$ is of the form stated in the theorem.

The condition is sufficient. For, if T is as indicated in the theorem, let $U = \{u \in E \mid \|A_i(u)\| \leq 1, i \in N\}$. Then U is an absolutely convex closed neighborhood of zero in E by the equicontinuity of $\{A_i\}$. Defining $\widetilde{A_i}$ $(i \in N)$ by $A_i = \widetilde{A_i} \circ \phi_U$ on E_U and the extension to $\widetilde{E_U}$, we obtain $\|\widetilde{A_i}\| \leq 1$ for all i. Then $\overline{T}_0$ is the mapping

$$\hat{u} \longrightarrow \sum_{i=1}^{\infty} \lambda_i (\psi_B^{-1} \circ B_i) \widetilde{A_i} \hat{u}$$

with

$$\sum_{i=1}^{\infty} |\lambda_i| \, \|(\psi_B^{-1} \circ B_i)\| \cdot \|\widetilde{A_i}\| < \infty.$$

Therefore $\overline{T}_0$ is a Z-nuclear mapping. The proof is complete.

COROLLARY. *Let E, F, G and H be locally convex spaces, let Z be a Banach space, let $S \in L(E, F)$, let $W \in L(G, H)$ and let T be a Z-nuclear mapping on F into G. Then $T \circ S$ and $W \circ T$ are Z-nuclear mappings.*

PROOF. It is evident from Theorem 7 that $T \circ S$ is Z-nuclear. By our assumption, we have a decomposition of T as a sequence

$$F \xrightarrow{\phi_U} F_U \xrightarrow{\overline{T}_0} G_B \xrightarrow{\psi_B} G,$$

where ϕ_U and ψ_B are the canonical mappings, $\overline{T}_0$ is Z-nuclear and B is an absolutely convex bounded set in G for which G_B is complete. Since G_B is a Banach space and since $C = W(B)$ is an absolutely convex bounded in H, H_C is a Banach space. In fact, the restriction $W|G_B$ of W to G_B induces an isometry J of the Banach space $G_B/(G_B \cap W^{-1}(0))$ onto H_C. Therefore H_C is a Banach space. Finally, we have a decomposition of $W \circ T$ into the sequence

$$F \xrightarrow{\phi_U} \tilde{F}_U \xrightarrow{\overline{T}_0} G_B \xrightarrow{W|G_B} H_C \xrightarrow{\psi_C} H.$$

Since $(W|G_B) \circ \overline{T}_0$ is Z-nuclear, so is $W \circ T$.

PROPOSITION 11. *Let E and F be locally convex spaces and let Z be a Banach space. Then if $T \in L(E, F)$ is Z-nuclear, T has a unique extension $\overline{T} \in L(\tilde{E}, F)$, where $\tilde{E}$ is the completion of E, and $\overline{T}$ is Z-nuclear.*

PROOF. By our assumption, we have a decomposition of T as follows.

$$E \xrightarrow{\phi_U} \widetilde{E_U} \xrightarrow{\overline{T}_0} F_B \xrightarrow{\psi_B} F,$$

where ϕ_U and ψ_B are the canonical mappings and $\overline{T}_0$ is a Z-nuclear mapping. Then ϕ_U has a unique continuous extension $\overline{\phi}_U$ on $\tilde{E}$ into $\widetilde{E_U}$. Therefore $\overline{T} = \psi_B \circ \overline{T}_0 \circ \overline{\phi}_U$ is a continuous extension of T on $\tilde{E}$ into F and this extension is unique. It is clear that $\overline{T}$ is Z-nuclear from the above corollary. The proof is complete.

DEFINITION 9. *Let Z be a Banach space. Then a locally convex space E is said to be Z-nuclear if for each absolutely convex neighborhood of zero U in E there exists another absolutely convex neighborhood of zero V with $U \supset V$ such that $\phi_{U,V}\colon \widetilde{E_V} \to \widetilde{E_U}$ is Z-nuclear.*

THEOREN 8. *Let E be a locally convex space and let Z be a Banach space. Then the following assertions are equivalent:*

(a) *E is Z-nuclear.*

(b) *There exists a base $\mathfrak{B}$ of absolutely convex neighborhoods of zero in E such that for each $V \in \mathfrak{B}$, the canonical mapping $\phi_V\colon E \to \widetilde{E_V}$ is Z-nuclear.*

(c) *Every continuous mapping of E into any Banach space is Z-nuclear.*

PROOF. (a)$\Rightarrow$(b): If U is a given absolutely convex neighborhood of zero in E, there exists another V with $U \supset V$ such that $\phi_{U,V}$ is Z-nuclear. Since $\phi_U = \phi_{U,V} \circ \phi_V$, it follows from the corollary of Theorem 7 that $\phi_U: E \to \widetilde{E_U}$ is Z-nuclear.

