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Lie Algebras in which Every Ascendant Subalgebra is a Subideal

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Introduction

The class $\mathfrak{D}$ of Lie algebras in which every subalgebra is a subideal and the class $\mathfrak{T}$ of Lie algebras in which every subideal is an ideal were investigated by Stewart and Amayo ([1], [2], [3]). In connection with these, it seems interesting to know the properties of Lie algebras L satisfying each of the following conditions:

(M) Every ascendant subalgebra of L is a subideal.

(M′) Every ascendant subalgebra of L is an ideal.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ denote the classes consisting of all Lie algebras which satisfy the conditions (M) and (M′) respectively. Then it is immediate that $\mathfrak{N}\leq\mathfrak{D}\leq\mathfrak{M}$ and $\mathfrak{A}\leq\mathfrak{M}'\leq\mathfrak{T}$. In this paper we shall investigate the calsses $\mathfrak{M}$, $\mathfrak{M}'$ and present several properties of Lie algebras belonging to these classes.

We shall show that Max$\leq$Max-asc$\leq\mathfrak{M}$ (Theorem 2.1). For a Lie algebra L over a field of characteristic 0 satisfying Min-asc, we shall obtain certain conditions which are equivalent to the condition (M) (Theorem 3.4). This will be applied to showing that $\mathrm{NL}\mathfrak{F}\cap$ Min-asc$\leq\mathfrak{M}$ (Theorem 4.4). We shall finally show that every solvable $\mathfrak{M}'$-algebra is either abelian or the split extension of an abelian Lie algebra by the 1-dimensional algebra of scalar multiplications and conversely (Theorem 5.2).

1.

In this preliminary section, we fix the notations and terminology, and recall a few fundamental results on locally nilpotent radicals.

Let L be a Lie algebra over a field Φ. When H is a subalgebra (resp. an ideal) of L, we write $H\leq L$ (resp. $H\lhd L$). For an ordinal σ, $H\leq L$ is a σ-step ascendant subalgebra of L if there is a series $(H_\alpha)_{\alpha\leq\sigma}$ of subalgebras of L such that

(1) $H_0=H$, $H_\sigma=L$,

(2) $H_\alpha\lhd H_{\alpha+1}$ for any ordinal $\alpha<\sigma$,

(3) $H_\lambda=\bigcup_{\alpha<\lambda}H_\alpha$ for any limit ordinal $\lambda\leq\sigma$.

We then write $H \lhd^{\sigma} L$. When σ is finite, H is a σ-step subideal of L. H is a subideal (resp. an ascendant subalgebra) of L when $H \lhd^{n} L$ (resp. $H \lhd^{\sigma} L$) for some $n \in \mathbf{N}$ (resp. σ). We then write H si L (resp. H asc L).

The Fitting radical $\nu(L)$ of L is the sum of all nilpotent ideals of L. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. Evidently $\nu(L) \leq \rho(L)$. If the basic field is of characteristic 0, the Baer radical $\beta(L)$ of L is the subalgebra generated by all nilpotent subideals of L and the Gruenberg radical $\gamma(L)$ of L is the subalgebra generated by all nilpotent ascendant subalgebras of L. Obviously $\nu(L) \leq \beta(L) \leq \gamma(L)$. $\beta(L)$ is a characteristic ideal of L, but $\gamma(L)$ is not an ideal of L generally.

The class $\mathfrak{A}$ consists of all abelian Lie algebras, the class $\mathfrak{F}$ consists of all finite-dimensional Lie algebras, and the class $\mathfrak{N}$ (resp. $\mathrm{E}\mathfrak{A}$) consists of all nilpotent (resp. solvable) Lie algebras. The class $\mathfrak{Z}$ consists of all hypercentral Lie algebras, and the class $\mathfrak{T}$ consists of all Lie algebras in which the relation $\lhd$ is transitive. $\mathrm{L}\mathfrak{F}$ is the class of all locally finite Lie algebras. For a class $\mathfrak{X}$, $\mathrm{N}\mathfrak{X}$ (resp. $\acute{\mathrm{N}}\mathfrak{X}$) consists of all Lie algebras generated by their $\mathfrak{X}$-subideals (resp. ascendant $\mathfrak{X}$-subalgebras).

