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On a p-Capacity of a Condenser and KD^p-Null Sets

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Introduction

Ahlfors and Beurling [1] introduced the notion of null sets of class N_D in the complex plane and characterized such null sets by means of the extremal length. Hedberg [6] considered a generalization of this notion, namely, removable sets for the class FD^p $(1<p<\infty)$ in an N-dimensional euclidean space R^N, and characterized such removable sets by means of condenser capacities. We can consider a class KD^p of p-precise functions on R^N $(N\geqq 3)$ and define KD^p-null sets. In the present paper, we shall show several relations between KD^p-null sets and p-capacities of a condenser.

A real valued function u defined in a domain D of R^N is called a p-precise function, if it is absolutely continuous along p-a.e. curve in D and $|\mathrm{grad}\, u|$ belongs to $L^p(D)$. A p-precise function u in D has a finite curvilinear limit $u(\gamma)$ along p-a.e. curve γ in D (see [9, Theorem 5.4]). Let α be a compact subset of ∂D and $\Gamma_D(\alpha)$ be the family of all locally rectifiable curves in D each of which starts from some point of D and tends to α. Let α_0, α_1 be non-empty compact subsets of ∂D such that $\alpha_0 \cap \alpha_1 = \emptyset$. We follow [9] in defining the p-capacity of condenser $(\alpha_0, \alpha_1; D)$:

$$C_p(\alpha_0, \alpha_1; D) = \inf_u \int_D |\mathrm{grad}\, u|^p dx,$$

where the infimum is taken over all p-precise functions u in D such that $u(\gamma)=0$ (resp. 1) for p-a.e. $\gamma\in\Gamma_D(\alpha_0)$ (resp. $\Gamma_D(\alpha_1)$). Denote by $\hat{D}$ the Kerékjártó-Stoïlow compactification of D. For a condenser $(\alpha_0, \alpha_1; D)$ such that α_0 and α_1 are two mutually disjoint closed subsets of $\hat{D}-D$ and a partition $\{\beta_\iota\}$ of $\hat{D}-D-\alpha_0-\alpha_1$, we shall consider a new kind of p-capacity $C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ as follows. Let the boundary components of D be divided into α_0, α_1 and $\{\beta_\iota\}$. We set

$$C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \inf_u \int_D |\mathrm{grad}\, u|^p dx,$$

where the infimum is taken over all p-precise functions u in D such that $u(\gamma)=0$ (resp. 1) for p-a.e. $\gamma\in\Gamma_D(\alpha_0)$ (resp. $\Gamma_D(\alpha_1)$) and $u(\gamma)=a_\iota$ for p-a.e. $\gamma\in\Gamma_D(\beta_\iota)$, where each a_ι is a constant depending on u. On the other hand, we take an exhaustion $\{D_n\}$ and set $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \lim_{n\to\infty} C_p^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_\iota^{(n)}\})$, where $\alpha_{in} = \partial D_n \cap \partial A_{in}$ $(i=0, 1)$, A_{in} being the component of $\bar{D} - D_n$ which contains α_i, and $\{\beta_\iota^{(n)}\}$ is some partition of $\partial D_n - \alpha_{0n} - \alpha_{1n}$ depending on $\{\beta_\iota\}$.

In § 2, we shall give a characterization of the extremal functions for $C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ and $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$. In § 3, for some condenser $(\alpha_0, \alpha_1; D)$ and some partition $\{\beta_\iota\}$ we shall relate $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ to the p-module of the family of curves each of which connects α_0 and α_1 in $\hat{D}$. This is a generalization of Gehring's result in [4].

A compact set E in R^N will be called a KD^p-null set with respect to an open set G containing E, if any function in $KD^p(G-E; E)$ can be extended to a function in $KD^p(G)$, where $KD^p(G)$ (resp. $KD^p(G-E; E)$) is the class of p-precise functions u in G (resp. $G-E$) satisfying the following condition:

$$\int_G |\mathrm{grad}\, u|^{p-2}(\mathrm{grad}\, u, \mathrm{grad}\, \phi)\, dx = 0$$

for all $\phi \in C_0^\infty(G)$ (resp. for all $\phi \in C_0^\infty(G)$ such that grad ϕ vanishes in some neighborhood of E).

In § 4, we shall give a necessary condition for a set to be KD^p-null in terms of p-capacities. In § 5, we observe some relations between KD^p-null sets and sets removable for the class FD^p. In § 6, we shall give a characterization of KD^2-null sets by means of 2-capacities.

§ 1. Preliminaries

We shall denote by $x = (x_1, x_2, \dots, x_N)$ a point in R^N, and set $|x| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$. For sets E and F in R^N, let $\mathrm{dist}(E, F)$ denote the distance between E and F. We denote by ∂E and $\bar{E}$ the boundary and the closure of E respectively. Let p be a finite number such that $p > 1$. For an open set G in R^N, let $L^p(G)$ be the family of functions f on G for which $|f|^p$ is integrable, and let $\|f\|_p$ be the L^p-norm. For a measurable vector field $v = (v_1, v_2, \dots, v_N)$ on G, we define $\|v\|_p$ by $\| |v| \|_p$. We denote by $C^\infty(G)$ the family of infinitely differentiable functions in G and by $C_0^\infty(G)$ the subfamily consisting of functions with compact support in G.

Let Γ be a family of locally rectifiable curves in R^N none of which is a point. A non-negative Borel measurable function f is called admissible in association with Γ if $\int_\gamma f ds \geqq 1$ for each $\gamma \in \Gamma$. The p-module $M_p(\Gamma)$ of Γ is defined by $\inf_f \int f^p dx$, where the infimum is taken over all functions f admissible in association with Γ. A property will be said to hold p-almost everywhere ($=p$-a.e.) on Γ if the p-module of the subfamily of exceptional curves is zero. The following properties are well known (see, e.g., [3, Chapter I] or [9, Chapter I]):

(1.1) If $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, then $M_p(\Gamma) \leqq \sum_{n=1}^{\infty} M_p(\Gamma_n)$.

(1.2) $M_p(\Gamma)=0$ if and only if there is a non-negative Borel measurable function $f \in L^p(R^N)$ such that $\int_\gamma f ds = \infty$ for every $\gamma \in \Gamma$.

(1.3) Every sequence $\{f_n\}$ of Borel measurable functions in an open set G such that $\int_G |f_n|^p dx$ tends to zero as $n \to \infty$ has a subsequence $\{f_{n_i}\}$ such that

$$\lim_{i\to\infty} \int_\gamma |f_{n_i}| ds = 0$$

for p-a.e. curve γ in G.

A real valued function u defined in an open set G is called a p-precise function, if (i) it is absolutely continuous along p-a.e. curve in G, and (ii) $|\nabla u|$ belongs to $L^p(G)$; from (i) it follows that the gradient ∇u exists almost everywhere in G. The following results are known:

(1.4) Let u be a p-precise function in G. Then

$$u(x^1) - u(x^0) = \int_{\widetilde{x^0x^1}} \left(\sum_{k=1}^{N} \frac{\partial u}{\partial x_k} \frac{dx_k}{ds} \right) ds$$

for any points x^0 and x^1 on p-a.e. curve γ in G, where $\widetilde{x^0x^1}$ is the subarc of γ connecting x^0 and x^1 (cf. [3, Chapter III, 2] or [9, Theorem 4.16]).

(1.5) Let $\{u_n\}$ be a sequence of p-precise functions in G and assume

$$\lim_{n,m\to\infty} \|\nabla(u_n - u_m)\|_p = 0.$$

Then there exists a p-precise function u in G such that $\|\nabla(u_n - u)\|_p \to 0$ as $n \to \infty$ (see [3, Theorem 14] or [9, Theorem 4.18]).

(1.6) Every p-precise function u in G has a finite curvilinear limit $u(\gamma)$ along p-a.e. curve γ in G (see [9, Theorem 5.4]).

(1.7) Let u be a p-precise function defined in G, and v a p-precise function defined in an open set $G' \subset G$ such that, for p-a.e. curve γ in G' terminating at a point x of $\partial G' \cap G$, $\lim v(y)$ exists and equals $u(x)$ as y tends to x along γ. Then the function w which is equal to v in G' and to u on $G - G'$ is a p-precise function in G (see [9, Theorem 5.5]).

Let D be a domain in R^N and denote by D^* the closure of D in the Aleksandrov compactification $R^N \cup \{\infty\}$. Let α be a closed subset of the boundary $D^* - D$. We shall denote by Γ_D (resp. $\Gamma_D(\alpha)$) the family of all locally rectifiable curves in D each of which starts from a point of D and tends to $D^* - D$ (resp. α). Let α_0, α_1 be non-empty closed subsets of $D^* - D$ such that $\alpha_0 \cap \alpha_1 = \emptyset$. We shall denote by $\mathscr{D}(\alpha_0, \alpha_1; D)$ the family of all p-precise functions u in D such that $u(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_D(\alpha_0)$ and $u(\gamma)=1$ for p-a.e. $\gamma \in \Gamma_D(\alpha_1)$. Following Ohtsuka [9, §6.2], we define the p-capacity of condenser $(\alpha_0, \alpha_1; D)$ as

$$C_p(\alpha_0, \alpha_1; D) = \inf_{u \in \mathscr{D}(\alpha_0, \alpha_1; D)} \int_D |\nabla u|^p dx.$$

If a p-precise function u in $\mathscr{D}(\alpha_0, \alpha_1; D)$ satisfies

$$C_p(\alpha_0, \alpha_1; D) = \int_D |\nabla u|^p dx,$$

then u is called an extremal function for $C_p(\alpha_0, \alpha_1; D)$.

Denote by $\hat{D}$ the Kerékjártó-Stoïlow compactification of D (see [11]). Throughout the rest of the paper let α_0 and α_1 be non-empty mutually disjoint closed sets consisting of boundary components. Divide the boundary components of $\hat{D}-D-\alpha_0-\alpha_1$ into mutually disjoint sets $\{\beta_\iota\}$, and let $\mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ be the family consisting of all $u \in \mathscr{D}(\alpha_0, \alpha_1; D)$ such that $u(\gamma)=a_\iota$ for p-a.e. $\gamma \in \Gamma_D(\beta_\iota)$, where each a_ι is a constant depending on u. We define the p-capacity of condenser $(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ as

$$C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \inf_{u \in \mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})} \int_D |\nabla u|^p dx.$$

If a p-precise function u in $\mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ satisfies

$$C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \int_D |\nabla u|^p dx,$$

then u is called an extremal function for $C_p^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$.

