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Optimal Balanced Fractional 3^m Factorial Designs of Resolution V and Balanced Third-Order Designs

Masahide KUWADA

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Contents

0. Introduction and summary

Fractional factorial designs were first introduced by Finney [18] in an agricultural setting. The theory has found increasing use in various fields of experimentations, and further developed in orthogonal fractions in which estimates of the effects of interest are mutually uncorrelated. However, orthogonal fractions are available only for special values of the number of assemblies (or treatment combinations) and are generally uneconomic in the sense that they involve more than the desirable number of assemblies. In this sense, one needs to consider non-orthogonal (or irregular) fractions as well (cf. [4]). The class of balanced designs considered here is the next wider class to be looked into.

As a generalization of orthogonal arrays, the concept of balanced arrays (B-arrays) was first introduced by Chakravarti [7], who called them "partially balanced" arrays. Several authors (cf. [8, 9, 17, 22, 33, 39, 46, 53]) have contributed to the development of such B-arrays, particularly, their constructions of 2, 3 and, in general, s levels. A connection between a B-array of strength 2ℓ and a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2\ell+1$ was given by Yamamoto, Shirakura and Kuwada [59]. Furthermore, the same authors [60] obtained an explicit expression for the characteristic polynomial of the information matrix of a 2^m-BFF design of resolution $2\ell+1$, by utilizing the decomposition of triangular multidimensional partially balanced (TMDPB) association algebra $\mathfrak{A}$ into its two-sided ideals $\mathfrak{A}_\beta$ for $\beta=0, 1,\dots, \ell$. This polynomial also yields the results obtained by Srivastava and Chopra [51]. The concept of MDPB association schemes was first introduced by Bose and Srivastava [5], as a generalization of association schemes. Optimal 2^m-BFF designs of resolutions V and VII with respect to popular criteria (e.g., the trace, determinant and maximum root) have been obtained by Shirakura [40–42], Srivastava and/or Chopra [10–15, 45, 52, 55], and others. Shirakura and/or Kuwada [39–44] made further investigation into 2^m-BFF designs on the basis of the above-mentioned results.

Saturated fractions, particularly, saturated main effect plans, which are special cases of fractional factorials, have been studied by several authors (cf. [31, 32, 34–38]). This theory is very useful in the sense that the number of assemblies is equal to the minimum value of estimating the unknown effects of interest. However, this case has no degree of freedom for error.

As mentioned above, most of the authors have dealt with a design of odd resolution. On the other hand, designs of even resolution have been studied by Margolin [27, 28], Shirakura [41, 42], Srivastava and/or Anderson [2, 47], Webb [57], and others.

Recently, an explicit expression for the trace of the covariance matrix of a balanced fractional (2, 0)-symmetric design of resolution V for 3^m factorials was obtained by Srivastava and Chopra [54]. Special cases of their results were also considered by Hoke [19, 20], who gave the characteristic polynomial of an information matrix and various properties of a design based on the second-order model for 3^m factorials. Kuwada [24] has obtained a connection between a B-array of strength 4, size N, m constraints, 3 levels and index set $\{\lambda_{i_0 i_1 i_2} | i_0+i_1+i_2=4\}$ (for brevity, B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$) and a 3^m-BFF design of resolution V. In general, a connection between a B-array of strength 2ℓ and an s^m-BFF design of resolution $2\ell+1$ has been given by Kuwada and/or Nishii [26, 30]. Furthermore, the above first author [25] has derived the characteristic polynomial of information matrix M_T of a 3^m-BFF design, T, of resolution V, by use of the algebraic structure of a multidimensional (MD) relationship. The concept of relationship was introduced by James [21] in an experimental design. The author believes that the theory of an MD relationship and its algebra for a 3^m-BFF design will be extensible to the theory for an s^m-BFF design by an argument similar to that for 3^m factorials.

This paper consists of three parts. In Part I, fractional 3^m factorial (3^m-FF) designs and their algebraic structure are discussed. Section 1 gives the linear model of a 3^m-FF design. Section 2 deals with a connection between indices $\lambda_{i_0 i_1 i_2}$ of a B-array $[N, m, 3, 4]$ and the elements of information matrix M_T of a 3^m-FF design, T. Section 3 gives the definition of a p-MDPB association scheme. Particularly, an $(\ell+1)$-TMDPB association scheme and its properties are discussed. As a generalization of an MDPB association scheme, Section 4 deals with an $\binom{\ell+2}{2}$-MD relationship defined among the sets of effects up to ℓ-factor interaction (particularly, the case $\ell=2$) and its algebra $\mathfrak{A}$ (called an MD relationship algebra). By use of the algebraic structure, $\mathfrak{A}$ is, for case $\ell=2$, decomposed into the direct sum of four two-sided ideals $\mathfrak{A}_\beta$ for $\beta=0, 1, 2, f$. The explicit expressions for irreducible representations K_β of M_T for B-array $[N, m, 3, 4]$ T with index set $\{\lambda_{i_0 i_1 i_2}\}$ with respect to ideals $\mathfrak{A}_\beta$ are also given.

In Part II, practical properties of a 3^m-BFF design of resolution V are discussed, and optimal 3^4-BFF designs of resolution V with respect to the trace and determinant criteria are presented for $33 \leqq N \leqq 56$. Furthermore, under some restrictions on indices $\lambda_{i_0 i_1 i_2}$, optimal 3^4- and 3^5-BFF designs of resolution V with respect to the above two criteria are presented for $57 \leqq N \leqq 81$ and $51 \leqq N \leqq 70$, respectively. Section 5 shows that a necessary and sufficient condition for design T to be balanced is that T is a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$, provided M_T is non-singular. In order to obtain an optimal design with respect to popular criteria, Section 6 gives an explicit expression for the characteristic polynomial of M_T (and hence that of V_T $(=M_T^{-1})$ for a 3^m-BFF design of resolution V) and for the trace and determinant of V_T. Furthermore, this section includes, as a by-product, a necessary condition for the existence of a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$, and a connection between a B-array and its (0, 2)-interchanged B-array. Utilizing the algebraic structure, Section 7 gives the elements of covariance matrix $V_T\sigma^2$ of a 3^m-BFF design, T, of resolution V. By use of the results obtained in the preceding sections, Sections 8 and 9 present optimal 3^4- and 3^5-BFF designs derived, respectively, from B-arrays $[N, 4, 3, 4]$ and B-arrays $[N, 5, 3, 5]$ with respect to the trace and determinant criteria, under some restrictions on $\lambda_{i_0 i_1 i_2}$.

In Part III, balanced third-order designs for 3^m factorials (briefly, 3^m-BTO designs) are discussed. As a generalization of the second-order model for 3^m factorials (cf. [38]), Section 10 suggests the definition of the third-order model, and also gives the relationship defined among the six sets of effects, i.e., the general mean, the linear and the quadratic components of the main effect, the linear by linear and the linear by quadratic of the two-factor interaction, and the 3-linear of the three-factor interaction. Section 11 gives a connection between a B-array of strength at least six and the elements of information matrix M_T^* for the third-order model. The explicit expression for irreducible representations K_β^* ($\beta=0$, 1, 2, 3, f) of M_T^* with respect to ideals $\mathfrak{A}_\beta^*$ of algebra $\mathfrak{A}^*$ is also given. Using the results mentioned above, this section also gives an explicit expression for the characteristic polynomial of M_T^* (hence that of V_T^* $(=M_T^{*-1})$) and for the trace and determinant of V_T^* for a 3^m-BTO design. Under some restrictions on $\lambda_{i_0 i_1 i_2}^*$, Section 12 presents the elements of V_T^* and optimal 3^6-BTO designs with respect to the trace and determinant criteria. As a by-product, this section also gives some existence condition for B-arrays based on the third-order model and for 3^m-BTO designs.

For convenience sake, the notations and symbols shown below are used throughout this paper. Unless stated otherwise, their meanings are as follows:

I_p	:	Identity matrix of order p.
$G_{p\times q}$	:	A $p\times q$ matrix all of whose elements are unity. As special cases, $G_{p\times p}$ and $G_{p\times 1}$ are denoted by G_p and $\boldsymbol{j}_p$, respectively.
$O_{p\times q}$	:	A $p\times q$ matrix all of whose elements are zero.
A'	:	Transpose of matrix A.
$\mathrm{tr}\,(A)$	:	Trace of matrix A.
$\det\,(A)$	:	Determinant of matrix A.
$\|S\|$	:	Cardinality of set S.
$\mathrm{Var}\,[\boldsymbol{y}]$	:	Covariance matrix of vector $\boldsymbol{y}$.
δ_{ij}	:	Kronecker's delta, i.e., $\delta_{ij}=1$ or 0 according as $i=j$ or not.
$\min\,(u, v)$	:	Minimum value of integers u and v.
$\binom{n}{m}$	:	Binomial coefficient. As a special case, $\binom{n}{m}=0$ if and only if $m>n$ or $m<0$.
$w_r(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_k)$	:	The number of r's in the vector $(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_k)$.
$[A_\alpha^{(ab,cd)}]$	:	Algebra generated by the linear closure of these matrices indicated in the bracket [ ].
$\mathrm{diag}\,\{\underbrace{a_1,\dots, a_1}_{p_1},\dots, \underbrace{a_s,\dots, a_s}_{p_s}\}$	:	A $p\times p$ diagonal matrix obtained by juxtaposing each a_i p_i-times $(i=1, 2,\dots, s)$ for $p=p_1+\cdots+p_s$.

Part I. 3^m-FF designs and their algebraic structure

1. Linear model of 3^m-FF designs

Consider a factorial experiment with m factors, $F_1, F_2, \ldots, F_m$, each at three levels. As usual, the assemblies or treatment combinations will be represented by $(j_1, j_2, \ldots, j_m)$, wherein factor F_r has level j_r for $r=1, 2, \ldots, m$ and each j_r is equal to 0, 1 or 2. Consider the observations $y(j_1, j_2, \ldots, j_m)$ corresponding to assemblies $(j_1, j_2, \ldots, j_m)$ and their expectations $\eta(j_1, j_2, \ldots, j_m)$. In this case, it is well known (cf. [24]) that various factorial effects can be expressed as a linear combination of all expectations $\eta(j_1, j_2, \ldots, j_m)$, i.e.,

$$\begin{aligned}(1.1)\qquad \theta(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = \sum_{j_1, j_2, \ldots, j_m} & 3^{-w_0(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)} 2^{-w_1(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)} 6^{-w_2(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)} \\ & \cdot d_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^{j_1 j_2 \cdots j_m} \eta(j_1, j_2, \ldots, j_m) \\ & \text{for} \quad \varepsilon_i = 0, 1, 2; \quad i = 1, 2, \ldots, m,\end{aligned}$$

where $d_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^{j_1 j_2 \cdots j_m}$ are given by

$$d_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_m}^{j_1 j_2 \cdots j_m} = d_{j_1}(\varepsilon_1) d_{j_2}(\varepsilon_2) \cdots d_{j_m}(\varepsilon_m),$$

in which elements $d_j(\varepsilon)$ are defined by

$$(1.2)\qquad D = \begin{bmatrix} d_0(0) & d_1(0) & d_2(0) \\ d_0(1) & d_1(1) & d_2(1) \\ d_0(2) & d_1(2) & d_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

Note that D is the coefficient matrix of an orthogonal polynomial of degree 3.

Let

$$\boldsymbol{\eta} = \begin{bmatrix} \eta(0, \ldots, 0, 0) \\ \eta(0, \ldots, 0, 1) \\ \eta(0, \ldots, 0, 2) \\ \vdots \\ \eta(2, \ldots, 2, 2) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} \theta(0, \ldots, 0, 0) \\ \theta(0, \ldots, 0, 1) \\ \theta(0, \ldots, 0, 2) \\ \vdots \\ \theta(2, \ldots, 2, 2) \end{bmatrix}$$

be, respectively, the $3^m \times 1$ vectors of all expectations of observations and effects arranged in the lexicographic order. Then, from (1.1), $\boldsymbol{\theta}$ can be written in matrix notation as

$$(1.3)\qquad \boldsymbol{\theta} = \Delta_{(m)}^{-2} D_{(m)} \boldsymbol{\eta},$$

where $D_{(m)} = D \otimes D \otimes \cdots \otimes D$ (m-times Kronecker products of D) and $\Delta_{(m)}^2 =$

$D_{(m)}D'_{(m)}$ is the diagonal matrix with diagonal elements $3^{w_0(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m)}2^{w_1(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m)}\cdot 6^{w_2(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m)}$ corresponding to the $(\varepsilon_1 3^{m-1}+\varepsilon_2 3^{m-2}+\cdots+\varepsilon_m+1,\ \varepsilon_1 3^{m-1}+\varepsilon_2 3^{m-2}+\cdots+\varepsilon_m+1)$ cell.

In particular, the general mean, denoted alternatively by $\theta(\phi)$, is represented by $\theta(0, 0,\dots, 0)$, and the linear (the quadratic) component of the main effect of factor F_t, denoted alternatively by $\theta(t^1)$ $(\theta(t^2))$, is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$ with $\varepsilon_t=1$ $(\varepsilon_t=2)$ and $\varepsilon_r=0$ for all $r\neq t$. The linear by linear (the quadratic by quadratic) component of the two-factor interaction of factors F_{t_1} and F_{t_2}, denoted alternatively by $\theta(t_1^1t_2^1)$ $(\theta(t_1^2t_2^2))$, is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$ with $\varepsilon_{t_1}=\varepsilon_{t_2}=1$ $(\varepsilon_{t_1}=\varepsilon_{t_2}=2)$ and $\varepsilon_r=0$ for all $r\neq t_1, t_2$, and the linear by quadratic of the two-factor interaction of factors F_{t_3} and F_{t_4}, denoted alternatively by $\theta(t_3^1t_4^2)$, is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$ with $\varepsilon_{t_3}=1$, $\varepsilon_{t_4}=2$ and $\varepsilon_r=0$ for all $r\neq t_3, t_4$. In general, for $\{t_1,\dots, t_{k_1}, t'_1,\dots, t'_{k_2}\}$ being a subset of $\{1, 2,\dots, m\}$, the k_1-linear by k_2-quadratic component of the k-factor interaction of factors $F_{t_1},\dots, F_{t_{k_1}}$ and $F_{t'_1},\dots, F_{t'_{k_2}}$, where $k=k_1+k_2$, denoted alternatively by $\theta(t_1^1\cdots t_{k_1}^1 t_1'^2\cdots t_{k_2}'^2)$, is represented by $\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m)$ with $\varepsilon_{t_1}=\cdots=\varepsilon_{t_{k_1}}=1$, $\varepsilon_{t'_1}=\cdots=\varepsilon_{t'_{k_2}}=2$ and the remaining ε_r being all equal to zero.

Solving (1.3) with respect to $\boldsymbol{\eta}$ yields $\boldsymbol{\eta}=D'_{(m)}\boldsymbol{\theta}$, i.e.,

$$\eta(j_1, j_2,\dots, j_m) = \sum_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m} d^{j_1j_2\cdots j_m}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_m}\theta(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_m).$$

We shall consider the situation where three-factor and higher-order interactions are assumed to be negligible. In this case, the number of unknown effects is given by $\nu_m=1+m+m+\binom{m}{2}+\binom{m}{2}+2\binom{m}{2}=1+2m^2$, and the $\nu_m\times 1$ vector of these effects $\boldsymbol{\theta}_{\nu_m}$ can be rearranged as follows:

$$\boldsymbol{\theta}'_{\nu_m} = (\{\theta(\phi)\};\ \{\theta(t^1)\};\ \{\theta(t^2)\};\ \{\theta(t_1^1t_2^1)\};\ \{\theta(t_1^2t_2^2)\};\ \{\theta(t_3^1t_4^2)\}),$$

where $t_1<t_2$ and $t_3\neq t_4$. Since $d_j(0)=1$ for all j, we can obtain the following model for the expectation of the observation corresponding to assembly $(j_1, j_2,\dots, j_m)$:

$$\begin{aligned}(1.4)\qquad \eta(j_1, j_2,\dots, j_m) = {} & \theta(\phi) + \sum_{\varepsilon=1}^{2}\sum_{t=1}^{m} d_{j_t}(\varepsilon)\theta(t^\varepsilon) \\ & + \sum_{\varepsilon=1}^{2}\sum_{t_1<t_2} d_{j_{t_1}}(\varepsilon)d_{j_{t_2}}(\varepsilon)\theta(t_1^\varepsilon t_2^\varepsilon) \\ & + \sum_{t_3\neq t_4} d_{j_{t_3}}(1)d_{j_{t_4}}(2)\theta(t_3^1t_4^2).\end{aligned}$$

2. B-arrays and information matrices

Let T be a fraction with N assemblies. Then T can be expressed as a matrix of size $N\times m$ whose α-th row $(j_1^{(\alpha)}, j_2^{(\alpha)},\dots, j_m^{(\alpha)})$ denotes the α-th assembly for

$\alpha=1, 2,\ldots, N$ and $j_i^{(\alpha)}$ are 0, 1 or 2 for $i=1, 2,\ldots, m$. Let $\boldsymbol{y}(T)$ and $\boldsymbol{\eta}(T)$ be the $N\times 1$ observation vector and the corresponding expectation vector with α-th elements $y(j_1^{(\alpha)}, j_2^{(\alpha)},\ldots, j_m^{(\alpha)})$ and $\eta(j_1^{(\alpha)}, j_2^{(\alpha)},\ldots, j_m^{(\alpha)})$, respectively. Consider the N observations in $\boldsymbol{y}(T)$ as independent random variables with common variance σ^2. Then, from (1.4), $\boldsymbol{\eta}(T)$ and Var $[\boldsymbol{y}(T)]$ can be expressed as follows:

$$\boldsymbol{\eta}(T) = E_T\boldsymbol{\theta}_{\nu_m} \quad \text{and} \quad \mathrm{Var}\,[\boldsymbol{y}(T)] = \sigma^2 I_N,$$

where E_T is called the $N\times \nu_m$ design matrix of T. It can be seen from (1.4) that the element corresponding to $\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$-column in the α-th row of E_T is given by $d_{j_{t_1}^{(\alpha)}}(\varepsilon_1)d_{j_{t_2}^{(\alpha)}}(\varepsilon_2)$. Here, in the symbol for an effect, when the exponent, ε_r, of t_r is zero, we shall omit this t_r from the symbol. In particular, $\theta(t_1^0t_2^0)$ means the general mean, $\theta(\phi)$.

The normal equation for estimating $\boldsymbol{\theta}_{\nu_m}$ can be written as

$$M_T\hat{\boldsymbol{\theta}}_{\nu_m} = E'_T\boldsymbol{y}(T), \tag{2.1}$$

where $M_T=E'_TE_T$ is the $\nu_m\times \nu_m$ information matrix whose row and column correspond to the elements of $\boldsymbol{\theta}_{\nu_m}$, respectively.

The term "resolution" was introduced by Box and Hunter [6] to classify the fractions. A fraction, T, is called a fractional 3^m factorial (3^m-FF) design of resolution V, if information matrix M_T is non-singular. In a 3^m-FF design of resolution V, all effects up to two-factor interaction are estimable if three-factor and higher-order interactions are negligible. For a 3^m-FF design of resolution V, the best linear unbiased estimate (BLUE) of $\boldsymbol{\theta}_{\nu_m}$ can be given by

$$\hat{\boldsymbol{\theta}}_{\nu_m} = V_TE'_T\boldsymbol{y}(T),$$

where $V_T=M_T^{-1}$. Note that $\hat{\boldsymbol{\theta}}_{\nu_m}$ is the unique solution of (2.1). It can also be shown that Var $[\hat{\boldsymbol{\theta}}_{\nu_m}]$ is given by

$$\mathrm{Var}\,[\hat{\boldsymbol{\theta}}_{\nu_m}] = V_T\sigma^2. \tag{2.2}$$

Let $\varepsilon(t_1^{\varepsilon_1}t_2^{\varepsilon_2};\ t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ be the element of information matrix M_T whose row and column correspond to $\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ and $\theta(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$, respectively. Then we have

$$\varepsilon(t_1^{\varepsilon_1}t_2^{\varepsilon_2};\ t_3^{\varepsilon_3}t_4^{\varepsilon_4}) = \sum_{\alpha=1}^{N} d_{j_{t_1}^{(\alpha)}}(\varepsilon_1)d_{j_{t_2}^{(\alpha)}}(\varepsilon_2)d_{j_{t_3}^{(\alpha)}}(\varepsilon_3)d_{j_{t_4}^{(\alpha)}}(\varepsilon_4), \tag{2.3}$$

which yields that $\varepsilon(t_1^{\varepsilon_1}t_2^{\varepsilon_2};\ t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ is invariant with respect to any permutation of $\{1, 2, 3, 4\}$. Thus we can write $\varepsilon(t_1^{\varepsilon_1}t_2^{\varepsilon_2};\ t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ as $\gamma_{t_1t_2t_3t_4}^{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}$ with the property

$$\gamma_{t_1t_2t_3t_4}^{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} = \gamma_{t_{i_1}t_{i_2}t_{i_3}t_{i_4}}^{\varepsilon_{i_1}\varepsilon_{i_2}\varepsilon_{i_3}\varepsilon_{i_4}} \qquad \text{for all} \quad \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}.$$

Let $\boldsymbol{d}'_0=(1, 1, 1)$, $\boldsymbol{d}'_1=(-1, 0, 1)$ and $\boldsymbol{d}'_2=(1, -2, 1)$ be the row vectors of matrix D as defined by (1.2). Then, since D is non-singular, we get

LEMMA 2.1 (Kuwada [24]).

$$\begin{aligned} &\boldsymbol{d}_0 * \boldsymbol{d}_r = \boldsymbol{d}_r * \boldsymbol{d}_0 = \boldsymbol{d}_r \quad \textit{for} \quad r = 0, 1, 2, \\ &\boldsymbol{d}_1 * \boldsymbol{d}_1 = (2\boldsymbol{d}_0 + \boldsymbol{d}_2)/3, \\ &\boldsymbol{d}_1 * \boldsymbol{d}_2 = \boldsymbol{d}_2 * \boldsymbol{d}_1 = \boldsymbol{d}_1, \\ &\boldsymbol{d}_2 * \boldsymbol{d}_2 = 2\boldsymbol{d}_0 - \boldsymbol{d}_2, \end{aligned} \tag{2.4}$$

where symbol $*$ *denotes Schur product* (*or Hadamard product*), *i.e.*, $(a_1, a_2, a_3)' * (b_1, b_2, b_3)' = (a_1 b_1, a_2 b_2, a_3 b_3)'$.

From (2.3) and (2.4), it is easily shown that $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_1^{\varepsilon_3} t_2^{\varepsilon_4})$, $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_2^{\varepsilon_3} t_1^{\varepsilon_4})$, $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_1^{\varepsilon_3} t_4^{\varepsilon_4})$, $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_3^{\varepsilon_3} t_1^{\varepsilon_4})$, $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_2^{\varepsilon_3} t_4^{\varepsilon_4})$ and $\varepsilon(t_1^{\varepsilon_1} t_2^{\varepsilon_2}; t_3^{\varepsilon_3} t_2^{\varepsilon_4})$ can be expressed as linear combinations of $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1' \varepsilon_2' 0\, 0}$, $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1' \varepsilon_2' 0\, 0}$, $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1' \varepsilon_2 0\, \varepsilon_4}$, $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1' \varepsilon_2 \varepsilon_3 0}$, $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1 \varepsilon_2' 0\, \varepsilon_4}$ and $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1 \varepsilon_2' \varepsilon_3 0}$, respectively, for $\varepsilon_r' = 0, 1, 2$ $(r = 1, 2)$.

We shall define a balanced array (B-array) of strength t.

DEFINITION 2.1. An $N \times m$ matrix, T, whose elements are 0, 1 or 2 is called a balanced array of strength t, size N, m constraints, 3 levels and index set $\{\lambda_{i_0 i_1 i_2} | i_0 + i_1 + i_2 = t,\ i_0, i_1, i_2 \geqq 0\}$ (for brevity, B-array $[N, m, 3, t]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$), if every subarray $T_{k_1 k_2 \cdots k_t}$ composed of k_1-th, k_2-th,..., k_t-th columns of T is such that every $1 \times t$ vector with $w_r(j_{k_1}, j_{k_2}, \ldots, j_{k_t}) = i_r$ $(r = 0, 1, 2)$ occurs exactly $\lambda_{i_0 i_1 i_2}$-times as a row of $T_{k_1 k_2 \cdots k_t}$. Clearly $N = \sum \{t!/(i_0! i_1! i_2!)\} \lambda_{i_0 i_1 i_2}$, where summation $\sum$ extends over $i_0 + i_1 + i_2 = t$ and $i_0, i_1, i_2 \geqq 0$.

Especially, a B-array $[N, m, 3, t]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$ is called an orthogonal array of strength t, size N, m constraints, 3 levels and index λ, if $\lambda_{i_0 i_1 i_2} = \lambda$ for any i_0, i_1, i_2 satisfying $i_0 + i_1 + i_2 = t$ and $i_0, i_1, i_2 \geqq 0$.

The following theorem has been established:

THEOREM 2.2 (Kuwada [24]). *A necessary and sufficient condition that, for any subarray* $T_{t_1 t_2 t_3 t_4}$ *of* T, *the elements* $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}$ *of information matrix* M_T *of a* 3^m-FF *design of resolution V depend on set* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ *only through* $w_r(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = p_r$ $(r = 0, 1, 2)$ *is that* T *is a B-array* $[N, m, 3, 4]$ *with index set* $\{\lambda_{i_0 i_1 i_2}\}$.

For $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with $w_r(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = p_r$ $(r = 0, 1, 2)$, we denote $\gamma_{t_1 t_2 t_3 t_4}^{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}$ by $\gamma_{p_0 p_1 p_2}$. A connection between fifteen possible distinct values $\gamma_{p_0 p_1 p_2}$ of the elements of M_T and indices $\lambda_{i_0 i_1 i_2}$ of a B-array can be expressed as

$$\begin{aligned} \gamma_{p_0 p_1 p_2} = &\sum \{p_0!/(i_0^0! i_1^0! i_2^0!)\} \{p_1!/(i_0^1! i_1^1! i_2^1!)\} \{p_2!/(i_0^2! i_1^2! i_2^2!)\} \\ &\cdot (-1)^{i_0^1} \delta_{0 i_1^1} (-2)^{i_1^2} \lambda_{i_0^0 + i_0^1 + i_0^2\ i_1^0 + i_1^1 + i_1^2\ i_2^0 + i_2^1 + i_2^2}, \end{aligned}$$

TABLE A.

$$\gamma\begin{bmatrix}400\\040\\004\\310\\301\\130\\031\\103\\013\\220\\202\\022\\211\\121\\112\end{bmatrix}=\begin{bmatrix}1&1&1&4&4&4&4&4&4&6&6&6&12&12&12\\1&0&1&0&-4&0&0&-4&0&0&6&0&0&0&0\\1&16&1&-8&4&-32&-32&4&-8&24&6&24&-24&48&-24\\-1&0&1&-3&-2&-1&1&2&3&-3&0&3&-3&0&3\\1&-2&1&1&4&-5&-5&4&1&-3&6&-3&3&-6&3\\-1&0&1&-1&2&0&0&-2&1&0&0&0&3&0&-3\\-1&0&1&2&2&0&0&-2&-2&0&0&0&-6&0&6\\1&-8&1&-5&4&4&4&4&-5&6&6&6&-15&12&-15\\-1&0&1&6&-2&8&-8&2&-6&-12&0&12&6&0&-6\\1&0&1&2&0&0&0&0&2&1&-2&1&-2&-2&-2\\1&4&1&-2&4&4&4&4&-2&-3&6&-3&-6&-6&-6\\1&0&1&-4&0&0&0&0&-4&4&-2&4&4&-8&4\\-1&0&1&0&-2&2&-2&2&0&3&0&-3&0&0&0\\1&0&1&-1&0&0&0&0&-1&-2&-2&-2&1&4&1\\-1&0&1&3&-2&-4&4&2&-3&0&0&0&3&0&-3\end{bmatrix}\lambda\begin{bmatrix}400\\040\\004\\310\\301\\130\\031\\103\\013\\220\\202\\022\\211\\121\\112\end{bmatrix}$$

where summation $\sum$ extends over $i_0^r+i_1^r+i_2^r=p_r$, $0\leqq i_q^r\leqq p_r$ $(r, q=0, 1, 2)$ and $p_0+p_1+p_2=4$. This relation is concretely written as Table A.

3. TMDPB association schemes

As a generalization of association schemes, multidimensional partially balanced (MDPB) association schemes were first introduced by Bose and Srivastava [5]. Several authors (cf. [1, 3, 48, 49, 59, 61]) have contributed to the development of the theory of such MDPB association schemes.

Let $S_1, S_2,\ldots, S_p$ be p mutually disjoint non-null finite sets of objects with $|S_i|=n_i$ each. Suppose that a relation of association is defined for each ordered pair of objects $x_{ia}\in S_i$ and $x_{jb}\in S_j$ and that x_{jb} is called the α-th associate of x_{ia} for some α belonging to a set of association indices, $\Pi^{(i,j)}$. As in the case of association schemes, every object is called the 0-th associate of itself and $0\notin \Pi^{(i,i)}$ is assumed. The following definition is due to Yamamoto, Shirakura and Kuwada [59]:

DEFINITION 3.1. The relation of association defined in a collection of sets $(S_1, S_2,\ldots, S_p)$ is called a p-MDPB association scheme if the following conditions are satisfied:

(i) The relation of association is symmetric, i.e., if x_{jb} is the α-th associate of x_{ia}, then x_{ia} is the α-th associate of x_{jb}.

(ii) With respect to any $x_{ia}\in S_i$, the objects of S_j, distinct from x_{ia}, can be divided into $n^{(i,j)}$ disjoint classes and the number of objects in the α-th associate class $S_j(\alpha; x_{ia})$ is $n_\alpha^{(i,j)}$ for $i, j=1, 2,\ldots, p$, the numbers, $n^{(i,j)}$ and $n_\alpha^{(i,j)}$

being independent of particular objects x_{ia} chosen in S_i.

(iii) Let S_i, S_j and S_k be any three sets which are not necessarily distinct. Further, let x_{jb} ($\in S_j$) be the α-th associate of x_{ia} ($\in S_i$) and consider sets $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$. Then the number of objects common to sets $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$ is a number, $p(i, j, \alpha; k, \beta, \gamma)$, depending on pair (x_{ia}, x_{jb}) and S_k only through i, j, α, k, β and γ.

Note that an association scheme is a special case of a p-MDPB association scheme, when $p=1$.