(b)$\Rightarrow$(c): Let F be any Banach space and $T \in L(E, F)$. Then there exists an absolutely convex neighborhood of zero V in E such that $\phi_V: E \to \widetilde{E_V}$ is Z-nuclear, and such that $T(V)$ is bounded in F. Since $\phi_V(E) = E_V$, T determines a unique $S \in L(\widetilde{E_V}, F)$ such that $T = S \circ \phi_V$. By the corollary of Theorem 7, T is Z-nuclear.

(c)$\Rightarrow$(a): Let U be any absolutely convex neighborhood of zero in E. By our assumption, the canonical mapping $\phi_U: E \to \widetilde{E_U}$ is Z-nuclear, and hence of the form

$$\phi_U = \textstyle\sum_{i=1}^{\infty} \lambda_i B_i A_i$$

with

$$\textstyle\sum_{i=1}^{\infty} |\lambda_i| < \infty, \quad \{A_i\} \subset L(E, Z) \quad \text{and} \quad \{B_i\} \subset L(Z, \widetilde{E_U})$$

as described in Theorem 7. Set $V = U \cap \{u \mid \|A_i u\| \leq 1,\ i \in N\}$; then $V \subset U$ is an absolutely convex neighborhood of zero in E. Now each A_i induces a continuous linear mapping on E_V into Z. Denote by C_i its continuous extension to $\widetilde{E_V}$. It is clear that the canonical mapping $\phi_{U,V}: \widetilde{E_V} \to \widetilde{E_U}$ is given by $\sum_{i=1}^{\infty} \lambda_i B_i C_i$ and hence Z-nuclear by Theorem 7. The proof is complete.

PROPOSITION 12. *Let Z be a Banach space, let E be a Z-nuclear locally convex space, let U be a given neighborhood of zero in E and let p be a number such that $1 \leq p \leq \infty$. Then there exists an absolutely convex neighborhood of zero $V \subset U$ for which $\widetilde{E_V}$ is norm isomorphic with a subspace of $l_p(Z)$.*

PROOF. We show that there exists a continuous linear mapping $T \in L(E, l_p(Z))$ such that $T^{-1}(B) \subset U$, where B is the open unit ball of $l_p(Z)$. $V = T^{-1}(B)$ will be the neighborhood in question. Assume without loss of generality that U is absolutely convex. The canonical mapping $\phi_U: E \to \widetilde{E_U}$ is Z-nuclear by the above theorem, and hence of the form

$$\phi_U = \textstyle\sum_{i=1}^{\infty} \lambda_i B_i A_i$$

with

$$\{A_i\} \subset L(E, Z) \quad \text{and} \quad \{B_i\} \subset L(Z, \widetilde{E_U}),$$

where we can assume that $\lambda_i>0$, $\sum_{i=1}^{\infty}\lambda_i=1$, $\|B_i\|=1$ and the sequence $\{A_i\}$ is equicontinuous. Define T by

$$T(u) = (\sqrt[p]{\lambda_1}A_1(u), \sqrt[p]{\lambda_2}A_2(u),\dots)$$

for each $u\in E$ (set $\sqrt[p]{\lambda_i}=1$ for all i if $p=\infty$). By the equicontinuity of the sequence $\{A_i\}$, we have $T(u)\in l_p(Z)$ and $T\in L(E, l_p(Z))$. Then we obtain

$$\begin{aligned}\|\phi_U(u)\| &= \|\textstyle\sum_{i=1}^{\infty}\lambda_i B_i A_i(u)\| \\ &\leq \textstyle\sum_{i=1}^{\infty}\lambda_i\|A_i(u)\| \leq \|T(u)\|_{l_p(Z)}.\end{aligned}$$

Therefore $T^{-1}(B)\subset U$. Letting $V=T^{-1}(B)$, the definition of T implies that E_V is norm isomorphic with $T(E)$. Hence $\widetilde{E_V}$ is norm isomorphic with the closed subspace $\overline{T(E)}$ of $l_p(Z)$. The proof is complete.