Max-asc (resp. Max, Max-$\lhd^{\sigma}$) is the maximal condition for ascendant subalgebras (resp. subalgebras, σ-step ascendant subalgebras). Min-asc, Min and Min-$\lhd^{\sigma}$ are similarly defined. Furthermore the same notations are used for the classes of Lie algebras satisfying the corresponding chain conditions.

2.

For a Lie algebra L over a field Φ and for an ordinal $\sigma \geq \omega$, we introduce the following conditions:

(M) Every ascendant subalgebra of L is a subideal.

(M_{σ}) Every σ-step ascendant subalgebra of L is a subideal.

We denote by $\mathfrak{M}$ and $\mathfrak{M}_{\sigma}$ the classes of Lie algebras satisfying the conditions (M) and (M_{σ}) respectively. Then for any ordinals $\rho \geq \sigma \geq \omega$, we have

$$\mathfrak{D} \leq \mathfrak{M} \leq \mathfrak{M}_{\rho} \leq \mathfrak{M}_{\sigma} \leq \mathfrak{M}_{\omega}.$$

First we show the following

THEOREM 2.1. (1) *Max-$\lhd^{\sigma} \leq \mathfrak{M}_{\sigma}$ for any $\sigma \geq \omega$.*
(2) *Max $\leq$ Max-asc $\leq \mathfrak{M}$.*

PROOF. (1) Assume that $L \notin \mathfrak{M}_{\sigma}$. Then there is a subalgebra H of L such that $H \lhd^{\sigma} L$ but H is not a subideal of L. Let $(H_{\alpha})_{\alpha \leq \sigma}$ be an ascending series for H in L. Since $\sigma \geq \omega$, we may assume that

$$H = H_0 \mathrel{\underset{\neq}{\lhd}} H_1 \mathrel{\underset{\neq}{\lhd}} \cdots \mathrel{\underset{\neq}{\lhd}} H_n \mathrel{\underset{\neq}{\lhd}} H_{n+1} \mathrel{\underset{\neq}{\lhd}} \cdots.$$

It is obvious that $H_n \lhd^\sigma L$ for each n. Therefore $L \notin \text{Max-}\lhd^\sigma$.

(2) Assume that $L \in \text{Max-asc}$. Suppose that H asc L. Then $H \lhd^\sigma L$ for some ordinal σ. Since $\text{Max-asc} \leq \text{Max-}\lhd^\sigma$, we have $L \in \text{Max-}\lhd^\sigma$. If $\sigma \geq \omega$, $L \in \mathfrak{M}_\sigma$ by the statement (1). It follows that H si L. Therefore $L \in \mathfrak{M}$.

LEMMA 2.2. *Let L be a hypercentral Lie algebra over Φ of central height $\leq \sigma$. If $H \leq L$, then $H \lhd^\sigma L$.*

PROOF. Let $(\zeta_\alpha(L))_{\alpha \leq \sigma}$ be the transfinite upper central series of L such that $\zeta_\sigma(L) = L$. If $H \leq L$, put $H_\alpha = H + \zeta_\alpha(L)$ for any $\alpha \leq \sigma$. Then the series $(H_\alpha)_{\alpha \leq \sigma}$ is an ascending series for H in L. Hence $H \lhd^\sigma L$.

As a consequence of the lemma we have the following

PROPOSITION 2.3. $\mathfrak{Z} \cap \mathfrak{M} = \mathfrak{Z} \cap \mathfrak{D}$.

PROOF. Let $L \in \mathfrak{Z} \cap \mathfrak{M}$. Assume that $H \leq L$. Since $L \in \mathfrak{Z}$, by Lemma 2.2 H asc L. Since $L \in \mathfrak{M}$, it follows that H si L. Hence $L \in \mathfrak{D}$. Thus $\mathfrak{Z} \cap \mathfrak{M} \leq \mathfrak{D}$ and therefore $\mathfrak{Z} \cap \mathfrak{M} = \mathfrak{Z} \cap \mathfrak{D}$.