Now let $S_0, S_1,\dots, S_\ell$ be the $\ell+1$ sets such that $S_0=\{(\phi)\}$, $S_1=\{(t_1)\},\dots, S_\ell=\{(t_1t_2\cdots t_\ell)\}$, where $\{t_1, t_2,\dots, t_k\}$ is a subset of $\{1, 2,\dots, m\}$ and $t_1<t_2<\cdots<t_k$ for $k=1, 2,\dots, \ell$, and $\ell\leqq[m/2]$, in which $[x]$ is the greatest integer not exceeding x. Then we have $|S_k|=n_k=\binom{m}{k}$. Suppose that a relation of association is defined among those sets such that $(t_1t_2\cdots t_u)\in S_u$ and $(t'_1t'_2\cdots t'_v)\in S_v$ are the α-th associates if

$$|\{t_1, t_2,\dots, t_u\} \cap \{t'_1, t'_2,\dots, t'_v\}| = \min(u, v) - \alpha. \tag{3.1}$$

In this case, the following theorem was shown by Yamamoto, Shirakura and Kuwada [59]:

THEOREM 3.1. *The relation of association as defined by* (3.1) *among* $\ell+1$ *sets*, $\{(\phi)\}, \{(t_1)\},\dots, \{(t_1t_2\cdots t_\ell)\}$, *is an* $(\ell+1)$-MDPB *association scheme with parameters*

$$\Pi^{(u,v)} = \begin{cases} \{1, 2,\dots, u\} & \text{if } u = v, \\ \{0, 1,\dots, \min(u, v)\} & \text{if } u \neq v, \end{cases} \tag{3.2}$$

$$n^{(u,v)} = \begin{cases} u & \text{if } u = v, \\ \min(u, v)+1 & \text{if } u \neq v, \end{cases} \tag{3.3}$$

$$n_\alpha^{(u,v)} = \binom{u}{\min(u,v)-\alpha}\binom{m-u}{v-\min(u,v)+\alpha}, \tag{3.4}$$

$$p(u, v, \alpha; w, \beta, \gamma) = \sum_{k=0}^{\min(u,v)-\alpha} \binom{\min(u, v)-\alpha}{k}\binom{u-\min(u, v)+\alpha}{\min(u, w)-\beta-k} \tag{3.5}$$

$$\cdot\binom{v-\min(u, v)+\alpha}{\min(v, w)-\gamma-k}\binom{m-u-v+\min(u, v)-\alpha}{w-\min(u, w)+\beta-\min(v, w)+\gamma+k}.$$

The scheme thus defined is called a triangular $(\ell+1)$-MDPB (TMDPB) association scheme which can be regarded as a generalization of triangular series of association schemes (cf. [58]).

The local association matrices, $A_\alpha^{(u,v)}=\|a_{t_1t_2\cdots t_u;\alpha}^{t'_1t'_2\cdots t'_v}\|$ ($\alpha=0, 1,\dots, \min(u, v)$; $u, v=0, 1,\dots, \ell$), of size $n_u\times n_v$ are defined as

$$a_{t_1t_2\cdots t_u;\alpha}^{t'_1t'_2\cdots t'_v} = \begin{cases} 1 & \text{if } (t'_1t'_2\cdots t'_v) \text{ is the } \alpha\text{-th associate of } (t_1t_2\cdots t_u), \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $n_u \times n_v$ matrices $A_\beta^{\#(u,v)}$ ($\beta=0, 1,\ldots, \min(u, v)$; $u, v=0, 1,\ldots, \ell$) which are linear combinations of the local association matrices, $A_\alpha^{(\alpha,v)}$, as follows (see [44, 60]):

$$\text{(3.6)} \qquad \begin{aligned} A_\alpha^{(u,v)} &= \sum_{\beta=0}^{u} z_{\beta\alpha}^{(u,v)} A_\beta^{\#(u,v)} && \text{for } 0 \leqq \alpha \leqq u \leqq v, \\ A_\beta^{\#(u,v)} &= \sum_{\alpha=0}^{u} z_{(u,v)}^{\beta\alpha} A_\alpha^{(u,v)} && \text{for } 0 \leqq \beta \leqq u \leqq v, \\ A_\beta^{\#(u,v)} &= (A_\beta^{\#(v,u)})' && \text{for } u > v, \end{aligned}$$

where $z_{\beta\alpha}^{(u,v)} = \sum_{b=0}^{\alpha} (-1)^{\alpha-b} \binom{u-\beta}{b} \binom{u-b}{u-\alpha} \binom{m-u-\beta+b}{b} \sqrt{\binom{m-u-\beta}{v-u} \binom{v-\beta}{v-u}} \Big/ \binom{v-u+b}{b}$, $z_{(u,v)}^{\beta\alpha} = \phi_\beta^* z_{\beta\alpha}^{(u,v)} \Big/ \left\{ \binom{m}{u} \binom{u}{\alpha} \binom{m-u}{v-u+\alpha} \right\}$ and $\phi_\beta^* = \binom{m}{\beta} - \binom{m}{\beta-1}$. Then matrices $A_\beta^{\#(u,v)}$ satisfy the following properties:

$$\text{(3.7)} \qquad \begin{aligned} &\sum_{\beta=0}^{u} A_\beta^{\#(u,u)} = I_{n_u}, \\ &A_0^{\#(u,v)} = \left\{ 1 \Big/ \sqrt{\binom{m}{u}\binom{m}{v}} \right\} G_{n_u \times n_v}, \\ &A_\alpha^{\#(u,w)} A_\beta^{\#(w,v)} = \delta_{\alpha\beta} A_\alpha^{\#(u,v)}, \\ &\operatorname{rank}(A_\beta^{\#(u,v)}) = \phi_\beta^*. \end{aligned}$$

The theory of TMDPB association schemes is powerful to obtain various properties of a 2^m-BFF design of resolutions 2ℓ and $2\ell+1$ (cf. [39–44, 59, 60]).

4. MD relationships and their algebra

As seen in the preceding section, the relation of a p-MDPB association scheme is symmetric. In this section, however, we shall consider the case in which the relation is not always symmetric. Nair [29] generalized PBIB designs by relaxing the condition of symmetry in the relation of association.

Consider p mutually disjoint non-null finite sets of objects, $S_1, S_2,\ldots, S_p$, where the number of objects in set S_i is $|S_i| = n_i$ ($i=1, 2,\ldots, p$). Suppose that relationship $R(\alpha; i, j)$ is defined for a set of each ordered pair (x_{ia}, x_{jb}) of objects $x_{ia} \in S_i$ and $x_{jb} \in S_j$. We say that x_{ia} is related to x_{jb} by relationship $R(\alpha; i, j)$, if ordered pair (x_{ia}, x_{jb}) of objects belongs to $R(\alpha; i, j)$, where index α belongs to a set of relationship indices, $\Omega(i, j)$.

DEFINITION 4.1. A collection, $\mathfrak{S}$, of sets, $S_1, S_2,\ldots, S_p$, is said to have a p sets multidimensional (p-MD) relationship if the following conditions are satisfied:

(i) Given any object $x_{ia} \in S_i$, the objects of S_j $(j=1, 2,\ldots, p)$, which are not necessarily distinct from $x_{ia} \in S_i$, can be partitioned into $n(i, j)$ disjoint subsets, where x_{ia} is related to each element of the α-th subset by relationship $R(\alpha; i, j)$. The number, $n(i, j)$, of such subsets, and the number, $n(\alpha; i, j)$, of objects in the α-th subset are independent of particular object x_{ia}, so long as $x_{ia} \in S_i$.

(ii) Let S_i, S_j and S_k be any three sets in $\mathfrak{S}$, where i, j and k are not necessarily distinct. Further, let x_{ia} $(\in S_i)$ be related to x_{jb} $(\in S_j)$ by $R(\alpha; i, j)$. Then the number of objects $x_{kc} \in S_k$ such that x_{ia} is related to x_{kc} by $R(\beta; i, k)$ and x_{kc} is also related to x_{jb} by $R(\gamma; k, j)$, is a constant, $q(i, j, \alpha; k, \beta, \gamma)$, which depends on ordered pair (x_{ia}, x_{jb}) and S_k only through i, j, k, $R(\alpha; i, j)$, $R(\beta; i, k)$ and $R(\gamma; k, j)$.

Let $\mathfrak{S}$ be a collection of $\binom{\ell+2}{2}$ sets of effects $S_{a_1a_2}=\{\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)\}$, where $0 \leqq a_1+a_2 \leqq \ell \leqq [m/2]$. Then the number of effects in set $S_{a_1a_2}$ is $n_{a_1a_2} = |S_{a_1a_2}| = \binom{m}{a_1}\binom{m-a_1}{a_2}$.

Define relationship $R(\boldsymbol{\alpha}; a_1a_2, b_1b_2)$ in such a way that $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ is related to $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ by $R(\boldsymbol{\alpha}; a_1a_2, b_1b_2)$, if components α_{ij} $(i, j=1, 2)$ of relationship indices $\boldsymbol{\alpha}$ satisfy the following formulas:

$$
\begin{aligned}
|\{t_1,\ldots, t_{a_1}\} \cap \{u_1,\ldots, u_{b_1}\}| &= \min(a_1, b_1) - \alpha_{11}, \\
|\{t_1,\ldots, t_{a_1}\} \cap \{u_1',\ldots, u_{b_2}'\}| &= \min(a_1, b_2) - \alpha_{12}, \\
|\{t_1',\ldots, t_{a_2}'\} \cap \{u_1,\ldots, u_{b_1}\}| &= \min(a_2, b_1) - \alpha_{21}, \\
|\{t_1',\ldots, t_{a_2}'\} \cap \{u_1',\ldots, u_{b_2}'\}| &= \min(a_2, b_2) - \alpha_{22},
\end{aligned} \tag{4.1}
$$

where $\boldsymbol{\alpha}=(\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22})$. Note, from (4.1), that if $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ is related to $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ by $R(\boldsymbol{\alpha}; a_1a_2, b_1b_2)$, then $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ is related to $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ by $R(\tilde{\boldsymbol{\alpha}}; b_1b_2, a_1a_2)$, where $\tilde{\boldsymbol{\alpha}}=(\alpha_{11}\alpha_{21}\alpha_{12}\alpha_{22})$. In this case, we have

THEOREM 4.1. *Among* $\binom{\ell+2}{2}$ *sets of effects* $S_{a_1a_2}$ $(0 \leqq a_1+a_2 \leqq \ell)$, *suppose that relationship indices* $\boldsymbol{\alpha}$ *are defined by* (4.1). *Then the collection of these sets of effects have a* $\binom{\ell+2}{2}$-*MD relationship with parameters*

$$
\begin{aligned}
\Omega(a_1a_2, b_1b_2) = \{(\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}) \mid\, & 0 \leqq \alpha_{11}^* \leqq \min(a_1, b_1), \\
& 0 \leqq \alpha_{12}^* \leqq \min(a_1 - \alpha_{11}^*, b_2), \\
& 0 \leqq \alpha_{21}^* \leqq \min(a_2, b_1 - \alpha_{11}^*), \\
& 0 \leqq \alpha_{22}^* \leqq \min(a_2 - \alpha_{21}^*, b_2 - \alpha_{12}^*)\},
\end{aligned} \tag{4.2}
$$

(4.3) $n(a_1a_2,\ b_1b_2)$

$$= \sum_{\alpha^*_{11}=0}^{\min(a_1,b_1)} \sum_{\alpha^*_{12}=0}^{\min(a_1-\alpha^*_{11},b_2)} \sum_{\alpha^*_{21}=0}^{\min(a_2,b_1-\alpha^*_{11})} \{\min(a_2-\alpha^*_{21},\ b_2-\alpha^*_{12})+1\},$$

(4.4) $n(\boldsymbol{\alpha};\ a_1a_2,\ b_1b_2)$

$$=\binom{a_1}{\alpha^*_{11}}\binom{a_1-\alpha^*_{11}}{\alpha^*_{12}}\binom{a_2}{\alpha^*_{21}}\binom{a_2-\alpha^*_{21}}{\alpha^*_{22}}\binom{m-a}{b_1-\alpha^*_{\cdot 1}}\binom{m-a-b_1+\alpha^*_{\cdot 1}}{b_2-\alpha^*_{\cdot 2}},$$

(4.5) $q(a_1a_2,\ b_1b_2,\ \boldsymbol{\alpha};\ c_1c_2,\ \boldsymbol{\beta},\ \boldsymbol{\gamma})$

$$= \sum_{k_{11}=0}^{\alpha^*_{11}} \sum_{k_{12}=0}^{\alpha^*_{12}} \sum_{k_{21}=0}^{\alpha^*_{21}} \sum_{k_{22}=0}^{\alpha^*_{22}} \sum_{h_{11}=0}^{\alpha^*_{11}-k_{11}} \sum_{h_{12}=0}^{\alpha^*_{12}-k_{12}} \sum_{h_{21}=0}^{\alpha^*_{21}-k_{21}} \sum_{h_{22}=0}^{\alpha^*_{22}-k_{22}} \binom{\alpha^*_{11}}{k_{11}}\binom{\alpha^*_{11}-k_{11}}{h_{11}}$$

$$\cdot\binom{\alpha^*_{12}}{k_{12}}\binom{\alpha^*_{12}-k_{12}}{h_{12}}\binom{\alpha^*_{21}}{k_{21}}\binom{\alpha^*_{21}-k_{21}}{h_{21}}\binom{\alpha^*_{22}}{k_{22}}\binom{\alpha^*_{22}-k_{22}}{h_{22}}\binom{a_1-\alpha^*_{1\cdot}}{\beta^*_{11}-k_{1\cdot}}$$

$$\cdot\binom{a_1-\alpha^*_{1\cdot}-\beta^*_{11}+k_{1\cdot}}{\beta^*_{12}-h_{1\cdot}}\binom{a_2-\alpha^*_{2\cdot}}{\beta^*_{21}-k_{2\cdot}}\binom{a_2-\alpha^*_{2\cdot}-\beta^*_{21}+k_{2\cdot}}{\beta^*_{22}-h_{2\cdot}}$$

$$\cdot\binom{b_1-\alpha^*_{\cdot 1}}{\gamma^*_{11}-k_{\cdot 1}}\binom{b_1-\alpha^*_{\cdot 1}-\gamma^*_{11}+k_{\cdot 1}}{\gamma^*_{21}-h_{\cdot 1}}\binom{b_2-\alpha^*_{\cdot 2}}{\gamma^*_{12}-k_{\cdot 2}}\binom{b_2-\alpha^*_{\cdot 2}-\gamma^*_{12}+k_{\cdot 2}}{\gamma^*_{22}-h_{\cdot 2}}$$

$$\cdot\binom{m-(a+b)+\alpha^*_{\cdot\cdot}}{c_1-(\beta^*_{\cdot 1}+\gamma^*_{1\cdot})+k_{\cdot\cdot}}\binom{m-(a+b)+\alpha^*_{\cdot\cdot}-c_1+\beta^*_{\cdot 1}+\gamma^*_{1\cdot}-k_{\cdot\cdot}}{c_2-(\beta^*_{\cdot 2}+\gamma^*_{2\cdot})+h_{\cdot\cdot}},$$

where $\alpha^*_{ij}=\min(a_i,\ b_j)-\alpha_{ij}$, $\beta^*_{ij}=\min(a_i,\ c_j)-\beta_{ij}$, $\gamma^*_{ij}=\min(c_i,\ b_j)-\gamma_{ij}$, $\alpha^*_{i\cdot}=\alpha^*_{i1}+\alpha^*_{i2}$, $\alpha^*_{\cdot j}=\alpha^*_{1j}+\alpha^*_{2j}$, $\alpha^*_{\cdot\cdot}=\alpha^*_{11}+\alpha^*_{12}+\alpha^*_{21}+\alpha^*_{22}$, $\beta^*_{\cdot j}=\beta^*_{1j}+\beta^*_{2j}$, $\gamma^*_{i\cdot}=\gamma^*_{i1}+\gamma^*_{i2}$, $k_{i\cdot}=k_{i1}+k_{i2}$, $k_{\cdot j}=k_{1j}+k_{2j}$, $k_{\cdot\cdot}=k_{11}+k_{12}+k_{21}+k_{22}$, $h_{i\cdot}=h_{i1}+h_{i2}$, $h_{\cdot j}=h_{1j}+h_{2j}$, $h_{\cdot\cdot}=h_{11}+h_{12}+h_{21}+h_{22}$, $a=a_1+a_2$ *and* $b=b_1+b_2$ *for* $i,\ j=1,\ 2$.

Proof. For any $\theta(t^1_1\cdots t^1_{a_1}t'^2_1\cdots t'^2_{a_2})\in S_{a_1a_2}$, consider the number of disjoint subsets $S_{b_1b_2}(\boldsymbol{\alpha};\ \theta(t^1_1\cdots t^1_{a_1}t'^2_1\cdots t'^2_{a_2}))$ into which the objects of $S_{b_1b_2}$ can be partitioned, where $\theta(t^1_1\cdots t^1_{a_1}t'^2_1\cdots t'^2_{a_2})$ is related to $\theta(u^1_1\cdots u^1_{b_1}u'^2_1\cdots u'^2_{b_2})$ $(\in S_{b_1b_2})$ by $R(\boldsymbol{\alpha};\ a_1a_2,\ b_1b_2)$. Let $\alpha^*_{11}=|\{t_1,\dots,t_{a_1}\}\cap\{u_1,\dots,u_{b_1}\}|$, $\alpha^*_{12}=|\{t_1,\dots,t_{a_1}\}\cap\{u'_1,\dots,u'_{b_2}\}|$, $\alpha^*_{21}=|\{t'_1,\dots,t'_{a_2}\}\cap\{u_1,\dots,u_{b_1}\}|$ and $\alpha^*_{22}=|\{t'_1,\dots,t'_{a_2}\}\cap\{u'_1,\dots,u'_{b_2}\}|$. Then there exist integers α^*_{11}, α^*_{12}, α^*_{21} and α^*_{22} such that $\alpha^*_{11}=\alpha^*_{12}=\alpha^*_{21}=\alpha^*_{22}=0$, since $0\leqq a_1+a_2\leqq\ell$ and $0\leqq b_1+b_2\leqq\ell$. Hence $0\leqq\alpha^*_{12}\leqq\min(a_1-\alpha^*_{11},\ b_2)$ and $0\leqq\alpha^*_{21}\leqq\min(a_2,\ b_1-\alpha^*_{11})$ for $\alpha^*_{11}=0,\ 1,\dots,\min(a_1,\ b_1)$. For any α^*_{11}, α^*_{12} and α^*_{21} satisfying the above conditions, it holds that $0\leqq\alpha^*_{22}\leqq\min(a_2-\alpha^*_{21},\ b_2-\alpha^*_{12})$. Therefore, we get (4.2) and (4.3). The cardinality, $n(\boldsymbol{\alpha};\ a_1a_2,\ b_1b_2)$, of set $S_{b_1b_2}(\boldsymbol{\alpha};\ \theta(t^1_1\cdots t^1_{a_1}t'^2_1\cdots t'^2_{a_2}))$ in (4.4) will be given by counting the number of sets $\{u_1,\dots,u_{b_1},\ u'_1,\dots,u'_{b_2}\}$ with cardinality b_1+b_2 satisfying $\alpha^*_{11}=|\{t_1,\dots,t_{a_1}\}\cap\{u_1,\dots,u_{b_1}\}|$, $\alpha^*_{12}=|\{t_1,\dots,t_{a_1}\}\cap\{u'_1,\dots,u'_{b_2}\}|$, $\alpha^*_{21}=|\{t'_1,\dots,t'_{a_2}\}\cap\{u_1,\dots,u_{b_1}\}|$ and

$\alpha_{22}^* = |\{t'_1,\dots, t'_{a_2}\} \cap \{u'_1,\dots, u'_{b_2}\}|$, where $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2) \in S_{b_1 b_2}$. Since those numbers are independent of the particular choice of object $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ in $S_{a_1 a_2}$, the condition (i) of the MD relationship is satisfied.

The condition (ii) of the MD relationship can be verified by counting the number of those $\theta(v_1^1\cdots v_{c_1}^1 v_1'^2\cdots v_{c_2}'^2)$ in $S_{c_1 c_2}$ such that $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ is related to $\theta(v_1^1\cdots v_{c_1}^1 v_1'^2\cdots v_{c_2}'^2)$ by $R(\boldsymbol{\beta}; a_1 a_2, c_1 c_2)$ and $\theta(v_1^1\cdots v_{c_1}^1 v_1'^2\cdots v_{c_2}'^2)$ is related to $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ by $R(\boldsymbol{\gamma}; c_1 c_2, b_1 b_2)$, where $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$ is related to $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ by $R(\boldsymbol{\alpha}; a_1 a_2, b_1 b_2)$. This is equivalent to the counting of the number of sets $\{v_1,\dots, v_{c_1}, v'_1,\dots, v'_{c_2}\}$ such that $\{t_1,\dots, t_{a_1}\}$, $\{t_1,\dots, t_{a_1}\}$, $\{t'_1,\dots, t'_{a_2}\}$ and $\{t'_1,\dots, t'_{a_2}\}$ have β_{11}^*, β_{12}^*, β_{21}^* and β_{22}^* intersections with $\{v_1,\dots, v_{c_1}\}$, $\{v'_1,\dots, v'_{c_2}\}$, $\{v_1,\dots, v_{c_1}\}$ and $\{v'_1,\dots, v'_{c_2}\}$, respectively, and $\{v_1,\dots, v_{c_1}\}$, $\{v_1,\dots, v_{c_1}\}$, $\{v'_1,\dots, v'_{c_2}\}$ and $\{v'_1,\dots, v'_{c_2}\}$ have γ_{11}^*, γ_{12}^*, γ_{21}^* and γ_{22}^* intersections with $\{u_1,\dots, u_{b_1}\}$, $\{u'_1,\dots, u'_{b_2}\}$, $\{u_1,\dots, u_{b_1}\}$ and $\{u'_1,\dots, u'_{b_2}\}$, respectively. The number, therefore, is given by (4.5). It can further be seen that the number, $q(a_1 a_2, b_1 b_2, \boldsymbol{\alpha}; c_1 c_2, \boldsymbol{\beta}, \boldsymbol{\gamma})$, is dependent on $\theta(t_1^1\cdots t_{a_1}^1 t_1'^2\cdots t_{a_2}'^2)$, $\theta(u_1^1\cdots u_{b_1}^1 u_1'^2\cdots u_{b_2}'^2)$ and $S_{c_1 c_2}$ only through $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, $R(\boldsymbol{\alpha}; a_1 a_2, b_1 b_2)$, $R(\boldsymbol{\beta}; a_1 a_2, c_1 c_2)$ and $R(\boldsymbol{\gamma}; c_1 c_2, b_1 b_2)$. Thus, the proof is completed.

Note that, for pairs (a_1, a_2), (b_1, b_2) and (c_1, c_2), if at least one of elements of each pair is zero, parameters (4.2), (4.3), (4.4) and (4.5) can be reducible to (3.2), (3.3), (3.4) and (3.5), respectively.

REMARK. For S_{00}, S_{10}, S_{01}, S_{20} and S_{02}, relationships $R(\boldsymbol{\alpha}; a_1 a_2, b_1 b_2)$ with relationship indices $\boldsymbol{\alpha}$ defined among those sets by (4.1) are similar to the relations of TMDPB association schemes defined by (3.1).

COROLLARY 4.2. *Among six sets of effects* $\{\theta(\phi)\}$, $\{\theta(t^1)\}$, $\{\theta(t^2)\}$, $\{\theta(t_1^1 t_2^1)\}$, $\{\theta(t_1^2 t_2^2)\}$ *and* $\{\theta(t_3^1 t_4^2)\}$, *suppose that relationship indices* $\boldsymbol{\alpha}$ *are given by* (4.1). *Then the collection of those sets of effects has a* 6-*MD relationship.*

Let $w_r(\varepsilon_1,\dots, \varepsilon_{k_1}) = a_r$ and $w_r(\varepsilon'_1,\dots, \varepsilon'_{k_2}) = b_r$ for $r = 1, 2$. Then the local relationship matrices, $A_{\boldsymbol{\alpha}}^{(a_1 a_2, b_1 b_2)} = \|a(t_1^{\varepsilon_1}\cdots t_{k_1}^{\varepsilon_{k_1}}; t_1'^{\varepsilon'_1}\cdots t_{k_2}'^{\varepsilon'_{k_2}})_{\boldsymbol{\alpha}}\|$, of size $n_{a_1 a_2} \times n_{b_1 b_2}$ are defined as

$$(4.6)\qquad a(t_1^{\varepsilon_1}\cdots t_{k_1}^{\varepsilon_{k_1}}; t_1'^{\varepsilon'_1}\cdots t_{k_2}'^{\varepsilon'_{k_2}})_{\boldsymbol{\alpha}} = \begin{cases} 1 & \text{if } \theta(t_1^{\varepsilon_1}\cdots t_{k_1}^{\varepsilon_{k_1}}) \text{ is related to } \theta(t_1'^{\varepsilon'_1}\cdots t_{k_2}'^{\varepsilon'_{k_2}}) \\ & \text{by } R(\boldsymbol{\alpha}; a_1 a_2, b_1 b_2), \\ 0 & \text{otherwise.} \end{cases}$$

In Parts I and II, we shall mainly discuss the case where $\ell = 2$, i.e., $S_{00} =$

$\{\theta(\phi)\}$, $S_{10}=\{\theta(t^1)\}$, $S_{01}=\{\theta(t^2)\}$, $S_{20}=\{\theta(t_1^1t_2^1)\}$, $S_{02}=\{\theta(t_1^2t_2^2)\}$ and $S_{11}=\{\theta(t_3^1t_4^2)\}$, with $n_{00}=1$, $n_{10}=n_{01}=\binom{m}{1}$, $n_{20}=n_{02}=\binom{m}{2}$ and $n_{11}=2\binom{m}{2}$, respectively. From (4.1) and Corollary 4.2, we have

$$
(4.7)\quad
\begin{aligned}
&A_{(0000)}^{(a_1a_2,a_1a_2)} = I_{\binom{m}{a_1+a_2}} \quad \text{for} \quad a_1a_2 = 00,\ 10,\ 01,\ 20,\ 02,\\
&A_{(0110)}^{(11,11)} = I_{2\binom{m}{2}},\\
&A_{\boldsymbol{\alpha}}^{(b_1b_2,a_1a_2)} = (A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)})',\\
&\sum_{\boldsymbol{\alpha}} A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} = G_{n_{a_1a_2}\times n_{b_1b_2}},\\
&A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} j_{n_{b_1b_2}} = n(\boldsymbol{\alpha};\, a_1a_2,\, b_1b_2)\, j_{n_{a_1a_2}},\\
&A_{\boldsymbol{\beta}}^{(a_1a_2,c_1c_2)} A_{\boldsymbol{\gamma}}^{(c_1c_2,b_1b_2)} = \sum_{\boldsymbol{\alpha}} q(a_1a_2,\, b_1b_2,\, \boldsymbol{\alpha};\, c_1c_2,\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}.
\end{aligned}
$$

The relationship matrices, $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$, of order v_m are defined such that every matrix has thirty-six submatrices $M^{(u_1u_2,v_1v_2)}$ of size $n_{u_1u_2}\times n_{v_1v_2}$, whose row and column blocks correspond, respectively, to sets $\{\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2})\}$ and $\{\theta(t_3^{\varepsilon_3}t_4^{\varepsilon_4})\}$, and such that all but $M^{(a_1a_2,b_1b_2)}=A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ are zero submatrices, where $u_r=w_r(\varepsilon_1,\varepsilon_2)$ and $v_r=w_r(\varepsilon_3,\varepsilon_4)$ for $r=1,2$. From (4.7), some properties of relationship matrices $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ are cited in the following:

$$
(4.8)\quad
\begin{aligned}
&D_{(0000)}^{(00,00)} + D_{(0000)}^{(10,10)} + D_{(0000)}^{(01,01)} + D_{(0000)}^{(20,20)} + D_{(0000)}^{(02,02)} + D_{(0110)}^{(11,11)} = I_{v_m},\\
&D_{\boldsymbol{\alpha}}^{(b_1b_2,a_1a_2)} = (D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)})',\\
&\sum_{a_1a_2}\sum_{b_1b_2}\sum_{\boldsymbol{\alpha}} D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} = G_{v_m},\\
&D_{\boldsymbol{\beta}}^{(a_1a_2,c_1c_2)} D_{\boldsymbol{\gamma}}^{(d_1d_2,b_1b_2)} = \delta_{c_1d_1}\delta_{c_2d_2}\sum_{\boldsymbol{\alpha}} q(a_1a_2,\, b_1b_2,\, \boldsymbol{\alpha};\, c_1c_2,\, \boldsymbol{\beta},\, \boldsymbol{\gamma}) D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}.
\end{aligned}
$$

The symmetric matrices, $B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$, of order v_m are defined as

$$
B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} =
\begin{cases}
D_{\boldsymbol{\alpha}}^{(a_1a_2,a_1a_2)} & \text{if } a_1a_2 = b_1b_2 = 00,\ 10,\ 01,\ 20,\ 02,\\
D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} + D_{\boldsymbol{\alpha}}^{(b_1b_2,a_1a_2)} & \\
& \text{if } a_1a_2 \neq b_1b_2 \text{ and } a_1a_2 = 00,\ 10,\ 01,\\
& \quad 20,\ 02;\ b_1b_2 = 00,\ 10,\ 01,\ 20,\ 02,\ 11,\\
D_{\boldsymbol{\alpha}}^{(11,11)} & \text{if } a_1a_2 = b_1b_2 = 11 \text{ and } \boldsymbol{\alpha} = (0110),\\
& \quad (1001),\ (0111),\ (1110),\ (1111),\\
D_{(1011)}^{(11,11)} + D_{(1101)}^{(11,11)} & \text{if } a_1a_2 = b_1b_2 = 11\\
& \quad \text{and } \boldsymbol{\alpha} = (1011) \text{ or } (1101).
\end{cases}
$$

Then we have the following:

THEOREM 4.3. *The algebra,* $\mathfrak{A}=\{B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}\}$, *generated by forty-nine symmetric matrices* $B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ *is a semi-simple, completely reducible matrix algebra containing* I_{v_m}. $\mathfrak{A}$ *can also be represented by* $[D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} | \boldsymbol{\alpha}\in\Omega(a_1a_2, b_1b_2); a_1a_2, b_1b_2=00, 10, 01, 20, 02, 11]$ *of all eighty-two relationship matrices* $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$.