THEOREM 9. *Let Z be a Banach space. Then the locally convex direct sum of a countable family of Z-nuclear spaces is a Z-nuclear space.*

PROOF. Let $E=\oplus_{i=1}^{\infty}E_i$, let E_i $(i\in N)$ be Z-nuclear spaces and let T be a continuous linear mapping of E into a given Banach space F. If T_i is the restriction of T to the subspace E_i of E, T_i is continuous and hence Z-nuclear, and thus of the form

$$T_i = \textstyle\sum_{n=1}^{\infty}\mu_{i,n}B_{i,n}A_{i,n}$$

with

$$\textstyle\sum_{n=1}^{\infty}|\mu_{i,n}| \leq i^{-2}\ (i\in N), \quad \{A_{i,n}\}\subset L(E_i, Z) \quad \text{and} \quad \{B_{i,n}\}\subset L(Z, F)$$

such that $\|B_{i,n}\|\leq 1$ for all $(i, n)\in N\times N$ and $\{A_{i,n}: n\in N\}$ is equicontinuous in $L(E_i, Z)$. Let us define the linear mapping $\bar{A}_{i,n}$ of E into Z in such a way that

$$\bar{A}_{i,n} = \begin{cases} A_{i,n} & \text{on } E_i \\ 0 & \text{on } \oplus_{j\neq i}E_j. \end{cases}$$

Then the family $\{\bar{A}_{i,n}: (i, n)\in N\times N\}$ is equicontinuous in $L(E, Z)$. Since T can be written in the form

$$T = \textstyle\sum_{i,n=1}^{\infty}\mu_{i,n}B_{i,n}\bar{A}_{i,n},$$

it follows from Theorem 7 that T is Z-nuclear. Hence by Theorem 8, E is Z-nuclear. The proof is complete.

THEOREM 10. *Let Z be a Banach space. Then the product of an arbitrary family of Z-nuclear spaces is Z-nuclear.*

PROOF. Let $\{E_\alpha : \alpha \in A\}$ be an arbitrary family of Z-nuclear spaces, let $E = \prod_\alpha E_\alpha$, and let T be a continuous linear mapping of E into a given Banach space F. Then there exists a neighborhood of zero V in E such that $T(V)$ is bounded in F, and by definition of the product topology, V contains a neighborhood of zero of the form $V_{\alpha_1} \times \cdots \times V_{\alpha_n} \times \prod_{\beta \neq \alpha_i} E_\beta$. It follows that T vanishes on the subspace $G = \prod_{\beta \neq \alpha_i} E_\beta$ of E. Since $E = \prod_{i=1}^n E_{\alpha_i} \times G$, it remains to show that the restriction of T to $\prod_{i=1}^n E_{\alpha_i}$ is Z-nuclear. Since $\prod_{i=1}^n E_{\alpha_i}$ can be identified with $\oplus_{i=1}^n E_{\alpha_i}$, this is clear from Theorem 9. The proof is complete.

THEOREM 11. *Let Z_1 and Z_2 be Banach spaces. Then if E is a Z_1-nuclear locally convex space and F is a Z_2-nuclear locally convex space, then the projective tensor product $E \otimes_\pi F$ is $Z_1 \widetilde{\otimes}_\pi Z_2$-nuclear and also $E \widetilde{\otimes}_\pi F$ is $Z_1 \widetilde{\otimes}_\pi Z_2$-nuclear.*

PROOF. Let U and V be absolutely convex neighborhoods of zero in E and F respectively, let $G = E \otimes_\pi F$ and let W be the absolutely convex hull of $U \otimes V$ in G. It is clear ([6, Ch. III, 6.3]) that G_W is identical with the normed space $(E_U \otimes F_V, r)$, where r is the tensor product of the respective norms of E_U and F_V. Hence if ϕ_U, ϕ_V and ϕ_W denote the respective canonical mappings $E \to \tilde{E}_U$, $F \to \tilde{F}_V$ and $G \to \tilde{G}_W$, we have $\phi_W = \phi_U \otimes \phi_V$. Since E and F are Z_1-nuclear and Z_2-nuclear respectively, Theorem 7 implies that

$$\phi_U = \textstyle\sum_{i=1}^\infty \lambda_i B_i A_i$$

with

$$\{A_i\} \subset L(E, Z_1) \quad \text{and} \quad \{B_i\} \subset L(Z_1, \widetilde{E_U})$$