3.

In order to investigate the condition (M), we further consider the following conditions for a Lie algebra L over a field of characteristic 0:

(A) For any I, H such that $I \lhd H$ asc L, $\beta(H/I) = \gamma(H/I)$.

(B) For any I, H such that $I \lhd H$ asc L, $\gamma(H/I) \lhd H/I$.

(C) For any I, H such that $I \lhd H$ asc L, $\gamma(H/I)$ si H/I.

(A_0) For any $I \lhd L$, $\beta(L/I) = \gamma(L/I)$.

Similarly we define (B_0) and (C_0).

Then we have the following

LEMMA 3.1. *For a Lie algebra L over a field of characteristic* 0,

$$\begin{array}{ccccccc}
(\mathrm{M}) & \Longrightarrow & (\mathrm{A}) & \Longrightarrow & (\mathrm{B}) & \Longrightarrow & (\mathrm{C}) \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & (\mathrm{A}_0) & \Longrightarrow & (\mathrm{B}_0) & \Longrightarrow & (\mathrm{C}_0).
\end{array}$$

PROOF. (M)$\Rightarrow$(A). Assume that L satisfies the condition (M) and $I \lhd H$ asc L. Suppose that K/I asc H/I and $K/I \in \mathfrak{N}$. Then K asc H. Hence K asc L. By the condition (M) for L, K si L. It follows that K si H. Hence K/I si H/I. This shows that $\gamma(H/I) \leq \beta(H/I)$ and therefore $\gamma(H/I) = \beta(H/I)$.

(A)$\Rightarrow$(B). This follows from the fact that $\beta(M) \lhd M$ for any Lie algebra M

over a field of characteristic 0.

The other implications are now evident.

Let us study the converse of some of the above implications under a certain assumption. To this end, we need the following

Lemma 3.2. *Let L be a Lie algebra over a field Φ. If $L \in$ Min-$\lhd^2$, then $\rho(L) \in \mathfrak{F} \cap \mathfrak{N}$.*

Proof. It is shown in [2, Lemma 8.1.3] that $\rho(L) \in \mathfrak{F}$. Hence $\rho(L) \in \mathfrak{F} \cap \mathrm{L}\mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$.

Theorem 3.3. *Let L be a Lie algebra over a field of characteristic 0.*

(1) *If $L \in$ Min-$\lhd^\omega$ and satisfies the condition (C_0), then $L \in \mathfrak{M}_\omega$.*

(2) *If $L \in$ Min-$\lhd^\sigma$ $(\sigma \geq \omega)$ and satisfies the condition (C), then $L \in \mathfrak{M}_\sigma$.*

Proof. (1) Assume that $L \in$ Min-$\lhd^\omega$ and satisfies the condition (C_0). Suppose that $H \lhd^\omega L$. Then $H^k \lhd H$ for any $k \in \mathbf{N}$. Hence $H^k \lhd^\omega L$. Since $L \in$ Min-$\lhd^\omega$, there is an $m \in \mathbf{N}$ such that $H^m = H^{m+1} = \cdots$. Therefore $H^\omega = \bigcap_{k=1}^{\infty} H^k = H^m$. On the other hand, it is obvious that $H^\omega \lhd L$. Hence $H/H^\omega \lhd^\omega L/H^\omega$ and $H/H^\omega \in \mathfrak{N}$. It follows that $H/H^\omega \leq \gamma(L/H^\omega)$. By the condition (C_0) for L, $\gamma(L/H^\omega)$ si L/H^ω. Since $L/H^\omega \in$ Min-$\lhd^\omega$, $\gamma(L/H^\omega) \in \mathrm{L}\mathfrak{N} \cap$ Min-$\lhd^2$. By Lemma 3.2, it follows that $\gamma(L/H^\omega) \in \mathfrak{F} \cap \mathfrak{N}$. If we write $\gamma(L/H^\omega) = I/H^\omega$, then we have $I^n \leq H^\omega$ for some n. Hence $I^n \leq H \leq I$. Since $I/I^n \in \mathfrak{N} \leq \mathfrak{D}$, H/I^n si I/I^n. Therefore H si I. Since I si L, H si L. Thus $L \in \mathfrak{M}_\omega$.