PROOF. Since all generators of matrix algebra $\mathfrak{A}$ are symmetric, $\mathfrak{A}$ is semi-simple and completely reducible. From definitions of $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ and $B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$,

$$(4.9)\quad D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}=\begin{cases} B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}B_{(0000)}^{(b_1b_2,b_1b_2)} & \text{for } a_1a_2=00,10,01,20,02,11 \\ & \text{and } b_1b_2=00,10,01,20,02, \\ B_{(0000)}^{(a_1a_2,a_1a_2)}B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} & \text{for } a_1a_2=00,10,01,20,02 \\ & \text{and } b_1b_2=00,10,01,20,02,11, \\ B_{\boldsymbol{\alpha}}^{(11,11)}B_{(0110)}^{(11,11)} & \text{for } a_1a_2=b_1b_2=11 \\ & \text{and } \boldsymbol{\alpha}=(0110),(1001),(0111),(1110),(1111), \\ B_{(1001)}^{(11,11)}B_{(1110)}^{(11,11)} & \text{for } a_1a_2=b_1b_2=11 \\ & \text{and } \boldsymbol{\alpha}=(1011), \\ B_{(1001)}^{(11,11)}B_{(0111)}^{(11,11)} & \text{for } a_1a_2=b_1b_2=11 \\ & \text{and } \boldsymbol{\alpha}=(1101), \end{cases}$$

$$(4.10)\quad B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}B_{\boldsymbol{\beta}}^{(c_1c_2,d_1d_2)}$$
$$= D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}D_{\boldsymbol{\beta}}^{(c_1c_2,d_1d_2)}+D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}D_{\underline{\boldsymbol{\beta}}}^{(d_1d_2,c_1c_2)}$$
$$+D_{\underline{\boldsymbol{\alpha}}}^{(b_1b_2,a_1a_2)}D_{\boldsymbol{\beta}}^{(c_1c_2,d_1d_2)}+D_{\underline{\boldsymbol{\alpha}}}^{(b_1b_2,a_1a_2)}D_{\underline{\boldsymbol{\beta}}}^{(d_1d_2,c_1c_2)}.$$

Relations (4.8) through (4.10) show that $\mathfrak{A}=[D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}]$.

Note (cf. [5]) that combining two matrices $D_{(1011)}^{(11,11)}$ and $D_{(1101)}^{(11,11)}$ as $D_{(1{*}{*}1)}^{(11,11)}=D_{(1011)}^{(11,11)}+D_{(1101)}^{(11,11)}$, six matrices $D_{\boldsymbol{\alpha}}^{(11,11)}$ ($\boldsymbol{\alpha}$=(0110), (1001), (0111), (1110), (1**1), (1111)), which are not the linear closure, turn out to be symmetric.

From Appendix I, let $D_{\beta}^{\#(a_1a_2,b_1b_2)}$ ($\beta=0, 1, 2$; $a_1a_2, b_1b_2=00, 10, 01, 20, 02, 11$) and $D_{fij}^{\#(u_1u_2,v_1v_2)}$ ($i, j=1, 2, 3, 4$; $u_1u_2, v_1v_2=10, 01, 20, 02, 11$) be, respectively, the matrices obtained by replacing only non-zero submatrices $A_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ of $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ by $A_{\beta}^{\#(a_1a_2,b_1b_2)}$ and $A_{fij}^{\#(u_1u_2,v_1v_2)}$. Then, from (A.4) we have

$$
(4.11)\quad \left\{
\begin{aligned}
&D^{(00,00)}_{(0000)} = D^{\#(00,00)}_{0},\\
&D^{(00,b_1b_2)}_{(0000)} = \sqrt{m}\, D^{\#(00,b_1b_2)}_{0},\\
&D^{(00,b'_1b'_2)}_{(0000)} = \sqrt{\binom{m}{2}}D^{\#(00,b'_1b'_2)}_{0},\\
&D^{(00,11)}_{(0000)} = \sqrt{2\binom{m}{2}}D^{\#(00,11)}_{0},\\
&D^{(a_1a_2,b_1b_2)}_{(0000)} = D^{\#(a_1a_2,b_1b_2)}_{0} + D^{\#(a_1a_2,b_1b_2)}_{f_{11}},\\
&D^{(a_1a_2,b_1b_2)}_{\alpha_1} = (m-1)D^{\#(a_1a_2,b_1b_2)}_{0} - D^{\#(a_1a_2,b_1b_2)}_{f_{11}},\\
&D^{(a_1a_2,b'_1b'_2)}_{(0000)} = \sqrt{2(m-1)}\,D^{\#(a_1a_2,b'_1b'_2)}_{0} + \sqrt{m-2}\,D^{\#(a_1a_2,b'_1b'_2)}_{f_{12}},\\
&D^{(a_1a_2,b'_1b'_2)}_{\beta_1} = (m-2)\sqrt{(m-1)/2}D^{\#(a_1a_2,b'_1b'_2)}_{0} - \sqrt{m-2}D^{\#(a_1a_2,b'_1b'_2)}_{f_{12}},\\
&D^{(a_1a_2,11)}_{\xi_0} = \sqrt{m-1}D^{\#(a_1a_2,11)}_{0} + \sqrt{m/2}D^{\#(a_1a_2,11)}_{f_{13}}\\
&\qquad\qquad + \sqrt{(m-2)/2}D^{\#(a_1a_2,11)}_{f_{14}},\\
&D^{(a_1a_2,11)}_{\xi_1} = \sqrt{m-1}D^{\#(a_1a_2,11)}_{0} - \sqrt{m/2}D^{\#(a_1a_2,11)}_{f_{13}}\\
&\qquad\qquad + \sqrt{(m-2)/2}D^{\#(a_1a_2,11)}_{f_{14}},\\
&D^{(a_1a_2,11)}_{\xi_2} = (m-2)\sqrt{m-1}D^{\#(a_1a_2,11)}_{0} - \sqrt{2(m-2)}D^{\#(a_1a_2,11)}_{f_{14}},\\
&D^{(a'_1a'_2,b'_1b'_2)}_{(0000)} = D^{\#(a'_1a'_2,b'_1b'_2)}_{0} + D^{\#(a'_1a'_2,b'_1b'_2)}_{1} + D^{\#(a'_1a'_2,b'_1b'_2)}_{f_{22}},\\
&D^{(a'_1a'_2,b'_1b'_2)}_{\gamma_1} = 2(m-2)D^{\#(a'_1a'_2,b'_1b'_2)}_{0} - 2D^{\#(a'_1a'_2,b'_1b'_2)}_{1} + (m-4)D^{\#(a'_1a'_2,b'_1b'_2)}_{f_{22}},\\
&D^{(a'_1a'_2,b'_1b'_2)}_{\gamma_2} = \binom{m-2}{2}D^{\#(a'_1a'_2,b'_1b'_2)}_{0} + D^{\#(a'_1a'_2,b'_1b'_2)}_{1} - (m-3)D^{\#(a'_1a'_2,b'_1b'_2)}_{f_{22}},\\
&D^{(a'_1a'_2,11)}_{(0000)} = \sqrt{2}\,\{D^{\#(a'_1a'_2,11)}_{0} + D^{\#(a'_1a'_2,11)}_{1} + D^{\#(a'_1a'_2,11)}_{f_{24}}\},\\
&D^{(a'_1a'_2,11)}_{\zeta_1} = \{1/\sqrt{2}\}\{2(m-2)D^{\#(a'_1a'_2,11)}_{0} - 2D^{\#(a'_1a'_2,11)}_{1}\\
&\qquad\qquad + \sqrt{m(m-2)}\,D^{\#(a'_1a'_2,11)}_{f_{23}} + (m-4)D^{\#(a'_1a'_2,11)}_{f_{24}}\},\\
&D^{(a'_1a'_2,11)}_{\zeta_2} = \{1/\sqrt{2}\}\{2(m-2)D^{\#(a'_1a'_2,11)}_{0} - 2D^{\#(a'_1a'_2,11)}_{1}\\
&\qquad\qquad - \sqrt{m(m-2)}\,D^{\#(a'_1a'_2,11)}_{f_{23}} + (m-4)D^{\#(a'_1a'_2,11)}_{f_{24}}\},\\
&D^{(a'_1a'_2,11)}_{\zeta_3} = \sqrt{2}\left\{\binom{m-2}{2}D^{\#(a'_1a'_2,11)}_{0} + D^{\#(a'_1a'_2,11)}_{1} - (m-3)D^{\#(a'_1a'_2,11)}_{f_{24}}\right\},
\end{aligned}
\right.
$$

$$\left\{\begin{aligned}
D_{(0110)}^{(11,11)} &= D_0^{\#(11,11)} + D_1^{\#(11,11)} + D_2^{\#(11,11)} + D_{f_{33}}^{\#(11,11)} + D_{f_{44}}^{\#(11,11)},\\
D_{(1001)}^{(11,11)} &= D_0^{\#(11,11)} + D_1^{\#(11,11)} - D_2^{\#(11,11)} - D_{f_{33}}^{\#(11,11)} + D_{f_{44}}^{\#(11,11)},\\
D_{(0111)}^{(11,11)} &= (m-2)D_0^{\#(11,11)} - D_1^{\#(11,11)} - D_2^{\#(11,11)}\\
&\quad + \{(m-2)/2\}D_{f_{33}}^{\#(11,11)} + \{\sqrt{m(m-2)}/2\}\{D_{f_{34}}^{\#(11,11)}\\
&\quad + D_{f_{43}}^{\#(11,11)}\} + \{(m-4)/2\}D_{f_{44}}^{\#(11,11)},\\
D_{(1110)}^{(11,11)} &= (m-2)D_0^{\#(11,11)} - D_1^{\#(11,11)} - D_2^{\#(11,11)}\\
&\quad + \{(m-2)/2\}D_{f_{33}}^{\#(11,11)} - \{\sqrt{m(m-2)}/2\}\{D_{f_{34}}^{\#(11,11)}\\
&\quad + D_{f_{43}}^{\#(11,11)}\} + \{(m-4)/2\}D_{f_{44}}^{\#(11,11)},\\
D_{(1011)}^{(11,11)} &= (m-2)D_0^{\#(11,11)} - D_1^{\#(11,11)} + D_2^{\#(11,11)}\\
&\quad - \{(m-2)/2\}D_{f_{33}}^{\#(11,11)} + \{\sqrt{m(m-2)}/2\}\{D_{f_{34}}^{\#(11,11)}\\
&\quad - D_{f_{43}}^{\#(11,11)}\} + \{(m-4)/2\}D_{f_{44}}^{\#(11,11)},\\
D_{(1101)}^{(11,11)} &= (m-2)D_0^{\#(11,11)} - D_1^{\#(11,11)} + D_2^{\#(11,11)}\\
&\quad - \{(m-2)/2\}D_{f_{33}}^{\#(11,11)} - \{\sqrt{m(m-2)}/2\}\{D_{f_{34}}^{\#(11,11)}\\
&\quad - D_{f_{43}}^{\#(11,11)}\} + \{(m-4)/2\}D_{f_{44}}^{\#(11,11)},\\
D_{(1111)}^{(11,11)} &= 2\binom{m-2}{2}D_0^{\#(11,11)} + 2D_1^{\#(11,11)} - 2(m-3)D_{f_{44}}^{\#(11,11)},
\end{aligned}\right.$$

where $\boldsymbol{\alpha}_1=(1000), (0100), (0001)$ according as $(a_1a_2,\ b_1b_2)=(10, 10), (10, 01), (01, 01)$; $\boldsymbol{\beta}_1=(1000), (0100), (0010), (0001)$ according as $(a_1a_2,\ b_1'b_2')=(10, 20), (10, 02), (01, 20), (01, 02)$; $\boldsymbol{\xi}_0=(0100), (0001)$, $\boldsymbol{\xi}_1=(1000), (0010)$, $\boldsymbol{\xi}_2=(1100), (0011)$ according as $a_1a_2=10, 01$; $\boldsymbol{\gamma}_r=(r000), (0r00), (000r)$ $(r=1, 2)$ according as $(a_1'a_2',\ b_1'b_2')=(20, 20), (20, 02), (02, 02)$; $\boldsymbol{\zeta}_1=(0100), (0001)$, $\boldsymbol{\zeta}_2=(1000), (0010)$, $\boldsymbol{\zeta}_3=(1100), (0011)$ according as $a_1'a_2'=20, 02$, respectively.

Note that matrices $D_\beta^{\#(a_1a_2,b_1b_2)}$ and $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ satisfy the following properties:

$$(4.12)\quad \left\{\begin{aligned}
&D_\alpha^{\#(a_1a_2,c_1c_2)}D_\beta^{\#(d_1d_2,b_1b_2)} = \delta_{c_1d_1}\delta_{c_2d_2}\delta_{\alpha\beta}D_\alpha^{\#(a_1a_2,b_1b_2)},\\
&D_\beta^{\#(a_1a_2,b_1b_2)}D_{f_{ij}}^{\#(u_1u_2,v_1v_2)} = D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}D_\beta^{\#(a_1a_2,b_1b_2)} = O_{v_m\times v_m},\\
&D_{f_{ik}}^{\#(u_1u_2,w_1w_2)}D_{f_{lj}}^{\#(s_1s_2,v_1v_2)} = \delta_{w_1s_1}\delta_{w_2s_2}\delta_{kl}D_{f_{ij}}^{\#(u_1u_2,v_1v_2)},\\
&D_0^{\#(00,00)} + D_0^{\#(10,10)} + D_0^{\#(01,01)} + D_0^{\#(20,20)} + D_0^{\#(02,02)} + D_0^{\#(11,11)}\\
&\quad + D_1^{\#(20,20)} + D_1^{\#(02,02)} + D_1^{\#(11,11)} + D_2^{\#(11,11)} + D_{f_{11}}^{\#(10,10)}\\
&\quad + D_{f_{11}}^{\#(01,01)} + D_{f_{22}}^{\#(20,20)} + D_{f_{22}}^{\#(02,02)} + D_{f_{33}}^{\#(11,11)} + D_{f_{44}}^{\#(11,11)} = I_{v_m},
\end{aligned}\right.$$

$$
\begin{aligned}
&\operatorname{rank}(D_\beta^{\#(a_1a_2,b_1b_2)}) = \phi_\beta = \begin{cases} 1 & \text{if } \beta = 0, \\ m(m-3)/2 & \text{if } \beta = 1, \\ \binom{m-1}{2} & \text{if } \beta = 2, \end{cases} \\
&\operatorname{rank}(D_{fij}^{\#(u_1u_2,v_1v_2)}) = \phi_f = m-1.
\end{aligned} \tag{4.13}
$$

Note that $\phi_0=\phi_0^*$, $\phi_1=\phi_2^*$ and $\phi_f=\phi_1^*$, where ϕ_β^* ($\beta=0, 1, 2$) are defined by (3.7).

Let $\mathfrak{A}_0=[D_0^{\#(a_1a_2,b_1b_2)}|a_1a_2, b_1b_2=00, 10, 01, 20, 02, 11]$, $\mathfrak{A}_1=[D_1^{\#(c_1c_2,d_1d_2)}|c_1c_2, d_1d_2=20, 02, 11]$, $\mathfrak{A}_2=[D_2^{\#(11,11)}]$ and $\mathfrak{A}_f=[D_{fij}^{\#(u_1u_2,v_1v_2)}|u_1u_2, v_1v_2=10, 01, 20, 02, 11; i, j=1, 2, 3, 4]$. Then, from (4.12), we have $\mathfrak{A}_\alpha\mathfrak{A}_\beta=\mathfrak{A}_\beta\mathfrak{A}_\alpha=\delta_{\alpha\beta}\mathfrak{A}_\alpha$ for $\alpha, \beta=0, 1, 2, f$. From Theorem 4.3 and (4.11) through (4.13), the following theorem can be established:

THEOREM 4.4 (Kuwada [25]). *The MD relationship algebra, $\mathfrak{A}$, which is generated by forty-nine symmetric matrices $B_\alpha^{(a_1a_2,b_1b_2)}$ and is expressed by the linear closure of all eighty-two relationship matrices $D_\alpha^{(a_1a_2,b_1b_2)}$, is also represented by the linear closure of all eighty-two matrices $D_\beta^{\#(a_1a_2,b_1b_2)}$ and $D_{fij}^{\#(u_1u_2,v_1v_2)}$. The algebra, $\mathfrak{A}$, is decomposed into the direct sum of four two-sided ideals $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_f$. The ideals, $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_f$, are isomorphic to the complete 6×6, 3×3, 1×1 and 6×6 matrix algebras, respectively. The multiplicities of these irreducible representations are given by ϕ_β ($\beta=0, 1, 2, f$).*

This theorem implies that for any symmetric matrix M belonging to $\mathfrak{A}$ as

$$
\begin{aligned}
M &= \sum_{a_1a_2}\sum_{b_1b_2}\sum_{\alpha} h_\alpha^{(a_1a_2,b_1b_2)} D_\alpha^{(a_1a_2,b_1b_2)} \\
&= \sum_{a_1a_2}{}^{*}\sum_{b_1b_2}{}^{*}\sum_{\alpha} h_\alpha^{(a_1a_2,b_1b_2)} B_\alpha^{(a_1a_2,b_1b_2)} \\
&= \sum_{a_1a_2}\sum_{b_1b_2} c_0^{(a_1a_2,b_1b_2)} D_0^{\#(a_1a_2,b_1b_2)} + \sum_{c_1c_2}\sum_{d_1d_2} c_1^{(c_1c_2,d_1d_2)} D_1^{\#(c_1c_2,d_1d_2)} \\
&\quad + c_2^{(11,11)} D_2^{\#(11,11)} + \sum_{u_1u_2}\sum_{v_1v_2}\sum_{i,j} c_{fij}^{(u_1u_2,v_1v_2)} D_{fij}^{\#(u_1u_2,v_1v_2)},
\end{aligned}
$$

there exists a matrix, P, of order v_m such that

$$
P'MP = \operatorname{diag}[\Lambda_0; \underbrace{\Lambda_1,\dots,\Lambda_1}_{\phi_1}; \underbrace{\Lambda_2,\dots,\Lambda_2}_{\phi_2}; \underbrace{\Lambda_f,\dots,\Lambda_f}_{\phi_f}],
$$

where $c_\beta^{(**,**)}$ and $c_{fij}^{(**,**)}$ are the linear combinations of $h_\alpha^{(**,**)}$, and diag $[\Lambda_0; \Lambda_1,\dots, \Lambda_1; \Lambda_2,\dots, \Lambda_2; \Lambda_f,\dots, \Lambda_f]$ denotes a $(\phi_0+\phi_1+\phi_2+\phi_f)\times(\phi_0+\phi_1$

$+\phi_2+\phi_f$) diagonal matrix whose diagonal positions are matrices Λ_0, Λ_1, Λ_2 and Λ_f such that

$$\Lambda_0 = \begin{bmatrix} c_0^{(00,00)} & c_0^{(00,10)} & c_0^{(00,01)} & c_0^{(00,20)} & c_0^{(00,02)} & c_0^{(00,11)} \\ & c_0^{(10,10)} & c_0^{(10,01)} & c_0^{(10,20)} & c_0^{(10,02)} & c_0^{(10,11)} \\ & & c_0^{(01,01)} & c_0^{(01,20)} & c_0^{(01,02)} & c_0^{(01,11)} \\ & & & c_0^{(20,20)} & c_0^{(20,02)} & c_0^{(20,11)} \\ & \text{Sym.} & & & c_0^{(02,02)} & c_0^{(02,11)} \\ & & & & & c_0^{(11,11)} \end{bmatrix},$$

$$\Lambda_1 = \begin{bmatrix} c_1^{(20,20)} & c_1^{(20,02)} & c_1^{(20,11)} \\ & c_1^{(02,02)} & c_1^{(02,11)} \\ \text{Sym.} & & c_1^{(11,11)} \end{bmatrix}, \quad \Lambda_2 = [c_2^{(11,11)}],$$

$$\Lambda_f = \begin{bmatrix} c_{f_{11}}^{(10,10)} & c_{f_{11}}^{(10,01)} & c_{f_{12}}^{(10,20)} & c_{f_{12}}^{(10,02)} & c_{f_{13}}^{(10,11)} & c_{f_{14}}^{(10,11)} \\ & c_{f_{11}}^{(01,01)} & c_{f_{12}}^{(01,20)} & c_{f_{12}}^{(01,02)} & c_{f_{13}}^{(01,11)} & c_{f_{14}}^{(01,11)} \\ & & c_{f_{22}}^{(20,20)} & c_{f_{22}}^{(20,02)} & c_{f_{23}}^{(20,11)} & c_{f_{24}}^{(20,11)} \\ & & & c_{f_{22}}^{(02,02)} & c_{f_{23}}^{(02,11)} & c_{f_{24}}^{(02,11)} \\ & \text{Sym.} & & & c_{f_{33}}^{(11,11)} & c_{f_{34}}^{(11,11)} \\ & & & & & c_{f_{44}}^{(11,11)} \end{bmatrix}.$$

The matrices, Λ_β ($\beta=0, 1, 2, f$), are called irreducible representations of M with respect to ideals $\mathfrak{A}_\beta$ and the following notation is used here:

$$\mathfrak{A}_\beta\colon M \longrightarrow \Lambda_\beta \qquad \text{for} \quad \beta = 0, 1, 2, f.$$

Part II. 3^m-BFF designs of resolution V and their optimal designs

5. 3^m-BFF designs of resolution V

We shall consider a 3^m-FF design of resolution V.

DEFINITION 5.1. A 3^m-FF design, T, of resolution V is said to be balanced (for brevity, 3^m-BFF design), if covariance matrix $\mathrm{Var}\,[\hat{\boldsymbol{\theta}}_{v_m}]$ given by (2.2) is invariant under any permutation of m factors, i.e., for any two estimates $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ and $\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ in the BLUE $\hat{\boldsymbol{\theta}}_{v_m}$,

$$\mathrm{Cov}\,(\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2});\,\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4})) = \mathrm{Cov}\,(\hat{\theta}(\tau(t_1)^{\varepsilon_1}\tau(t_2)^{\varepsilon_2});\,\hat{\theta}(\tau(t_3)^{\varepsilon_3}\tau(t_4)^{\varepsilon_4})),$$

where $\mathrm{Cov}\,(\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2});\,\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4}))$ denotes an element of V_T whose row and column correspond to estimates $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ and $\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$, respectively, and τ is any element of a permutation group, $\left\{\tau \,\middle|\, \tau=\begin{pmatrix} 1 & 2 & \cdots & m \\ \tau(1) & \tau(2) & \cdots & \tau(m)\end{pmatrix}\right\}$.

LEMMA 5.1 (Kuwada [24]). *A maximal invariant of the function of two sets* $\{t_1^1,\dots, t_{a_1}^1, t_1'^2,\dots, t_{a_2}'^2\}$ *and* $\{u_1^1,\dots, u_{b_1}^1, u_1'^2,\dots, u_{b_2}'^2\}$ *with respect to the permutation group is* $(a_1, a_2, b_1, b_2, |\{t_1,\dots, t_{a_1}\}\cap\{u_1,\dots, u_{b_1}\}|, |\{t_1,\dots, t_{a_1}\}\cap\{u_1',\dots, u_{b_2}'\}|, |\{t_1',\dots, t_{a_2}'\}\cap\{u_1,\dots, u_{b_1}\}|, |\{t_1',\dots, t_{a_2}'\}\cap\{u_1',\dots, u_{b_2}'\}|)$, *where* $\{t_1,\dots, t_{a_1}, t_1',\dots, t_{a_2}'\}$ *and* $\{u_1,\dots, u_{b_1}, u_1',\dots, u_{b_2}'\}$ *are subsets of* $\{1, 2,\dots, m\}$.

This lemma yields

THEOREM 5.2. *For a 3^m-FF design, T, of resolution V, a necessary and sufficient condition for T to be balanced is that $V_T = V_{T_Q}$ holds for design T_Q obtained from T by $T_Q = TQ$, where Q is any permutation matrix of order m.*

PROOF. From Definition 5.1 and Lemma 5.1, it can easily be shown that for any permutation group τ of degree m, there exists a matrix, Q, of order m such that

$$(\tau(1), \tau(2),\dots, \tau(m)) = (1, 2,\dots, m)Q.$$

Hence, the proof is completed.

THEOREM 5.3 (Kuwada [24]). *A necessary and sufficient condition for a 3^m-FF design, T, of resolution V to be balanced is that T is a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$, provided information matrix M_T is non-singular.*

6. Characteristic polynomials of information matrices

There are in general a large number of possible 3^m-BFF designs of resolution V with N assemblies. Out of these, one must choose a design which allows estimates of all ν_m effects and, further, maximizes the information in some sense. For such purpose, there are many criteria (e.g., the trace, determinant and maximum root) which are functions of characteristic roots of information matrix M_T. Thus, in order to obtain a design which is optimal in a class of balanced designs with given m and N, the derivation of the characteristic polynomial of M_T (hence the characteristic roots of M_T) is a basic first step.

Let T be a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$. Then it follows from Theorem 5.3 that T is a 3^m-BFF design of resolution V with N assemblies, provided M_T is non-singular.

When $\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ is related to $\theta(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ by $R(\boldsymbol{\alpha};\, a_1a_2,\, b_1b_2)$, we denote the element of M_T corresponding to $\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$-row and $\theta(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$-column by $p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$, respectively, where $a_r = w_r(\varepsilon_1, \varepsilon_2)$ and $b_r = w_r(\varepsilon_3, \varepsilon_4)$ $(r=1, 2)$. Then a connection between $p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ and $\gamma_{p_0p_1p_2}$ of information matrix M_T is given as follows:

$$
(6.1)\quad \left\{\begin{array}{l}
p_{(0000)}^{(00,00)} = \gamma_{400} = N, \quad p_{(0000)}^{(00,10)} = p_{(0000)}^{(10,01)} = \gamma_{310}, \quad p_{(0000)}^{(00,01)} = \gamma_{301}, \\
p_{(0000)}^{(00,20)} = p_{(1000)}^{(10,10)} = p_{(1000)}^{(10,11)} = p_{(0000)}^{(01,20)} = p_{(0000)}^{(20,02)} = p_{(1001)}^{(11,11)} = \gamma_{220}, \\
p_{(0000)}^{(00,02)} = p_{(0001)}^{(01,01)} = \gamma_{202}, \quad p_{(0000)}^{(00,11)} = p_{(0100)}^{(10,01)} = p_{(0000)}^{(10,02)} = p_{(0001)}^{(01,11)} = \gamma_{211}, \\
p_{(0000)}^{(10,10)} = (2N + \gamma_{301})/3, \quad p_{(0000)}^{(10,20)} = p_{(0000)}^{(20,11)} = (2\gamma_{310} + \gamma_{211})/3, \\
p_{(1000)}^{(10,20)} = p_{(1000)}^{(20,11)} = \gamma_{130}, \quad p_{(0100)}^{(10,02)} = p_{(0011)}^{(01,11)} = p_{(0001)}^{(02,11)} = \gamma_{112}, \\
p_{(0100)}^{(10,11)} = (2\gamma_{301} + \gamma_{202})/3, \quad p_{(1100)}^{(10,11)} = p_{(0010)}^{(01,20)} = p_{(0100)}^{(20,02)} = p_{(1011)}^{(11,11)} \\
= p_{(1101)}^{(11,11)} = \gamma_{121}, \quad p_{(0000)}^{(01,01)} = 2N - \gamma_{301}, \quad p_{(0000)}^{(01,02)} = 2\gamma_{301} - \gamma_{202}, \\
p_{(0001)}^{(01,02)} = \gamma_{103}, \quad p_{(0010)}^{(01,11)} = p_{(0000)}^{(02,11)} = 2\gamma_{310} - \gamma_{211}, \quad p_{(0000)}^{(20,20)} \\
= (4N + 4\gamma_{301} + \gamma_{202})/9, \quad p_{(1000)}^{(20,20)} = (2\gamma_{220} + \gamma_{121})/3, \quad p_{(2000)}^{(20,20)} = \gamma_{040}, \\
p_{(0200)}^{(20,02)} = p_{(1111)}^{(11,11)} = \gamma_{022}, \quad p_{(0100)}^{(20,11)} = (2\gamma_{211} + \gamma_{112})/3, \quad p_{(1100)}^{(20,11)} = \gamma_{031}, \\
p_{(0000)}^{(02,02)} = 4N - 4\gamma_{301} + \gamma_{202}, \quad p_{(0001)}^{(02,02)} = 2\gamma_{202} - \gamma_{103}, \quad p_{(0002)}^{(02,02)} = \gamma_{004}, \\
p_{(0010)}^{(02,11)} = 2\gamma_{211} - \gamma_{112}, \quad p_{(0011)}^{(02,11)} = \gamma_{013}, \quad p_{(0110)}^{(11,11)} = (4N - \gamma_{202})/3, \\
p_{(0111)}^{(11,11)} = (2\gamma_{202} + \gamma_{103})/3, \quad p_{(1110)}^{(11,11)} = 2\gamma_{220} - \gamma_{121},
\end{array}\right.
$$

where $p_{\tilde{\boldsymbol{\alpha}}}^{(b_1b_2,a_1a_2)} = p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ and a connection between $\gamma_{p_0p_1p_2}$ and indices $\lambda_{i_0i_1i_2}$ of a B-array is always given by Table A. From the definition of relationship matrices $D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$, information matrix M_T of T can be expressed as

$$
\begin{aligned}
M_T &= \sum_{a_1a_2}\sum_{b_1b_2}\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} D_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} \\
&= \sum_{a_1a_2}{}^{*}\sum_{b_1b_2}{}^{*}\sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)} B_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}.
\end{aligned}
$$