We shall give another definition of p-capacity. Let $\{\beta_\iota\}$ be as above. Let $\{D_n\}$ be an exhaustion of D, that is, each D_n is a bounded subdomain of D, each ∂D_n consists of a finite number of C^1-surfaces, $\bar{D}_n \subset D_{n+1}$ $(n=1, 2,\dots)$ and $\cup_{n=1}^{\infty} D_n = D$. Let A_{0n} (resp. A_{1n}) consist of the components of $\hat{D}-D_n$ each of which meets α_0 (resp. α_1). We may assume $A_{01} \cap A_{11} = \emptyset$. Set $\alpha_{in} = \partial D_n \cap \partial A_{in}$ $(i=0, 1)$. Take any boundary components β and β' in $\partial D_n - \partial A_{0n} - \partial A_{1n}$, and let A and A' be the components of $\hat{D}-D_n$ such that $\partial A=\beta$ and $\partial A'=\beta'$. We say that β and β' are in the same class if there exists some β_ι such that $\beta_\iota \cap A \neq \emptyset$ and $\beta_\iota \cap A' \neq \emptyset$. We classify the boundary components of $\partial D_n - \partial A_{0n} - \partial A_{1n}$ in this way and denote them by $\{\beta_j^{(n)}\}$; these are naturally finite in number. Let $B_j^{(n)}$ consist of the components of $\hat{D}-D_n$ such that $\partial B_j^{(n)} = \beta_j^{(n)}$. We suppose that $\{\beta_\iota\}$ has the following property:

(1.8) We can take an exhaustion $\{D_n\}$ such that for each β_ι and D_n, if $\beta_\iota \cap \cup_{j=1}^{j(n)} B_j^{(n)} \neq \emptyset$ then $\beta_\iota \cap A_{0n} = \emptyset$ and $\beta_\iota \cap A_{1n} = \emptyset$.

Let $\{D_n\}$ be an exhaustion of the type considered in (1.8). By property (1.7) for any u in $\mathscr{D}^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\})$, the function $\tilde{u}$ in D_{n+1} which is an extension of u with a suitable constant on each component of $D_{n+1}-D_n$ belongs to $\mathscr{D}^*(\alpha_{0(n+1)}, \alpha_{1(n+1)}; D_{n+1}, \{\beta_j^{(n+1)}\})$. Therefore $C_p^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\}) \geqq$

$C_p^*(\alpha_{0(n+1)}, \alpha_{1(n+1)}; D_{n+1}, \{\beta_j^{(n+1)}\})$ $(n=1, 2,\dots)$. We set

$$C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \lim_{n\to\infty} C_p^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\}).$$

We note that $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ does not depend on the choice of exhaustion of the type considered in (1.8). When $\{\beta_\iota\}$ is the canonical partition, we write $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$ for $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$.

§2. Extremal functions for the p-capacity of a condenser

We begin with

LEMMA 1. *Let Γ be a family of curves in a domain D in R^N, and $\{\phi_n\}$ be a sequence of functions defined p-a.e. and tending to a finite-valued function ϕ p-a.e. on Γ. Let $u_0, u_1, u_2,\dots$ be p-precise functions such that $u_n(\gamma)=\phi_n(\gamma)$ for each $n\geqq 1$ and p-a.e. $\gamma\in\Gamma$ and $\|\nabla(u_n-u_0)\|_p\to 0$ as $n\to\infty$. Then there exists a constant c such that $u_0(\gamma)=\phi(\gamma)-c$ for p-a.e. $\gamma\in\Gamma$.*

PROOF. We may assume that $M_p(\Gamma)>0$. In view of properties (1.1) and (1.6) we may assume furthermore that ϕ_n and ϕ are defined everywhere on Γ, $u_0(\gamma)$, $u_1(\gamma),\dots$ exist and are finite everywhere on Γ and $u_n(\gamma)=\phi_n(\gamma)$ for all $n\geqq 1$ and $\gamma\in\Gamma$. By properties (1.3) and (1.4) there is a family Γ' of curves in D with $M_p(\Gamma')=0$ and having the following properties:

(1) There exists a subsequence $\{u_{n_i}\}$ such that

$$\lim_{i\to\infty}\int_\gamma |\nabla(u_{n_i}-u_0)|ds = 0$$

for all $\gamma\notin\Gamma'$.

(2) $$u_n(x^1) - u_n(x^0) = \int_{\widetilde{x^0x^1}}\left(\sum_{k=1}^N \frac{\partial u_n}{\partial x_k}\frac{dx_k}{ds}\right)ds$$

for each $n=0, 1,\dots$, for all $\gamma\notin\Gamma'$ and for arbitrary points x^0 and x^1 on γ.

We shall denote $\{u_{n_i}\}$ again by $\{u_n\}$.

By property (1.2) there exists a non-negative Borel measurable function h in $L^p(D)$ such that $\int_\gamma h\,ds=\infty$ for every $\gamma\in\Gamma'$. We can find a subset D_h of D containing almost all points of D such that for any two points x and y in D_h there exists a curve γ which passes through x and y and along which $\int_\gamma h\,ds<\infty$ and such that for p-a.e. curve γ' in D it is contained in D_h and $\int_{\gamma'} h\,ds<\infty$ (see [9, Lemma 4.6]). Take $x^0\in D_h$ at which all u_n and u_0 are finite, and take any $\gamma\in\Gamma-\Gamma'$ such that γ is contained in D_h and $\int_\gamma h\,ds<\infty$. Then we can find a curve γ_0 in D_h which contains x^0 and some end part of γ and for which $\int_{\gamma_0} h\,ds<\infty$. Let

$x(t)$, $0<t<1$, be a representation of γ_0. Since $\gamma_0 \notin \Gamma'$, by (2)

$$u_n(x(t)) - u_n(x^0) = \int_{\widetilde{x^0x(t)}} \left(\sum_{k=1}^N \frac{\partial u_n}{\partial x_k} \frac{dx_k}{ds}\right) ds$$

for any $t \in (0, 1)$ and $n=0, 1,\dots$. It follows that

$$\begin{aligned} &|u_n(x^0) - \phi_n(\gamma) - u_0(x^0) + u_0(\gamma)| \\ &= \lim_{t\to 1} |u_n(x^0) - u_n(x(t)) - u_0(x^0) + u_0(x(t))| \\ &\leqq \int_{\gamma_0} |\nabla(u_n - u_0)| ds \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

Since $\phi_n(\gamma) \to \phi(\gamma)$ (=a finite value), $u_n(x^0)$ tends to $\phi(\gamma)+u_0(x^0)-u_0(\gamma)$. Set $c_0=\phi(\gamma)-u_0(\gamma)$. Then $u_n(x^0)$ tends to $u_0(x^0)+c_0$. Thus c_0 does not depend on γ. This proves our lemma.

Let α_0, α_1, $\{\beta_\iota\}$ be as in § 1. We denote by $\mathscr{A}^*(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ the family of all p-precise functions v in D such that $v(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_D(\alpha_0) \cup \Gamma_D(\alpha_1)$ and $v(\gamma)=a_\iota$ for p-a.e. $\gamma \in \Gamma_D(\beta_\iota)$, where each a_ι is a constant depending on v.

First we shall show the following theorem.

THEOREM 1. *Let D be a domain and divide its boundary components into α_0, α_1, $\{\beta_j\}_{j=1}^\infty$. Then there exists an extremal function u^* for $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ and it is characterized by the condition that*

$$\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla v) dx = 0$$

for every v in $\mathscr{A}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Here $(\nabla u^*, \nabla v)$ means the inner product of ∇u^* and ∇v, and at a point x^0 where $|\nabla u^*(x^0)|=0$ we set

$$|\nabla u^*(x^0)|^{p-2}(\nabla u^*(x^0), \nabla v(x^0)) = 0.$$

The difference of two extremal functions is constant a.e. in D.

PROOF. In this proof we write C_p^* and $\mathscr{D}^*$ for $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ and $\mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ respectively. For the existence of u^*, we may assume that $M_p(\Gamma_D(\alpha_i))>0$ $(i=0, 1)$, for, otherwise, the constant 0 or 1 belongs to $\mathscr{D}^*$ so that the assertion is trivial. Choose a sequence $\{u_n\}$ in $\mathscr{D}^*$ such that $\|\nabla u_n\|_p^p$ tends to C_p^* as $n\to\infty$. By using Clarkson's inequality (see [2] or [9, Lemma 1.1]) and the fact $(u_n+u_m)/2 \in \mathscr{D}^*$, we see that $\lim_{n,m\to\infty}\|\nabla(u_n-u_m)\|_p=0$. Applying property (1.5) we have a p-precise function u_0 such that $C_p^*=\|\nabla u_0\|_p^p$.

We observe that $u_n'=\max(0, \min(u_n, 1))$ belongs to $\mathscr{D}^*$ (see [9, Theorem 4.15]) and $\|\nabla u_n'\|_p \leqq \|\nabla u_n\|_p$. Hence we may assume that $0\leqq u_n \leqq 1$ for all n. By Lemma 1 there exists a constant c such that $u_0(\gamma)+c=0$ (resp. 1) p-a.e. on

$\Gamma_D(\alpha_0)$ (resp. $\Gamma_D(\alpha_1)$). We write u_0 for u_0+c. Suppose $u_n(\gamma)=a_j^n$ for p-a.e. $\gamma \in \Gamma_D(\beta_j)$. By choosing a suitable subsequence we may assume that $\{a_j^n\}_{n=1}^{\infty}$ converges to a_j. By Lemma 1 again $u_0(\gamma)=a_j$ p-a.e. on $\Gamma_D(\beta_j)$. Thus $u_0 \in \mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$, and hence u_0 is an extremal function.

For the latter half, let u^* be any extremal function for C_p^*. For any $v \in \mathscr{A}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$, there exists an integrable function $f(x)$ in D such that

$$\left|\frac{|\nabla(u^* \pm \varepsilon v)(x)|^p - |\nabla u^*(x)|^p}{\varepsilon}\right| \leqq f(x) \qquad \text{for all } \varepsilon \in (0, 1).$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D \frac{|\nabla(u^* \pm \varepsilon v)|^p - |\nabla u^*|^p}{\varepsilon}dx &= \int_D \lim_{\varepsilon\to 0}\frac{|\nabla(u^* \pm \varepsilon v)|^p - |\nabla u^*|^p}{\varepsilon}dx \\ &= \pm p\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla v)dx.\end{aligned}$$

Since $u^* \pm \varepsilon v \in \mathscr{D}^*$,

$$\int_D |\nabla(u^* \pm \varepsilon v)|^p dx \geqq \int_D |\nabla u^*|^p dx.$$

Hence we have

$$\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla v)dx = 0.$$

Conversely, suppose that $u \in \mathscr{D}^*$ satisfies the equation

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla v)dx = 0$$

for every $v \in \mathscr{A}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Since $u^*-u \in \mathscr{A}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$,

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla(u^* - u))dx = 0.$$

By using Hölder's inequality, we derive that

$$\int_D |\nabla u|^p dx \leqq \int_D |\nabla u^*|^p dx = C_p^*.$$

This implies that u is an extremal function for C_p^*.

Finally, let u^*, v^* be extremal. Since $(u^*+v^*)/2 \in \mathscr{D}^*$, $\|\nabla(u^*-v^*)\|_p=0$ by Clarkson's inequality so that $u^*-v^*=$const. a.e. in D. This completes the proof of our theorem.