and

$$\phi_V = \textstyle\sum_{j=1}^\infty \mu_j D_j C_j$$

with

$$\{C_j\} \subset L(F, Z_2) \quad \text{and} \quad \{D_j\} \subset L(Z_2, \tilde{F}_V),$$

where $\{\lambda_i\}$, $\{\mu_j\}$, etc., have the properties enumerated in Theorem 7. For $u \in E$ and $v \in F$ we have by definition

$$\phi_U \otimes \phi_V(u \otimes v) = (\textstyle\sum_{i=1}^\infty \lambda_i B_i A_i(u)) \otimes (\sum_{j=1}^\infty \mu_j D_j C_j(v)).$$

Hence we have

$$\phi_W(u \otimes v) = \textstyle\sum_{i,j} \lambda_i \mu_j (B_i \otimes D_j)(A_i(u) \otimes C_j(v))$$

so that

$$\phi_W = \sum_{i,j} \lambda_i \mu_j (B_i \otimes D_j)(A_i \otimes C_j).$$

Now $\{\lambda_i \mu_j: (i, j) \in N \times N\}$ is a summable family, $\{A_i \otimes C_j\} \subset L(E \otimes_\pi F, Z_1 \widetilde{\otimes}_\pi Z_2)$ and $\{B_i \otimes D_j\} \subset L(Z_1 \widetilde{\otimes}_\pi Z_2, \tilde{G}_W)$. By Theorem 7 ϕ_W is $Z_1 \widetilde{\otimes}_\pi Z_2$-nuclear. The nuclearity of $E \widetilde{\otimes}_\pi F$ follows from Proposition 11. The proof is complete.

COROLLARY. *Let E be a locally convex space and let Z be a Banach space. If E is nuclear, then $E \otimes_\pi Z$ is a Z-nuclear space and also $E \widetilde{\otimes}_\pi Z$ is a Z-nuclear space.*

PROOF. This follows from the above theorem.

DEFINITION 10. *Let Z be a topological vector space and let $C^\infty(R^n, Z)$ be the space of infinitely differentiable functions, defined in R^n and valued in Z. Then we denote by $\mathfrak{S}(R^n, Z)$ the space of functions $u \in C^\infty(R^n, Z)$ such that, for all pairs of polynomials P, Q in n variables, with complex coefficients, $P(x)Q(\partial/\partial x)u(x)$ remains in a bounded subset of Z as x varies over R^n. We equip $\mathfrak{S}(R^n, Z)$ with the topology of uniform convergence of the functions $P(x)Q(\partial/\partial x)u(x)$ over the whole of R^n, for all possible P and Q.*

If Z is a Banach space, by [7] we have $\mathfrak{S}(R^n, Z) = \mathfrak{S}(R^n) \widetilde{\otimes}_\pi Z$. Since the space $\mathfrak{S}(R^n)$ of rapidly decreasing functions is a nuclear space, by virtue of the above corollary $\mathfrak{S}(R^n, Z)$ is a Z-nuclear space.

THEOREM 12. *Let Z be a finite dimensional Banach space and let E be a Z-nuclear space. Then every bounded subset of E is precompact.*

PROOF. Let B be a bounded subset of E. Then for each absolutely convex neighborhood of zero U in E there exists another absolutely convex neighborhood of zero V with $U \supset V$ such that $\phi_{U,V}: \tilde{E}_V \to \tilde{E}_U$ is Z-nuclear. The canonical mapping ϕ_U of E into $\tilde{E}_U$ can be decomposed into $\phi_{U,V} \circ \phi_V$. Since ϕ_V and $\phi_{U,V}$ are continuous linear mappings and since $\phi_{U,V}$ is a compact mapping, the canonical image of B in $\tilde{E}_U$ is precompact. Since U is arbitrary, B is precompact in E. The proof is complete.

COROLLARY 1. *Let Z be finite dimensional and let E be a quasi-complete Z-nuclear space. Then every closed bounded subset of E is compact.*

The proof is simple.

COROLLARY 2. *Let Z be finite dimensional. Then a normable space E is Z-nuclear if and only if it is finite dimensional.*

PROOF. If E is finite dimensional, it is clear that E is Z-nuclear. Conversely, if E is Z-nuclear, then $\tilde{E}$ is Z-nuclear. Since $\tilde{E}$ is quasi-complete, the unit ball in $\tilde{E}$ is compact. Therefore E is finite dimensional. The proof is complete.

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