Hence, it follows from Theorem 4.3 that M_T belongs to MD relationship algebra $\mathfrak{A}$. From (4.11), M_T can also be expressed as

$$
\begin{aligned}
M_T = &\sum_{a_1a_2}\sum_{b_1b_2}\sum_{\beta=0}^{2} \kappa_{\beta}^{a_1a_2,b_1b_2} D_{\beta}^{\#(a_1a_2,b_1b_2)} \\
&+ \sum_{u_1u_2}\sum_{v_1v_2}\sum_{i,j} \kappa_{f_{ij}}^{u_1u_2,v_1v_2} D_{f_{ij}}^{\#(u_1u_2,v_1v_2)},
\end{aligned}
$$

where

$$
(6.2)\quad \left\{
\begin{aligned}
&\kappa_0^{00,00} = p_{(0000)}^{(00,00)}, \quad \kappa_0^{00,b_1b_2} = \sqrt{m}\, p_{(0000)}^{(00,b_1b_2)}, \quad \kappa_0^{00,b_1'b_2'} = \sqrt{\binom{m}{2}}\, p_{(0000)}^{(00,b_1'b_2')}, \\
&\kappa_0^{00,11} = \sqrt{2\binom{m}{2}}\, p_{(0000)}^{(00,11)}, \quad \kappa_0^{a_1a_2,b_1b_2} = p_{(0000)}^{(a_1a_2,b_1b_2)} + (m-1)p_{\alpha_1}^{(a_1a_2,b_1b_2)}, \\
&\kappa_0^{a_1a_2,b_1'b_2'} = \sqrt{(m-1)/2}\,\{2p_{(0000)}^{(a_1a_2,b_1'b_2')} + (m-2)p_{\beta_1}^{(a_1a_2,b_1'b_2')}\}, \\
&\kappa_0^{a_1a_2,11} = \sqrt{m-1}\,\{p_{\zeta_0}^{(a_1a_2,11)} + p_{\zeta_1}^{(a_1a_2,11)} + (m-2)p_{\zeta_2}^{(a_1a_2,11)}\}, \\
&\kappa_0^{a_1'a_2',b_1'b_2'} = p_{(0000)}^{(a_1'a_2',b_1'b_2')} + 2(m-2)p_{\gamma_1}^{(a_1'a_2',b_1'b_2')} + \binom{m-2}{2}p_{\gamma_2}^{(a_1'a_2',b_1'b_2')}, \\
&\kappa_0^{a_1'a_2',11} = \sqrt{2}\,\{p_{(0000)}^{(a_1'a_2',11)} + (m-2)(p_{\zeta_1}^{(a_1'a_2',11)} + p_{\zeta_2}^{(a_1'a_2',11)}) \\
&+ \binom{m-2}{2}p_{\zeta_3}^{(a_1'a_2',11)}\}, \quad \kappa_0^{11,11} = p_{(0110)}^{(11,11)} + p_{(1001)}^{(11,11)} + (m-2) \\
&\cdot\{p_{(0111)}^{(11,11)} + p_{(1110)}^{(11,11)} + p_{(1011)}^{(11,11)} + p_{(1101)}^{(11,11)}\} + 2\binom{m-2}{2}p_{(1111)}^{(11,11)}, \\
&\kappa_1^{a_1'a_2',b_1'b_2'} = p_{(0000)}^{(a_1'a_2',b_1'b_2')} - 2p_{\gamma_1}^{(a_1'a_2',b_1'b_2')} + p_{\gamma_2}^{(a_1'a_2',b_1'b_2')}, \\
&\kappa_1^{a_1'a_2',11} = \sqrt{2}\,\{p_{(0000)}^{(a_1'a_2',11)} - p_{\zeta_1}^{(a_1'a_2',11)} - p_{\zeta_2}^{(a_1'a_2',11)} + p_{\zeta_3}^{(a_1'a_2',11)}\}, \\
&\kappa_1^{11,11} = p_{(0110)}^{(11,11)} + p_{(1001)}^{(11,11)} - p_{(0111)}^{(11,11)} - p_{(1110)}^{(11,11)} - p_{(1011)}^{(11,11)} - p_{(1101)}^{(11,11)} \\
&+ 2p_{(1111)}^{(11,11)}, \quad \kappa_2^{11,11} = p_{(0110)}^{(11,11)} - p_{(1001)}^{(11,11)} - p_{(0111)}^{(11,11)} - p_{(1110)}^{(11,11)} \\
&+ p_{(1011)}^{(11,11)} + p_{(1101)}^{(11,11)}, \quad \kappa_{f_{11}}^{a_1a_2,b_1b_2} = p_{(0000)}^{(a_1a_2,b_1b_2)} - p_{\alpha_1}^{(a_1a_2,b_1b_2)}, \\
&\kappa_{f_{12}}^{a_1a_2,b_1'b_2'} = \sqrt{m-2}\,\{p_{(0000)}^{(a_1a_2,b_1'b_2')} - p_{\beta_1}^{(a_1a_2,b_1'b_2')}\}, \quad \kappa_{f_{13}}^{a_1a_2,11} \\
&= \sqrt{m/2}\,\{p_{\zeta_0}^{(a_1a_2,11)} - p_{\zeta_1}^{(a_1a_2,11)}\}, \quad \kappa_{f_{14}}^{a_1a_2,11} = \sqrt{(m-2)/2}\,\{p_{\zeta_0}^{(a_1a_2,11)} \\
&+ p_{\zeta_1}^{(a_1a_2,11)} - 2p_{\zeta_2}^{(a_1a_2,11)}\}, \quad \kappa_{f_{22}}^{a_1'a_2',b_1'b_2'} = p_{(0000)}^{(a_1'a_2',b_1'b_2')} + (m-4)\,p_{\gamma_1}^{(a_1'a_2',b_1'b_2')} \\
&- (m-3)p_{\gamma_2}^{(a_1'a_2',b_1'b_2')}, \quad \kappa_{f_{23}}^{a_1'a_2',11} = \sqrt{m(m-2)/2}\,\{p_{\zeta_1}^{(a_1'a_2',11)} - p_{\zeta_2}^{(a_1'a_2',11)}\}, \\
&\kappa_{f_{24}}^{a_1'a_2',11} = \{1/\sqrt{2}\}\{2p_{(0000)}^{(a_1'a_2',11)} + (m-4)(p_{\zeta_1}^{(a_1'a_2',11)} + p_{\zeta_2}^{(a_1'a_2',11)}) \\
&- 2(m-3)p_{\zeta_3}^{(a_1'a_2',11)}\}, \quad \kappa_{f_{33}}^{11,11} = \{1/2\}\{2(p_{(0110)}^{(11,11)} - p_{(1001)}^{(11,11)}) \\
&+ (m-2)(p_{(0111)}^{(11,11)} + p_{(1110)}^{(11,11)} - p_{(1011)}^{(11,11)} - p_{(1101)}^{(11,11)})\},
\end{aligned}
\right.
$$

$$\left\{\begin{aligned} &\kappa_{f34}^{11,11} = \{\sqrt{m(m-2)}/2\}\{p_{(0111)}^{(11,11)} - p_{(1110)}^{(11,11)}\}, \quad \kappa_{f44}^{11,11} \\ &= \{1/2\}\{2(p_{(0110)}^{(11,11)} + p_{(1001)}^{(11,11)}) + (m-4)(p_{(0111)}^{(11,11)} + p_{(1110)}^{(11,11)} + p_{(1011)}^{(11,11)} \\ &+ p_{(1101)}^{(11,11)}) - 4(m-3)p_{(1111)}^{(11,11)}\}. \end{aligned}\right.$$

Here a_1a_2, b_1b_2, $a_1'a_2'$, $b_1'b_2'$, α_1, β_1, γ_1, γ_2, ξ_0, ξ_1, ξ_2, ζ_1, ζ_2 and ζ_3 are the same as those given by (4.11). Therefore, Theorem 4.4 yields a 6×6 matrix, K_0, a 3×3, K_1, a 1×1, K_2, and a 6×6, K_f, such that

$$\mathfrak{A}_\beta\colon M_T \longrightarrow K_\beta \quad \text{for} \quad \beta = 0, 1, 2, f,$$

where

$$(6.3)\qquad \begin{aligned} &K_0 = \|\kappa_0^{a_1a_2,b_1b_2}\|, \quad K_1 = \|\kappa_1^{c_1c_2,d_1d_2}\|, \quad K_2 = \|\kappa_2^{11,11}\|, \\ &K_f = \|\kappa_{fij}^{u_1u_2,v_1v_2}\|. \end{aligned}$$

In this case, since I_{ν_m} belongs to $\mathfrak{A}$, we have

THEOREM 6.1 (Kuwada [25]). *The characteristic polynomial, $\Psi_T(x)$, of information matrix M_T of a 3^m-BFF design, T, of resolution V is given by*

$$\begin{aligned} \Psi_T(x) &= \det(M_T - xI_{\nu_m}) \\ &= \{\det(K_0 - xI_6)\}^{\phi_0}\{\det(K_1 - xI_3)\}^{\phi_1}\{\det(K_2 - x)\}^{\phi_2} \\ &\cdot\{\det(K_f - xI_6)\}^{\phi_f}, \end{aligned}$$

where multiplicities ϕ_β ($\beta=0, 1, 2, f$) are given by (4.13).

Let $K_0^{-1} = \|\kappa_{a_1a_2,b_1b_2}^0\|$, $K_1^{-1} = \|\kappa_{c_1c_2,d_1d_2}^1\|$, $K_2^{-1} = \|\kappa_{11,11}^2\|$ and $K_f^{-1} = \|\kappa_{u_1u_2,v_1v_2}^{fij}\|$. Then from the above theorem and $\mathfrak{A}$ (containing I_{ν_m}), we have

COROLLARY 6.2. *For T being a design of Theorem 6.1, characteristic polynomial $\chi_T(x)$ of V_T $(=M_T^{-1})$ is given by*

$$\begin{aligned} \chi_T(x) &= \det(V_T - xI_{\nu_m}) \\ &= \{\det(K_0^{-1} - xI_6)\}^{\phi_0}\{\det(K_1^{-1} - xI_3)\}^{\phi_1}\{\det(K_2^{-1} - x)\}^{\phi_2} \\ &\cdot\{\det(K_f^{-1} - xI_6)\}^{\phi_f}. \end{aligned}$$

THEOREM 6.3 (Kuwada [25]). *For T being a design of Theorem* 6.1,

$$(6.4)\qquad \begin{aligned} &\operatorname{tr}(V_T) \equiv \operatorname{tr}(M_T^{-1}) = \phi_0 \operatorname{tr}(K_0^{-1}) + \phi_1 \operatorname{tr}(K_1^{-1}) + \phi_2 \operatorname{tr}(K_2^{-1}) + \phi_f \operatorname{tr}(K_f^{-1}), \\ &\det(V_T) \equiv \det(M_T^{-1}) = \{\det(K_0^{-1})\}^{\phi_0}\{\det(K_1^{-1})\}^{\phi_1}\{\det(K_2^{-1})\}^{\phi_2}\{\det(K_f^{-1})\}^{\phi_f}. \end{aligned}$$

THEOREM 6.4 (Kuwada [25]). *Let T be a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$. Then a necessary condition for the existence of T is that every irreducible representation K_β of information matrix M_T corresponding to T is positive semidefinite for $\beta=0, 1, 2, f$.*

THEOREM 6.5 (Kuwada [25]). *A necessary and sufficient condition for the array of Theorem 6.4 to be a 3^m-BFF design of resolution V is that every irreducible representation K_β of M_T is positive definite for $\beta=0, 1, 2, f$.*

From (6.3) and Theorem 6.4, we get

COROLLARY 6.6. *A necessary condition for the existence of the B-array of Theorem 6.4 is that the following relations hold:*

$$\lambda_{211} + \lambda_{121} + \lambda_{112} \geqq 0, \tag{6.5}$$

$$\lambda_{220} + \lambda_{022} + \lambda_{121} \geqq 0, \tag{6.6a}$$

$$\lambda_{220} + \lambda_{022} + \lambda_{211} + \lambda_{112} \geqq 0, \tag{6.6b}$$

$$\begin{aligned} &m(\lambda_{400} + \lambda_{004}) + (2m+1)(\lambda_{310} + \lambda_{013}) + 4(\lambda_{301} + \lambda_{103}) + \lambda_{130} \\ &+ \lambda_{031} + (m+2)(\lambda_{220} + \lambda_{022}) - \{2(m-4)(\lambda_{202} + \lambda_{121}) \\ &+ (2m-11)(\lambda_{211} + \lambda_{112})\} \geqq 0, \end{aligned} \tag{6.7a}$$

$$\begin{aligned} &m(\lambda_{400} + 4\lambda_{040} + \lambda_{004}) - (2m-9)(\lambda_{310} + \lambda_{013}) + 4m(\lambda_{301} + \lambda_{103}) \\ &+ (4m+9)(\lambda_{130} + \lambda_{031}) - 3(m-6)(\lambda_{220} + \lambda_{022}) + 6m\lambda_{202} \\ &- \{3(2m-9)(\lambda_{211} + \lambda_{112}) + 6(m-6)\lambda_{121}\} \geqq 0, \end{aligned} \tag{6.7b}$$

$$\begin{aligned} &\binom{m}{2}(\lambda_{400} + \lambda_{004}) + 2(m-1)(\lambda_{310} + \lambda_{013}) - 2(m-1)(m-4)(\lambda_{301} \\ &+ \lambda_{103}) + \lambda_{220} + \lambda_{022} + (3m^2 - 19m + 32)\lambda_{202} - 2(m-5)(\lambda_{211} \\ &+ \lambda_{112}) + 2\lambda_{121} \geqq 0, \end{aligned} \tag{6.7c}$$

$$\begin{aligned} &\binom{m}{2}\{\lambda_{400} + 16\lambda_{040} + \lambda_{004} + 4(\lambda_{301} + \lambda_{103}) + 6\lambda_{202}\} \\ &- 2(m-1)(2m-9)(\lambda_{310} + \lambda_{013}) - 8(m-1)(2m-9)(\lambda_{130} \\ &+ \lambda_{031}) + 3(4m^2 - 28m + 51)(\lambda_{220} + \lambda_{022} + 2\lambda_{121}) \\ &- 6(m-1)(2m-9)(\lambda_{211} + \lambda_{112}) \geqq 0, \end{aligned} \tag{6.7d}$$

(6.7e) $\binom{m}{2}(\lambda_{400}+\lambda_{004})-2(m-1)(m-4)(\lambda_{310}+\lambda_{013})+2(m-1)(\lambda_{301}$
$$+\lambda_{130}+\lambda_{031}+\lambda_{103})+(2m^2-10m+17)(\lambda_{220}+\lambda_{022})$$
$$-(m-1)(m-4)(\lambda_{202}+4\lambda_{121})+4(m-4)^2(\lambda_{211}+\lambda_{112})\geqq 0,$$

(6.8a) $(m-2)\{\lambda_{310}+\lambda_{013}+4(\lambda_{130}+\lambda_{031})+3(\lambda_{211}+\lambda_{112})\}$
$$-(4m-17)(\lambda_{220}+\lambda_{022}+2\lambda_{121})\geqq 0,$$

(6.8b) $m\{\lambda_{310}+\lambda_{301}+\lambda_{130}+\lambda_{031}+\lambda_{103}+\lambda_{013}+2(\lambda_{220}+\lambda_{202}+\lambda_{022})\}$
$$-2(2m-9)(\lambda_{211}+\lambda_{121}+\lambda_{112})\geqq 0,$$

(6.8c) $(m-2)\{4(\lambda_{310}+\lambda_{013})+\lambda_{301}+\lambda_{130}+\lambda_{031}+\lambda_{103}+2\lambda_{202}+8\lambda_{121}\}$
$$-(4m-17)(\lambda_{220}+\lambda_{022})-(7m-32)(\lambda_{211}+\lambda_{112})\geqq 0.$$

COROLLARY 6.7. *A necessary condition for the B-array of Theorem* 6.4 *to be a* 3^m*-BFF design of resolution V is that relations* (6.5) *through* (6.8) *hold with strict inequalities.*

It was shown by Srivastava, Raktoe and Pesotan [56] that the characteristic roots of the information matrix of a design in the general factorials relative to an admissible vector of effects remain invariant under a permutation of levels. Their approach was essentially based on the orthonormal matrix, i.e.,

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix},$$

for D as defined by (1.2).

Let $\bar{T}$ be an array obtained from T by interchanging all of symbols 0 and 2. Then it is easily shown that if T is a B-array $[N, m, 3, 4]$ with index set $\{\lambda_{i_0i_1i_2}\}$, then $\bar{T}$ is also a B-array $[N, m, 3, 4]$ with index set $\{\bar{\lambda}_{i_0i_1i_2}\}$, where $\bar{\lambda}_{i_0i_1i_2}=\lambda_{i_2i_1i_0}$. Therefore, $\bar{T}$ is called a (0, 2)-interchanged balanced array ((0, 2)IB-array) of B-array T. Let $\bar{p}_\alpha^{(a_1a_2,b_1b_2)}$ and $\bar{\gamma}_{p_0p_1p_2}$ be the values of $\bar{T}$ corresponding to $p_\alpha^{(a_1a_2,b_1b_2)}$ and $\gamma_{p_0p_1p_2}$ of T, respectively. Then we get

LEMMA 6.8. *For a B-array and its* (0, 2)*IB-array,*

(6.9) $$\bar{\gamma}_{p_0p_1p_2}=(-1)^{p_1}\gamma_{p_0p_1p_2} \quad \text{for } p_0+p_1+p_2=4 \text{ and } p_0, p_1, p_2\geqq 0.$$

From (6.1) and (6.9), we have

LEMMA 6.9.

$$
(6.10)\quad \bar{p}_{\alpha}^{(a_1a_2,b_1b_2)} = \begin{cases} -p_{\alpha}^{(a_1a_2,b_1b_2)} & \textit{if}\ \ (a_1a_2, b_1b_2) = (00, 10), (00, 11), (10, 01), \\ & \quad (10, 20), (10,02), (01, 11), (20, 11), (02, 11), \\ p_{\alpha}^{(a_1a_2,b_1b_2)} & \textit{otherwise}. \end{cases}
$$

THEOREM 6.10. *For a B-array* $[N, m, 3, 4]$, T, *with index set* $\{\lambda_{i_0i_1i_2}\}$ *and its* $(0, 2)$*IB-array*, $\bar{T}$,

$$
(6.11)\qquad \Psi_T(x) = \Psi_T(x),
$$

where $\Psi_T(x)$ *is the characteristic polynomial of information matrix* M_T *based on* $\bar{T}$.

PROOF. From (6.1) and (6.10), there exists a matrix, U, of order v_m such that

$$
M_T = UM_TU,
$$

where $U = \operatorname{diag}\{1, \underbrace{-1, \dots, -1}_{m}, \underbrace{1, \dots, 1}_{m}, \underbrace{1, \dots, 1}_{\binom{m}{2}}, \underbrace{1, \dots, 1}_{\binom{m}{2}}, \underbrace{-1, \dots, -1}_{2\binom{m}{2}}\}$. Hence, (6.11) can easily be proved.

The above theorem is powerful for finding optimal balanced designs with respect to popular criteria. For given values of m and N, there exists, in general, a large number of B-arrays of strength 4 whose indices satisfy $N = \lambda_{400} + \lambda_{040} + \lambda_{004} + 4(\lambda_{310} + \lambda_{301} + \lambda_{130} + \lambda_{031} + \lambda_{103} + \lambda_{013}) + 6(\lambda_{220} + \lambda_{202} + \lambda_{022}) + 12(\lambda_{211} + \lambda_{121} + \lambda_{112})$. Out of these, we choose a design which is optimal in a sense. Theorem 6.10, however, implies that it is enough to select a design such that (a) $\lambda_{400} < \lambda_{004}$, if $\lambda_{400} \neq \lambda_{004}$, (b) $\lambda_{310} > \lambda_{013}$, if $\lambda_{400} = \lambda_{004}$ and $\lambda_{310} \neq \lambda_{013}$, (c) $\lambda_{301} > \lambda_{103}$, if $\lambda_{400} = \lambda_{004}$, $\lambda_{310} = \lambda_{013}$ and $\lambda_{301} \neq \lambda_{103}$, (d) $\lambda_{130} > \lambda_{031}$, if $\lambda_{400} = \lambda_{004}$, $\lambda_{310} = \lambda_{013}$, $\lambda_{301} = \lambda_{103}$ and $\lambda_{130} \neq \lambda_{031}$, (e) $\lambda_{220} < \lambda_{022}$, if $\lambda_{400} = \lambda_{004}$, $\lambda_{310} = \lambda_{013}$, $\lambda_{301} = \lambda_{103}$, $\lambda_{130} = \lambda_{031}$ and $\lambda_{220} \neq \lambda_{022}$ or (f) $\lambda_{211} > \lambda_{112}$, if $\lambda_{400} = \lambda_{004}$, $\lambda_{310} = \lambda_{013}$, $\lambda_{301} = \lambda_{103}$, $\lambda_{130} = \lambda_{031}$, $\lambda_{220} = \lambda_{022}$ and $\lambda_{211} \neq \lambda_{112}$.

It is easily shown from Theorem 2.2 that if T is a 3^m-BFF design of resolution V, $\bar{T}$ is also so and is called a (0, 2)-interchanged 3^m-BFF (3^m-(0, 2)IBFF) design of T. Theorem 6.10 yields

COROLLARY 6.11. *For a* 3^m*-BFF design*, T, *of resolution V and its* 3^m*-*(0, 2)*IBFF design*, $\bar{T}$,

$$
\operatorname{tr}(V_T) = \operatorname{tr}(V_T),
$$

$$
\det(V_T) = \det(V_T).
$$

7. Covariance matrices of 3^m-BFF designs

Using inverse matrices K_β^{-1} of K_β ($\beta=0, 1, 2, f$), we shall obtain explicit expressions for all distinct elements of V_T. Let $V_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ be the elements of V_T corresponding to $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ and $\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$, where $w_r(\varepsilon_1, \varepsilon_2)=a_r$ and $w_r(\varepsilon_3, \varepsilon_4)=b_r$ ($r=1, 2$), and $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2})$ is related to $\hat{\theta}(t_3^{\varepsilon_3}t_4^{\varepsilon_4})$ by $R(\boldsymbol{\alpha};\, a_1a_2,\, b_1b_2)$. Then we have

THEOREM 7.1. *For a 3^m-BFF design of resolution V,*

$$
(7.1)\quad \left\{
\begin{aligned}
& V_{(0000)}^{(00,00)} = \kappa_{00,00}^{0}, \quad V_{(0000)}^{(00,b_1b_2)} = \{1/\sqrt{m}\}\kappa_{00,b_1b_2}^{0}, \\
& V_{(0000)}^{(00,b_1'b_2')} = \left\{1\Big/\sqrt{\binom{m}{2}}\right\}\kappa_{00,b_1'b_2'}^{0}, \quad V_{(0000)}^{(00,11)} = \left\{1\Big/\sqrt{2\binom{m}{2}}\right\}\kappa_{00,11}^{0}, \\
& V_{(0000)}^{(a_1a_2,b_1b_2)} = \{1/m\}\{\kappa_{a_1a_2,b_1b_2}^{0} + (m-1)\kappa_{a_1a_2,b_1b_2}^{f_{11}}\}, \\
& V_{\alpha_1}^{(a_1a_2,b_1b_2)} = \{1/m\}\{\kappa_{a_1a_2,b_1b_2}^{0} - \kappa_{a_1a_2,b_1b_2}^{f_{11}}\}, \quad V_{(0000)}^{(a_1a_2,b_1'b_2')} \\
& = \left[1\Big/\left\{m\sqrt{\binom{m-1}{2}}\right\}\right]\{\sqrt{m-2}\,\kappa_{a_1a_2,b_1'b_2'}^{0} + (m-2)\sqrt{(m-1)/2}\,\kappa_{a_1a_2,b_1'b_2'}^{f_{12}}\}, \\
& V_{\beta_1}^{(a_1a_2,b_1'b_2')} = \left[1\Big/\left\{m\sqrt{\binom{m-1}{2}}\right\}\right]\{\sqrt{m-2}\,\kappa_{a_1a_2,b_1'b_2'}^{0} - \sqrt{2(m-1)}\,\kappa_{a_1a_2,b_1'b_2'}^{f_{12}}\}, \\
& V_{\xi_0}^{(a_1a_2,11)} = [1/\{m\sqrt{m-1}\}]\{\kappa_{a_1a_2,11}^{0} + \sqrt{\binom{m}{2}}\kappa_{a_1a_2,11}^{f_{13}} \\
& + \sqrt{\binom{m-1}{2}}\kappa_{a_1a_2,11}^{f_{14}}\}, \quad V_{\xi_1}^{(a_1a_2,11)} = [1/\{m\sqrt{m-1}\}]\Big\{\kappa_{a_1a_2,11}^{0} \\
& - \sqrt{\binom{m}{2}}\kappa_{a_1a_2,11}^{f_{13}} + \sqrt{\binom{m-1}{2}}\kappa_{a_1a_2,11}^{f_{14}}\Big\}, \quad V_{\xi_2}^{(a_1a_2,11)} \\
& = \left[1\Big/\left\{m\sqrt{2\binom{m-1}{2}}\right\}\right]\{\sqrt{m-2}\,\kappa_{a_1a_2,11}^{0} - \sqrt{2(m-1)}\,\kappa_{a_1a_2,11}^{f_{14}}\}, \\
& V_{(0000)}^{(a_1'a_2',b_1'b_2')} = \left[1\Big/\left\{2\binom{m}{2}\right\}\right]\{2\kappa_{a_1'a_2',b_1'b_2'}^{0} + m(m-3)\kappa_{a_1'a_2',b_1'b_2'}^{1} \\
& + 2(m-1)\kappa_{a_1'a_2',b_1'b_2'}^{f_{22}}\}, \quad V_{\gamma_1}^{(a_1'a_2',b_1'b_2')} = \left[1\Big/\left\{6\binom{m}{3}\right\}\right]\{2(m-2)\kappa_{a_1'a_2',b_1'b_2'}^{0} \\
& - m(m-3)\kappa_{a_1'a_2',b_1'b_2'}^{1} + (m-1)(m-4)\kappa_{a_1'a_2',b_1'b_2'}^{f_{22}}\}, \quad V_{\gamma_2}^{(a_1'a_2',b_1'b_2')} \\
& = \left[1\Big/\left\{3\binom{m}{3}\right\}\right]\{(m-2)\kappa_{a_1'a_2',b_1'b_2'}^{0} + m\kappa_{a_1'a_2',b_1'b_2'}^{1} - 2(m-1)\kappa_{a_1'a_2',b_1'b_2'}^{f_{22}}\},
\end{aligned}
\right.
$$

$$V_{(0000)}^{(a_1'a_2',11)} = \left[1\Big/\left\{2\binom{m}{2}\sqrt{2}\right\}\right]\{2\kappa^0_{a_1'a_2',11} + m(m-3)\kappa^1_{a_1'a_2',11}$$

$$+ 2(m-1)\kappa^{f_{24}}_{a_1'a_2',11}\},\quad V_{\zeta_1}^{(a_1'a_2'11)} = \left[1\Big/\left\{6\binom{m}{3}\sqrt{2}\right\}\right]\{2(m-2)\kappa^0_{a_1'a_2',11}$$

$$- m(m-3)\kappa^1_{a_1'a_2',11} + (m-1)(m-4)\kappa^{f_{24}}_{a_1'a_2',11}\} + \{1/\sqrt{2m(m-2)}\}\kappa^{f_{23}}_{a_1'a_2',11},$$

$$V_{\zeta_2}^{(a_1'a_2',11)} = \left[1\Big/\left\{6\binom{m}{3}\sqrt{2}\right\}\right]\{2(m-2)\kappa^0_{a_1'a_2',11} - m(m-3)\kappa^1_{a_1'a_2',11}$$

$$+ (m-1)(m-4)\kappa^{f_{24}}_{a_1'a_2',11}\} - \{1/\sqrt{2m(m-2)}\}\kappa^{f_{23}}_{a_1'a_2',11},$$

$$V_{\zeta_3}^{(a_1'a_2',11)} = \left[1\Big/\left\{6\binom{m}{3}\sqrt{2}\right\}\right]\{2(m-2)\kappa^0_{a_1'a_2',11} + 2m\kappa^1_{a_1'a_2',11}$$

$$- 4(m-1)\kappa^{f_{24}}_{a_1'a_2',11}\},\quad V_{(0110)}^{(11,11)} = \left[1\Big/\left\{4\binom{m}{2}\right\}\right]\{2\kappa^1_{11,11}$$

$$+ m(m-3)\kappa^1_{11,11} + 2\binom{m-1}{2}\kappa^2_{11,11} + 2(m-1)(\kappa^{f_{33}}_{11,11} + \kappa^{f_{44}}_{11,11})\},$$

$$V_{(1001)}^{(11,11)} = \left[1\Big/\left\{4\binom{m}{2}\right\}\right]\{2\kappa^0_{11,11} + m(m-3)\kappa^1_{11,11} - 2\binom{m-1}{2}\kappa^2_{11,11}$$

$$- 2(m-1)(\kappa^{f_{33}}_{11,11} - \kappa^{f_{44}}_{11,11})\},\quad V_{(0111)}^{(11,11)} = \left[1\Big/\left\{12\binom{m}{3}\right\}\right]\{2(m-2)\kappa^0_{11,11}$$

$$- m(m-3)\kappa^1_{11,11} - 2\binom{m-1}{2}\kappa^2_{11,11} + 2\binom{m-1}{2}\kappa^{f_{33}}_{11,11}$$

$$+ (m-1)(m-4)\kappa^{f_{44}}_{11,11}\} + [1/\{2\sqrt{m(m-2)}\}](\kappa^{f_{34}}_{11,11} + \kappa^{f_{43}}_{11,11}),$$

$$V_{(1110)}^{(11,11)} = \left[1\Big/\left\{12\binom{m}{3}\right\}\right]\{2(m-2)\kappa^0_{11,11} - m(m-3)\kappa^1_{11,11}$$

$$- 2\binom{m-1}{2}\kappa^2_{11,11} + 2\binom{m-1}{2}\kappa^{f_{33}}_{11,11} + (m-1)(m-4)\kappa^{f_{44}}_{11,11}\}$$

$$- [1/\{2\sqrt{m(m-2)}\}](\kappa^{f_{34}}_{11,11} + \kappa^{f_{43}}_{11,11}),\quad V_{(1011)}^{(11,11)} = V_{(1101)}^{(11,11)}$$

$$= \left[1\Big/\left\{12\binom{m}{3}\right\}\right]\{2(m-2)\kappa^0_{11,11} - m(m-3)\kappa^1_{11,11} + 2\binom{m-1}{2}\kappa^2_{11,11}$$

$$- 2\binom{m-1}{2}\kappa^{f_{33}}_{11,11} + (m-1)(m-4)\kappa^{f_{44}}_{11,11}\},\quad V_{(1111)}^{(11,11)}$$

$$= \left[1\Big/\left\{6\binom{m}{3}\right\}\right]\{(m-2)\kappa^0_{11,11} + m\kappa^1_{11,11} - 2(m-1)\kappa^{f_{44}}_{11,11}\},$$

where a_1a_2, $b_1b_2=10$, 01 *and* $a'_1a'_2$, $b'_1b'_2=20$, 02, *and* $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ *and* $\boldsymbol{\zeta}_3$ *are the same as those given by* (4.11).

8. Optimal 3^4-BFF designs of resolution V

We here consider a 3^4-BFF design of resolution V with given N. Generally, there are many such designs. Out of these, we choose designs which minimize $\mathrm{tr}(V_T)$ or $\det(V_T)$. We assume, throughout Sections 8 and 9, that M_T is positive definite. In this case, we have

THEOREM 8.1. *For any* N, *there does not exist any B-array* $[N, m, 3, 4]$ *with* (i) *at least two of* λ_{220}, λ_{202} *and* λ_{022} *being zero and at least two of* λ_{211}, λ_{121} *and* λ_{112} *being zero, and* (ii) $\lambda_{220}=\lambda_{202}=\lambda_{022}=0$ *and at least one of* λ_{211}, λ_{121} *and* λ_{112} *being zero.*

PROOF. It follows from (6.3) that

$$\begin{aligned}\det(K_1) = 6^6[&2\lambda_{220}\lambda_{202}\lambda_{022} + (\lambda_{220}\lambda_{202} + \lambda_{202}\lambda_{022} + \lambda_{022}\lambda_{220}) \\ &\cdot(\lambda_{211} + \lambda_{121} + \lambda_{112}) + 2\{\lambda_{220}(\lambda_{121}\lambda_{112} + \lambda_{211}\lambda_{112}) \\ &+ \lambda_{202}(\lambda_{211}\lambda_{121} + \lambda_{121}\lambda_{112}) + \lambda_{022}(\lambda_{211}\lambda_{121} + \lambda_{211}\lambda_{112}) \\ &+ 4\lambda_{211}\lambda_{121}\lambda_{112}\}],\end{aligned}$$

which, from the assumptions, is zero. This completes the proof.