Let D' be a relatively compact subdomain of D with C^1 boundary such that no component of $D-D'$ is relatively compact in D. We classify the boundary components of $\partial D'$ into $\alpha_0^{(D')}$, $\alpha_1^{(D')}$ and $\{\beta_j^{(D')}\}$ as we did to D_n in § 1. We extend each function of $\mathscr{D}^*(\alpha_0^{(D')}, \alpha_1^{(D')}; D', \{\beta_j^{(D')}\})$ by suitable constants to a p-precise function on D, and denote by $\mathscr{D}^*(D')$ the family of all such functions on D. Let $A_0^{(D')}$, $A_1^{(D')}$ and $B_j^{(D')}$ be the unions of $\hat{D}-D'$ such that $\partial A_0^{(D')}=\alpha_0^{(D')}$, $\partial A_1^{(D')}=\alpha_1^{(D')}$ and $\partial B_j^{(D')}=\beta_j^{(D')}$. Let $\{\beta_\iota\}$ be a partition with property (1.8). We can take some D' such that for each β_ι, if $\beta_\iota \cap \bigcup_{j=1}^{j(D')} B_j^{(D')} \neq \emptyset$ then $\beta_\iota \cap A_0^{(D')}=\emptyset$ and $\beta_\iota \cap A_1^{(D')}=\emptyset$. Let D'' be such a domain. Set

$$\tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \bigcup_{D''} \mathscr{D}^*(D''),$$

and denote by $\mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ the family of all p-precise functions u in D with the following properties:

(1) For each u, there exists a sequence $\{u_n\}$ in $\tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ such that $\lim_{n\to\infty}\|\nabla(u_n-u)\|_p=0$.

(2) $u(\gamma)=0$ (resp. 1) for p-a.e. $\gamma \in \Gamma_D(\alpha_0)$ (resp. $\Gamma_D(\alpha_1)$).

We observe that

$$C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\}) = \inf_{u\in\mathscr{D}^{**}(\alpha_0,\alpha_1;D,\{\beta_\iota\})} \int_D |\nabla u|^p dx.$$

We call u in $\mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ an extremal function for $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ if $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})=\int_D |\nabla u|^p dx$. We denote by $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$ the family of all $C^\infty(D)$-functions ϕ in D such that the support of $|\nabla\phi|$ is compact in D, $\phi=0$ on $U\cap D$ for some neighborhood U of $\alpha_0 \cup \alpha_1$. Denote by $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ the family consisting of all $\phi\in\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$ such that $\phi=$const. on each $B_j^{(D'')}\cap D$ and $\phi=0$ on $(A_0^{(D'')}\cup A_1^{(D'')})\cap D$ for some D''.

Now we prove

THEOREM 2. a) *There exists an extremal function u^* for $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$ and it is characterized by the condition that*

$$\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla\phi)dx = 0$$

for every ϕ in $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$. The difference of two extremal functions is constant a.e. in D.

b) *Let $\{\beta_\iota\}$ be a partition with property* (1.8). *Then there exists an extremal function u^* for $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ and it is characterized by the condition that*

$$\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla\phi)dx = 0$$

for every ϕ in $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$. The difference of two extremal functions is constant a.e. in D.

PROOF. a) Let $\{D_n\}$ be an exhaustion of D, and u_n^* be an extremal function for $C_p^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\})$. We have seen near the end of §1 that $\|\nabla u_n^*\|_p^p = C_p^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\})$ decreases as $n\to\infty$. As in the proof of Theorem 1, we see that there exists a p-precise function u^* in D such that

$$\lim_{n\to\infty} \|\nabla(u_n^* - u^*)\|_p = 0$$

and

$$u^*(\gamma) = 0 \text{ (resp. 1)} \quad \text{for} \quad p\text{-a.e. } \gamma\in\Gamma_D(\alpha_0) \text{ (resp. } \Gamma_D(\alpha_1)).$$

It is easy to see that u^* is an extremal function for $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$.

Let v^* be another extremal function. Then there exists a sequence $\{v_n\}$ in $\tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ such that $\|\nabla(v_n - v^*)\|_p \to 0$. Since $(u^*+v^*)/2 \in \mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$, by Clarkson's inequality we see that $\|\nabla(u^*-v^*)\|_p = 0$ so that $u^*-v^* = \text{const. a.e. in } D$.

Next, let u^* be any extremal function for $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$. Then there is a sequence $\{u_n\}$ in $\tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ such that $\lim_{n\to\infty}\|\nabla(u_n - u^*)\|_p = 0$. For any ε $(0<\varepsilon<1)$ and ϕ in $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$, we see that $u_n \pm \varepsilon\phi \in \tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$. It follows that $u^* \pm \varepsilon\phi \in \mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$. Hence, as in the latter half of the proof of Theorem 1, we have

$$\int_D |\nabla u^*|^{p-2}(\nabla u^*, \nabla\phi)dx = 0.$$

Conversely, let $u \in \mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ satisfy the equation

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla\phi)dx = 0$$

for every ϕ in $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$. Let u^* be an extremal function for $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$. Then there are two sequences $\{u_n\}$ and $\{\tilde{u}_n\}$ in $\tilde{\mathscr{D}}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ such that $\lim_{n\to\infty}\|\nabla(u_n - u)\|_p = 0$ and $\lim_{n\to\infty}\|\nabla(\tilde{u}_n - u^*)\|_p = 0$. Set $f_n = u_n - \tilde{u}_n$. It vanishes in a neighborhood U of $\alpha_0 \cup \alpha_1$ with compact relative boundary $\partial U \subset D$. We can find a function h_n in $C^\infty(D)$ such that $h_n = 0$ on $U \cap D$ and $h_n - f_n = 0$ on $U' \cap D$ for a neighborhood U' of $D^* - D - U$. Then $f_n - h_n$ has a compact support in D. There exists a sequence $\{f_n^i\}_{i=1}^\infty$ in $C_0^\infty(D)$ such that

$$\lim_{i\to\infty} \|\nabla(f_n - h_n - f_n^i)\|_p = 0.$$

Since $h_n+f_n^i \in \mathscr{A}^{**}(\alpha_0, \alpha_1; D, \beta_Q)$, we see that

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla(h_n + f_n^i))dx = 0$$

for all i. Using Hölder's inequality and letting $i \to \infty$, we obtain

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla f_n)dx = 0$$

for all n. Since $\lim_{n\to\infty}\|\nabla(u-u^*)-\nabla f_n\|_p=0$, Hölder's inequality again gives

$$\int_D |\nabla u|^{p-2}(\nabla u, \nabla(u - u^*))dx = 0.$$

It follows from this equality and Hölder's inequality that

$$\|\nabla u\|_p^p \leqq \|\nabla u^*\|_p^p = C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q).$$

This implies that u is an extremal function for $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q)$.

b) We note that $(u+v)/2$ and $u \pm \varepsilon\phi$ belong to $\mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ for any u, v in $\mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ and ϕ in $\mathscr{A}^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$. Then we can complete the proof in the same way as a).

REMARK. In case $M_p(\Gamma_D(\alpha_0) \cup \Gamma_D(\alpha_1))>0$ two extremal functions coincide a.e. in D.

Let us compare the results in Theorems 1 and 2. Let $\{\beta_j\}_{j=1}^\infty$ be a partition with property (1.8). Let u_1 and u_2 be extremal functions in Theorems 1 and 2 respectively. By Lemma 1 we see that $u_2 \in \mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ so that $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\}) \leqq C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_j\})$. We obtain the equality in a special case. We give first

LEMMA 2 ([9, Theorem 6.16]). *Let u be a p-precise function in D whose limit along p-a.e. $\gamma \in \Gamma_D$ vanishes. Then there exists a sequence $\{u_n\}$ in $C_0^\infty(D)$ such that $\|\nabla(u-u_n)\|_p \to 0$ as $n \to \infty$.*

We shall prove

THEOREM 3. *Suppose the boundary of D in $R^N \cup \{\infty\}$ consists of mutually disjoint closed sets $\alpha_0, \alpha_1, \beta_1, \ldots, \beta_k$. Then*

$$C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\}) = C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_j\}).$$

PROOF. Take any $u \in \mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Let $\{D_n\}$ be an exhaustion. If n is large, then each component of $\hat{D}-D_n$ contains points of only one of α_0, $\alpha_1, \beta_1, \ldots, \beta_k$. Hence, we may assume that it is so for $n=1$. Denote by A_i

(resp. B_j) the union of the components of $\hat{D}-D_1$ such that $A_i \cap (\hat{D}-D)=\alpha_i$ (resp. $B_j \cap (\hat{D}-D)=\beta_j$). For each j, $1 \leqq j \leqq k$, there is a value b_j such that $u(\gamma)=b_j$ p-a.e. on $\Gamma_D(\beta_j)$. We can find a C^∞ function v in D which is equal to i $(=0, 1)$ (resp. b_j) on $A_i-\alpha_i$ (resp. $B_j-\beta_j$). Then $u(\gamma)-v(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_D$. By Lemma 2 there exists $\{u_n\}$ in $C_0^\infty(D)$ such that $\|\nabla(u-v-u_n)\|_p \to 0$ as $n\to\infty$. Since $v+u_n \in \mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_j\})$, $u \in \mathscr{D}^{**}(\alpha_0, \alpha_1; D, \{\beta_j\})$ so that $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_j\}) \leqq \|\nabla u\|_p^p$ as was observed before Theorem 2. We obtain the inequality $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_j\}) \leqq C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ and hence the equality.

We obtain the following theorem with respect to $C_p(\alpha_0, \alpha_1; D)$, which is proved in the same way as Theorem 1.

THEOREM 4 (cf. [5, Theorem 1]). *Let D be a domain in R^N and α_0, α_1 be non-empty compact subsets of $\partial D \cup \{\infty\}$ such that $\alpha_0 \cap \alpha_1 = \emptyset$. Then there exists an extremal function u_0 for $C_p(\alpha_0, \alpha_1; D)$ and it is characterized by the condition that*

$$\int_D |\nabla u_0|^{p-2}(\nabla u_0, \nabla v)dx = 0$$

for every p-precise function v in D such that $v(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_D(\alpha_0) \cup \Gamma_D(\alpha_1)$. The difference of two extremal functions is constant a.e. in D.

REMARK 1. Weyl's lemma shows that each of extremal functions for $C_2(\alpha_0, \alpha_1; D)$, $C_2^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ and $C_2^{**}(\alpha_0, \alpha_1; D, \{\beta_\iota\})$ is equal to a harmonic function a.e. in D.