(2) Assume that $L \in$ Min-$\lhd^\sigma$ and satisfies the condition (C). Suppose that $L \notin \mathfrak{M}_\sigma$. Then there is a subalgebra H of L such that $H \lhd^\sigma L$ but H is not a subideal of L. Let $(H_\alpha)_{\alpha \leq \sigma}$ be an ascending series for H in L. Since $\sigma \geq \omega$, we may assume that H is not a subideal of H_ω. Then we assert that $H_\omega \in$ Min-$\lhd^\omega$ and satisfies (C_0). In fact, if $K_1 \geq K_2 \geq \cdots$ and $K_n \lhd^\omega H_\omega$ for any $n \in \mathbf{N}$, then $K_n \lhd^\sigma L$. Since $L \in$ Min-$\lhd^\sigma$, there is an $m \in \mathbf{N}$ such that $K_m = K_{m+1} = \cdots$. If $I \lhd H_\omega$, then $I \lhd H_\omega$ asc L. Since L satisfies (C), $\gamma(H_\omega/I)$ si H_ω/I. Thus we can apply the statement (1) for H_ω to see that H si H_ω, which is a contradiction.

By using Theorem 3.3, we now show the following

Theorem 3.4. *Let L be a Lie algebra over a field of characteristic 0. If $L \in$ Min-asc,*

$$(\mathrm{M}) \Longleftrightarrow (\mathrm{A}) \Longleftrightarrow (\mathrm{B}) \Longleftrightarrow (\mathrm{C}).$$

Proof. Assume that $L \in$ Min-asc and satisfies the condition (C). Suppose that H asc L. Then $H \lhd^\sigma L$ for some ordinal σ. Since Min-asc $\leq$ Min-$\lhd^\sigma$, $L \in$ Min-$\lhd^\sigma$. If $\sigma \geq \omega$, $L \in \mathfrak{M}_\sigma$ by Theorem 3.3. Hence H si L. Therefore

L satisfies the condition (M). The statement now follows from Lemma 3.1.

4.

In this section by using Theorem 3.4 we shall show that certain subclasses of Min-$\lhd^{\omega}$ are contained in $\mathfrak{M}$ and $\mathfrak{M}_{\omega}$.

LEMMA 4.1. *Let L be a Lie algebra over a field of characteristic* 0. *If $L \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F}$, then $\gamma(L) \lhd L$.*

This is [2, Corollary 6.3.5] and can be shown by using the fact that $\gamma(L)$ is invariant under every locally finite derivation of L and by observing that $\mathrm{ad}_L x$ is a locally finite derivation of L for any element x of an ascendant $\mathrm{L}\mathfrak{F}$-subalgebra of L.

LEMMA 4.2. *Let L be a Lie algebra over a field of characteristic* 0. *If $L \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2$, then*

$$\nu(L) = \beta(L) = \gamma(L) = \rho(L) \in \mathfrak{F} \cap \mathfrak{N}.$$

PROOF. By Lemma 4.1 $\gamma(L)$ is a locally nilpotent ideal of L. Hence

$$\nu(L) \leq \beta(L) \leq \gamma(L) \leq \rho(L).$$

But by Lemma 3.2 $\rho(L) \in \mathfrak{F} \cap \mathfrak{N}$. Therefore we have $\rho(L) \leq \nu(L)$.

LEMMA 4.3. *Let L be a Lie algebra over a field of characteristic* 0.

(1) *If $L \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2$, then L satisfies the condition* (A_0).

(2) *If $L \in \mathrm{NL}\mathfrak{F} \cap \text{Min-asc}$, then L satisfies the condition* (A).