THEOREM 8.2. *For any* N, *there does not exist any B-array* $[N, 4, 3, 4]$ *with* (i) $\lambda_{310}=\lambda_{301}=\lambda_{130}=\lambda_{031}=\lambda_{103}=\lambda_{013}=0$ *and* $\lambda_{211}+\lambda_{121}+\lambda_{112}=1$, *and* (ii) $\lambda_{400}=\lambda_{004}=\lambda_{301}=\lambda_{130}=\lambda_{031}=\lambda_{103}=\lambda_{220}=\lambda_{022}=0$.

PROOF. For case (i), if at least one of λ_{220}, λ_{202} and λ_{022} is zero, the number of possible distinct assemblies is less than ν_4 $(=33)$. Hence $\det(M_T)=0$. On the other hand, if $\lambda_{220}, \lambda_{202}, \lambda_{022}\geqq 1$, it holds that $\det(K_f)=0$. For case (ii), from (6.7e), $\kappa_0^{11,11}=0$. Hence, it follows from Corollary 6.7 that there does not exist any B-array with the above indices.

Srivastava and Anderson [50] gave a comparison of the determinant, maximum root and trace optimal criteria. In general, there are many optimal criteria (e.g., [23]). In this paper, we shall deal with the trace and determinant criteria for a given value of N. The trace criterion is proportional to the average of variance of all normalized linear parametric functions. On the other hand, the determinant criterion is proportional to the volume of the ellipsoid of concentration, that is, in a sense, it refers to the volume of the region within which the true parametric point may lie with a certain probability (see [16, 45]).

Consider 3^4-BFF designs of resolution V with ν_4 $(=33) \leqq N \leqq 56$. In Table 1, optimal designs with respect to the trace criterion are given with values of $\mathrm{tr}(V_T)$ and indices $\lambda_{i_0 i_1 i_2}$ of B-arrays for each value of N in the above range. Furthermore, from (7.1), we can easily obtain forty-nine possible distinct elements $V_\alpha^{(a_1 a_2, b_1 b_2)}$ of V_T for each of optimal designs. These are also given in Table 1. Next, we consider optimal designs with respect to the determinant criterion. These designs are given in Table 2 together with elements $V_\alpha^{(a_1 a_2, b_1 b_2)}$ of V_T for each of optimal designs.

As seen in Tables 1 and 2, optimal 3^4-BFF designs of resolution V with $33 \leqq N \leqq 56$ are B-arrays $[N, 4, 3, 4]$ with indices $\lambda_{i_0 i_1 i_2}$ such that λ_{400}, λ_{040} and λ_{004} are 0, 1 or 2, and the remaining $\lambda_{i_0 i_1 i_2}$ are either 0 or 1. Therefore, we shall consider a B-array $[N, 4, 3, 4]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$ for $57 \leqq N \leqq 81$, where λ_{400}, λ_{040} and λ_{004} are 0, 1 or 2, and the remaining $\lambda_{i_0 i_1 i_2}$ are either 0 or 1. Optimal 3^4-BFF designs with respect to the trace and determinant criteria are, respectively, given in Tables 3 and 4, where indices $\lambda_{i_0 i_1 i_2}$ satisfy the above-mentioned restrictions. Furthermore, $V_\alpha^{(a_1 a_2, b_1 b_2)}$ of V_T for each of optimal designs are given in Tables 3 and 4.

9. Optimal 3^5-BFF designs of resolution V derivable from B-arrays of strength 5

Let T be a B-array $[N, m, 3, t]$ with index set $\{\lambda_{i_0 i_1 i_2} | i_0 + i_1 + i_2 = t\}$. Then for any non-negative integer k $(\leqq t)$, T is also a B-array $[N, m, 3, t-k]$ with index set $\{\lambda'_{i'_0 i'_1 i'_2} | i'_0 + i'_1 + i'_2 = t-k\}$.

THEOREM 9.1. *A connection between indices* $\lambda_{i_0 i_1 i_2}$ $(i_0 + i_1 + i_2 = t)$ *and* $\lambda'_{i'_0 i'_1 i'_2}$ $(i'_0 + i'_1 + i'_2 = t-k)$ *is given by*

$$\lambda'_{i'_0 i'_1 i'_2} = \sum_{k_0, k_1, k_2} \{k!/(k_0! k_1! k_2!)\} \lambda'_{i'_0 + k_0\, i'_1 + k_1\, i'_2 + k_2}, \tag{9.1}$$

where k_0, k_1 *and* k_2 *are non-negative integers satisfying* $k_0 + k_1 + k_2 = k$, *and* $i_r = i'_r + k_r$ *for* $r = 0, 1, 2$.

In general, there always exists a B-array $[N, m, 3, m]$ with index set $\{\lambda_{i_0 i_1 i_2}\}$ satisfying $N = \sum \{m!/(i_0! i_1! i_2!)\} \lambda_{i_0 i_1 i_2}$. For 2^m factorials, such a B-array $[N, m, 2, m]$ with indices λ_i $(i = 0, 1, \ldots, m)$ is called a "simple" array (S-array) by Shirakura [39]. We consider a case in which $m = t = 5$ (hence $k = 1$). Then, from (9.1), we have

$$(9.2)\quad \begin{aligned}
&\lambda_{400}=\lambda_{500}+\lambda_{410}+\lambda_{401},\ \lambda_{040}=\lambda_{140}+\lambda_{050}+\lambda_{041},\\
&\lambda_{004}=\lambda_{104}+\lambda_{014}+\lambda_{005},\ \lambda_{310}=\lambda_{410}+\lambda_{320}+\lambda_{311},\\
&\lambda_{301}=\lambda_{401}+\lambda_{311}+\lambda_{302},\ \lambda_{130}=\lambda_{230}+\lambda_{140}+\lambda_{131},\\
&\lambda_{031}=\lambda_{131}+\lambda_{041}+\lambda_{032},\ \lambda_{103}=\lambda_{203}+\lambda_{113}+\lambda_{104},\\
&\lambda_{013}=\lambda_{113}+\lambda_{023}+\lambda_{014},\ \lambda_{220}=\lambda_{320}+\lambda_{230}+\lambda_{221},\\
&\lambda_{202}=\lambda_{302}+\lambda_{212}+\lambda_{203},\ \lambda_{022}=\lambda_{122}+\lambda_{032}+\lambda_{023},\\
&\lambda_{211}=\lambda_{311}+\lambda_{221}+\lambda_{212},\ \lambda_{121}=\lambda_{221}+\lambda_{131}+\lambda_{122},\\
&\lambda_{112}=\lambda_{212}+\lambda_{122}+\lambda_{113}.
\end{aligned}$$

From (9.2) and Theorem 6.10, we obtain

THEOREM 9.2. *For a B-array* $[N, m, 3, 5]$, T, *with index set* $\{\lambda_{i_0 i_1 i_2} | i_0+i_1+i_2=5\}$ *and its* (0, 2)*IB-array*, $\bar{T}$,

$$\Psi_T(x) = \Psi_{\bar{T}}(x).$$

Let T be a B-array $[N, 5, 3, 5]$ with index set $\{\lambda_{i_0 i_1 i_2} | i_0+i_1+i_2=5\}$, where λ_{500}, λ_{050} and λ_{005} are 0, 1 or 2, and the remaining $\lambda_{i_0 i_1 i_2}$ $(i_0+i_1+i_2=5)$ are either 0 or 1. Then in Tables 5 and 6, optimal 3^5-BFF designs of resolution V with respect to the trace and determinant criteria are, respectively, presented for ν_5 $(=51) \leqq N \leqq 70$, where indices $\lambda_{i_0 i_1 i_2}$ satisfy the above-mentioned restrictions. For each of optimal designs of Tables 5 and 6, forty-nine possible distinct elements of V_T are also presented.

Part III. Balanced third-order designs for 3^m factorials and their optimal designs

10. Third-order model and relationship

The general second-order model (cf. [38]) in m factors is presented by

$$(10.1)\quad \eta(j_1, j_2, \dots, j_m) = \theta(\phi) + \sum_{\varepsilon=1}^{2}\sum_{t=1}^{m} d_{j_t}(\varepsilon)\theta(t^{\varepsilon}) + \sum_{t_1<t_2} d_{j_{t_1}}(1)d_{j_{t_2}}(1)\theta(t_1^1 t_2^1)$$

in which the number of unknown effects is $\nu_m^* = 1+m+m+\binom{m}{2} = \binom{m+2}{2}$, where $d_j(\varepsilon)$ are given by (1.2). As its generalization, the third-order model for 3^m factorials can be given by

(10.2) $$\begin{aligned}\eta(j_1, j_2, \ldots, j_m) &= \theta(\phi) + \sum_{\varepsilon=1}^{2}\sum_{t=1}^{m} d_{j_t}(\varepsilon)\theta(t^{\varepsilon}) + \sum_{t_1<t_2} d_{j_{t_1}}(1)d_{j_{t_2}}(1)\theta(t_1^1 t_2^1)\\ &+ \sum_{t_3\neq t_4} d_{j_{t_3}}(1)d_{j_{t_4}}(2)\theta(t_3^1 t_4^2)\\ &+ \sum_{t_5<t_6<t_7} d_{j_{t_5}}(1)d_{j_{t_6}}(1)d_{j_{t_7}}(1)\theta(t_5^1 t_6^1 t_7^1).\end{aligned}$$

Four-factor and higher-order interactions are assumed to be negligible. Further, suppose that the quadratic by quadratic component, $\theta(t_1^2 t_2^2)$, of two-factor interaction, and the 2-linear by quadratic, $\theta(t_3^1 t_4^1 t_5^2)$, the linear by 2-quadratic, $\theta(t_5^1 t_3^2 t_4^2)$, and the 3-quadratic, $\theta(t_6^2 t_7^2 t_8^2)$, of three-factor interaction are negligible, where $t_1<t_2$, $t_3<t_4$, $t_3\neq t_5$, $t_4\neq t_5$ and $t_6<t_7<t_8$.

Let T be a fraction with N assemblies such that $\mathrm{Var}\,[\boldsymbol{y}(T)]=\sigma^2 I_N$. Then (10.2) can also be written in matrix notation as

$$\boldsymbol{\eta}(T) = E_T^*\boldsymbol{\theta}_{\mu_m},$$

where E_T^* is the design matrix of size $N\times\mu_m$, $\boldsymbol{\theta}'_{\mu_m}=(\{\theta(\phi)\};\ \{\theta(t^1)\};\ \{\theta(t^2)\};\ \{\theta(t_1^1 t_2^1)\};\ \{\theta(t_3^1 t_4^2)\};\ \{\theta(t_5^1 t_6^1 t_7^1)\})$ and $\mu_m=1+m+m+\binom{m}{2}+2\binom{m}{2}+\binom{m}{3}=1+m(m+1)(m+5)/6$ being the number of elements of effect vector $\boldsymbol{\theta}_{\mu_m}$.

The normal equation for estimating $\boldsymbol{\theta}_{\mu_m}$ can be written as

$$M_T^*\hat{\boldsymbol{\theta}}_{\mu_m} = E_T^{*\prime}\boldsymbol{y}(T),$$

where $M_T^*=E_T^{*\prime}E_T^*$ is the information matrix of order μ_m. If M_T^* is non-singular, T is called a third-order design for 3^m factorials. For a third-order design, the BLUE $\hat{\boldsymbol{\theta}}_{\mu_m}$ of $\boldsymbol{\theta}_{\mu_m}$ and $\mathrm{Var}\,[\hat{\boldsymbol{\theta}}_{\mu_m}]$ are given by $\hat{\boldsymbol{\theta}}_{\mu_m}=V_T^*E_T^{*\prime}\boldsymbol{y}(T)$ and $\mathrm{Var}\,[\hat{\boldsymbol{\theta}}_{\mu_m}]=V_T^*\sigma^2$, respectively, where $V_T^*=M_T^{*-1}$.

Let $R(\boldsymbol{\alpha};\ a_1a_2,\ b_1b_2)$ be the relationship defined between the sets of effects $\theta(t_1^{\varepsilon_1}t_2^{\varepsilon_2}t_3^{\varepsilon_3})$ and $\theta(t_4^{\varepsilon_4}t_5^{\varepsilon_5}t_6^{\varepsilon_6})$, where $w_r(\varepsilon_1, \varepsilon_2, \varepsilon_3)=a_r$ and $w_r(\varepsilon_4, \varepsilon_5, \varepsilon_6)=b_r$ for $r=1, 2$. Among the six sets of effects $\{\theta(\phi)\}$, $\{\theta(t^1)\}$, $\{\theta(t^2)\}$, $\{\theta(t_1^1 t_2^1)\}$, $\{\theta(t_3^1 t_4^2)\}$ and $\{\theta(t_5^1 t_6^1 t_7^1)\}$, suppose that relationship indices $\boldsymbol{\alpha}$ are given by (4.1). Then the collection of these sets of effects has a 6-MD relationship.

11. Characteristic polynomials of information matrices of balanced third-order designs

We assume throughout that T is a B-array of strength at least six, and that M_T^* is positive definite. This guarantees that M_T^* is invariant under any permutation of m factors, which implies that T is a balanced design (for brevity, 3^m-BTO design).

REMARK. A sufficient condition for T to be a 3^m-BTO design is that T is a B-array $[N, m, 3, 6]$ with index set $\{\lambda^*_{i_0 i_1 i_2} | i_0 + i_1 + i_2 = 6\}$, provided M^*_T is non-singular. However, this is not a necessary condition.

Let T be a B-array $[N, m, 3, 6]$ with index set $\{\lambda^*_{i_0 i_1 i_2} | i_0 + i_1 + i_2 = 6\}$ and $\gamma^*_{p_0 p_1 p_2}$ $(p_0 + p_1 + p_2 = 6)$ be the elements of information matrix M^*_T whose row and column correspond to $\theta(t_1^{\varepsilon_1} t_2^{\varepsilon_2} t_3^{\varepsilon_3})$ and $\theta(t_4^{\varepsilon_4} t_5^{\varepsilon_5} t_6^{\varepsilon_6})$, respectively, where $w_r(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_6) = p_r$ for $r = 0, 1, 2$ and $\{t_1, t_2, \dots, t_6\}$ is a subset of $\{1, 2, \dots, m\}$. Then a connection between $\gamma^*_{p_0 p_1 p_2}$ and indices $\lambda^*_{i_0 i_1 i_2}$ of B-array T is given by Table B.

For $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(\varepsilon_4, \varepsilon_5, \varepsilon_6)$ with $w_r(\varepsilon_1, \varepsilon_2, \varepsilon_3) = a_r$ and $w_r(\varepsilon_4, \varepsilon_5, \varepsilon_6) = b_r$ for $r = 1, 2$, the element of M^*_T whose row and column correspond to $\theta(t_1^{\varepsilon_1} t_2^{\varepsilon_2} t_3^{\varepsilon_3})$ and $\theta(t_4^{\varepsilon_4} t_5^{\varepsilon_5} t_6^{\varepsilon_6})$, respectively, is denoted by $p^{*(a_1 a_2, b_1 b_2)}_{\boldsymbol{\alpha}}$, where $\theta(t_1^{\varepsilon_1} t_2^{\varepsilon_2} t_3^{\varepsilon_3})$ is related to $\theta(t_4^{\varepsilon_4} t_5^{\varepsilon_5} t_6^{\varepsilon_6})$ by $R(\boldsymbol{\alpha}; a_1 a_2, b_1 b_2)$. Then information matrix M^*_T includes

TABLE B.

γ^*																
600	=	1	1	1	6	6	6	6	6	6	15	15	15	15	15	15
060		1	0	1	0	−6	0	0	−6	0	0	15	0	0	15	0
006		1	64	1	−12	6	−192	−192	6	−12	60	15	240	240	15	60
510		−1	0	1	−5	−4	−1	1	4	5	−10	−5	−5	5	5	10
501		1	−2	1	3	6	−9	−9	6	3	0	15	−15	−15	15	0
150		−1	0	1	−1	4	0	0	−4	1	0	−5	0	0	5	0
051		−1	0	1	2	4	0	0	−4	−2	0	−5	0	0	5	0
105		1	−32	1	−9	6	48	48	6	−9	30	15	0	0	15	30
015		−1	0	1	10	−4	32	−32	4	−10	−40	−5	−80	80	5	40
420		1	0	1	4	2	0	0	2	4	6	−1	1	1	−1	6
402		1	4	1	0	6	12	12	6	0	−6	15	9	9	15	−6
240		1	0	1	2	−2	0	0	−2	2	1	−1	0	0	−1	1
042		1	0	1	−4	−2	0	0	−2	−4	4	−1	0	0	−1	4
204		1	16	1	−6	6	0	0	6	−6	9	15	−24	−24	15	9
024		1	0	1	−8	2	0	0	2	−8	24	−1	16	16	−1	24
330		−1	0	1	−3	0	0	0	0	3	−3	3	0	0	−3	3
303		1	−8	1	−3	6	−12	−12	6	−3	−3	15	6	6	15	−3
033		−1	0	1	6	0	0	0	0	−6	−12	3	0	0	−3	12
411		−1	0	1	−2	−4	2	−2	4	2	2	−5	7	−7	5	−2
141		1	0	1	−1	−2	0	0	−2	−1	−2	−1	0	0	−1	−2
114		−1	0	1	7	−4	−16	16	4	−7	−16	−5	16	−16	5	16
321		1	0	1	1	2	0	0	2	1	−3	−1	−2	−2	−1	−3
312		−1	0	1	1	−4	−4	4	4	−1	5	−5	−8	8	5	−5
231		−1	0	1	0	0	0	0	0	0	3	3	0	0	−3	−3
132		−1	0	1	3	0	0	0	0	−3	0	3	0	0	−3	0
213		−1	0	1	4	−4	8	−8	4	−4	−1	−5	4	−4	5	1
123		1	0	1	−5	2	0	0	2	−5	6	−1	−8	−8	−1	6
222		1	0	1	−2	2	0	0	2	−2	−3	−1	4	4	−1	−3

at least thirty-two possible distinct elements as follows:

$$
(11.1)\quad \begin{cases}
p^{*(00,00)}_{(0000)} = \gamma^*_{600} = N, \quad p^{*(00,10)}_{(0000)} = p^{*(10,01)}_{(0000)} = \gamma^*_{510}, \quad p^{*(00,01)}_{(0000)} = \gamma^*_{501}, \\
p^{*(00,20)}_{(0000)} = p^{*(10,10)}_{(1000)} = p^{*(10,11)}_{(1000)} = p^{*(01,20)}_{(0000)} = p^{*(11,11)}_{(1001)} = \gamma^*_{420}, \\
p^{*(00,11)}_{(0000)} = p^{*(10,01)}_{(0100)} = p^{*(01,11)}_{(0001)} = \gamma^*_{411}, \quad p^{*(00,30)}_{(0000)} = p^{*(10,20)}_{(1000)} = p^{*(01,30)}_{(0000)} \\
= p^{*(20,11)}_{(1000)} = \gamma^*_{330}, \quad p^{*(10,10)}_{(0000)} = (2N + \gamma^*_{501})/3, \quad p^{*(10,20)}_{(0000)} = p^{*(20,11)}_{(0000)} \\
= (2\gamma^*_{510} + \gamma^*_{411})/3, \quad p^{*(10,11)}_{(0100)} = (2\gamma^*_{501} + \gamma^*_{402})/3, \quad p^{*(10,11)}_{(1100)} = p^{*(01,20)}_{(0010)} \\
= p^{*(11,11)}_{(1011)} = p^{*(11,11)}_{(1101)} = \gamma^*_{321}, \quad p^{*(10,30)}_{(0000)} = p^{*(20,20)}_{(1000)} = p^{*(11,30)}_{(0000)} \\
= (2\gamma^*_{420} + \gamma^*_{321})/3, \quad p^{*(10,30)}_{(1000)} = p^{*(20,20)}_{(2000)} = p^{*(11,30)}_{(1000)} = \gamma^*_{240},
\end{cases}
$$

TABLE B. (continued)

20	20	20	30	30	30	60	60	60	60	60	60	90	λ^*	600
0	−20	0	0	0	0	0	0	0	0	0	0	0		060
−160	20	−160	−60	480	−60	240	−120	−480	−480	−120	240	360		006
−10	0	10	−15	0	15	−20	−10	−10	10	10	20	0		510
−10	20	−10	15	−30	15	0	30	−30	−30	30	0	0		501
0	0	0	5	0	−5	0	−10	0	0	10	0	0		150
0	0	0	−10	0	10	0	20	0	0	−20	0	0		051
−40	20	−40	−45	0	−45	120	−90	−120	−120	−90	120	180		105
80	0	−80	30	0	−30	−80	20	80	−80	−20	80	0		015
4	−4	4	4	−2	4	0	−8	−4	−4	−8	0	−12		420
−4	20	−4	0	18	0	−24	0	−12	−12	0	−24	−36		402
0	4	0	−6	0	−6	−4	4	0	0	4	−4	6		240
0	4	0	12	0	12	−16	−8	0	0	−8	−16	24		042
8	20	8	−30	−48	−30	36	−60	24	24	−60	36	54		204
−32	−4	−32	−8	−32	−8	0	16	32	32	16	0	−48		024
−1	0	1	3	0	−3	6	6	3	−3	−6	−6	0		330
11	20	11	−15	12	−15	−12	−30	33	33	−30	−12	−18		303
8	0	−8	−6	0	6	24	−12	−24	24	12	−24	0		033
8	0	−8	−6	0	6	4	−4	8	−8	4	−4	0		411
0	4	0	3	0	3	8	−2	0	0	−2	8	−12		141
8	0	−8	21	0	−21	−32	14	8	−8	−14	32	0		114
−5	−4	−5	1	4	1	0	−2	5	5	−2	0	6		321
−1	0	1	3	0	−3	10	2	−1	1	−2	−10	0		312
2	0	−2	0	0	0	−6	0	−6	6	0	6	0		231
−4	0	4	−3	0	3	0	−6	12	−12	6	0	0		132
−10	0	10	12	0	−12	−2	8	−10	10	−8	2	0		213
4	−4	4	−5	16	−5	0	10	−4	−4	10	0	−12		123
4	−4	4	−2	−8	−2	0	4	−4	−4	4	0	6		222

$$\left\{\begin{array}{l}
p^{*(01,01)}_{(0000)} = 2N - \gamma^*_{501}, \quad p^{*(01,01)}_{(0001)} = \gamma^*_{402}, \quad p^{*(01,11)}_{(0010)} = 2\gamma^*_{510} - \gamma^*_{411}, \\
p^{*(01,11)}_{(0011)} = \gamma^*_{312}, \quad p^{*(01,30)}_{(0010)} = p^{*(20,11)}_{(1100)} = \gamma^*_{231}, \quad p^{*(20,20)}_{(0000)} \\
= (4N + 4\gamma^*_{501} + \gamma^*_{402})/9, \quad p^{*(20,11)}_{(0100)} = (2\gamma^*_{411} + \gamma^*_{312})/3, \quad p^{*(20,30)}_{(0000)} \\
= (4\gamma^*_{510} + 4\gamma^*_{411} + \gamma^*_{312})/9, \quad p^{*(20,30)}_{(1000)} = (2\gamma^*_{330} + \gamma^*_{231})/3, \\
p^{*(20,30)}_{(2000)} = \gamma^*_{150}, \quad p^{*(11,11)}_{(0110)} = (4N - \gamma^*_{402})/3, \quad p^{*(11,11)}_{(0111)} = (2\gamma^*_{402} + \gamma^*_{303})/3, \\
p^{*(11,11)}_{(1110)} = 2\gamma^*_{420} - \gamma^*_{321}, \quad p^{*(11,11)}_{(1111)} = \gamma^*_{222}, \quad p^{*(11,30)}_{(0010)} = (2\gamma^*_{321} + \gamma^*_{222})/3, \\
p^{*(11,30)}_{(1010)} = \gamma^*_{141}, \quad p^{*(30,30)}_{(0000)} = (8N + 12\gamma^*_{501} + 6\gamma^*_{402} + \gamma^*_{303})/27, \\
p^{*(30,30)}_{(1000)} = (4\gamma^*_{420} + 4\gamma^*_{321} + \gamma^*_{222})/9, \quad p^{*(30,30)}_{(2000)} = (2\gamma^*_{240} + \gamma^*_{141})/3, \\
p^{*(30,30)}_{(3000)} = \gamma^*_{060}.
\end{array}\right.$$

The local relationship matrices, $A^{(a_1a_2,b_1b_2)}_{\alpha}$, are defined by (4.6), and the relationship matrices, $D^{(a_1a_2,b_1b_2)}_{\alpha}$, are defined by the same way as shown in Section 4. Let $B^{(a_1a_2,b_1b_2)}_{\alpha}$ be the symmetric matrices of order μ_m such that

$$B^{(a_1a_2,b_1b_2)}_{\alpha} = \begin{cases}
D^{(a_1a_2,a_1a_2)}_{\alpha} & \text{if } a_1a_2 = b_1b_2 = 00, 10, 01, 20, 30, \\
D^{(a_1a_2,b_1b_2)}_{\alpha} + D^{(b_1b_2,a_1a_2)}_{\alpha} & \text{if } a_1a_2 \neq b_1b_2 \text{ and } a_1a_2 \\
\quad = 00, 10, 01, 20, 30;\ b_1b_2 = 00, 10, 01, 20, 11, 30, & \\
D^{(11,11)}_{\alpha} & \text{if } a_1a_2 = b_1b_2 = 11 \text{ and } \alpha = (0110), \\
\quad (1001), (0111), (1110), (1111), & \\
D^{(11,11)}_{(1011)} + D^{(11,11)}_{(1101)} & \text{if } a_1a_2 = b_1b_2 = 11 \text{ and } \\
 & \alpha = (1011) \text{ or } (1101).
\end{cases}$$

From Appendix II, we have

LEMMA 11.1. *Matrices* $D^{\#(a_1a_2,b_1b_2)}_{\beta}$ *and* $D^{\#(u_1u_2,v_1v_2)}_{fij}$ *satisfy the following properties*:

$$D^{\#(a_1a_2,c_1c_2)}_{\alpha} D^{\#(d_1d_2,b_1b_2)}_{\beta} = \delta_{c_1d_1}\delta_{c_2d_2}\delta_{\alpha\beta} D^{\#(a_1a_2,b_1b_2)}_{\alpha},$$

$$D^{\#(a_1a_2,b_1b_2)}_{\beta} D^{\#(u_1u_2,v_1v_2)}_{fij} = D^{\#(u_1u_2,v_1v_2)}_{fij} D^{\#(a_1a_2,b_1b_2)}_{\beta} = O_{\mu_m\times\mu_m},$$

$$D^{\#(u_1u_2,w_1w_2)}_{fik} D^{\#(s_1s_2,v_1v_2)}_{flj} = \delta_{w_1s_1}\delta_{w_2s_2}\delta_{kl} D^{\#(u_1u_2,v_1v_2)}_{fij}.$$

Let $\mathfrak{A}^*$ be a relationship algebra generated by eighty-three relationship matrices $D^{(a_1a_2,b_1b_2)}_{\alpha}$ which is also generated by fifty symmetric matrices

$B_\alpha^{(a_1a_2,b_1b_2)}$. Let $\mathfrak{A}_0^*$, $\mathfrak{A}_1^*$, $\mathfrak{A}_2^*$, $\mathfrak{A}_3^*$ and $\mathfrak{A}_f^*$ be the matrix algebras generated by 6^2 matrices $D_0^{\#(a_1a_2,b_1b_2)}$, 3^2 $D_1^{\#(c_1c_2,d_1d_2)}$, a $D_2^{\#(11,11)}$, a $D_3^{\#(30,30)}$ and 6^2 $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$, respectively, for a_1a_2, $b_1b_2=00$, 10, 01, 20, 11, 30; c_1c_2, $d_1d_2=20$, 11, 30; u_1u_2, $v_1v_2=10$, 01, 20, 11, 30; $i,j=1, 2, 3, 4, 5$. Then $\mathfrak{A}_\alpha^*\mathfrak{A}_\beta^*=\mathfrak{A}_\beta^*\mathfrak{A}_\alpha^*=\delta_{\alpha\beta}\mathfrak{A}_\alpha^*$ for α, $\beta=0, 1, 2, 3, f$, that is, these matrix algebras are mutually annihilated.

THEOREM 11.2. (i) *The algebra*, $\mathfrak{A}^*$, *is represented by the linear closure of eighty-three matrices* $D_\beta^{\#(a_1a_2,b_1b_2)}$ *and* $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ *for* $\beta=0, 1, 2, 3$ *and* i, $j=1, 2, 3, 4, 5$.

(ii) *The algebra*, $\mathfrak{A}^*$, *is decomposed into the direct sum of five ideals* $\mathfrak{A}_\beta^*$ $(\beta=0, 1, 2, 3, f)$, *i.e.*,

$$\mathfrak{A}^* = \mathfrak{A}_0^* \oplus \mathfrak{A}_1^* \oplus \mathfrak{A}_2^* \oplus \mathfrak{A}_3^* \oplus \mathfrak{A}_f^*.$$

(iii) *The ideals*, $\mathfrak{A}_0^*$, $\mathfrak{A}_1^*$, $\mathfrak{A}_2^*$, $\mathfrak{A}_3^*$ *and* $\mathfrak{A}_f^*$, *have* $D_0^{\#(a_1a_2,b_1b_2)}$, $D_1^{\#(c_1c_2,d_1d_2)}$, $D_2^{\#(11,11)}$, $D_3^{\#(30,30)}$ *and* $D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ *as their bases, respectively, which are isomorphic to the complete* 6×6, 3×3, 1×1, 1×1 *and* 6×6 *matrix algebras with multiplicities* ϕ_β $(\beta=0, 1, 2, 3, f)$, *respectively, where* $\phi_0=1$, $\phi_1=m(m-3)/2$, $\phi_2=\binom{m-1}{2}$, $\phi_3=m(m-1)(m-5)/6$ *and* $\phi_f=m-1$.