REMARK 2. Let α_0, α_1 be disjoint boundary components of a domain D. In general $C_p(\alpha_0, \alpha_1; D) \leqq C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Now we shall give an example in which $C_p(\alpha_0, \alpha_1; D) < C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Let $\Omega=\{x;\ 1<|x|<2\}$ and E be a closed ball in Ω. Set $D=\Omega-E$, $\alpha_0=\{x;\ |x|=1\}$, $\alpha_1=\{x;\ |x|=2\}$ and $\beta=\partial E$. Suppose $C_p(\alpha_0, \alpha_1; D)=C_p^*(\alpha_0, \alpha_1; D, \beta)$. Let u^* and u_0 be extremal functions for $C_p^*(\alpha_0, \alpha_1; D, \beta)$ and $C_p(\alpha_0, \alpha_1; D)$ respectively. Since $u^* \in \mathscr{D}(\alpha_0, \alpha_1; D)$, from Theorem 4 it follows that $u^*=u_0$ except on a set of measure zero in D. Then the extension $\tilde{u}_0$ of u_0 by a suitable constant on E belongs to $\mathscr{D}(\alpha_0, \alpha_1; \Omega)$ by (1.7). Since

$$C_p(\alpha_0, \alpha_1; \Omega) \geqq C_p(\alpha_0, \alpha_1; D) = \int_D |\nabla u_0|^p dx = \int_\Omega |\nabla \tilde{u}_0|^p dx,$$

$\tilde{u}_0$ is an extremal function for $C_p(\alpha_0, \alpha_1; \Omega)$. It is well known that an extremal function for $C_p(\alpha_0, \alpha_1; \Omega)$ is given by

$$g(x) = \begin{cases} (|x|^{\frac{p-N}{p-1}} - 1)/(2^{\frac{p-N}{p-1}} - 1) & \text{if } p \neq N \\ (\log|x|)/\log 2 & \text{if } p = N. \end{cases}$$

By Theorem 4, $g=\tilde{u}_0$ except on a set of measure zero in Ω, which is impossible since $\tilde{u}_0=$const. on E. Hence $C_p(\alpha_0, \alpha_1; D)<C_p^*(\alpha_0, \alpha_1; D, \beta)$.

§3. Reation between the p-capacity and the p-module

Let D be a domain in R^N. By a locally rectifiable chain in D we mean a countable formal sum $\gamma=\Sigma\gamma_i$, where each γ_i is a locally rectifiable curve in D. If f is a non-negative Borel measurable function defined in D and $\gamma=\Sigma\gamma_i$ is a locally rectifiable chain in D, then we set $\int_\gamma fds=\Sigma\int_{\gamma_i} fds$. Let Γ be a family of locally rectifiable chains in D. A non-negative Borel measurable function f defined in D is called admissible in association with Γ if $\int_\gamma fds\geqq 1$ for every $\gamma\in\Gamma$. The p-module $M_p(\Gamma)$ is defined by $\inf_f\int_D f^p dx$, where the infimum is taken over all admissible functions f in association with Γ; if there is no such a function, then $M_p(\Gamma)$ is set to be ∞.

Suppose that the boundary components of D are partitioned into non-empty mutually disjoint closed sets $\alpha_0, \alpha_1, \beta_1,\dots, \beta_k$. Let $\beta=\cup_{j=1}^k\beta_j$. Each β_j is called a part of β. Let $\Gamma^*=\Gamma^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ be the family of all chains γ in $\bar{D}$ such that:

(1) γ is a continuous mapping from a union of closed intervals $[t_1, t_2]\cup[t_3, t_4]\cup\cdots\cup[t_{2n-1}, t_{2n}]$ into $\bar{D}$ with $t_1<t_2<\cdots<t_{2n}$.

(2) $\gamma(t_1)\in\alpha_0$, $\gamma(t_{2n})\in\alpha_1$ and for each $i=1, 2,\dots, n-1$, $\gamma(t_{2i})$ and $\gamma(t_{2i+1})$ belong to the same part of β.

(3) $\gamma(t)\in D$ if $t\in\cup_{i=1}^n(t_{2i-1}, t_{2i})$.

(4) $\gamma\cap D$ is a locally rectifiable chain in D, where $\gamma\cap D$ is the restriction of γ to D.

We define the p-module $M_p(\Gamma^*)$ to be the p-module of the family of locally rectifiable chains obtained by restricting each chain in Γ^* to D.

Now we prove

THEOREM 5 (cf. [9, Theorem 6.10]). *Suppose that the boundary components of D are partitioned into non-empty mutually disjoint closed sets α_0, $\alpha_1, \beta_1,\dots, \beta_k$. Let Γ^* be the family defined as above. Then $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})=M_p(\Gamma^*)$.*

PROOF. In this proof we write $\mathscr{D}^*$ and C_p^* for $\mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ and $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ respectively. Since $\operatorname{dist}(\alpha_0, \alpha_1)>0$, we see that $\mathscr{D}^*\neq\emptyset$, and hence $C_p^*<\infty$. Take any function u in $\mathscr{D}^*$. Then from property (1.4) we see easily that

$$\int_{\gamma\cap D}|\nabla u|ds\geqq 1 \qquad \text{for} \quad p\text{-a.e. } \gamma\in\Gamma^*.$$

It follows that $C_p^* \geqq M_p(\Gamma^*)$. Hence $M_p(\Gamma^*) < \infty$.

We note that we may restrict admissible f to belong to $L^p(D)$ and to be continuous in defining $M_p(\Gamma^*)$ (see [9, Theorem 2.8]). Let f be such a function. Given $x \in D$, denote by $\Gamma^*(x)$ the family of all chains γ in $\bar{D}$ of type given in the definition of Γ^*, condition $\gamma(t_{2n}) \in \alpha_1$ being replaced by $\gamma(t_{2n}) = x$. Set

$$g(x) = \inf_{\gamma \in \Gamma^*(x)} \int_{\gamma \cap D} f\, ds.$$

We know that $\int_\gamma f\, ds < \infty$ for p-a.e. curve γ in D. If γ is such a curve, then

$$|g(x) - g(x^0)| \leqq \int_{\widetilde{xx^0}} f\, ds$$

for any points x and x^0 on γ, where $\widetilde{xx^0} \subset \gamma$. It follows that g is absolutely continuous along p-a.e. curve in D. By Rademacher-Stepanov's theorem we have $|\nabla g(x)| \leqq f(x)$ a.e. in D. Thus g is a p-precise function in D. As in the proof of [9, Theorem 6.10] (also cf. the arguments below) we see that

$$g(\gamma) = 0 \qquad \text{for} \quad p\text{-a.e.} \quad \gamma \in \Gamma_D(\alpha_0)$$

and

$$g(\gamma) \geqq 1 \qquad \text{for} \quad p\text{-a.e.} \quad \gamma \in \Gamma_D(\alpha_1).$$

Let us show that g has the same curvilinear limit along p-a.e. curve in $\Gamma_D(\beta_j)$. For this, assume $M_p(\Gamma_D(\beta_j)) > 0$. Denote by $\Gamma'_D(\beta_j)$ the subfamily of $\Gamma_D(\beta_j)$ consisting of curves γ such that $\int_\gamma f\, ds < \infty$, γ tends to a point on β_j and g has a finite curvilinear limit $g(\gamma)$ along γ. Since $M_p(\Gamma_D(\beta_j) - \Gamma'_D(\beta_j)) = 0$, it suffices to show that $g(\gamma_1) = g(\gamma_2)$ for any curves γ_1 and γ_2 in $\Gamma'_D(\beta_j)$. For any $\varepsilon > 0$, we can take two points $x^1 \in \gamma_1$ and $x^2 \in \gamma_2$ such that

$$|g(\gamma_i) - g(x^i)| < \varepsilon \qquad (i = 1, 2)$$

and

$$\int_{\gamma'_i} f\, ds < \varepsilon \qquad (i = 1, 2),$$

where γ'_i is the part of γ_i starting at x^i and tending to β_j. Since each γ'_i tends to a point in β_j, by adding these limiting points to γ'_i, we can regard $\gamma + \gamma'_1 + (-\gamma'_2)$ as an element of $\Gamma^*(x^2)$ for each $\gamma \in \Gamma^*(x^1)$, and we have

$$g(x^2) \leqq g(x^1) + \int_{\gamma'_1} f\, ds + \int_{\gamma'_2} f\, ds < g(x^1) + 2\varepsilon.$$

Similarly $g(x^1)<g(x^2)+2\varepsilon$, and hence $|g(x^1)-g(x^2)|<2\varepsilon$. It follows that

$$|g(\gamma_1)-g(\gamma_2)| \leqq |g(\gamma_1)-g(x^1)| + |g(x^1)-g(x^2)| + |g(x^2)-g(\gamma_2)| < 4\varepsilon.$$

Therefore $g(\gamma_1)=g(\gamma_2)$. Thus we see that $\min(g, 1)$ belongs to $\mathscr{D}^*$. Hence

$$C_p^* \leqq \int_D |\nabla[\min(g, 1)]|^p dx \leqq \int_D f^p dx.$$

It follows that $C_p^* \leqq M_p(\Gamma^*)$.

REMARK. On a compact bordered Riemann surface, Minda [8] showed that the extremal distances are computed in terms of principal functions having prescribed boundary behavior (see [8, Theorem 1]). We shall show later in Theorem 12 that a principal function is extremal for $C_2^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ with respect to a regular domain D. Thus, Theorem 5 is a euclidean space version of Minda's result.

Marden and Rodin [7] gave a useful continuity lemma for extremal length on Riemann surfaces. Here we shall establish a similar continuity lemma for extremal length of order p on a domain D in R^N.

Let D be a domain in R^N and partition its boundary components into non-empty mutually disjoint sets α_0, α_1 and β such that α_0 and α_1 are closed sets in $\hat{D}$. Let the boundary components in β be divided into mutually disjoint closed sets $\{\beta_j\}_{j=1}^{\infty}$ with property (1.8). Let $\{D_n\}$ be an exhaustion of D of the type considered in (1.8) such that each ∂D_n consists of a finite number of C^∞-surfaces. Then, as in § 1, the boundary components of D_n are divided into α_{0n}, α_{1n} and $\{\beta_j^{(n)}\}_{j=1}^{j(n)}$. Let $\tilde{\beta}_n=\cup_{j=1}^{j(n)}\beta_j^{(n)}$ and $\Gamma_n^*=\Gamma^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\})$. Given $\gamma\in\Gamma_n^*$ and $m\leqq n$, as in the proof of [7, Lemma III.2.1] we obtain a "sequence" $C_0, C_1,\dots, C_k$ such that $C_0=\alpha_{0m}$, $C_k=\alpha_{1m}$ and $C_1,\dots, C_{k-1}$ are distinct parts of $\tilde{\beta}_m$ and a sequence of "stopping times" $t'_1<t'_2<\cdots<t'_{2k}$ such that $\gamma(t'_{2i-1})\in C_{i-1}$, $\gamma(t'_{2i})\in C_i$ $(i=1,\dots, k)$ and $\gamma(t)\in D_m$ if $t\in\cup_{i=1}^k(t'_{2i-1}, t'_{2i})$. We define $\gamma\|D_m$ to be the restriction of γ to $[t'_1, t'_2]\cup[t'_3, t'_4]\cup\cdots\cup[t'_{2k-1}, t'_{2k}]$, which we call the domain of $\gamma\|D_m$. $\gamma(t'_i)$, $i=1,\dots, 2k$, are called stopping points for $\gamma\|D_m$.

Let $\hat{\Gamma}$ be the family of all locally rectifiable chains γ in D such that:

(1) γ is a continuous mapping of an open dense subset J_γ of (0, 1) into D.