PROOF. (1) Assume that $L \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^2$. If $I \lhd L$, then $L/I \in \mathrm{Q}\acute{\mathrm{N}}\mathrm{L}\mathfrak{F} = \acute{\mathrm{N}}\mathrm{L}\mathfrak{F}$. Evidently $L/I \in \text{Min-}\lhd^2$. Therefore by Lemma 4.2 we have $\beta(L/I) = \gamma(L/I)$.

(2) Assume that $L \in \mathrm{NL}\mathfrak{F} \cap \text{Min-asc}$ and that $I \lhd H \operatorname{asc} L$. From the fact that $\mathrm{L}\mathfrak{F}$ is locally coalescent, it follows that $\mathrm{NL}\mathfrak{F}$ is s-closed. Hence $H \in \mathrm{NL}\mathfrak{F} \leq \acute{\mathrm{N}}\mathrm{L}\mathfrak{F}$ and therefore $H/I \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F}$. It is immediate that $H \in \text{Min-}\lhd^2$ and therefore $H/I \in \text{Min-}\lhd^2$. Hence by Lemma 4.2 we have $\beta(H/I) = \gamma(H/I)$.

We are now in a position to show the following

THEOREM 4.4. *For fields of characteristic* 0,

(1) $\acute{\mathrm{N}}\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^{\omega} \leq \mathfrak{M}_{\omega}$.

(2) $\mathrm{NL}\mathfrak{F} \cap \text{Min-asc} \leq \mathfrak{M}$.

PROOF. (1) Assume that $L \in \acute{\mathrm{N}}\mathrm{L}\mathfrak{F} \cap \text{Min-}\lhd^{\omega}$. Then by Lemma 4.3 L

satisfies the condition (A_0). It follows from Lemma 3.1 that L satisfies the condition (C_0). Hence by Theorem 3.3 we see that $L \in \mathfrak{M}_\omega$.

(2) Assume that $L \in \mathrm{NL}\mathfrak{F} \cap \text{Min-asc}$. Then by Lemma 4.3 L satisfies the condition (A). Theorem 3.4 now tells us that L satisfies the condition (M), that is, $L \in \mathfrak{M}$.

5.

In this section, we introduce the following conditions for a Lie algebra L which are stronger than the conditions (M) and (M_σ) $(\sigma \geq \omega)$:

(M′) Every ascendant subalgebra of L is an ideal.

(M'_σ) Every σ-step ascendant subalgebra of L is an ideal.

We denote by $\mathfrak{M}'$ and $\mathfrak{M}'_\sigma$ the classes of Lie algebras satisfying the condition (M′) and (M'_σ) respectively. Then for any ordinals $\rho \geq \sigma \geq \omega$, we have

$$\mathfrak{A} \leq \mathfrak{M}' \leq \mathfrak{M}'_\rho \leq \mathfrak{M}'_\sigma \leq \mathfrak{M}'_\omega \leq \mathfrak{T}.$$

Moreover we have

$$\mathfrak{T} \cap \mathfrak{D} = \mathfrak{A}, \quad \mathfrak{T} \cap \mathfrak{M} = \mathfrak{M}', \quad \mathfrak{T} \cap \mathfrak{M}_\sigma = \mathfrak{M}'_\sigma \qquad (\sigma \geq \omega).$$

In fact, let $L \in \mathfrak{T} \cap \mathfrak{D}$. Then every subalgebra of L is an ideal. For any $x, y \in L$, $\langle x \rangle \lhd L$ and $\langle y \rangle \lhd L$. If x and y are linearly independent, then $[x, y] \in \langle x \rangle \cap \langle y \rangle = (0)$. If x and y are linearly dependent, then obviously $[x, y] = 0$. Therefore $L \in \mathfrak{A}$. Thus we have $\mathfrak{T} \cap \mathfrak{D} = \mathfrak{A}$. The other formulas are evident.

It can be shown as in Lemma 3.1 that every $\mathfrak{M}'$-algebra L over a field of characteristic 0 satisfies the condition:

(A′) For any I, H such that $I \lhd H$ asc L, $\nu(H/I) = \gamma(H/I)$.