Note that $\phi_0=\phi_0^*$, $\phi_1=\phi_2^*$, $\phi_3=\phi_3^*$ and $\phi_f=\phi_1^*$, where ϕ_β^* $(\beta=0, 1, 2, 3)$ are defined by (3.7).

From the above theorem, information matrix M_T^* can be expressed as

$$\begin{aligned} M_T^* &= \sum_{a_1a_2}\sum_{b_1b_2}\sum_{\alpha} p_\alpha^{*(a_1a_2,b_1b_2)} D_\alpha^{(a_1a_2,b_1b_2)} \\ &= \sum_{a_1a_2}{}^*\sum_{b_1b_2}{}^*\sum_{\alpha} p_\alpha^{*(a_1a_2,b_1b_2)} B_\alpha^{(a_1a_2,b_1b_2)} \\ &= \sum_{c_1c_2}\sum_{d_1d_2}\sum_{\beta=0}^{3} \kappa_\beta^{*c_1c_2,d_1d_2} D_\beta^{\#(c_1c_2,d_1d_2)} \\ &\quad + \sum_{u_1u_2}\sum_{v_1v_2}\sum_{i,j} \kappa_{f_{ij}}^{*u_1u_2,v_1v_2} D_{f_{ij}}^{\#(u_1u_2,v_1v_2)}, \end{aligned}$$

where

$$\left\{\begin{aligned} &\kappa_0^{*00,30} = \sqrt{\binom{m}{3}}\,p_{(0000)}^{*(00,30)}, \quad \kappa_0^{*a_1a_2,30} = \sqrt{\binom{m-1}{2}\Big/3}\,\{3p_{(0000)}^{*(a_1a_2,30)} \\ &+ (m-3)p_{\delta_1}^{*(a_1a_2,30)}\}, \quad \kappa_0^{*20,30} = \sqrt{(m-2)/3}\,\{3p_{(0000)}^{*(20,30)} \\ &+ 3(m-3)p_{(1000)}^{*(20,30)} + \binom{m-3}{2}p_{(2000)}^{*(20,30)}\}, \quad \kappa_0^{*11,30} = \sqrt{(m-2)/6} \\ &\cdot\{6p_{(0000)}^{*(11,30)} + 3(m-3)(p_{(0010)}^{*(11,30)} + p_{(1000)}^{*(11,30)}) + 2\binom{m-3}{2}p_{(1010)}^{*(11,30)}\}, \end{aligned}\right.$$

$$
(11.2)\left\{
\begin{aligned}
&\kappa_0^{*30,30} = p^{*(30,30)}_{(0000)} + 3(m-3)p^{*(30,30)}_{(1000)} + 3\binom{m-3}{2}p^{*(30,30)}_{(2000)} \\
&+\binom{m-3}{3}p^{*(30,30)}_{(3000)}, \quad \kappa_1^{*20,30} = \sqrt{m-4}\,\{p^{*(20,30)}_{(0000)} - 2p^{*(20,30)}_{(1000)} \\
&+ p^{*(20,30)}_{(2000)}\}, \quad \kappa_1^{*11,30} = \sqrt{2(m-4)}\,\{p^{*(11,30)}_{(0000)} - p^{*(11,30)}_{(0010)} - p^{*(11,30)}_{(1000)} \\
&+ p^{*(11,30)}_{(1010)}\}, \quad \kappa_1^{*30,30} = p^{*(30,30)}_{(0000)} + (m-7)p^{*(30,30)}_{(1000)} \\
&- (2m-11)p^{*(30,30)}_{(2000)} + (m-5)p^{*(30,30)}_{(3000)}, \quad \kappa_3^{*30,30} = p^{*(30,30)}_{(0000)} \\
&- 3(p^{*(30,30)}_{(1000)} - p^{*(30,30)}_{(2000)}) - p^{*(30,30)}_{(3000)}, \quad \kappa_{f_{15}}^{*a_1a_2,30} = \sqrt{\binom{m-2}{2}}\{p^{*(a_1a_2,30)}_{(0000)} \\
&- p^{*(a_1a_2,30)}_{\boldsymbol{\delta}_1}\}, \quad \kappa_{f_{25}}^{*20,30} = \sqrt{(m-3)/2}\,\{2p^{*(20,30)}_{(0000)} + (m-6)p^{*(20,30)}_{(1000)} \\
&- (m-4)p^{*(20,30)}_{(2000)}\}, \quad \kappa_{f_{35}}^{*11,30} = \{\sqrt{m(m-2)(m-3)}/2\}\,\{p^{*(11,30)}_{(0010)} \\
&- p^{*(11,30)}_{(1000)}\}, \quad \kappa_{f_{45}}^{*11,30} = \{\sqrt{m-3}/2\}\,\{4p^{*(11,30)}_{(0000)} + (m-6)(p^{*(11,30)}_{(0010)} \\
&+ p^{*(11,30)}_{(1000)}) - 2(m-4)p^{*(11,30)}_{(1010)}\}, \quad \kappa_{f_{55}}^{*30,30} = p^{*(30,30)}_{(0000)} \\
&+ (2m-9)p^{*(30,30)}_{(1000)} + \{(m-4)(m-9)/2\}p^{*(30,30)}_{(2000)} - \binom{m-4}{2}p^{*(30,30)}_{(3000)},
\end{aligned}
\right.
$$

and the remaining $\kappa_\beta^{*a_1a_2,b_1b_2}$ and $\kappa_{f_{ij}}^{*u_1u_2,v_1v_2}$ are the same as those obtained by replacing $p_{\boldsymbol{\alpha}}^{(a_1a_2,b_1b_2)}$ (in (6.2)) by $p_{\boldsymbol{\alpha}}^{*(a_1a_2,b_1b_2)}$. Here $\boldsymbol{\delta}_1=(1000)$, (0010) according as $a_1a_2=10$, 01, and connections between $\gamma^*_{p_0p_1p_2}$ and $\lambda^*_{i_0i_1i_2}$ and between $\gamma^*_{p_0p_1p_2}$ and $p_{\boldsymbol{\alpha}}^{*(a_1a_2,b_1b_2)}$ are given by Table B and (11.1), respectively. Note that $\kappa_\beta^{*a_1a_2,b_1b_2}=\kappa_\beta^{*b_1b_2,a_1a_2}$ and $\kappa_{f_{ij}}^{*u_1u_2,v_1v_2}=\kappa_{f_{ji}}^{*v_1v_2,u_1u_2}$. From Theorem 11.2, we can obtain 6×6 matrix K_0^*, 3×3 K_1^*, 1×1 K_2^*, 1×1 K_3^* and 6×6 K_f^* such that

$$\mathfrak{A}_\beta^*\colon M_T^* \longrightarrow K_\beta^* \qquad \text{for} \quad \beta = 0, 1, 2, 3, f,$$

where

$$
K_0^* = \begin{bmatrix}
\kappa_0^{*00,00} & \kappa_0^{*00,10} & \kappa_0^{*00,01} & \kappa_0^{*00,20} & \kappa_0^{*00,11} & \kappa_0^{*00,30} \\
 & \kappa_0^{*10,10} & \kappa_0^{*10,01} & \kappa_0^{*10,20} & \kappa_0^{*10,11} & \kappa_0^{*10,30} \\
 & & \kappa_0^{*01,01} & \kappa_0^{*01,20} & \kappa_0^{*01,11} & \kappa_0^{*01,30} \\
 & & & \kappa_0^{*20,20} & \kappa_0^{*20,11} & \kappa_0^{*20,30} \\
 & \text{Sym.} & & & \kappa_0^{*11,11} & \kappa_0^{*11,30} \\
 & & & & & \kappa_0^{*30,30}
\end{bmatrix},
$$

$$K_1^* = \begin{bmatrix} \kappa_1^{*20,20} & \kappa_1^{*20,11} & \kappa_1^{*20,30} \\ & \kappa_1^{*11,11} & \kappa_1^{*11,30} \\ \text{Sym.} & & \kappa_1^{*30,30} \end{bmatrix}, \quad K_2^* = [\kappa_2^{*11,11}], \quad K_3^* = [\kappa_3^{*30,30}],$$

$$K_f^* = \begin{bmatrix} \kappa_{f_{11}}^{*10,01} & \kappa_{f_{11}}^{*10,01} & \kappa_{f_{12}}^{*10,20} & \kappa_{f_{13}}^{*10,11} & \kappa_{f_{14}}^{*10,11} & \kappa_{f_{15}}^{*10,30} \\ & \kappa_{f_{11}}^{*01,01} & \kappa_{f_{12}}^{*01,20} & \kappa_{f_{13}}^{*01,11} & \kappa_{f_{14}}^{*01,11} & \kappa_{f_{15}}^{*01,30} \\ & & \kappa_{f_{22}}^{*20,20} & \kappa_{f_{23}}^{*20,11} & \kappa_{f_{24}}^{*20,11} & \kappa_{f_{25}}^{*20,30} \\ & & & \kappa_{f_{33}}^{*11,11} & \kappa_{f_{34}}^{*11,11} & \kappa_{f_{35}}^{*11,30} \\ & \text{Sym.} & & & \kappa_{f_{44}}^{*11,11} & \kappa_{f_{45}}^{*11,30} \\ & & & & & \kappa_{f_{55}}^{*30,30} \end{bmatrix}.$$

Since I_{μ_m} belongs to $\mathfrak{A}^*$, we have

THEOREM 11.3. *The characteristic polynomial, $\Psi_T^*(x)$, of information matrix M_T^* of a 3^m-BTO design, T, is given by*

$$\begin{aligned} \psi_T^*(x) &= \det(M_T^* - xI_{\mu_m}) \\ &= \{\det(K_0^* - xI_6)\}^{\phi_0}\{\det(K_1^* - xI_3)\}^{\phi_1}\{\det(K_2^* - x)\}^{\phi_2} \\ &\quad \cdot\{\det(K_3^* - x)\}^{\phi_3}\{\det(K_f^* - xI_6)\}^{\phi_f}, \end{aligned}$$

where ϕ_β ($\beta=0, 1, 2, 3, f$) are given in Theorem 11.2.

Let T be a B-array of strength at least four. Then, for the second-order model defined by (10.1), the characteristic polynomial of information matrix M_T^{**} of T (cf. [20]) can also be obtained by use of the algebraic structure as follows:

$$\begin{aligned} \Psi_T^{**}(x) &= \det(M_T^{**} - xI_{\nu_m^*}) \\ &= \{\det(K_0^{**} - xI_4)\}^{\phi_0}\{\det(K_1^{**} - x)\}^{\phi_1}\{\det(K_f^{**} - xI_3)\}^{\phi_f}, \end{aligned}$$

where $\Psi_T^{**}(x)$ is a characteristic polynomial of M_T^{**} for the second-order model, and

$$K_0^{**} = \begin{bmatrix} \kappa_0^{**00,00} & \kappa_0^{**00,10} & \kappa_0^{**00,01} & \kappa_0^{**00,20} \\ & \kappa_0^{**10,10} & \kappa_0^{**10,01} & \kappa_0^{**10,20} \\ & & \kappa_0^{**01,01} & \kappa_0^{**01,20} \\ \text{Sym.} & & & \kappa_0^{**20,20} \end{bmatrix}, \tag{11.3}$$

$$K_1^{**} = [\kappa_1^{**20,20}],$$

$$K_f^{**} = \begin{bmatrix} \kappa_{f_{11}}^{**10,10} & \kappa_{f_{11}}^{**10,01} & \kappa_{f_{12}}^{**10,20} \\ & \kappa_{f_{11}}^{**01,01} & \kappa_{f_{12}}^{**01,20} \\ \text{Sym.} & & \kappa_{f_{22}}^{**20,20} \end{bmatrix}$$

in which $\kappa_\beta^{**a_1a_2,b_1b_2}$ ($\beta=0, 1$) and $\kappa_{f_{ij}}^{**u_1u_2,v_1v_2}$ ($i, j=1, 2$) are the same as those obtained by replacing $\kappa_\beta^{a_1a_2,b_1b_2}$ and $\kappa_{f_{ij}}^{u_1u_2,v_1v_2}$ (in (6.2)) by $\kappa_\beta^{**a_1a_2,b_1b_2}$ and $\kappa_{f_{ij}}^{**u_1u_2,v_1v_2}$, respectively, and ϕ_β ($\beta=0, 1, f$) are given by (4.13).

Let $K_0^{*-1}=\|\kappa_{a_1a_2,b_1b_2}^{*0}\|$, $K_1^{*-1}=\|\kappa_{c_1c_2,d_1d_2}^{*1}\|$, $K_2^{*-1}=\|\kappa_{11,11}^{*2}\|$, $K_3^{*-1}=\|\kappa_{30,30}^{*3}\|$ and $K_f^{*-1}=\|\kappa_{u_1u_2,v_1v_2}^{*f_{ij}}\|$. Then characteristic polynomial, $\chi_T^*(x)$, of covariance matrix $V_T^*\sigma^2$ of BLUE $\hat{\theta}_{\mu_m}$ can be obtained by getting that of V_T^* as follows:

COROLLARY 11.4. *When T is a design of Theorem 11.3, $\chi_T^*(x)$ is given by*

$$\begin{aligned} \chi_T^*(x) &= \det(V_T^* - xI_{\mu_m}) \\ &= \{\det(K_0^{*-1} - xI_6)\}^{\phi_0}\{\det(K_1^{*-1} - xI_3)\}^{\phi_1} \\ &\quad \cdot\{\det(K_2^{*-1} - x)\}^{\phi_2}\{\det(K_3^{*-1} - x)\}^{\phi_3}\{\det(K_f^{*-1} - xI_6)\}^{\phi_f}. \end{aligned}$$

Theorem 11.2 yields

THEOREM 11.5. *For T being a design of Theorem 11.3,*

$$\begin{aligned} \operatorname{tr}(V_T^*) &= \phi_0\operatorname{tr}(K_0^{*-1}) + \phi_1\operatorname{tr}(K_1^{*-1}) + \phi_2\operatorname{tr}(K_2^{*-1}) \\ &\quad + \phi_3\operatorname{tr}(K_3^{*-1}) + \phi_f\operatorname{tr}(K_f^{*-1}), \\ \det(V_T^*) &= \{\det(K_0^{*-1})\}^{\phi_0}\{\det(K_1^{*-1})\}^{\phi_1}\{\det(K_2^{*-1})\}^{\phi_2} \\ &\quad \cdot\{\det(K_3^{*-1})\}^{\phi_3}\{\det(K_f^{*-1})\}^{\phi_f}. \end{aligned}$$

COROLLARY 11.6. *For T being a B-array of strength at least four, when information matrix M_T^{**} for the second-order model is non-singular,*

$$\operatorname{tr}(V_T^{**}) = \phi_0\operatorname{tr}(K_0^{**-1}) + \phi_1\operatorname{tr}(K_1^{**-1}) + \phi_f\operatorname{tr}(K_f^{**-1}),$$

$$\det(V_T^{**}) = \{\det(K_0^{**-1})^{\phi_0}\{\det(K_1^{**-1})\}^{\phi_1}\{\det(K_f^{**-1})\}^{\phi_f},$$

where K_β^{**} ($\beta=0, 1, f$) are given by (11.3).

If T is a 3^m-BTO design, a (0, 2)IB-array, $\bar{T}$, say, of T is called a 3^m-(0, 2)IBTO design.

THEOREM 11.7. *For a B-array, T, and its* (0, 2)*IB-array, $\bar{T}$,*

$$\Psi_T^*(x) = \Psi_{\bar{T}}^*(x).$$

PROOF. Let $E_{\bar{T}}^*$ be the design matrix of $\bar{T}$. Then $E_{\bar{T}}^* = E_T^* W$, where E_T^* is the design matrix of T and W is the diagonal matrix of order μ_m such that

$$W = \operatorname{diag}\{1, \underbrace{-1, \ldots, -1}_{m}, \underbrace{1, \ldots, 1}_{m}, \underbrace{1, \ldots, 1}_{\binom{m}{2}}, \underbrace{-1, \ldots, -1}_{2\binom{m}{2}}, \underbrace{-1, \ldots, -1}_{\binom{m}{3}}\}.$$

Hence $M_{\bar{T}}^* = W M_T^* W$ which yields that

$$\begin{aligned}\Psi_{\bar{T}}^*(x) &= \det(M_{\bar{T}}^* - x I_{\mu_m}) \\ &= \det(W(M_T^* - x I_{\mu_m})W) \\ &= \det(M_T^* - x I_{\mu_m}) \\ &= \Psi_T^*(x).\end{aligned}$$

COROLLARY 11.8. *For a 3^m-BTO design, T, and its 3^m-(0, 2)IBTO design, $\bar{T}$,*

$$\operatorname{tr}(V_{\bar{T}}^*) = \operatorname{tr}(V_T^*),$$

$$\det(V_*^{\bar{T}}) = \det(V_T^*).$$

12. Covariance matrices and optimal balanced third-order designs for 3^6 factorials

Let T be a B-array $[N, m, 3, 6]$ with index set $\{\lambda_{i_0 i_1 i_2}^* | i_0 + i_1 + i_2 = 6\}$. Then we obviously obtain the following:

THEOREM 12.1. *A necessary and sufficient condition for T to be a 3^m-BTO design is that every irreducible representation K_β^* $(\beta = 0, 1, 2, 3, f)$ of M_T^* is positive definite.*

Since M_T^* is positive semidefinite, K_β^* $(\beta = 0, 1, 2, 3, f)$ are also so. Therefore, Table B, (11.1) and (11.2) imply

COROLLARY 12.2. *A necessary condition for the existence of a B-array $[N, m, 3, 6]$ with index set $\{\lambda_{i_0 i_1 i_2}^*\}$ based on the third-order model is that the following relations hold:*

(12.1) $\lambda_{411}^* + \lambda_{141}^* + \lambda_{114}^* + \lambda_{321}^* + \lambda_{312}^* + \lambda_{231}^* + \lambda_{132}^* + \lambda_{213}^* + \lambda_{123}^* + \lambda_{222}^* \geqq 0,$

(12.2) $\lambda_{330}^* + \lambda_{303}^* + \lambda_{033}^* + \lambda_{321}^* + \lambda_{312}^* + \lambda_{231}^* + \lambda_{132}^* + \lambda_{213}^* + \lambda_{123}^* + \lambda_{222}^* \geqq 0,$

(12.3a) $\lambda_{420}^* + \lambda_{240}^* + \lambda_{042}^* + \lambda_{024}^* + \lambda_{330}^* + \lambda_{033}^* + \lambda_{411}^*$

$$+ \lambda_{114}^* + \lambda_{321}^* + \lambda_{312}^* + \lambda_{231}^* + \lambda_{132}^* + \lambda_{213}^* + \lambda_{123}^* + \lambda_{222}^* \geqq 0,$$

(12.3b) $$(m-4)\{\lambda^*_{420}+16(\lambda^*_{402}+\lambda^*_{204})+\lambda^*_{024}\}+\lambda^*_{330}+\lambda^*_{033}$$
$$-32(m-6)\lambda^*_{303}+8(m-4)(\lambda^*_{411}+\lambda^*_{114})+12(\lambda^*_{321}+\lambda^*_{123})$$
$$-8(m-10)(\lambda^*_{312}+\lambda^*_{213})+3(\lambda^*_{231}+\lambda^*_{132})-2(m-16)\lambda^*_{222}\geqq 0,$$

(12.4a) $$(m-2)\{\lambda^*_{510}+4(\lambda^*_{501}+\lambda^*_{105})+\lambda^*_{015}\}+(2m-3)(\lambda^*_{420}+\lambda^*_{024})$$
$$+16(\lambda^*_{402}+\lambda^*_{204})+\lambda^*_{240}+\lambda^*_{042}+m(\lambda^*_{330}+\lambda^*_{033})-8(m-6)\lambda^*_{303}$$
$$+(9m-10)(\lambda^*_{411}+\lambda^*_{114})+2\lambda^*_{141}+4(m+3)(\lambda^*_{321}+\lambda^*_{123})$$
$$-2(5m-38)(\lambda^*_{312}+\lambda^*_{213})-(m-16)(\lambda^*_{231}+\lambda^*_{132})$$
$$-6(2m-13)\lambda^*_{222}\geqq 0,$$

(12.4b) $$(5m-6)(\lambda^*_{510}+\lambda^*_{015})+4(2m-3)(\lambda^*_{501}+\lambda^*_{105})+2m(\lambda^*_{150}+\lambda^*_{051})$$
$$-3(m-8)(\lambda^*_{420}+\lambda^*_{024})+(5m+6)(\lambda^*_{402}+\lambda^*_{204})$$
$$+2(m+6)(\lambda^*_{240}+\lambda^*_{042})-3(m-10)(\lambda^*_{330}+\lambda^*_{033})+24\lambda^*_{303}$$
$$+3(m+10)(\lambda^*_{411}+\lambda^*_{114})+12(m+1)\lambda^*_{141}$$
$$-2(5m-54)(\lambda^*_{321}+\lambda^*_{123})-8(2m-15)(\lambda^*_{312}+\lambda^*_{213})$$
$$+(5m+78)(\lambda^*_{231}+\lambda^*_{132})-18(m-10)\lambda^*_{222}\geqq 0,$$

(12.5a) $$m(\lambda^*_{600}+\lambda^*_{006})+(4m+1)(\lambda^*_{510}+\lambda^*_{015})+2(m+2)(\lambda^*_{501}+\lambda^*_{105})$$
$$+\lambda^*_{150}+\lambda^*_{051}+2(3m+2)(\lambda^*_{420}+\lambda^*_{024})-(m-16)(\lambda^*_{402}+\lambda^*_{204})$$
$$+(m+4)(\lambda^*_{240}+\lambda^*_{042})+2(2m+3)(\lambda^*_{330}+\lambda^*_{033})-4(m-6)\lambda^*_{303}$$
$$+(4m+21)(\lambda^*_{411}+\lambda^*_{114})-2(m-6)\lambda^*_{141}+40(\lambda^*_{321}+\lambda^*_{123})$$
$$-2(4m-29)(\lambda^*_{312}+\lambda^*_{213})-(2m-17)(\lambda^*_{231}+\lambda^*_{132})$$
$$-12(m-6)\lambda^*_{222}\geqq 0,$$

(12.5b) $$m(\lambda^*_{600}+4\lambda^*_{060}+\lambda^*_{006})+9(\lambda^*_{510}+\lambda^*_{015})+6m(\lambda^*_{150}+\lambda^*_{051})$$
$$+3(4m+3)(\lambda^*_{150}+\lambda^*_{051})-6(m-6)(\lambda^*_{420}+\lambda^*_{024})$$
$$+15m(\lambda^*_{402}+\lambda^*_{204})+9(m+4)(\lambda^*_{240}+\lambda^*_{042})$$
$$-2(m-27)(\lambda^*_{330}+\lambda^*_{033})+20m\lambda^*_{303}+45(\lambda^*_{411}+\lambda^*_{114})$$
$$+18(m+4)\lambda^*_{141}-24(m-6)(\lambda^*_{321}+\lambda^*_{123})+90(\lambda^*_{312}+\lambda^*_{213})$$
$$-6(2m-27)(\lambda^*_{231}+\lambda^*_{132})-36(m-6)\lambda^*_{222}\geqq 0.$$

COROLLARY 12.3. *A necessary condition for the existence of a 3^m-BTO design is that relations* (12.1) *through* (12.5) *hold with strict inequalities.*

Let $V_{\boldsymbol{\alpha}}^{*(a_1a_2,b_1b_2)}$ be the elements of V_T^* corresponding to $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2}t_3^{\varepsilon_3})$ and $\hat{\theta}(t_4^{\varepsilon_4}t_5^{\varepsilon_5}t_6^{\varepsilon_6})$, where $w_r(\varepsilon_1, \varepsilon_2, \varepsilon_3)=a_r$ and $w_r(\varepsilon_4, \varepsilon_5, \varepsilon_6)=b_r$ for $r=1, 2$, and $\hat{\theta}(t_1^{\varepsilon_1}t_2^{\varepsilon_2}t_3^{\varepsilon_3})$ is related to $\hat{\theta}(t_4^{\varepsilon_4}t_5^{\varepsilon_5}t_6^{\varepsilon_6})$ by $R(\boldsymbol{\alpha};\, a_1a_2,\, b_1b_2)$. Then we obtain

THEOREM 12.4. *For a 3^m-BTO design, T, there are the following fifty possible distinct elements, $V_{\boldsymbol{\alpha}}^{*(a_1a_2,b_1b_2)}$, of V_T^*:*

$$V_{(0000)}^{*(00,30)} = \left\{1\Big/\sqrt{\binom{m}{3}}\right\}\kappa_{00,30}^{*0}, \quad V_{(0000)}^{*(a_1a_2,30)} = \left[1\Big/\left\{m\sqrt{\binom{m-1}{2}}\right\}\right]$$

$$\cdot\{\sqrt{3}\,\kappa_{a_1a_2,30}^{*0} + \sqrt{(m-1)(m-3)}\,\kappa_{a_1a_2,30}^{*f_{15}}\},$$

$$V_{\delta_1}^{*(a_1a_2,30)} = \left[1\Big/\left\{m\sqrt{\binom{m-1}{3}}\right\}\right]\{\sqrt{m-3}\,\kappa_{a_1a_2,30}^{*0}$$

$$-\sqrt{3(m-1)}\,\kappa_{a_1a_2,30}^{*f_{15}}\}, \quad V_{(0000)}^{*(20,30)} = \left[1\Big/\left\{6\binom{m}{3}\right\}\right]\{2\sqrt{3(m-2)}\,\kappa_{20,30}^{*0}$$

$$+ m(m-3)\sqrt{m-4}\,\kappa_{20,30}^{*1} + 2(m-1)\sqrt{2(m-3)}\,\kappa_{20,30}^{*f_{25}}\},$$

$$V_{(1000)}^{*(20,30)} = \left[1\Big/\left\{6\binom{m}{3}\sqrt{m-3}\right\}\right]\left\{2\sqrt{6\binom{m-2}{2}}\,\kappa_{20,30}^{*0}\right.$$

$$\left.-2m\sqrt{2\binom{m-3}{2}}\,\kappa_{20,30}^{*1} + \sqrt{2}\,(m-1)(m-6)\,\kappa_{20,30}^{*f_{25}}\right\}, \quad V_{(2000)}^{*(20,30)}$$

$$= \left[1\Big/\left\{\binom{m}{3}\sqrt{2\binom{m-3}{2}}\right\}\right]\left\{\sqrt{2\binom{m-2}{3}}\,\kappa_{20,30}^{*0} + m\sqrt{m-3}\,\kappa_{20,30}^{*1}\right.$$

$$\left.-(m-1)\sqrt{2(m-4)}\,\kappa_{20,30}^{*f_{25}}\right\}, \quad V_{(0000)}^{*(11,30)} = \left[1\Big/\left\{12\binom{m}{3}\right\}\right]$$

$$\cdot\{2\sqrt{6(m-2)}\,\kappa_{11,30}^{*0} + m(m-3)\sqrt{2(m-4)}\,\kappa_{11,30}^{*1}$$

$$+ 4(m-1)\sqrt{m-3}\,\kappa_{11,30}^{*f_{45}}\}, \quad V_{(0010)}^{*(11,30)}$$

$$= \left[1\Big/\left\{6\binom{m}{3}\sqrt{m-3}\right\}\right]\left\{2\sqrt{3\binom{m-2}{2}}\,\kappa_{11,30}^{*0} - 2m\sqrt{\binom{m-3}{2}}\,\kappa_{11,30}^{*1}\right.$$

$$\left.+ (m-1)\sqrt{m(m-2)}\,\kappa_{11,30}^{*f_{35}} + (m-1)(m-6)\,\kappa_{11,30}^{*f_{45}}\right\}, \quad V_{(1000)}^{*(11,30)}$$

$$= \left[1\Big/\left\{6\binom{m}{3}\sqrt{m-3}\right\}\right]\left\{2\sqrt{3\binom{m-2}{2}}\,\kappa_{11,30}^{*0} - 2m\sqrt{\binom{m-3}{2}}\,\kappa_{11,30}^{*1}\right.$$

$$\left.-(m-1)\sqrt{m(m-2)}\,\kappa_{11,30}^{*f_{35}} + (m-1)(m-6)\,\kappa_{11,30}^{*f_{45}}\right\}, \quad V_{(1010)}^{*(11,30)}$$

$$= \left[1\Big/\left\{2\binom{m}{3}\sqrt{\binom{m-3}{2}}\right\}\right]\left\{\sqrt{2\binom{m-2}{3}}\,\kappa_{11,30}^{*0} + m\sqrt{m-3}\,\kappa_{11,30}^{*1}\right.$$

$$- (m-1)\sqrt{2(m-4)}\,\kappa^{*f_{45}}_{11,30}\Big\},\quad V^{*(30,30)}_{(0000)} = \Big[1\Big/\Big\{6\binom{m}{3}\Big\}\Big]\{6\kappa^{*0}_{30,30}$$

$$+ 3m(m-3)\kappa^{*1}_{30,30} + m(m-1)(m-5)\kappa^{*3}_{30,30} + 6(m-1)\kappa^{*f_{55}}_{30,30}\},$$

$$V^{*(30,30)}_{(1000)} = \Big[1\Big/\Big\{24\binom{m}{4}\Big\}\Big]\{6(m-3)\kappa^{*0}_{30,30} + m(m-3)(m-7)\kappa^{*1}_{30,30}$$

$$- m(m-1)(m-5)\kappa^{*3}_{30,30} + 2(m-1)(2m-9)\kappa^{*f_{55}}_{30,30}\},\quad V^{*(30,30)}_{(2000)}$$

$$= \Big[1\Big/\Big\{60\binom{m}{5}\Big\}\Big]\Big\{6\binom{m-3}{2}\kappa^{*0}_{30,30} - m(m-3)(2m-11)\kappa^{*1}_{30,30}$$

$$+ m(m-1)(m-5)\kappa^{*3}_{30,30} + (m-1)(m-4)(m-9)\kappa^{*f_{55}}_{30,30}\Big\},$$

$$V^{*(30,30)}_{(3000)} = \Big[1\Big/\Big\{20\binom{m}{5}\Big\}\Big]\Big\{2\binom{m-3}{2}\kappa^{*0}_{30,30} + 3m(m-3)\kappa^{*1}_{30,30}$$

$$- 2\binom{m}{2}\kappa^{*3}_{30,30} - 3(m-1)(m-4)\kappa^{*f_{55}}_{30,30}\Big\},$$

and the remaining $V^{*(c_1c_2,d_1d_2)}_{\alpha}$ *are the same as those obtained by replacing* $\kappa^{\beta}_{c_1c_2,d_1d_2}$ *and* $\kappa^{f_{ij}}_{u_1u_2,v_1v_2}$ *(in (7.1)) by* $\kappa^{*\beta}_{c_1c_2,d_1d_2}$ *and* $\kappa^{*f_{ij}}_{u_1u_2,v_1v_2}$, *respectively.*

Let T be a B-array $[N, 6, 3, 6]$ with index set $\{\lambda^*_{i_0i_1i_2} | i_0+i_1+i_2=6\}$. Then, in Table 7, optimal 3^6-BTO designs with respect to the trace criterion are presented for $\mu_6\ (=78) \leqq N \leqq 100$, where indices λ^*_{600}, λ^*_{060} and λ^*_{006} are 0, 1 or 2, and the remaining $\lambda^*_{i_0i_1i_2}$ are either 0 or 1.