(2) If $t_0\notin J_\gamma$ and $0<t_0<1$, then there exists a part β_j of β such that $\lim_{t\to t_0}\gamma(t)$ belongs to β_j.

(3) $\lim_{t\to 0}\gamma(t)$ (resp. $\lim_{t\to 1}\gamma(t)$) belongs to α_0 (resp. α_1).

Next, we shall define a family Γ^* following Marden and Rodin [7]. A locally rectifiable chain γ in D belongs to Γ^* if either γ is some chain in $\hat{\Gamma}$ or if γ is a continuous mapping of an open dense subset J_γ of (0, 1) into D such that:

(1) If $t_0\notin J_\gamma$ and $0<t_0<1$, then there exist sequences $\{r_n\}$, $\{s_n\}$ in J_γ and a

part β_j of β such that $r_n \uparrow t_0$, $s_n \downarrow t_0$ and $\gamma(r_n)\to\beta_j$, $\gamma(s_n)\to\beta_j$. If $t_0=0$ (resp. 1), we require only a sequence $\{s_n\}$ (resp. $\{r_n\}$) from J_γ with $s_n \downarrow 0$ (resp. $r_n \uparrow 1$) and $\gamma(s_n)\to\alpha_0$ (resp. $\gamma(r_n)\to\alpha_1$).

(2) There is an exhaustion $\{D_n\}$ of D such that the restriction of γ to $\gamma^{-1}[\gamma(J_\gamma)\cap\bar{D}_n]$, which we denote by $\gamma|\bar{D}_n$, is a chain in $\bar{D}_n$ and $\gamma\|\bar{D}_n\equiv(\gamma|\bar{D}_n)\|D_n\in\Gamma_n^*$ for each $n\geqq 1$.

(3) If $t\in J_\gamma$, then there is n_0 such that t belongs to the domain of $\gamma\|D_n$ for all $n\geqq n_0$.

Lemma 3 (cf. the proof of [7, Lemma III.2.1]). *Let f be a non-negative continuous function on D, and $\{D_n\}$ be an exhaustion of D. If $\gamma_n\in\Gamma_n^*=\Gamma^*(\alpha_{0n}, \alpha_{1n}; D_n, \{\beta_j^{(n)}\})$ for each n, then given $\varepsilon>0$, there exists $\gamma(\varepsilon)$ in Γ^* satisfying*

$$\int_{\gamma(\varepsilon)} f\,ds \leqq \liminf_{n\to\infty}\int_{\gamma_n} f\,ds+\varepsilon.$$

Proof. We may assume that $\lim_{n\to\infty}\int_{\gamma_n} f\,ds$ exists and is finite. As in the proof of [7, Lemma III.2.1] we can find a subsequence of $\{\gamma_n\}$, which we again denote by $\{\gamma_n\}$, such that for each m, all $\gamma_n\|D_m$ $(n\geqq m)$ have the same sequence of boundary components on ∂D_m and $\lim_{n\to\infty}x_{n,m}^i=x_m^i\in\partial D_m$ for all i and m, where $x_{n,m}^i$ $(i=1,\dots,k(m))$ are the stopping points for $\gamma_n\|D_m$. Let $S(x,r)$ denote the closed N-ball of radius r and centered at x. Since ∂D_m is smooth and f is continuous, we can take $r_{i,m}>0$ $(i=1,\dots,k(m))$ with the following properties:

(1) For each i, $S(x_m^i, r_{i,m})\subset D_{m+1}$ and $S(x_m^i, r_{i,m})\cap(\partial D_m-C_m^i)=\emptyset$, where C_m^i is the boundary component of D_m such that $x_m^i\in C_m^i$.

(2) Any $y\in\partial S(x_m^i, r_{i,m})\cap D_m$ (resp. $\partial S(x_m^i, r_{i,m})-\bar{D}_m$, $S(x_m^i, r_{i,m})\cap\partial D_m$) and x_m^i can be joined by a curve in $S(x_m^i, r_{i,m})\cap D_m$ (resp. $S(x_m^i, r_{i,m})-\bar{D}_m$, $S(x_m^i, r_{i,m})\cap\partial D_m$) along which $\int f\,ds<\varepsilon/2^{m+2}k(m)$.

By taking a subsequence again we may assume that $|x_{n,m}^i-x_m^i|<r_{i,m}$ for all i, n and m with $n\geqq m$. Denote by $\gamma_{n,m}^i$ the subarc of $\gamma_n\|D_m$ connecting $x_{n,m}^i$ and a point $y_{n,m}^i\in\gamma_n\|D_m\cap\partial S(x_m^i, r_{i,m})$ in $S(x_m^i, r_{i,m})\cap D_m$ and by $\tilde{\gamma}_{n,m}^i$ the subarc of $\gamma_n-\gamma_n\|D_m$ connecting $x_{n,m}^i$ and a point $\tilde{y}_{n,m}^i\in(\gamma_n-\gamma_n\|D_m)\cap\partial S(x_m^i, r_{i,m})$ in $S(x_m^i, r_{i,m})$ for each $m\leqq n$ and $i=1,\dots,k(m)$. For each n, we modify γ_n as follows: for each $m<n$ and $i=1,\dots,k(m)$, replace a subarc $\gamma_{n,m}^i+\tilde{\gamma}_{n,m}^i$ of γ_n by a curve in $S(x_m^i, r_{i,m})$ which passes through x_m^i and connects $y_{n,m}^i$ and $\tilde{y}_{n,m}^i$ and along which $\int f\,ds<\varepsilon/2^{m+1}k(m)$, for each $i=1,\dots,k(n)$, replace $\gamma_{n,n}^i$ by a curve in $S(x_n^i, r_{i,n})\cap D_n$ which connects x_n^i and $y_{n,n}^i$ and along which $\int f\,ds<\varepsilon/2^{n+2}k(n)$. The modified curve will be denoted by γ_n^*. We have

$$\int_{\gamma_n^*} f\,ds \leqq \int_{\gamma_n} f\,ds+\frac{\varepsilon}{2}.$$

Let $\tilde{\Gamma}_m = \{\gamma_n^* \| D_m - \gamma_n^* \| D_{m-1};\ n \geqq m\}$ $(m=1, 2,\dots)$, where $D_0 = \emptyset$, and choose $\tilde{\gamma}_m \in \tilde{\Gamma}_m$ such that

$$\int_{\tilde{\gamma}_m} f ds < \inf_{\gamma \in \tilde{\Gamma}_m} \int_\gamma f ds + \frac{\varepsilon}{2^{m+1}}.$$

Then, we see

$$\int_{\tilde{\gamma}_1+\tilde{\gamma}_2+\cdots+\tilde{\gamma}_n} f ds < \sum_{m=1}^{n} \inf_{\gamma \in \tilde{\Gamma}_m} \int_\gamma f ds + \sum_{m=1}^{n} \frac{\varepsilon}{2^{m+1}} < \int_{\gamma_n^*} f ds + \frac{\varepsilon}{2} < \int_{\gamma_n} f ds + \varepsilon.$$

The chain $\gamma(\varepsilon) = \sum_{n=1}^{\infty} \tilde{\gamma}_n$ can be regarded as an element of Γ^* by a suitable parametrization (cf. the proof of [7, Lemma III.2.1]). From the above inequalities we have

$$\int_{\gamma(\varepsilon)} f ds = \lim_{n\to\infty} \int_{\tilde{\gamma}_1+\cdots+\tilde{\gamma}_n} f ds \leqq \lim_{n\to\infty} \int_{\gamma_n} f ds + \varepsilon.$$

Thus $\gamma(\varepsilon)$ satisfies all the requirements.

LEMMA 4 (cf. [7, Lemma III.2.1] and [9, Theorem 2.6]).

$$\lim_{n\to\infty} M_p(\Gamma_n^*) = M_p(\hat{\Gamma}).$$

PROOF. In general, $M_p(\hat{\Gamma}) \leqq M_p(\Gamma_n^*)$. So assume $M_p(\hat{\Gamma}) < \infty$. As in the proof of [7, Lemma III.2.1] we have $M_p(\hat{\Gamma}) = M_p(\Gamma^*)$. We may restrict admissible f to be continuous in D in defining $M_p(\Gamma^*)$ (cf. [9, Theorem 2.8]). Given ε, $0<\varepsilon<1$, choose a continuous function f in D which is admissible in association with Γ^* such that $\int_D f^p dx < M_p(\Gamma^*) + \varepsilon$. We infer that there is n_0 such that if $n \geqq n_0$ then $\int_\gamma f ds \geqq 1-\varepsilon$ for every γ in Γ_n^*. In fact, otherwise there would be $n_1 < n_2 < \cdots$ and $\gamma_{n_j} \in \Gamma_{n_j}^*$, $j=1, 2,\dots$, such that $\int_{\gamma_{n_j}} f ds < 1-\varepsilon$ for each j. We apply Lemma 3 and find $\gamma(\varepsilon)$ in Γ^* which satisfies $\int_{\gamma(\varepsilon)} f ds \leqq 1-\varepsilon$. This is a contradiction. Thus $f/(1-\varepsilon)$ is admissible in association with Γ_n^*, and hence

$$M_p(\Gamma_n^*) \leqq \frac{1}{(1-\varepsilon)^p} \int_D f^p dx < \frac{1}{(1-\varepsilon)^p} (M_p(\Gamma^*) + \varepsilon)$$

for $n \geqq n_0$. It follows that $\lim_{n\to\infty} M_p(\Gamma_n^*) = M_p(\Gamma^*)$. Hence we have $\lim_{n\to\infty} M_p(\Gamma_n^*) = M_p(\hat{\Gamma})$.

On account of Theorem 5 and Lemma 4, we have

THEOREM 6. *Suppose that the boundary components of D are partitioned* *into mutually disjoint sets* α_0, α_1 *and* β *such that* α_0 *and* α_1 *are closed sets in* $\hat{D}$. *Let the boundary components in* β *be divided into mutually disjoint closed sets* $\{\beta_j\}$ *with property* (1.8). *Then* $C_p^{**}(\alpha_0, \alpha_1; D, \{\beta_j\}) = M_p(\hat{\Gamma})$.

REMARK. As to C_p, the following result is well-known (see, e.g., [9, Theorem 6.10] or [13, Theorem 3.8]): Let D be a domain and α_0, α_1 be non-empty compact subsets of ∂D such that $\alpha_0 \cap \alpha_1 = \emptyset$. Let Γ be the family of all curves connecting α_0 and α_1 in D. Then $C_p(\alpha_0, \alpha_1; D) = M_p(\Gamma)$.

Finally we are concerned with the case that β is given the canonical partition throughout the rest of this paper. Let $\hat{\Gamma}^*$ be the family of all arcs in $\hat{D}$ connecting α_0 and α_1. $M_p(\hat{\Gamma}^*)$ is the p-module of the family of locally rectifiable chains in D obtained by restricting each arc in $\hat{\Gamma}^*$ to D. Since each γ in $\hat{\Gamma}$ can be extended continuously to $[0, 1]$ with values in $\hat{D}$, $M_p(\hat{\Gamma}^*) = M_p(\hat{\Gamma})$. Thus we have

THEOREM 7. *Let* $\hat{\Gamma}^*$ *be the family of all arcs in* $\hat{D}$ *connecting* α_0 *and* α_1. *Then* $C_p^{**}(\alpha_0, \alpha_1; D, \beta_Q) = M_p(\hat{\Gamma}^*)$.