We shall determine the structure of Lie algebras belonging to some special subclasses of $\mathfrak{M}'$.

PROPOSITION 5.1. $\mathfrak{Z} \cap \mathfrak{M}' = \mathfrak{A}$.

PROOF. Let $L \in \mathfrak{Z} \cap \mathfrak{M}'$. Then by using Lemma 2.2 we see that every subalgebra of L is an ideal. It follows that L is abelian. Therefore $\mathfrak{Z} \cap \mathfrak{M}' \leq \mathfrak{A}$.

THEOREM 5.2. *Every* $\mathrm{E}\mathfrak{A} \cap \mathfrak{M}'$*-algebra is either abelian or the split extension of an abelian Lie algebra by the* 1*-dimensional algebra of scalar multiplications, and conversely. Furthermore*

$$\mathrm{E}\mathfrak{A} \cap \mathfrak{M}' = \mathrm{E}\mathfrak{A} \cap \mathfrak{M}'_\sigma = \mathrm{E}\mathfrak{A} \cap \mathfrak{T}.$$

PROOF. (1) Assume that $L \in \mathrm{E}\mathfrak{A} \cap \mathfrak{T}$. It is stated without proof in

[2, p.167] that L is then either abelian, or the split extension of an abelian Lie algebra by the 1-dimensional algebra of scalar multiplications. We shall give the outline of its proof. Let L be not abelian. Then there is an integer $k>0$ such that $L^{(k)}\neq(0)$ and $L^{(k+1)}=(0)$. For any $x\in L^{(k)}$, $<x>$ si L and therefore $<x>\lhd L$. It follows that $\mathrm{ad}_{L^{(k)}}z=\alpha 1_{L^{(k)}}$ for any $z\in L$, where $\mathrm{ad}_{L^{(k)}}z$ is the restriction of $\mathrm{ad}_L z$ to $L^{(k)}$. If $\mathrm{ad}_{L^{(k)}}L=0$, for every $u\in L^{(k-1)}$ $<u>$ si L and therefore $<u>\lhd L$. It follows that $L^{(k)}=(0)$, which contradicts the choice of k. So $\dim \mathrm{ad}_{L^{(k)}}L=1$ and

$$L = \mathrm{Ker}\,\mathrm{ad}_{L^{(k)}} + <z_0>, \quad \mathrm{ad}_{L^{(k)}}z_0 = 1_{L^{(k)}}.$$

We assert that $k=1$. In fact, if not, for every $u\in L^{(k-1)}$ $<u>$ si L since $L^{(k-1)} \leq L^{(1)}\leq \mathrm{Ker}\,\mathrm{ad}_{L^{(k)}}$. Hence $<u>\lhd L$ and therefore $L^{(k)}=(0)$, contradicting the choice of k. We next assert that $\mathrm{Ker}\,\mathrm{ad}_{L^{(1)}}=L^{(1)}$. In fact, if not, take a subspace U of $\mathrm{Ker}\,\mathrm{ad}_{L^{(1)}}$ complementary to $L^{(1)}$. For any $u\in U$, $<u>$ si L, whence $<u>\lhd L$. It follows that U is contained in the center of L. Take an element $y\neq 0$ in $L^{(1)}$ and a subspace V of $L^{(1)}$ complementary to $<y>$. Put $H=V+<y+u>$ with $0\neq u\in U$. Then H si L but H is not an ideal of L. This contradicts the assumption that $L\in\mathfrak{T}$. Thus we conclude that $L=L^{(1)} +<z_0>$.