In Table 8, optimal 3^6-BTO designs with respect to the determinant criterion are presented for $78 \leqq N \leqq 100$, where $\lambda^*_{i_0i_1i_2}$ has the same restrictions as the trace criterion. Furthermore, in Tables 7 and 8, possible distinct elements of V^*_T for each of optimal designs are presented.

Appendix I. Connection between $A^{(a_1a_2,b_1b_2)}_{\alpha}$, and $A^{\#(c_1c_2,d_1d_2)}_{\beta}$ and $A^{\#(u_1u_2,v_1v_2)}_{f_{ij}}$

Since, for a_1a_2, b_1b_2, c_1c_2, $d_1d_2=00$, 10, 01, 20, 02; u_1u_2, $v_1v_2=10$, 01, 20, 02, MD relationships are similar to TMDPB association schemes, it follows from (3.6) that $n_{c_1c_2}\times n_{d_1d_2}$ matrices $A^{\#(c_1c_2,d_1d_2)}_{\beta}$ $(\beta=0, 1)$ and $n_{u_1u_2}\times n_{v_1v_2}$ matrices $A^{\#(u_1u_2,v_1v_2)}_{f_{ij}}$ are, respectively, linear combinations of local relationship matrices $A^{(a_1a_2,b_1b_2)}_{\alpha}$ as follows (cf. [60]):

$$\begin{cases} A^{\#(00,00)}_0 = A^{(00,00)}_{(0000)} = 1, \quad A^{\#(00,b_1b_2)}_0 = \{1/\sqrt{m}\}\, A^{(00,b_1b_2)}_{(0000)}, \\ A^{\#(00,b'_1b'_2)}_0 = \Big\{1\Big/\sqrt{\binom{m}{2}}\Big\} A^{(00,b'_1b'_2)}_{(0000)}, \quad A^{\#(a_1a_2,b_1b_2)}_0 = \{1/m\}\{A^{(a_1a_2,b_1b_2)}_{(0000)} \end{cases}$$

$$
\text{(A.1)} \quad \left|\begin{array}{l}
+ A_{\boldsymbol{\alpha}_1}^{(a_1a_2,b_1b_2)}\}, \quad A_{f_{11}}^{\#(a_1a_2,b_1b_2)} = \{1/m\}\{(m-1)A_{(0000)}^{(a_1a_2,b_1b_2)} \\
- A_{\boldsymbol{\alpha}_1}^{(a_1a_2,b_1b_2)}\}, \quad A_0^{\#(a_1a_2,b_1'b_2')} = [\sqrt{2}/\{m\sqrt{m-1}\}]\{A_{(0000)}^{(a_1a_2,b_1'b_2')} \\
+ A_{\boldsymbol{\beta}_1}^{(a_1a_2,b_1'b_2')}\}, \quad A_{f_{12}}^{\#(a_1a_2,b_1'b_2')} = [1/\{m\sqrt{m-2}\}]\{(m-2)A_{(0000)}^{(a_1a_2,b_1'b_2')} \\
- 2A_{\boldsymbol{\beta}_1}^{(a_1a_2,b_1'b_2')}\}, \quad A_0^{\#(a_1'a_2',b_1'b_2')} = \{1/\binom{m}{2}\}\{A_{(0000)}^{(a_1'a_2',b_1'b_2')} + A_{\boldsymbol{\gamma}_1}^{(a_1'a_2',b_1'b_2')} \\
+ A_{\boldsymbol{\gamma}_2}^{(a_1'a_2',b_1'b_2')}\}, \quad A_{f_{22}}^{\#(a_1'a_2',b_1'b_2')} = [1/\{m(m-2)\}]\{2(m-2)A_{(0000)}^{(a_1'a_2',b_1'b_2')} \\
+ (m-4)A_{\boldsymbol{\gamma}_1}^{(a_1'a_2',b_1'b_2')} - 4A_{\boldsymbol{\gamma}_2}^{(a_1'a_2',b_1'b_2')}\}, \quad A_1^{\#(a_1'a_2',b_1'b_2')} \\
= \left[1\Big/\left\{2\binom{m-1}{2}\right\}\right]\left\{2\binom{m-2}{2}A_{(0000)}^{(a_1'a_2',b_1'b_2')} - (m-3)A_{\boldsymbol{\gamma}_1}^{(a_1'a_2',b_1'b_2')} \right. \\
\left. + 2A_{\boldsymbol{\gamma}_2}^{(a_1'a_2',b_1'b_2')}\right\},
\end{array}\right.
$$

where a_1a_2, $b_1b_2=10$, 01 and $a_1'a_2'$, $b_1'b_2'=20$, 02. Here $\boldsymbol{\alpha}_1=(1000)$, (0100), (0001) according as $(a_1a_2, b_1b_2)=(10, 10)$, (10, 01), (01, 01); $\boldsymbol{\beta}_1=(1000)$, (0100), (0010), (0001) according as $(a_1a_2, b_1'b_2')=(10, 20)$, (10, 02), (01, 20), (01, 02); $\boldsymbol{\gamma}_r=(r000)$, $(0r00)$, $(000r)$ according as $(a_1'a_2', b_1'b_2')=(20, 20)$, (20, 02), (02, 02) for $r=1$, 2, respectively.

Kuwada [25] has obtained matrices $A_\beta^{\#(a_1a_2,11)}$ and $A_{f_{ij}}^{\#(u_1u_2,11)}$ by multiplying $A_\gamma^{\#(11,11)}$ and $A_{f_{kl}}^{\#(11,11)}$ by $A_{\boldsymbol{\alpha}}^{(a_1a_2,11)}$ ($\beta, \gamma=0, 1, 2$; $i, j, k, l=1, 2, 3, 4$), respectively. In this appendix, however, we shall obtain $A_\beta^{\#(a_1a_2,11)}$ and $A_{f_{ij}}^{\#(u_1u_2,11)}$ by a method different from the approach of [25] as follows:

$$
\left(\begin{array}{l}
A_0^{\#(00,11)} = \left\{1\Big/\sqrt{2\binom{m}{2}}\right\}A_0^{\#(00,00)}A_{(0000)}^{(00,11)} = \left\{1\Big/\sqrt{2\binom{m}{2}}\right\}A_{(0000)}^{(00,11)}, \\
A_0^{\#(a_1a_2,11)} = \{1/\sqrt{m-1}\}A_0^{\#(a_1a_2,a_1a_2)}A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} \\
= [1/\{m\sqrt{m-1}\}]\{A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} + A_{\boldsymbol{\xi}_1}^{(a_1a_2,11)} + A_{\boldsymbol{\xi}_2}^{(a_1a_2,11)}\}, \\
A_{f_{13}}^{\#(a_1a_2,11)} = \{1/\sqrt{2m}\}A_{f_{11}}^{\#(a_1a_2,a_1a_2)}\{A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} - A_{\boldsymbol{\xi}_1}^{(a_1a_2,11)}\} \\
= \{1/\sqrt{2m}\}\{A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} - A_{\boldsymbol{\xi}_1}^{(a_1a_2,11)}\}, \quad A_{f_{14}}^{\#(a_1a_2,11)} \\
= \{1/\sqrt{2(m-2)}\}A_{f_{11}}^{\#(a_1a_2,a_1a_2)}\{A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} + A_{\boldsymbol{\xi}_1}^{(a_1a_2,11)}\} \\
= [1/\{m\sqrt{2(m-2)}\}]\{(m-2)A_{\boldsymbol{\xi}_0}^{(a_1a_2,11)} + A_{\boldsymbol{\xi}_1}^{(a_1a_2,11)} - 2A_{\boldsymbol{\xi}_2}^{(a_1a_2,11)}\}, \\
A_0^{\#(a_1'a_2',11)} = \{1/\sqrt{2}\}A_0^{\#(a_1'a_2',a_1'a_2')}A_{(0000)}^{(a_1'a_2',11)} = \left[1\Big/\left\{2\binom{m}{2}\right\}\right] \\
\cdot\{A_{(0000)}^{(a_1a_2,11)} + A_{\boldsymbol{\zeta}_1}^{(a_1a_2,11)} + A_{\boldsymbol{\zeta}_2}^{(a_1a_2,11)} + A_{\boldsymbol{\zeta}_3}^{(a_1a_2,11)}\}, \quad A_1^{\#(a_1'a_2',11)}
\end{array}\right.
$$

$$
(\mathrm{A}.2)\quad
\left\{
\begin{aligned}
&= \{1/\sqrt{2}\}A_1^{\#(a'_1a'_2,a'_1a'_2)}A_{(0000)}^{(a'_1a'_2,11)} = \left[1\Big/\left\{2\binom{m-1}{2}\sqrt{2}\right\}\right] \\
&\cdot\left\{2\binom{m-2}{2}A_{(0000)}^{(a'_1a'_2,11)} - (m-3)(A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}) + 2A_{\zeta_3}^{(a'_1a'_2,11)}\right\}, \\
&A_{f_{23}}^{\#(a'_1a'_2,11)} = \{1/\sqrt{2m(m-2)}\}A_{f_{22}}^{\#(a'_1a'_2,a'_1a'_2)}\{A_{\zeta_1}^{(a'_1a'_2,11)} \\
&- A_{\zeta_2}^{(a'_1a'_2,11)}\} = \{1/\sqrt{2m(m-2)}\}\,\{A_{\zeta_1}^{(a'_1a'_2,11)} - A_{\zeta_2}^{(a'_1a'_2,11)}\}, \\
&A_{f_{24}}^{\#(a'_1a'_2,11)} = [1/\{(m-3)\sqrt{2}\}]A_{f_{22}}^{\#(a'_1a'_2,a'_1a'_2)}\{A_{(0000)}^{(a'_1a'_2,11)} \\
&+ A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}\} = [1/\{m(m-2)\sqrt{2}\}]\{2(m-2)A_{(000)}^{(a'_1a'_2,11)} \\
&+ (m-4)(A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}) - 4A_{\zeta_3}^{(a'_1a'_2,11)}\}, \quad A_0^{\#(11,11)} \\
&= \{1/2\}A_{(0000)}^{(11,a'_1a'_2)}A_0^{\#(a'_1a'_2,a'_1a'_2)}A_{(0000)}^{(a'_1a'_2,11)} = \left[1\Big/\left\{2\binom{m}{2}\right\}\right] \\
&\cdot\{A_{(0110)}^{(11,11)} + A_{(1001)}^{(11,11)} + A_{(0111)}^{(11,11)} + A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} + A_{(1101)}^{(11,11)} \\
&+ A_{(1111)}^{(11,11)}\}, \quad A_1^{\#(11,11)} = \{1/2\}A_{(0000)}^{(11,a'_1a'_2)}A_1^{\#(a'_1a'_2,a'_1a'_2)}A_{(0000)}^{(a'_1a'_2,11)} \\
&= \left[1\Big/\left\{4\binom{m-1}{2}\right\}\right]\left\{2\binom{m-2}{2}(A_{(0110)}^{(11,11)} + A_{(1001)}^{(11,11)}) - (m-3)(A_{(0111)}^{(11,11)}\right. \\
&\left.+ A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} + A_{(1101)}^{(11,11)}) + 2A_{(1111)}^{(11,11)}\right\}, \quad A_{f_{33}}^{\#(11,11)} \\
&= [1/\{2m(m-2)\}]\{A_{\bar{\zeta}_1}^{(11,a'_1a'_2)} - A_{\bar{\zeta}_2}^{(11,a'_1a'_2)}\} \\
&\cdot A_{f_{22}}^{\#(a'_1a'_2,a'_1a'_2)}\{A_{\zeta_1}^{(a'_1a'_2,11)} - A_{\zeta_2}^{(a'_1a'_211)}\} = [1/\{2m\}]\{2(A_{(0110)}^{(11,11)} \\
&- A_{(1001)}^{(11,11)}) + A_{(0111)}^{(11,11)} + A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} + A_{(1101)}^{(11,11)}\}, \quad A_{f_{34}}^{\#(11,11)} \\
&= [1/\{2(m-3)\sqrt{m(m-2)}\}]\,\{A_{\bar{\zeta}_1}^{(11,a'_1a'_2)} - A_{\bar{\zeta}_2}^{(11,a'_1a'_2)}\}A_{f_{22}}^{\#(a'_1a'_2,a'_1a'_2)} \\
&\cdot\{A_{(0000)}^{(a'_1a'_2,11)} + A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}\} = [1/\{2\sqrt{m(m-2)}\}] \\
&\cdot\{A_{(0111)}^{(11,11)} - A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} - A_{(1101)}^{(11,11)}\} = \{A_{f_{43}}^{\#(11,11)}\}', \\
&A_{f_{44}}^{\#(11,11)} = [1/\{2(m-3)^2\}]\,\{A_{(0000)}^{(11,a'_1a'_2)} + A_{\bar{\zeta}_1}^{(11,a'_1a'_2)} + A_{\bar{\zeta}_2}^{(11,a'_1a'_2)}\} \\
&\cdot A_{f_{22}}^{\#(a'_1a'_2,a'_1a'_2)}\{A_{(0000)}^{(a'_1a'_2a11)} + A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}\} \\
&= [1/\{2m(m-2)\}]\{2(m-2)(A_{(0110)}^{(11,11)} + A_{(1001)}^{(11,11)}) + (m-4)(A_{(0111)}^{(11,11)} \\
&+ A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} + A_{(1101)}^{(11,11)}) - 4A_{(1111)}^{(11,11)}\}, \quad A_2^{\#(11,11)}
\end{aligned}
\right.
$$

$$
\begin{aligned}
&= I_{2\binom{m}{2}} - A_0^{\#(11,11)} - A_1^{\#(11,11)} - A_{f_{33}}^{\#(11,11)} - A_{f_{44}}^{\#(11,11)} = [1/\{2m\}] \\
&\cdot \{(m-2)(A_{(0110)}^{(11,11)} - A_{(1001)}^{(11,11)}) - A_{(0111)}^{(11,11)} - A_{(1110)}^{(11,11)} + A_{(1011)}^{(11,11)} \\
&+ A_{(1101)}^{(11,11)}\},
\end{aligned}
$$

where $\xi_0=(0100), (0001)$, $\xi_1=(1000), (0010)$, $\xi_2=(1100), (0011)$ according as $a_1a_2=10, 01$, and $\zeta_1=(0100), (0001)$, $\zeta_2=(1000), (0010)$, $\zeta_3=(1100), (0011)$ according as $a'_1a'_2=20, 02$, respectively.

For matrices $A_{(0000)}^{(00,11)}, A_{\xi_0}^{(a_1a_211)}, A_{\xi_1}^{(a_1a_2,11)}, A_{(0000)}^{(a'_1a'_2,11)}, A_{\zeta_1}^{(a'_1a'_2,11)}$ and $A_{\zeta_2}^{(a'_1a'_2,11)}$, the following relations hold:

$$
\begin{aligned}
&A_{(0000)}^{(00,11)} A_{(0000)}^{(11,00)} = 2\binom{m}{2} A_0^{\#(00,00)}, \quad A_{(0000)}^{(00,11)} A_{\xi_0}^{(11,b_1b_2)} \\
&= (m-1)\sqrt{m}\, A_0^{\#(00,b_1b_2)}, \quad A_{(0000)}^{(00,11)}\{A_{\xi_0}^{(11,b_1b_2)} - A_{\xi_1}^{(11,b_1b_2)}\} \\
&= O_{1\times m}, \quad A_{(0000)}^{(00,11)}\{A_{\xi_0}^{(11,b_1b_2)} + A_{\xi_1}^{(11,b_1b_2)}\} = 2(m-1)\sqrt{m}\, A_0^{\#(00,b_1b_2)}, \\
&A_{(0000)}^{(00,11)} A_{(0000)}^{(11,b'_1b'_2)} = 2\sqrt{\binom{m}{2}} A_0^{\#(00,b'_1b'_2)}, \quad A_{(0000)}^{(00,11)}\{A_{\zeta_1}^{(11,b'_1b'_2)} - A_{\zeta_2}^{(11,b'_1b'_2)}\} \\
&= O_{1\times\binom{m}{2}}, \quad A_{(0000)}^{(00,11)}\{A_{(0000)}^{(11,b'_1b'_2)} + A_{\zeta_1}^{(11,b'_1b'_2)} + A_{\zeta_2}^{(11,b'_1b'_2)}\} \\
&= 2(2m-3)\sqrt{\binom{m}{2}} A_0^{\#(00,b'_1b'_2)}, \quad A_{\xi_0}^{(a_1a_2,11)} A_{\xi_0}^{(11,b_1b_2)} \\
&= (m-1)\{A_0^{\#(a_1a_2,b_1b_2)} + A_{f_{11}}^{\#(a_1a_2,b_1b_2)}\}, \quad A_{\xi_0}^{(a_1a_2,11)}\{A_{\xi_0}^{(11,b_1b_2)} \\
&- A_{\xi_1}^{(11,b_1b_2)}\} = mA_{f_{11}}^{\#(a_1a_2,b_1b_2)}, \quad A_{\xi_0}^{(a_1a_2,11)}\{A_{\xi_0}^{(11,b_1b_2)} + A_{\xi_1}^{(11,b_1b_2)}\} \\
&= 2(m-1)A_0^{\#(a_1a_2,b_1b_2)} + (m-2)A_{f_{11}}^{\#(a_1a_2,b_1b_2)}, \quad A_{\xi_0}^{(a_1a_2,11)} A_{(0000)}^{(11,b'_1b'_2)} \\
&= \sqrt{2(m-1)}\, A_0^{\#(a_1a_2,b'_1b'_2)} + \sqrt{m-2}\, A_{f_{12}}^{\#(a_1a_2,b'_1b'_2)}, \\
&A_{\xi_0}^{(a_1a_2,11)}\{A_{\zeta_1}^{(11,b'_11'_2)} - A_{\zeta_2}^{(11,b'_1b'_2)}\} = m\sqrt{m-2}\, A_{f_{12}}^{\#(a_1a_2,b'_1b'_2)}, \\
&A_{\xi_0}^{(a_1a_2,11)}\{A_{(0000)}^{(11,b'_1b'_2)} + A_{\zeta_1}^{(11,b'_1b'_2)} + A_{\zeta_2}^{(11,b'_1b'_2)}\} \\
&= 2\binom{m}{2}\sqrt{(m-1)/2}\, A_0^{\#(a_1a_2,b'_1b'_2)}, \quad \{A_{\xi_0}^{(a_1a_2,11)} - A_{\xi_1}^{(a_1a_2,11)}\} \\
&\cdot \{A_{\xi_0}^{(11,b_1b_2)} - A_{\xi_1}^{(11,b_1b_2)}\} = 2mA_{f_{11}}^{\#(a_1a_2,b_1b_2)}, \quad \{A_{\xi_0}^{(a_1a_2,11)} \\
&- A_{\xi_1}^{(a_1a_2,11)}\}\{A_{\xi_0}^{(11,b_1b_2)} + A_{\xi_1}^{(11,b_1b_2)}\} = 2\{2(m-1)A_0^{\#(a_1a_2,b_1b_2)}
\end{aligned}
$$

$$
\text{(A.3)}\quad \left\{
\begin{aligned}
&+ (m-2)A_{f_{11}}^{\#(a_1a_2,b_1b_2)}\}, \quad \{A_{\xi_0}^{(a_1a_2,11)} - A_{\xi_1}^{(a_1a_2,11)}\}A_{(0000)}^{(11,b'_1b'_2)}\\
&= O_{m\times\binom{m}{2}}, \quad \{A_{\xi_0}^{(a_1a_2,11)} - A_{\xi_1}^{(a_1a_2,11)}\}\{A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} - A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= 2m\sqrt{m-2}\, A_{f_{12}}^{\#(a_1a_2,b'_1b'_2)}, \quad \{A_{\xi_0}^{(a_1a_2,11)} - A_{\xi_1}^{(a_1a_2,11)}\}\{A_{(0000)}^{(11,b'_1b'_2)}\\
&+ A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\} = O_{m\times\binom{m}{2}}, \quad \{A_{\xi_0}^{(a_1a_2,11)} + A_{\xi_1}^{(a_1a_2,11)}\}\\
&\cdot\{A_{\tilde{\xi}_0}^{(11,b_1b_2)} + A_{\tilde{\xi}_1}^{(11b_1,b_2)}\} = 2\{2(m-1)A_0^{\#(a_1a_2,b_1b_2)}\\
&+ (m-2)A_{f_{11}}^{\#(a_1a_2,b_1b_2)}\}, \quad \{A_{\xi_0}^{(a_1a_2,11)} + A_{\xi_1}^{(a_1a_2,11)}\}A_{(0000)}^{(11,b'_1b'_2)}\\
&= 2\sqrt{2(m-1)}A_0^{\#(a_1a_2,b'_1b'_2)}, \quad \{A_{\xi_0}^{(a_1a_2,11)} + A_{\xi_1}^{(a_1a_2,11)}\}\\
&\cdot\{A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} - A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\} = O_{m\times\binom{m}{2}}, \quad \{A_{\xi_0}^{(a_1a_2,11)}\\
&+ A_{\xi_1}^{(a_1a_2,11)}\}\{A_{(0000)}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= 2\{(2m-3)\sqrt{2(m-1)}\, A_0^{\#(a_1a_2,b'_1b'_2)} - \sqrt{m-2}\, A_{f_{12}}^{\#(a_1a_2,b'_1b'_2)}\},\\
&A_{(0000)}^{(a'_1a'_2,11)}A_{(0000)}^{(11,b'_1b'_2)} = 2\{A_0^{\#(a'_1a'_2,b'_1b'_2)} + A_1^{\#(a'_1a'_2,b'_1b'_2)}\\
&+ A_{f_{22}}^{\#(a'_1a'_2,b'_1b'_2)}\}, \quad A_{(0000)}^{(a'_1a'_2,11)}\{A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} - A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= O_{\binom{m}{2}\times\binom{m}{2}}, \quad A_{(0000)}^{(a'_1a'_2,11)}\{A_{(0000)}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= 2\{(2m-3)A_0^{\#(a'_1a'_2,b'_1b'_2)} - A_1^{\#(a'_1a'_2,b'_1b'_2)} + (m-3)A_{f_{22}}^{\#(a'_1a'_2,b'_1b'_2)}\},\\
&\{A_{\zeta_1}^{(a'_1a'_2,11)} - A_{\zeta_2}^{(a'_1a'_2,11)}\}\{A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} - A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= 2m(m-2)A_{f_{22}}^{\#(a'_1a'_2,b'_1b'_2)}, \quad \{A_{\zeta_1}^{(a'_1a'_2,11)} - A_{\zeta_2}^{(a'_1a'_2,11)}\}\{A_{(0000)}^{(11,b'_1b'_2)}\\
&+ A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\} = O_{\binom{m}{2}\times\binom{m}{2}}, \quad \{A_{(0000)}^{(a'_1a'_2,11)}\\
&+ A_{\zeta_1}^{(a'_1a'_2,11)} + A_{\zeta_2}^{(a'_1a'_2,11)}\}\{A_{(0000)}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_1}^{(11,b'_1b'_2)} + A_{\tilde{\zeta}_2}^{(11,b'_1b'_2)}\}\\
&= 2\{(2m-3)A_0^{\#(a'_1a'_2,b'_1b'_2)} + A_1^{\#(a'_1a'_2,b'_1b'_2)} + (m-3)A_{f_{22}}^{\#(a'_1a'_2,b'_1b'_2)}\},
\end{aligned}\right.
$$

where a_1a_2, $b_1b_2=10$, 01 and $a'_1a'_2$, $b'_1b'_2=20$, 02.

From (3.6), (3.7) and (A.3), matrices $A_\beta^{\#(a_1a_2,b_1b_2)}$ and $A_{f_{ij}}^{\#(u_1u_2,v_1v_2)}$ satisfy the following properties (cf. [25]):

$$A_{\alpha}^{\#(a_1a_2,c_1c_2)}A_{\beta}^{\#(c_1c_2,b_1b_2)}=\delta_{\alpha\beta}A_{\alpha}^{\#(a_1a_2,b_1b_2)},$$

$$A_{fij}^{\#(u_1u_2,w_1w_2)}A_{\alpha}^{\#(w_1w_2,v_1v_2)}=A_{\alpha}^{\#(u_1u_2,w_1w_2)}A_{fij}^{\#(w_1w_2,v_1v_2)}=O_{n_{u_1u_2}\times n_{v_1v_2}},$$

$$A_{fik}^{\#(u_1u_2,w_1w_2)}A_{fij}^{\#(w_1w_2,v_1v_2)}=\delta_{kl}A_{fij}^{\#(u_1u_2,v_1v_2)}.$$

Solving (A.1) and (A.2) with respect to $A_{\alpha}^{(a_1a_2,b_1b_2)}$, we get

$$\text{(A.4)}\quad\left\{\begin{aligned}
&A_{(0000)}^{(00,00)}=A_0^{\#(00,00)},\ A_{(0000)}^{(00,b_1b_2)}=\sqrt{m}\,A_0^{\#(00,b_1b_2)},\\
&A_{(0000)}^{(00,b_1'b_2')}=\sqrt{\binom{m}{2}}\,A_0^{\#(00,b_1'b_2')},\quad A_{(0000)}^{(00,11)}=\sqrt{2\binom{m}{2}}\,A_0^{\#(00,11)},\\
&A_{(0000)}^{(a_1a_2,b_1b_2)}=A_0^{\#(a_1a_2,b_1b_2)}+A_{f_{11}}^{\#(a_1a_2,b_1b_2)},\quad A_{\alpha_1}^{(a_1a_2,b_1b_2)}\\
&=(m-1)A_0^{\#(a_1a_2,b_1b_2)}-A_{f_{11}}^{\#(a_1a_2,b_1b_2)},\quad A_{(0000)}^{(a_1a_2,b_1'b_2')}\\
&=\sqrt{2(m-1)}A_0^{\#(a_1a_2,b_1'b_2')}+\sqrt{m-2}\,A_{f_{12}}^{\#(a_1a_2,b_1'b_2')},\quad A_{\beta_1}^{(a_1a_2,b_1'b_2')}\\
&=(m-2)\sqrt{(m-1)/2}\,A_0^{\#(a_1a_2,b_1'b_2')}-\sqrt{m-2}\,A_{f_{12}}^{\#(a_1a_2,b_1'b_2')},\\
&A_{\xi_0}^{(a_1a_2,11)}=\sqrt{m-1}\,A_0^{\#(a_1a_2,11)}+\sqrt{m/2}A_{f_{13}}^{\#(a_1a_2,11)}\\
&+\sqrt{(m-2)/2}\,A_{f_{14}}^{\#(a_1a_2,11)},\quad A_{\xi_1}^{(a_1a_2,11)}=\sqrt{m-1}\,A_0^{\#(a_1a_2,11)}\\
&-\sqrt{m/2}\,A_{f_{13}}^{\#(a_1a_2,11)}+\sqrt{(m-2)/2}\,A_{f_{14}}^{\#(a_1a_2,11)},\quad A_{\xi_2}^{(a_1a_2,11)}\\
&=(m-2)\sqrt{m-1}\,A_0^{\#(a_1a_2,11)}-\sqrt{2(m-2)}\,A_{f_{14}}^{\#(a_1a_2,11)},\quad A_{(0000)}^{(a_1'a_2',b_1'b_2')}\\
&=A_0^{\#(a_1'a_2',b_1'b_2')}+A_1^{\#(a_1'a_2',b_1'b_2')}+A_{f_{22}}^{\#(a_1'a_2',b_1'b_2')},\quad A_{\gamma_1}^{(a_1'a_2',b_1'b_2')}\\
&=2(m-2)A_0^{\#(a_1'a_2',b_1'b_2')}-2A_1^{\#(a_1'a_2',b_1'b_2')}+(m-4)\,A_{f_{22}}^{\#(a_1'a_2',b_1'b_2')},\\
&A_{\gamma_2}^{(a_1'a_2',b_1'b_2')}=\binom{m-2}{2}A_0^{\#(a_1'a_2',b_1'b_2')}+A_1^{\#(a_1'a_2',b_1'b_2')}\\
&-(m-3)A_{f_{22}}^{\#(a_1'a_2',b_1'b_2')},\quad A_{(0000)}^{(a_1'a_2',11)}=\sqrt{2}\,\{A_0^{\#(a_1'a_2',11)}\\
&+A_1^{\#(a_1'a_2',11)}+A_{f_{24}}^{\#(a_1'a_2',11)}\},\quad A_{\zeta_1}^{(a_1'a_2',11)}=\sqrt{2}\,[(m-2)A_0^{\#(a_1'a_2',11)}\\
&-A_1^{\#(a_1'a_2',11)}+\{\sqrt{m(m-2)}/2\}\,A_{f_{23}}^{\#(a_1'a_2',11)}+\{(m-4)/2\}A_{f_{24}}^{\#(a_1'a_2',11)}],\\
&A_{\zeta_2}^{(a_1'a_2',11)}=\sqrt{2}\,[(m-2)\,A_0^{\#(a_1'a_2',11)}-A_1^{\#(a_1'a_2',11)}\\
&-\{\sqrt{m(m-2)}/2\}\,A_{f_{23}}^{\#(a_1'a_2',11)}+\{(m-4)/2\}A_{f_{24}}^{\#(a_1'a_2',11)}],\quad A_{\zeta_3}^{(a_1'a_2',11)}\\
&=\sqrt{2}\left\{\binom{m-2}{2}A_0^{\#(a_1'a_2',11)}+A_1^{\#(a_1'a_2',11)}-(m-3)A_{f_{24}}^{\#(a_1'a_2',11)}\right\},
\end{aligned}\right.$$

$$
\begin{aligned}
&A^{(11,11)}_{(0110)} = A^{\#(11,11)}_0 + A^{\#(11,11)}_1 + A^{\#(11,11)}_2 + A^{\#(11,11)}_{f_{33}} + A^{\#(11,11)}_{f_{44}},\\
&A^{(11,11)}_{(1001)} = A^{\#(11,11)}_0 + A^{\#(11,11)}_1 - A^{\#(11,11)}_2 - A^{\#(11,11)}_{f_{33}} + A^{\#(11,11)}_{f_{44}},\\
&A^{(11,11)}_{(0111)} = (m-2)A^{\#(11,11)}_0 - A^{\#(11,11)}_1 - A^{\#(11,11)}_2\\
&+ \{(m-2)/2\}A^{\#(11,11)}_{f_{33}} + \{\sqrt{m(m-2)}/2\}(A^{\#(11,11)}_{f_{34}} + A^{\#(11,11)}_{f_{43}})\\
&+ \{(m-4)/2\}A^{\#(11,11)}_{f_{44}}, \quad A^{(11,11)}_{(1110)} = (m-2)A^{\#(11,11)}_0 - A^{\#(11,11)}_1\\
&- A^{\#(11,11)}_2 + \{(m-2)/2\}A^{\#(11,11)}_{f_{33}} - \{\sqrt{m(m-2)}/2\}(A^{\#(11,11)}_{f_{34}}\\
&+ A^{\#(11,11)}_{f_{43}}) + \{(m-4)/2\}A^{\#(11,11)}_{f_{44}}, \quad A^{(11,11)}_{(1011)} = (m-2)A^{\#(11,11)}_0\\
&- A^{\#(11,11)}_1 + A^{\#(11,11)}_2 - \{(m-2)/2\}A^{\#(11,11)}_{f_{33}}\\
&+ \{\sqrt{m(m-2)}/2\}(A^{\#(11,11)}_{f_{34}} - A^{\#(11,11)}_{f_{43}}) + \{(m-4)/2\}A^{\#(11,11)}_{f_{44}},\\
&A^{(11,11)}_{(1101)} = (m-2)A^{\#(11,11)}_0 - A^{\#(11,11)}_1 + A^{\#(11,11)}_2\\
&- \{(m-2)/2\}A^{\#(11,11)}_{f_{33}} - \{\sqrt{m(m-2)}/2\}(A^{\#(11,11)}_{f_{34}} - A^{\#(11,11)}_{f_{43}})\\
&+ \{(m-4)/2\}A^{\#(11,11)}_{f_{44}}, \quad A^{(11,11)}_{(1111)} = 2\binom{m-2}{2}A^{\#(11,11)}_0 + 2A^{\#(11,11)}_1\\
&- 2(m-3)A^{\#(11,11)}_{f_{44}}.
\end{aligned}
$$

REMARK. Let $\mathfrak{B}=[A^{(11,11)}_{\alpha}|\alpha\in\Omega(11, 11)]=[A^{\#(11,11)}_{\beta}, A^{\#(11,11)}_{f_{ij}}|\beta=0, 1, 2; i, j=3, 4]$. Then algebra $\mathfrak{B}$ is a special case of the relationship algebra of a BIB design with parameters $v=m$, $b=\binom{m}{2}$, $r=m-1$, $k=2$ and $\lambda=1$ as shown by James [21].