§4. KD^p-null sets

Let E be a compact set in R^N and G be a bounded open set which contains E. We denote by $C_1^\infty(G; E)$ the family of all functions ϕ in $C_0^\infty(G)$ such that $\nabla\phi$ vanishes in some neighborhood of E. Let $KD^p(G)$ (resp. $KD^p(G-E; E)$) be the class of p-precise functions u in G (resp. $G-E$) satisfying the condition that

$$\int_G |\nabla u|^{p-2}(\nabla u, \nabla\phi)\, dx = 0$$

for every ϕ in $C_0^\infty(G)$ (resp. $C_1^\infty(G; E)$). We say that a compact set E is a KD^p-null set with respect to G if every function u in $KD^p(G-E; E)$ can be extended to a function belonging to $KD^p(G)$. The class of KD^p-null sets with respect to G is denoted by $N_{KD^p}^G$. The following lemma is an easy consequence of the definition.

LEMMA 5. *If* $E \in N_{KD^p}^G$, *then* $E \in N_{KD^p}^{G_1}$ *for any bounded open set* G_1 *containing* G.

Next we prove

LEMMA 6. *If* $E \in N_{KD^p}^G$, *then* $R^N - E$ *is a domain.*

PROOF. Suppose $R^N - E$ is not a domain, and denote by Ω the union of all bounded components of R^N-E. Take a ring domain $G_1=\{x; r_1<|x-x^0|<r_2\}$ such that $G_1 \supset G \cup \Omega$. Let $\alpha_0=\{x; |x-x^0|=r_1\}$ and $\alpha_1=\{x; |x-x^0|=r_2\}$. Let u_0 be an extremal function for $C_p(\alpha_0, \alpha_1; G_1-E \cup \Omega)$. Setting $\tilde{u}=u_0$ on $G_1-E \cup \Omega$ and $\tilde{u}=0$ on Ω, we easily see that $\tilde{u} \in KD^p(G_1-E; E)$ by Theorem 4. By Lemma 5, $E \in N_{KD^p}^{G_1}$, so that there exists a p-precise function u_1 in $KD^p(G_1)$ such that $u_1=\tilde{u}$ in G_1-E. Obviously u_1 belongs to $\mathscr{D}(\alpha_0, \alpha_1; G_1)$. Since $u_1 \in KD^p(G_1)$, by using Lemma 2 and Hölder's inequality we see that

$$\int_{G_1} |\nabla u_1|^{p-2}(\nabla u_1, \nabla v)dx = 0$$

for every p-precise function v in G_1 such that $v(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_{G_1}$. From Theorem 4 it follows that u_1 is extremal for $C_p(\alpha_0, \alpha_1; G_1)$. It is known that an extremal function for $C_p(\alpha_0, \alpha_1; G_1)$ is given by

$$(4.1) \qquad g(x) = \begin{cases} \left(|x-x^0|^{\frac{p-N}{p-1}} - r_1^{\frac{p-N}{p-1}}\right) \Big/ \left(r_2^{\frac{p-N}{p-1}} - r_1^{\frac{p-N}{p-1}}\right) & \text{if } p \neq N \\ \left(\log \dfrac{|x-x^0|}{r_1}\right) \Big/ \log \dfrac{r_2}{r_1} & \text{if } p = N. \end{cases}$$

By Theorem 4, $g=u_1$ except for x in a set of measure zero in G_1. This is a contradiction since $u_1=0$ on Ω. Thus we see that R^N-E is a domain.

A bounded domain D is called a ring domain if it has two boundary components. We shall show a necessary condition for $E \in N_{KD^p}^G$.

THEOREM 8. *If* $E \in N_{KD^p}^G$, *then* $C_p(\alpha_0, \alpha_1; D-E)=C_p^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ *for every ring domain D containing G, where* α_0 *and* α_1 *are two boundary components of D and* $\beta=\partial E$.

PROOF. By Lemma 6 we note that $C_p(\alpha_0, \alpha_1; D-E)$ and $C_p^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ are well-defined. Let u_0 and u^* be extremal functions for $C_p(\alpha_0, \alpha_1; D-E)$ and $C_p^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ respectively. By Lemma 5, $E \in N_{KD^p}^D$. Hence there exist two functions $\tilde{u}_0$ and $\tilde{u}^*$ in $KD^p(D)$ such that $\tilde{u}_0=u_0$ in $D-E$ and $\tilde{u}^*=u^*$ in $D-E$. These imply that $\tilde{u}_0, \tilde{u}^* \in \mathscr{D}(\alpha_0, \alpha_1; D)$,

$$\int_D |\nabla \tilde{u}_0|^{p-2}(\nabla \tilde{u}_0, \nabla \phi)dx = 0$$

for every ϕ in $C_0^\infty(D)$ and

$$\int_D |\nabla \tilde{u}^*|^{p-2}(\nabla \tilde{u}^*, \nabla \phi)dx = 0$$

for every ϕ in $C_0^\infty(D)$. As in the proof of Lemma 6, we conclude that $\tilde{u}_0$ and $\tilde{u}^*$ are extremal for $C_p(\alpha_0, \alpha_1; D)$. By Theorem 4, $\tilde{u}_0=\tilde{u}^*$ a.e. in D. Hence $C_p(\alpha_0,$

$\alpha_1; D-E)=C_p^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$.

COROLLARY 1. *If $E \in N_{KD^p}^G$, then the N-dimensional Lebesgue measure of E is equal to zero.*

PROOF. Take a ring domain $D=\{x; r_1<|x-x^0|<r_2\}$ such that $D\supset G$. Let $\alpha_0=\{x; |x-x^0|=r_1\}$, $\alpha_1=\{x, |x-x^0|=r_2\}$ and $\beta=\partial E$. In general $C_p(\alpha_0, \alpha_1; D-E)\leqq C_p(\alpha_0, \alpha_1; D)\leqq C_p^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$. By Theorem 8, we see $C_p(\alpha_0, \alpha_1; D-E)=C_p(\alpha_0, \alpha_1; D)$. Let u_1 be the function defined by the right hand side of (4.1). Then u_1 is an extremal function for $C_p(\alpha_0, \alpha_1; D)$ and its restriction to $D-E$ belongs to $\mathscr{D}(\alpha_0, \alpha_1; D-E)$. Hence

$$\int_D |\nabla u_1|^p dx = C_p(\alpha_0, \alpha_1; D) = C_p(\alpha_0, \alpha_1; D - E) \leqq \int_{D-E} |\nabla u_1|^p dx,$$

which implies

$$\int_E |\nabla u_1|^p dx = 0.$$

Since $|\nabla u_1|\neq 0$ on D, we conclude that the N-dimensional Lebesgue measure of E is equal to zero.

§5. Relations between KD^p-null sets and FD^p-null sets

In [6], Hedberg considered the following notion of null sets. For an open set G in R^N, denote by $FD^p(G)$ the class of real valued harmonic functions u in G such that $|\nabla u|$ belongs to $L^p(G)$ and u has no flux, i.e., $\int_C \partial u/\partial v dS=0$ for all $(N-1)$-cycles C in G. A compact set E is said to be removable for FD^p if for some open set G containing E every function in $FD^p(G-E)$ can be extended to a function in $FD^p(G)$. The class of removable sets for FD^p is denoted by N_{FD^p}. Denote by $W_1^p(G)$ the Sobolev space of real valued functions f in $L^p(G)$ whose derivatives in the distribution sense are functions in $L^p(G)$. When G is bounded $\|\nabla f\|_p$ is a norm on $C_0^\infty(G)$ by the Poincaré inequality, and the closure in $W_1^p(G)$ of $C_0^\infty(G)$ with respect to this norm is denoted by $\mathring{W}_1^p(G)$. Hedberg proved

THEOREM A ([6, Theorem 1, b]). *$E\in N_{FD^p}$ if and only if $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^q(G)$ for some bounded open set $G\supset E$, where $q=p/(p-1)$.*

Let D be an N-dimensional open rectangle with sides parallel to the coordinate planes, E be a compact set in D (possibly an empty set) and G_1 be a bounded open set containing $\bar{D}$. We set

$$M_p^i(D - E) = \inf_\psi \int_{D-E} |\nabla\psi|^p dx \qquad (i = 1,\dots, N),$$

where the infimum is taken over all $\psi \in C_1^\infty(G_1; E)$ such that $\psi(x)=0$ on α_0^i which is one of the sides of D parallel to the coordinate plane $x_i=0$, and $\psi(x)=1$ on α_1^i which is the opposite side of α_0^i. Obviously $M_p^i(D-E)$ does not depend on the choice of G_1.

THEOREM B ([6, Theorem 4]). *$E \in N_{FD^q}$ if and only if the equalities $M_p^i(D-E)=M_p^i(D)$, $i=1,\dots, N$, hold for some open rectangle $D \supset E$.*

By using these theorems we shall give some results on KD^p-null sets.

LEMMA 7. *If $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^p(G)$ for a bounded open set G, then the N-dimensional Lebesgue measure of E is zero and R^N-E is a domain.*

PROOF. Choose a function $\phi \in C_0^\infty(G)$ such that $\phi(x)=x_1$ on a neighborhood of E for $x=(x_1,\dots, x_N)$. By the assumption of the lemma there is a sequence $\{\phi_n\}$ in $C_1^\infty(G; E)$ such that

$$\lim_{n\to\infty}\int_G |\nabla(\phi - \phi_n)|^p dx = 0.$$

Then

$$\int_E dx = \int_E |\nabla(\phi - \phi_n)|^p dx \leqq \int_G |\nabla(\phi - \phi_n)|^p dx.$$

Hence $\int_E dx=0$.

Next, suppose R^N-E is not a domain. Then there is a non-empty bounded domain $\Omega \subset R^N-E$ such that $\partial\Omega \subset E$. Take a bounded open ball G_1 containing G and a function $\psi \in C_0^\infty(G_1)$ such that $\psi=1$ on Ω. Let $\phi(x)=x_1\psi(x)$ for $x=(x_1,\dots, x_N)$. By the assumption of the lemma, we easily see that $C_1^\infty(G_1; E)$ is dense in $\mathring{W}_1^p(G_1)$. Since $\phi(x) \in C_0^\infty(G_1)$, there is a sequence $\{\phi_n\}$ in $C_1^\infty(G_1; E)$ such that

$$\lim_{n\to\infty}\int_{G_1} |\nabla(\phi - \phi_n)|^p dx = 0.$$

We take a subdomain Ω' of Ω such that $\partial\Omega'$ consists of a finite number of C^1-surfaces β_j $(j=1,\dots, m)$ and $\phi_n=$const. on each β_j. By using Stokes' theorem, we have

$$\int_\Omega \frac{\partial\phi_n}{\partial x_1} dx = \int_{\Omega'} \frac{\partial\phi_n}{\partial x_1} dx = 0.$$

It follows that

$$\int_\Omega dx = \int_\Omega \frac{\partial(\phi - \phi_n)}{\partial x_1} dx.$$

By Hölder's inequality, we have

$$\int_\Omega dx \leqq C\left\{\int_\Omega |\nabla(\phi - \phi_n)|^p dx\right\}^{1/p}.$$

Since the right-hand side tends to zero as $n\to\infty$, we obtain a contradiction. Therefore R^N-E is a domain.