(2) We shall show that $\mathrm{E}\mathfrak{A}\cap\mathfrak{T}\leq\mathfrak{M}'$. Let $L\in\mathrm{E}\mathfrak{A}\cap\mathfrak{T}$. If L is abelian, then $L\in\mathfrak{M}'$. So assume that L is the split extension

$$L = A + <z> \quad \text{with } A\in\mathfrak{A} \text{ and } z = 1_A.$$

Suppose that H asc L and $H\neq L$. Then there is an ascending series $(H_\alpha)_{\alpha\leq\sigma}$ for H in L. Let ρ be the first ordinal such that $z\in H_\rho$. Then ρ is not a limit ordinal. If $\rho<\sigma$, then $H_\rho\lhd H_{\rho+1}$. For any element $x=a+\beta z\in H_{\rho+1}(a\in A)$, $a=[x, z] \in H_\rho$ and therefore $x\in H_\rho$. Hence $H_\rho=H_{\rho+1}$. Thus $H_\rho=H_\sigma=L$ and we may assume that $\rho=\sigma$. Since $\sigma\geq 1$, $H_{\sigma-1}\lhd L$. If $x=a+\beta z\in H_{\sigma-1}(a\in A, \beta\neq 0)$, then $a=[x, z]\in H_{\sigma-1}$ and therefore $z\in H_{\sigma-1}$, which contradicts the choice of ρ. Therefore $H_{\sigma-1}\leq A$. It follows that $H\leq A$ and therefore $H\lhd L$. Thus $L\in\mathfrak{M}'$.

6.

In this section, we shall show by examples that some of the classes of Lie algebras observed in the preceding sections are really distinct.

EXAMPLE 1. We illustrate that $\mathfrak{M}_{\omega+1}\lneqq\mathfrak{M}_\omega$. Let A be an abelian Lie algebra with basis $e_0, e_1,\dots$, let y be the derivation of A defined by $e_0\mapsto 0$ and $e_i\mapsto e_{i-1}(i=1, 2,\dots)$, and let $z=1_A$. Let $L=A+<y, z>$ be a split extension. Then we show that $L\in\mathfrak{M}_\omega\backslash\mathfrak{M}_{\omega+1}$.

If $H\lhd^\omega L$, take an ascending series $(H_\alpha)_{\alpha\leq\omega}$ for H in L. There is an $n\in\mathbf{N}$

such that $y, z \in H_n$. It is then easy to show that $H_n = H_{n+1}$. Hence $H_n = L$ and therefore $H \lhd^n L$. Thus $L \in \mathfrak{M}_\omega$.

Put $K = \langle y \rangle$. If we set

$$K_0 = K, \quad K_i = \langle e_0, e_1, \dots, e_{i-1}, y \rangle \qquad (i = 1, 2, \dots),$$

$$K_\omega = A + \langle y \rangle \qquad \text{and} \qquad K_{\omega+1} = L,$$

then $(K_\alpha)_{\alpha \le \omega+1}$ is an ascending series for K in L. Hence $K \lhd^{\omega+1} L$. If $K \lhd^n L$, then $e_0 = [e_n, {}_n y] \in K$, which is a contradiction. Therefore K is not a subideal of L. Thus $L \notin \mathfrak{M}_{\omega+1}$.

Example 2. Let $L = A + \langle z \rangle$ with the notations in Example 1. Then in Theorem 5.2 it has been shown that L belongs to $\mathfrak{T}$ and $\mathfrak{M}'$. Therefore $L \in \mathfrak{M}$. Since the idealizer of $\langle z \rangle$ in L is $\langle z \rangle$ itself, $\langle z \rangle$ is not a subideal. Hence $L \notin \mathfrak{D}$. Thus $\mathfrak{T} \nleq \mathfrak{D}$, $\mathfrak{M}' \nleq \mathfrak{D}$ and $\mathfrak{D} \lneqq \mathfrak{M}$.

Example 3. Let $L = \langle e_0, e_1 \rangle + \langle y \rangle$ with the notations in Example 1. Since L is nilpotent, $L \in \mathfrak{D}$. However $L \notin \mathfrak{T}$, for $\langle y \rangle \lhd^2 L$ but $\langle y \rangle$ is not an ideal. Therefore $\mathfrak{D} \nleq \mathfrak{T}$. Consequently $\mathfrak{D} \nleq \mathfrak{M}'$, $\mathfrak{M} \nleq \mathfrak{T}$, $\mathfrak{M}' \lneqq \mathfrak{M}$ and $\mathfrak{M}'_\sigma \lneqq \mathfrak{M}_\sigma$.

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