THEOREM 9. *Let $q=p/(p-1)$. If $C_1^\infty(G; E)$ is dense both in $\mathring{W}_1^p(G)$ and in $\mathring{W}_1^q(G)$ for a bounded open set G, then E belongs to $N_{KD^p}^G$.*

PROOF. By Lemma 7, R^N-E is a domain and the N-dimensional Lebesgue measure of E is equal to zero. Moreover, since $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^q(G)$, as in the first half of the proof of [6, Theorem 1], we see that for any u in $KD^p(G-E; E)$ there is a function in $W_1^p(G)$ which is equal to u in $G-E$. Hence, by [9, Theorem 4.21], there is a p-precise function u_0 in G such that $u_0=u$ and $\partial u_0/\partial x_i=\partial u/\partial x_i$ $(i=1,\dots, N)$ except on a set of measure zero in $G-E$. Next, since $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^p(G)$, for any ψ in $C_0^\infty(G)$ there is a sequence $\{\phi_n\}$ in $C_1^\infty(G; E)$ such that

$$\lim_{n\to\infty}\int_G |\nabla(\psi - \phi_n)|^p dx = 0.$$

Then, by Hölder's inequality we have

$$\begin{aligned}
&\int_G |\nabla u_0|^{p-2}(\nabla u_0, \nabla\psi)dx \\
&= \lim_{n\to\infty}\int_G |\nabla u_0|^{p-2}(\nabla u_0, \nabla\phi_n)dx \\
&= \lim_{n\to\infty}\int_{G-E} |\nabla u|^{p-2}(\nabla u, \nabla\phi_n)dx \\
&= 0.
\end{aligned}$$

Hence $u_0\in KD^p(G)$, so that $E\in N_{KD^p}^G$.

THEOREM 10. *If $E\in N_{KD^p}^G$, then $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^p(G)$.*

PROOF. By Theorems A and B it is enough to show that

$$M_p^i(D - E) = M_p^i(D) \qquad (i = 1,\dots, N)$$

for some open rectangle D containing G. Take a bounded open set $G_1\supset\bar{D}$. First we observe by using Lemma 2 that

$$M_p^i(D - E) = \inf_u \int_{D-E} |\nabla u|^p dx,$$

where the infimum is taken over all p-precise functions u defined in $G_0=G_1-E-\alpha_0^i-\alpha_1^i$ such that $u(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_{G_0}(\alpha_0^i)\cup\Gamma_{G_0}(\partial G_1)$, $u(\gamma)=1$ for p-a.e. $\gamma\in\Gamma_{G_0}(\alpha_1^i)$ and $u=$const. on each component of some neighborhood of E. Moreover, in the same way as in Theorem 2, we have a p-precise function u_0 defined in $D-E$ such that $u_0(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_{D-E}(\alpha_0^i)$, $u_0(\gamma)=1$ for p-a.e. $\gamma\in\Gamma_{D-E}(\alpha_1^i)$, $M_p^i(D-E)=\int_{D-E}|\nabla u_0|^p dx$ and $\int_{D-E}|\nabla u_0|^{p-2}(\nabla u_0, \nabla\psi)dx=0$ for every ψ in $C_1^\infty(D; E)$. By Lemma 5, we see that $E\in N_{KD^p}^D$. Since $u_0\in KD^p(D-E; E)$, there exists a function $\tilde{u}_0$ in $KD^p(D)$ such that $\tilde{u}_0=u_0$ in $D-E$. On the other hand $M_p^i(D)=C_p(\alpha_0^i, \alpha_1^i; D)$. Obviously $\tilde{u}_0\in\mathscr{D}(\alpha_0^i, \alpha_1^i; D)$. Take ϕ_0 in $C_0^\infty(D)$ such that $\phi_0=1$ on a neighborhood of E. For any p-precise function v in D such that $v(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_D(\alpha_0^i)\cup\Gamma_D(\alpha_1^i)$, we have

$$\begin{aligned}&\int_D|\nabla\tilde{u}_0|^{p-2}(\nabla\tilde{u}_0, \nabla v)dx\\&=\int_D|\nabla\tilde{u}_0|^{p-2}(\nabla\tilde{u}_0, \nabla(\phi_0 v))dx+\int_D|\nabla\tilde{u}_0|^{p-2}(\nabla\tilde{u}_0, \nabla(v(1-\phi_0)))dx.\end{aligned}$$

Using Lemma 2 and the fact $\tilde{u}_0\in KD^p(D)$ we conclude that

$$\int_D|\nabla\tilde{u}_0|^{p-2}(\nabla\tilde{u}_0, \nabla v)dx=0.$$

From Theorem 4 it follows that $\tilde{u}_0$ is an extremal function for $M_p^i(D)$. By Corollary 1, we have that $M_p^i(D-E)=M_p^i(D)$ for all $i=1,\dots, N$. The proof is completed.

COROLLARY 2. *If $p\geqq 2$, then $E\in N_{KD^p}^G$ if and only if $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^p(G)$.*

COROLLARY 3. *If $p\geqq 2$, then the property $E\in N_{KD^p}^G$ does not depend on the choice of G.*

By virtue of Corollary 3, in case $p\geqq 2$ we may omit the suffix G in the notation $N_{KD^p}^G$ and have a notion of KD^p-null sets. We combine these results with Theorem A and have the following theorem.

THEOREM 11. *If $p\geqq 2$, then a compact set E is a KD^p-null set if and only if E is removable for FD^q, where $q=p/(p-1)$.*

REMARK. In case $p\geqq 2$, by Corollary 2 any compact subset of a KD^p-null set is a KD^p-null set. If $E_1,\dots, E_n$ are totally disconnected and KD^p-null sets, then so is $E_i\cap E_j$. Hence we see that $\cup_{i=1}^n E_i\in N_{KD^p}$.

§6. The case $p=2$

Here we shall give a characterization of KD^2-null sets. Let D be a bounded domain with a finite number of boundary components α_0, α_1 and β_j $(j=1,\dots,k)$. Denote by $\mathscr{D}'=\mathscr{D}'(\alpha_0, \alpha_1; D, \{\beta_j\})$ the family of all $C^\infty(D)$-functions u in D each of which is identically equal to 0 (resp. 1, a constant a_j, $j=1,\dots,k$) in the intersections with D of some neighborhoods of α_0 (resp. α_1, β_j, $j=1,\dots,k$).

LEMMA 8. $C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})=\inf_{u\in\mathscr{D}'}\int_D|\nabla u|^p dx.$

PROOF. Put $C_p'=\inf_{u\in\mathscr{D}'}\int_D|\nabla u|^p dx$ and $C_p^*=C_p^*(\alpha_0, \alpha_1; D, \{\beta_j\})$. Obviously, $C_p^*\leqq C_p'$. For any $u\in\mathscr{D}^*(\alpha_0, \alpha_1; D, \{\beta_j\})$ there is $f\in\mathscr{D}'$ such that $(u-f)(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_D$. By Lemma 2 we can take $\{f_n\}_{n=1}^\infty$ in $C_0^\infty(D)$ such that $\lim_{n\to\infty}\|\nabla(u-f-f_n)\|_p=0$. Therefore $\lim_{n\to\infty}\|\nabla(f+f_n)\|_p=\|\nabla u\|_p$. Since $f+f_n\in\mathscr{D}'$, $C_p'\leqq C_p^*$.

In the same way as Lemma 8, we have

LEMMA 9 (cf. [9, Theorems 6.13 and 6.14]).

$$C_p(\alpha_0, \alpha_1; D) = \inf_u\int_D|\nabla u|^p dx,$$

where the infimum is taken over all $C^\infty(D)$-functions u each of which is identically equal to 0 *and* 1 *in the intersections with D of some neighborhoods of α_0 and α_1 respectively.*

Let D be a regular domain, that is a domain for which ∂D consists of a finite number of compact C^1-surfaces α_0, α_1 and β_j $(j=1,\dots,k)$. We know (cf. [11]) that there exist principal functions h_i $(i=0, 1)$ with respect to α_0, α_1 and D, which are characterized by the following properties:

(1) h_i is harmonic in D and is continuous on $\bar{D}$;

(2) $h_i=0$ on α_0 and $h_i=1$ on α_1;

(3) $\partial h_0/\partial\nu=0$ on each β_j, $h_1=$const. on each β_j and $\int_{\beta_j}\partial h_1/\partial\nu dS=0$ for $j=1,\dots,k$, where $\partial/\partial\nu$ indicates the normal derivative and dS is the surface element.

In case $p=2$, by Green's formula and Lemmas 8 and 9, we have

THEOREM 12. *Let D be a regular domain with $\partial D=\alpha_0\cup\alpha_1\cup\beta_1\cup\cdots\cup\beta_k$. Then $C_2^*(\alpha_0, \alpha_1; D, \{\beta_j\})=\int_D|\nabla h_1|^2dx$ and $C_2(\alpha_0, \alpha_1; D)=\int_D|\nabla h_0|^2dx$.*

We note by Theorem 11 that the notion of KD^2-null sets coincides with the notion of KD-null sets defined in [12]. The author showed in [12, Theorem 3] a relation between N_{KD} and the span for the canonical partition of E. By this result and Theorem 12, we obtain the following theorem.

THEOREM 13. *$E \in N_{KD^2}$ if and only if $C_2(\alpha_0, \alpha_1; D-E) = C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ for every unbounded domain D such that $D \supset E$ and ∂D consists of two disjoint compact boundary components α_0, α_1, where $\beta = \partial E \cup \{\infty\}$.*

PROOF. Suppose $E \in N_{KD^2}$. Let D be an unbounded domain such that $D \supset E$ and ∂D consists of two disjoint compact boundary components α_0, α_1. Let u_0 and u^* be the extremal functions for $C_2(\alpha_0, \alpha_1; D-E)$ and $C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ respectively. Take a bounded domain G such that $G \supset E$ and $R^N - G \supset \alpha_0, \alpha_1$. Since $u_0, u^* \in KD^2(G-E; E)$, there exist 2-precise functions $\hat{u}_0, \hat{u}^*$ in $KD^2(G)$ such that $u_0 = \hat{u}_0$ in $G-E$ and $u^* = \hat{u}^*$ in $G-E$. Let

$$\tilde{u}_0 = \begin{cases} \hat{u}_0 & \text{in } G \\ u_0 & \text{in } D-G \end{cases}$$

and

$$\tilde{u}^* = \begin{cases} \hat{u}^* & \text{in } G \\ u^* & \text{in } D-G. \end{cases}$$

We take $\psi_0 \in C_0^\infty(G)$ such that $\psi_0 = 1$ on a neighborhood of E. We extend ψ_0 by 0 to $R^N - G$. Let ψ be any function in $C^\infty(D)$ such that the support of $|\nabla \psi|$ is bounded and $\psi = 0$ on $\alpha_0 \cup \alpha_1$. Then we have

$$\begin{aligned} &\int_D (\nabla \tilde{u}^*, \nabla \psi) dx \\ &= \int_D (\nabla \tilde{u}^*, \nabla(\psi(1-\psi_0))) dx + \int_D (\nabla \tilde{u}^*, \nabla(\psi\psi_0)) dx \\ &= \int_{D-E} (\nabla u^*, \nabla(\psi(1-\psi_0))) dx + \int_G (\nabla \hat{u}^*, \nabla(\psi\psi_0)) dx. \end{aligned}$$

Since $\psi\psi_0 \in C_0^\infty(G)$, the last integral vanishes. Since $\psi(1-\psi_0)$ is a function in $C^\infty(D-E)$ such that the support of $|\nabla(\psi(1-\psi_0))|$ is bounded, $\psi(1-\psi_0) = 0$ on $\alpha_0 \cup \alpha_1$ and $\psi(1-\psi_0) = 0$ on a neighborhood of E, we have $\int_{D-E} (\nabla u^*, \nabla(\psi(1-\psi_0))) dx = 0$. Hence

$$\int_D (\nabla \tilde{u}^*, \nabla \psi) dx = 0.$$

Let $\Gamma_D(\infty)$ be the family of all locally rectifiable curves in D each of which starts from a point of D and tends to the point at infinity. By [9, Theorem 9.12], $\tilde{u}^*-\tilde{u}_0$ has a finite constant limit along 2-a.e. curve in $\Gamma_D(\infty)$. By using Lemma 2 and Hölder's inequality, we have

$$\int_D (\nabla\tilde{u}^*, \nabla(\tilde{u}^* - \tilde{u}_0))dx = 0.$$

From this we see that

$$\int_D |\nabla\tilde{u}^*|^2 dx \leqq \int_D |\nabla\tilde{u}_0|^2 dx.$$

By Corollary 1 to Theorem 8,

$$\int_{D-E} |\nabla u^*|^2 dx \leqq \int_{D-E} |\nabla u_0|^2 dx.$$

Since the converse inequality is trivial, we conclude that

$$C_2(\alpha_0, \alpha_1; D - E) = C_2^{**}(\alpha_0, \alpha_1; D - E, \beta_Q).$$

Conversely we suppose that $C_2(\alpha_0, \alpha_1; D-E)=C_2^{**}(\alpha_0, \alpha_1; D-E, \beta_Q)$ for every D as in the theorem. Take distinct two points x^0, x^1 in the domain R^N-E $(=E^c)$ and balls S_r^0, S_r^1 of radius r, with centers at x^0, x^1 and with disjoint closures in E^c. Let $\{D_n\}$ be an exhaustion of E^c such that $D_1 \supset S_r^0, S_r^1$. Denote by β_j $(j=1,\dots, j(n))$ the boundary components of D_n. We know (cf. [11]) that there exist principal functions $P_{i,n}$ $(i=0, 1)$ with respect to x^0, x^1 and D_n, which are characterized by the following properties:

(1) $P_{i,n}$ is harmonic in $D_n-(\{x^0\} \cup \{x^1\})$;

(2) $$P_{i,n} = \frac{1}{\sigma|x - x^0|^{N-2}} + h_{i,n} \quad \text{on} \quad S_r^0,$$

$$P_{i,n} = \frac{-1}{\sigma|x - x^1|^{N-2}} + f_{i,n} \quad \text{on} \quad S_r^1,$$

where σ is the surface area of unit sphere in R^N, and $h_{i,n}$ and $f_{i,n}$ are harmonic in S_r^0 and S_r^1 respectively and $f_{i,n}(x^1)=0$;

(3) $\partial P_{0,n}/\partial\nu=0$ on ∂D_n, $P_{1,n}=$const. on each β_j and $\int_{\beta_j}\partial P_{1,n}/\partial\nu\, dS=0$ for $j=1,\dots, j(n)$.

We see that the limits

$$h_i = \lim_{n\to\infty} h_{i,n}, \quad f_i = \lim_{n\to\infty} f_{i,n} \qquad (i = 0, 1)$$

exist and the convergences are uniform on every compact subset of E^c. Set

$$\alpha_0 = \partial S_r^0,\ \alpha_1 = \partial S_r^1;$$

$$a_n = \max_{x\in\alpha_0} P_{0,n}(x),\ a'_n = \min_{x\in\alpha_0} P_{0,n}(x),$$

$$b'_n = \max_{x\in\alpha_1} P_{0,n}(x),\ b_n = \min_{x\in\alpha_1} P_{0,n}(x);$$

$$A_n = \{x;\ P_{0,n}(x) \geqq a_n\},\ A'_n = \{x;\ P_{0,n}(x) \geqq a'_n\},$$

$$B'_n = \{x;\ P_{0,n}(x) \leqq b'_n\},\ B_n = \{x;\ P_{0,n}(x) \leqq b_n\}$$

and

$$\alpha_{0n} = \partial A_n,\ \alpha'_{0n} = \partial A'_n,\ \alpha_{1n} = \partial B_n,\ \alpha'_{1n} = \partial B'_n.$$

For sufficiently small r, we easily see that

$$C_2(\alpha_{0n}, \alpha_{1n}; D_n - A_n - B_n) \leqq C_2(\alpha_0, \alpha_1; D_n - \overline{S_r^0} - \overline{S_r^1})$$
$$\leqq C_2(\alpha'_{0n}, \alpha'_{1n}; D_n - A'_n - B'_n).$$

By Theorem 12, $(a_n - P_{0,n})/(a_n - b_n)$ is extremal for $C_2(\alpha_{0n}, \alpha_{1n}; D_n - A_n - B_n)$. Therefore we have

$$C_2(\alpha_{0n}, \alpha_{1n}; D_n - A_n - B_n) = \frac{N-2}{a_n - b_n}.$$

From this we derive that

$$\max_{x\in\alpha_0} h_{0,n} - \min_{x\in\alpha_1} f_{0,n} = a_n - b_n - \frac{2}{\sigma r^{N-2}}$$
$$= \frac{N-2}{C_2(\alpha_{0n}, \alpha_{1n}; D_n - A_n - B_n)} - \frac{2}{\sigma r^{N-2}}.$$

Similarly,

$$\min_{x\in\alpha_0} h_{0,n} - \max_{x\in\alpha_1} f_{0,n} = \frac{N-2}{C_2(\alpha'_{0n}, \alpha'_{1n}; D_n - A'_n - B'_n)} - \frac{2}{\sigma r^{N-2}}.$$

From the above inequalities we see

$$\max_{x\in\alpha_0} h_{0,n} - \min_{x\in\alpha_1} f_{0,n} \geqq \frac{N-2}{C_2(\alpha_0, \alpha_1; D_n - \overline{S_r^0} - \overline{S_r^1})} - \frac{2}{\sigma r^{N-2}}$$
$$\geqq \min_{x\in\alpha_0} h_{0,n} - \max_{x\in\alpha_1} f_{0,n}.$$

Letting $n\to\infty$, we have

$$\max_{x\in\alpha_0} h_0 - \min_{x\in\alpha_1} f_0 \geqq \frac{N-2}{C_2(\alpha_0, \alpha_1; E^c - \overline{S_r^0} - \overline{S_r^1})} - \frac{2}{\sigma r^{N-2}} \geqq \min_{x\in\alpha_0} h_0 - \max_{x\in\alpha_1} f_0.$$

In the same way we have

$$\max_{x\in\alpha_0} h_1 - \min_{x\in\alpha_1} f_1 \geqq \frac{N-2}{C_2^{**}(\alpha_0, \alpha_1; E^c - \overline{S_r^0} - \overline{S_r^1}, \beta_Q)} - \frac{2}{\sigma r^{N-2}} \geqq \min_{x\in\alpha_0} h_1 - \max_{x\in\alpha_1} f_1.$$

By assumption the equality

$$C_2(\alpha_0, \alpha_1; E^c - \overline{S_r^0} - \overline{S_r^1}) = C_2^{**}(\alpha_0, \alpha_1; E^c - \overline{S_r^0} - \overline{S_r^1}, \beta_Q)$$

holds for every small $r>0$. Hence

$$\max_{x\in\alpha_0} h_0 - \min_{x\in\alpha_1} f_0 \geqq \min_{x\in\alpha_0} h_1 - \max_{x\in\alpha_1} f_1$$

and

$$\max_{x\in\alpha_0} h_1 - \min_{x\in\alpha_1} f_1 \geqq \min_{x\in\alpha_0} h_0 - \max_{x\in\alpha_1} f_0.$$

Since $f_i(x^1)=0$ $(i=0, 1)$, letting $r\to 0$ we have that $h_0(x^0)=h_1(x^0)$. This means that the span is equal to zero for all couples (x^0, x^1) of distinct points in E^c, so that by [12, Theorem 3], we conclude that $E\in N_{KD^2}$. The proof is completed.

REMARK. This theorem is a euclidean space version of Rodin's result on Riemann surfaces in [10].

References

[1] L. Ahlfors and A. Beurling: Conformal invariants and function-theoretic null-sets, Acta Math. **83** (1950), 101–129.

[2] J. A. Clarkson: Uniformly convex spaces, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[3] B. Fuglede: Extremal length and functional completion, Acta Math. **98** (1957), 171–219.

[4] F. W. Gehring: Extremal length definitions for the conformal capacity of rings in space, Michigan Math. J. **9** (1962), 137–150.

[5] F. W. Gehring: Rings and quasiconformal mappings in space, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[6] L. I. Hedberg: Removable singularities and condenser capacities, Ark. Mat. **12** (1974), 181–201.

[7] A. Marden and B. Rodin: Extremal and conjugate extremal distance on open Riemann surfaces with applications to circular-radial slit mappings, Acta Math. **115** (1966), 237–269.

[8] C. D. Minda: Extremal length and harmonic functions on Riemann surfaces, Trans. Amer. Math. Soc. **171** (1972), 1–22.

[9] M. Ohtsuka: Extremal length and precise functions in 3-space, Lecture Notes, Hiroshima Univ., 1973.

[10] B. Rodin: Extremal length and removable boundaries of Riemann surfaces, Bull. Amer. Math. Soc. **72** (1966), 274–276.

[11] B. Rodin and L. Sario: Principal functions, D. Van Nostrand, Princeton, 1968.
[12] H. Yamamoto: On KD-null sets in N-dimensional cuclidean space, J. Sci. Hiroshima Univ. Ser. A-1 Math. **34** (1970), 59–68.
[13] W. P. Ziemer: Extremal length and p-capacity, Michigan Math. J. **16** (1969), 43–51.

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