Appendix II. MD relationship algebra for balanced third-order designs

Let $D^{(a_1a_2,b_1b_2)}_{\alpha}$ be the relationship matrices defined among the sets of effects $\{\theta(\phi)\}$, $\{\theta(t^1)\}$, $\{\theta(t^2)\}$, $\{\theta(t^1_1t^1_2)\}$, $\{\theta(t^1_3t^2_4)\}$, $\{\theta(t^1_5t^1_6t^1_7)\}$. Then, from Section 4, $[D^{(a_1a_2,b_1b_2)}_{\alpha}|\alpha\in\Omega(a_1a_2, b_1b_2); (a_1a_2, b_1b_2), (b_1b_2, a_1a_2)=(00, 10), (00, 01), (00, 20), (00, 11), (10, 01), (10, 20), (10, 11), (01, 20), (01, 11), (20, 11); a_1a_2=b_1b_2=00, 10, 01, 20, 11]$ is the algebra generated by fifty-five relationship matrices and is also generated by thirty-four symmetric matrices $B^{(a_1a_2,b_1b_2)}_{\alpha}$. A connection between $D^{(a_1a_2,b_1b_2)}_{\alpha}$, and $D^{\#(c_1c_2,d_1d_2)}_{\beta}$ and $D^{\#(u_1u_2,v_1v_2)}_{f_{ij}}$ is referred to Section 4 for $(a_1a_2, b_1b_2), (c_1c_2, d_1d_2)=(00, 00), (00, 10), (00, 01), (00, 20), (00, 11), (10, 10), (10, 01), (10, 20), (10, 11), (01, 01), (01, 20), (01, 11), (20, 20), (20, 11), (11, 11)$; $(u_1u_2, v_1v_2)=(10, 10), (10, 01), (10, 20), (10, 11), (01, 01), (01, 20), (01, 11), (20, 20), (20, 11), (11, 11)$. The relation between $\theta(t^{\varepsilon_1}_1t^{\varepsilon_2}_2t^{\varepsilon_3}_3)$ and $\theta(t^1_4t^1_5t^1_6)$ is similar to that of the TMDPB association scheme (see [59]), where $(w_1(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$w_2(\varepsilon_1, \varepsilon_2, \varepsilon_3))=(00), (10), (01), (20), (30)$. In this appendix, therefore, we mainly discuss the properties of relationship matrices $D_{\boldsymbol{\alpha}}^{(11,30)}$, where $\boldsymbol{\alpha}=(0000), (0010), (1000), (1010)$. From (4.5), we then obtain the following:

$$D_{\boldsymbol{\beta}}^{(11,30)} D_{\boldsymbol{\gamma}}^{(30,11)} = \sum_{\boldsymbol{\alpha}} q(11, 11, \boldsymbol{\alpha}; 30, \boldsymbol{\beta}, \boldsymbol{\gamma}) D_{\boldsymbol{\alpha}}^{(11,11)},$$

$$D_{\boldsymbol{\beta}}^{(30,11)} D_{\boldsymbol{\gamma}}^{(11,30)} = \sum_{\boldsymbol{\alpha}} q(30, 30, \boldsymbol{\alpha}; 11, \boldsymbol{\beta}, \boldsymbol{\gamma}) D_{\boldsymbol{\alpha}}^{(30,30)}.$$

Especially, we have

$$\text{(A.5)} \quad D_{(0000)}^{(11,30)} D_{(0000)}^{(30,11)} = 6(m-2) D_0^{\#(11,11)} + 2(m-4) D_1^{\#(11,11)} + 4(m-3) D_{f_{44}}^{\#(11,11)}.$$

We shall define matrices $D_\beta^{\#(11,30)}$ $(\beta=0, 1)$ and $D_{i5}^{\#(11,30)}$ $(i=3, 4)$ as follows:

$$\text{(A.6)} \quad \begin{cases} D_0^{\#(11,30)} = \{1/\sqrt{6(m-2)}\} D_0^{\#(11,11)} D_{(0000)}^{(11,30)} = [\sqrt{6}/\{2\binom{m}{2}\sqrt{m-2}\}] \\ \cdot\{\sum_{\boldsymbol{\alpha}} D_{\boldsymbol{\alpha}}^{(11,30)}\}, \quad D_1^{\#(11,30)} = \{1/\sqrt{2(m-4)}\} D_1^{\#(11,11)} D_{(0000)}^{(11,30)} \\ = \left[1\Big/\left\{\binom{m-1}{2}\sqrt{2(m-4)}\right\}\right]\left\{\binom{m-3}{2} D_{(0000)}^{(11,30)} - (m-4)(D_{(0010)}^{(11,30)}\right. \\ \left. + D_{(1000)}^{(11,30)}) + 3D_{(1010)}^{(11,30)}\right\}, \quad D_{f_{35}}^{\#(11,30)} = [1/\{2\sqrt{m-3}\}] D_{f_{34}}^{\#(11,11)} D_{(0000)}^{(11,30)} \\ = \left\{1\Big/\sqrt{2m\binom{m-2}{2}}\right\}\{D_{(0010)}^{(11,30)} - D_{(1000)}^{(11,30)}\}, \quad D_{f_{45}}^{\#(11,30)} = [1/\{2\sqrt{m-3}\}] \\ \cdot D_{f_{44}}^{\#(11,11)} D_{(0000)}^{(11,30)} = [1/\{m(m-2)\sqrt{m-3}\}]\{2(m-3) D_{(0000)}^{(11,30)} \\ + (m-6)(D_{(0010)}^{(11,30)} + D_{(1000)}^{(11,30)}) - 6D_{(1010)}^{(11,30)}\}. \end{cases}$$

Note that $D_\beta^{\#(30,11)}=\{D_\beta^{\#(11,30)}\}'$ for $\beta=0, 1$, and $D_{f_{5i}}^{\#(30,11)}=\{D_{f_{i5}}^{\#(11,30)}\}'$ for $i=3, 4$. From (A.3) and (A.5), matrices $D_\beta^{\#(11,30)}$ $(\beta=0, 1)$ and $D_{f_{i5}}^{\#(11,30)}$ $(i=3, 4)$ satisfy the following:

$$D_\alpha^{\#(11,30)} D_\beta^{\#(a_1a_2,b_1b_2)} = \delta_{3a_1}\delta_{0a_2}\delta_{\alpha\beta} D_\alpha^{\#(11,b_1b_2)},$$

$$D_\alpha^{\#(a_1a_2,b_1b_2)} D_\beta^{\#(11,30)} = \delta_{b_11}\delta_{b_21}\delta_{\alpha\beta} D_\alpha^{\#(a_1a_2,30)},$$

$$D_\alpha^{\#(11,30)} D_{f_{jk}}^{\#(u_1u_2,v_1v_2)} = D_{f_{jk}}^{\#(u_1u_2,v_1v_2)} D_\alpha^{\#(11,30)}$$

$$= D_{f_{i5}}^{\#(11,30)} D_\alpha^{\#(a_1a_2,b_1b_2)} = D_\alpha^{\#(a_1a_2,b_1b_2)} D_{f_{i5}}^{\#(11,30)} = O_{\mu_m\times\mu_m},$$

$$D_{f_{i5}}^{\#(11,30)} D_{f_{kj}}^{\#(u_1u_2,v_1v_2)} = \delta_{3u_1}\delta_{0u_2}\delta_{5k} D_{f_{ij}}^{\#(11,v_1v_2)},$$

$$D_{f_{jk}}^{\#(u_1u_2,v_1v_2)} D_{f_{i5}}^{\#(11,30)} = \delta_{v_11}\delta_{v_21}\delta_{ki} D_{f_{j5}}^{\#(u_1u_2,30)}.$$

Solving (A.6) with respect to $D_{\alpha}^{(11,30)}$, we have

$$D_{(0000)}^{(11,30)} = \sqrt{6(m-2)}D_0^{\#(11,30)} + \sqrt{2(m-4)}D_1^{\#(11,30)}$$

$$+ 2\sqrt{m-3}D_{f_{45}}^{\#(11,30)}, \quad D_{(0010)}^{(11,30)} = \{(m-3)\sqrt{6(m-2)}/2\}D_0^{\#(11,30)}$$

$$- \sqrt{2(m-4)}D_1^{\#(11,30)} + \left\{\sqrt{2m\binom{m-2}{2}}\Big/2\right\}D_{f_{35}}^{\#(11,30)}$$

$$+ \{(m-6)\sqrt{m-3}/2\}D_{f_{45}}^{\#(11,30)}, \quad D_{(1000)}^{(11,30)} = \{(m-3)\sqrt{6(m-2)}/2\}$$

$$\cdot D_0^{\#(11,30)} - \sqrt{2(m-4)}D_1^{\#(11,30)} - \left\{\sqrt{2m\binom{m-2}{2}}\Big/2\right\}D_{f_{35}}^{\#(11,30)}$$

$$+ \{(m-6)\sqrt{m-3}/2\}D_{f_{45}}^{\#(11,30)}, \quad D_{(1010)}^{(11,30)} = \left\{\binom{m-3}{2}\sqrt{6(m-2)}/3\right\}$$

$$\cdot D_0^{\#(11,30)} + \sqrt{2(m-4)}D_1^{\#(11,30)} - (m-4)\sqrt{m-3}\, D_{f_{45}}^{\#(11,30)}.$$

Appendix III. Tabulation

TABLE 1. Trace-optimal 3^4-BFF designs of resolution V

N	λ'	$\mathrm{tr}(V_T)$	$V^{(00,00)}_{(0000)}$	$V^{(00,10)}_{(0000)}$	$V^{(00,01)}_{(0000)}$	$V^{(00,20)}_{(0000)}$	$V^{(00,02)}_{(0000)}$	$V^{(00,11)}_{(0000)}$	$V^{(10,10)}_{(0000)}$
33	001101000011100	1.70163	0.03288	0.00567	−0.00142	−0.00213	−0.00128	0.00151	0.06803
34	101101000011100	1.60364	0.03262	0.00446	−0.00127	−0.00096	−0.00057	0.00068	0.06240
35	102101000011100	1.58312	0.03162	0.00319	−0.00185	−0.00210	−0.00050	0.00011	0.06079
36	000101010011100	1.37867	0.03048	0.00405	−0.00212	0.00029	−0.00125	0.00087	0.05469
37	001010101110001	1.29300	0.03045	−0.00370	−0.00206	0.00062	−0.00103	−0.00062	0.05000
38	101101010011100	1.23115	0.02983	0.00230	−0.00256	−0.00133	−0.00080	−0.00010	0.04685
39	102101010011100	1.21787	0.02970	0.00200	−0.00267	−0.00175	−0.00075	−0.00025	0.04617
40	000111010011010	1.14988	0.02819	0.00114	0.00197	0.00104	−0.00249	0.00169	0.03916
41	010110011011010	1.07656	0.02950	−0.00371	−0.00191	−0.00141	−0.00079	0.00263	0.03828
42	101001000011101	0.99869	0.02728	−0.00322	−0.00182	0.00384	0.00001	0.00058	0.03839
43	102001000011101	0.97693	0.02658	−0.00386	−0.00230	0.00259	0.00008	0.00008	0.03781
44	101011010001101	0.92413	0.02572	−0.00309	−0.00257	0.00309	−0.00051	0.00077	0.03704
45	111011010001101	0.88563	0.02534	−0.00382	−0.00203	0.00413	−0.00126	0.00111	0.03564
46	101101000011101	0.84141	0.02405	0.00080	−0.00244	0.00049	0.00112	−0.00014	0.03175
47	001111010001101	0.78995	0.02372	0.00016	−0.00275	−0.00044	0.00112	−0.00045	0.03052
48	101111010001101	0.77324	0.02347	0.00041	−0.00285	−0.00114	0.00097	−0.00019	0.03026
49	102111010001101	0.76011	0.02340	0.00024	−0.00293	−0.00146	0.00099	−0.00028	0.02981
50	101101010011101	0.72352	0.02303	−0.00024	−0.00314	0.00014	0.00084	−0.00056	0.03028
51	102101010011101	0.71037	0.02299	−0.00040	−0.00320	−0.00012	0.00086	−0.00064	0.02969
52	101111110001101	0.69826	0.02016	−0.00066	−0.00098	−0.00047	−0.00022	0.00079	0.02883
53	001111010011101	0.67592	0.02250	0.00009	−0.00332	0.00101	0.00047	−0.00024	0.02912
54	101111010011101	0.65889	0.02227	0.00033	−0.00342	0.00033	0.00033	0.00000	0.02886
55	102101110011101	0.64663	0.01938	−0.00140	−0.00136	0.00004	−0.00047	0.00050	0.02770
56	011110010111011	0.63060	0.01841	−0.00031	−0.00097	0.00084	0.00029	−0.00030	0.02685

$\lambda' = (\lambda_{400}, \lambda_{040}, \lambda_{004}, \lambda_{310}, \lambda_{301}, \lambda_{130}, \lambda_{031}, \lambda_{103}, \lambda_{013}, \lambda_{220}, \lambda_{202}, \lambda_{022}, \lambda_{211}, \lambda_{121}, \lambda_{112})$.

TABLE 1. (continued-1)

$V^{(10,10)}_{(1000)}$	$V^{(10,01)}_{(0000)}$	$V^{(10,01)}_{(0100)}$	$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$	$V^{(01,01)}_{(0000)}$
−0.00142	−0.00312	0.00151	0.01616	−0.02551	−0.00605	−0.00142	−0.00269	−0.00269	0.01120	0.01814
−0.00704	−0.00245	0.00218	0.02160	−0.02007	−0.00278	0.00185	−0.00656	−0.00656	0.00733	0.01806
−0.00865	−0.00319	0.00144	0.02015	−0.02152	−0.00269	0.00194	−0.00729	−0.00729	0.00660	0.01773
0.00376	−0.00087	0.00068	0.00622	−0.00535	−0.00690	−0.00149	−0.00448	0.00014	0.01172	0.01765
−0.00093	0.00000	−0.00154	−0.01065	0.00093	0.00386	−0.00154	−0.00787	−0.00324	0.00833	0.01749
−0.00408	−0.00113	0.00041	0.00626	−0.00531	−0.00334	0.00206	−0.00948	−0.00485	0.00673	0.01708
−0.00475	−0.00138	0.00017	0.00532	−0.00626	−0.00323	0.00217	−0.00982	−0.00519	0.00638	0.01699
0.00270	−0.00023	0.00035	0.00459	0.00459	0.00039	0.00097	−0.00128	0.00046	0.00046	0.02130
−0.00131	−0.00153	0.00157	−0.00244	0.00105	0.00155	−0.00117	−0.00354	−0.00021	0.00012	0.01517
0.00556	−0.00151	0.00017	−0.00563	0.00005	0.00111	0.00216	−0.00339	0.00166	−0.00087	0.01613
0.00499	−0.00194	−0.00026	−0.00676	−0.00107	0.00116	0.00222	−0.00384	0.00121	−0.00131	0.01580
0.00463	−0.00154	0.00000	−0.00694	−0.00463	0.00231	0.00309	−0.00463	0.00000	−0.00231	0.01569
0.00323	−0.00051	0.00103	−0.00495	−0.00264	0.00089	0.00166	−0.00398	0.00065	−0.00167	0.01493
0.00109	0.00100	0.00036	0.00066	0.00210	−0.00014	0.00066	−0.00260	0.00027	0.00123	0.01415
−0.00011	0.00038	−0.00020	−0.00026	0.00040	−0.00013	0.00060	−0.00313	−0.00028	0.00059	0.01380
−0.00036	0.00049	−0.00009	0.00045	0.00110	0.00002	0.00075	−0.00338	−0.00054	0.00033	0.01376
−0.00081	0.00029	−0.00029	−0.00040	0.00026	0.00006	0.00079	−0.00361	−0.00076	0.00011	0.01367
0.00018	0.00053	−0.00043	−0.00105	0.00310	−0.00063	0.00057	−0.00385	−0.00020	0.00124	0.01353
−0.00041	0.00031	−0.00065	−0.00196	0.00219	−0.00056	0.00064	−0.00416	−0.00051	0.00093	0.01345
−0.00035	−0.00039	0.00101	0.00105	0.00093	0.00039	−0.00039	−0.00147	−0.00006	−0.00023	0.01063
0.00036	0.00065	−0.00034	−0.00040	0.00071	−0.00042	0.00081	−0.00488	−0.00008	0.00139	0.01347
0.00010	0.00075	−0.00023	0.00032	0.00143	−0.00027	0.00096	−0.00513	−0.00034	0.00113	0.01343
−0.00012	−0.00103	0.00048	−0.00169	0.00201	−0.00002	−0.00063	−0.00220	0.00004	0.00031	0.01050
0.00064	−0.00014	−0.00014	−0.00077	0.00337	−0.00006	−0.00006	−0.00103	−0.00024	0.00045	0.00975

TABLE 1. (continued-2)

$V^{(01,01)}_{(0001)}$	$V^{(01,20)}_{(0000)}$	$V^{(01,20)}_{(0010)}$	$V^{(01,02)}_{(0000)}$	$V^{(01,02)}_{(0001)}$	$V^{(01,11)}_{(0001)}$	$V^{(01,11)}_{(0010)}$	$V^{(01,11)}_{(0011)}$	$V^{(20,20)}_{(0000)}$	$V^{(20,20)}_{(1000)}$	$V^{(20,20)}_{(2000)}$
−0.00038	−0.00404	0.00638	0.00035	−0.00080	0.00125	0.00357	−0.00222	0.11894	0.00175	−0.05293
−0.00046	−0.00468	0.00573	−0.00003	−0.00119	0.00171	0.00402	−0.00176	0.11368	−0.00351	−0.05820
−0.00079	−0.00534	0.00507	0.00001	−0.00115	0.00138	0.00369	−0.00209	0.11238	−0.00481	−0.05950
−0.00035	0.00005	0.00545	0.00031	−0.00098	0.00169	0.00323	−0.00217	0.07501	0.00673	0.00094
−0.00051	−0.00077	0.00463	−0.00026	−0.00154	−0.00231	−0.00386	0.00154	0.07083	0.00255	−0.00324
−0.00092	−0.00235	0.00305	−0.00007	−0.00136	0.00174	0.00328	−0.00212	0.06472	−0.00356	−0.00935
−0.00101	−0.00269	0.00271	−0.00003	−0.00132	0.00161	0.00316	−0.00224	0.06341	−0.00487	−0.01066
−0.00157	−0.00167	0.00110	0.00252	−0.00647	0.00211	0.00385	−0.00078	0.06231	0.00606	0.01231
−0.00069	−0.00186	0.00114	0.00067	−0.00166	0.00046	−0.00198	−0.00037	0.05887	−0.00026	0.00310
0.00014	0.00231	0.00042	−0.00196	0.00078	−0.00089	−0.00258	0.00289	0.06601	0.00493	−0.02490
−0.00019	0.00146	−0.00043	−0.00192	0.00082	−0.00123	−0.00291	0.00256	0.06380	0.00272	−0.02711
−0.00026	0.00077	−0.00000	−0.00206	0.00077	−0.00116	−0.00270	0.00270	0.05556	0.00579	0.01852
−0.00102	−0.00070	−0.00147	−0.00100	0.00183	−0.00164	−0.00318	0.00222	0.05271	0.00294	0.01567
0.00064	−0.00059	0.00204	−0.00188	0.00112	−0.00092	−0.00028	0.00148	0.05425	0.00145	−0.02010
0.00039	−0.00045	0.00075	−0.00183	0.00105	−0.00108	−0.00050	0.00111	0.04803	−0.00020	0.01408
0.00034	−0.00074	0.00046	−0.00190	0.00098	−0.00097	−0.00039	0.00121	0.04609	−0.00213	0.01214
0.00026	−0.00111	0.00009	−0.00188	0.00100	−0.00107	−0.00049	0.00112	0.04450	−0.00373	0.01054
0.00020	−0.00005	0.00100	−0.00196	0.00081	−0.00072	−0.00054	0.00094	0.04073	0.00133	−0.00683
0.00012	−0.00038	0.00067	−0.00193	0.00084	−0.00084	−0.00065	0.00082	0.03934	−0.00006	−0.00822
−0.00004	−0.00058	−0.00046	−0.00017	0.00061	0.00068	−0.00073	−0.00056	0.04554	−0.00227	0.01243
0.00014	0.00028	0.00139	−0.00209	0.00069	−0.00061	−0.00044	0.00103	0.03578	0.00329	0.00204
0.00010	−0.00001	0.00110	−0.00215	0.00063	−0.00050	−0.00034	0.00113	0.03377	0.00128	0.00003
−0.00016	−0.00022	0.00034	−0.00027	0.00053	0.00035	−0.00099	−0.00078	0.03924	−0.00007	−0.00814
0.00049	0.00038	0.00038	−0.00049	−0.00049	−0.00013	−0.00013	−0.00013	0.04265	0.00142	0.00187

TABLE 1. (continued-3)

$V^{(20,02)}_{(0000)}$	$V^{(20,02)}_{(0100)}$	$V^{(20,02)}_{(0200)}$	$V^{(20,11)}_{(0000)}$	$V^{(20,11)}_{(0100)}$	$V^{(20,11)}_{(1000)}$	$V^{(20,11)}_{(1100)}$	$V^{(02,02)}_{(0000)}$	$V^{(02,02)}_{(0001)}$	$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$
−0.00641	0.00661	−0.00120	−0.00680	−0.00420	−0.00941	0.01403	0.01961	−0.00036	0.00514	−0.00038
−0.00957	0.00345	−0.00436	−0.00306	−0.00045	−0.00566	0.01778	0.01771	−0.00225	0.00325	0.00187
−0.00949	0.00353	−0.00429	−0.00371	−0.00110	−0.00631	0.01713	0.01771	−0.00226	0.00324	0.00191
−0.00870	0.00557	−0.00099	−0.00412	−0.00528	−0.00297	0.01671	0.01972	−0.00022	0.00531	−0.00065
−0.01157	0.00270	−0.00386	0.00093	0.00208	−0.00023	−0.01991	0.01775	−0.00219	0.00334	−0.00154
−0.01086	0.00342	−0.00314	−0.00316	−0.00432	−0.00201	0.01767	0.01766	−0.00227	0.00326	0.00181
−0.01070	0.00357	−0.00299	−0.00364	−0.00480	−0.00249	0.01719	0.01764	−0.00229	0.00324	0.00186
0.00142	0.00073	0.00698	−0.00676	0.00365	0.00365	−0.00676	0.01388	0.00142	−0.00409	−0.00119
−0.00447	−0.00015	0.01112	−0.00689	0.00534	0.00259	−0.00602	0.01399	−0.00018	−0.00740	−0.00365
0.00117	−0.00088	−0.00640	0.00130	0.00414	−0.00154	0.00130	0.01110	−0.00060	0.00352	−0.00014
0.00129	−0.00077	−0.00629	0.00043	0.00327	−0.00241	0.00043	0.01109	−0.00061	0.00351	−0.00010
−0.00463	−0.00270	−0.00772	0.00231	0.00347	0.00116	0.00231	0.01157	−0.00013	0.00437	−0.00039
−0.00259	−0.00066	−0.00567	0.00139	0.00255	0.00023	0.00139	0.01011	−0.00159	0.00291	0.00028
0.00183	0.00027	−0.00476	0.00522	0.00127	−0.00017	−0.00412	0.01069	−0.00098	0.00317	0.00038
−0.00146	0.00051	−0.00447	0.00193	0.00013	−0.00053	−0.00233	0.01068	−0.00102	0.00348	0.00016
−0.00188	0.00009	−0.00489	0.00264	0.00083	0.00018	−0.00163	0.01059	−0.00111	0.00339	0.00031
−0.00180	0.00017	−0.00481	0.00222	0.00041	−0.00024	−0.00205	0.01059	−0.00112	0.00339	0.00033
−0.00025	0.00017	−0.00288	0.00078	−0.00080	0.00161	0.00003	0.01032	−0.00106	0.00339	−0.00037
−0.00014	0.00029	−0.00276	0.00030	−0.00128	0.00113	−0.00045	0.01031	−0.00107	0.00338	−0.00033
−0.00246	0.00033	−0.00382	0.00149	0.00025	0.00037	−0.00087	0.00856	−0.00154	0.00455	−0.00119
0.00035	0.00071	−0.00241	0.00130	−0.00036	−0.00054	−0.00220	0.01006	−0.00131	0.00313	−0.00016
−0.00005	0.00030	−0.00282	0.00202	0.00036	0.00017	−0.00148	0.00998	−0.00139	0.00305	−0.00001
−0.00044	0.00034	−0.00235	−0.00012	−0.00147	0.00122	−0.00012	0.00837	−0.00156	0.00434	−0.00140
0.00015	0.00015	0.00015	−0.00069	−0.00062	0.00145	−0.00542	0.00643	0.00026	−0.00283	−0.00005

TABLE 1. (continued–4)

$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$	$V^{(11,11)}_{(0111)}$	$V^{(11,11)}_{(1110)}$	$V^{(11,11)}_{(1011)}=V^{(11,11)}_{(1101)}$	$V^{(11,11)}_{(1111)}$
−0.00067	−0.00009	−0.01658	0.04106	0.01676	0.00027	0.00027	0.00200	−0.00581
0.00158	0.00216	−0.01433	0.03840	0.01410	−0.00240	−0.00240	−0.00066	−0.00847
0.00162	0.00220	−0.01430	0.03808	0.01377	−0.00272	−0.00272	−0.00099	−0.00880
−0.00104	−0.00027	−0.01685	0.04087	0.01772	0.00036	0.00036	0.00268	−0.00543
−0.00116	−0.00193	0.01466	0.03843	0.01528	−0.00208	−0.00208	0.00023	−0.00787
0.00142	0.00219	−0.01440	0.03761	0.01446	−0.00290	−0.00290	−0.00059	−0.00869
0.00148	0.00225	−0.01434	0.03743	0.01428	−0.00308	−0.00308	−0.00076	−0.00887
−0.00003	0.00171	−0.00871	0.03522	0.01265	−0.00992	−0.00471	−0.00471	0.01265
0.00073	0.00286	−0.00433	0.03301	0.01056	−0.00749	−0.00666	−0.00441	0.01534
0.00038	−0.00067	−0.00477	0.02423	0.00529	0.00213	0.00213	−0.00039	−0.00608
0.00043	−0.00063	−0.00473	0.02388	0.00494	0.00178	0.00178	−0.00074	−0.00642
0.00000	−0.00077	−0.00502	0.02373	0.00521	0.00174	0.00174	−0.00058	−0.00637
0.00066	−0.00011	−0.00435	0.02343	0.00491	0.00144	0.00144	−0.00088	−0.00667
0.00050	−0.00030	−0.00481	0.02034	0.00358	0.00382	−0.00097	−0.00001	−0.00360
0.00047	−0.00026	−0.00458	0.01973	0.00299	0.00343	−0.00116	−0.00029	−0.00357
0.00062	−0.00011	−0.00443	0.01947	0.00273	0.00317	−0.00142	−0.00055	−0.00383
0.00064	−0.00009	−0.00441	0.01936	0.00262	0.00306	−0.00153	−0.00066	−0.00394
0.00010	−0.00013	−0.00429	0.01825	0.00259	0.00214	−0.00018	0.00010	−0.00239
0.00014	−0.00009	−0.00425	0.01808	0.00242	0.00198	−0.00035	−0.00007	−0.00256
0.00021	0.00100	−0.00223	0.01631	0.00101	0.00076	−0.00031	−0.00048	−0.00197
0.00031	0.00011	−0.00405	0.01733	0.00265	0.00123	−0.00006	0.00019	−0.00227
0.00046	0.00025	−0.00390	0.01708	0.00239	0.00098	−0.00031	−0.00007	−0.00252
0.00000	0.00090	−0.00233	0.01573	0.00095	0.00023	0.00023	−0.00022	−0.00138
−0.00005	−0.00005	−0.00005	0.01428	0.00078	−0.00076	0.00033	−0.00002	0.00034

TABLE 2. Determinant-optimal 3^4-BFF designs of resolution V

N	λ'	det (V_T)	$V^{(00,00)}_{(0000)}$	$V^{(00,10)}_{(0000)}$	$V^{(00,01)}_{(0000)}$
33a	001101000011010	0.82576E–50	0.03288	0.00071	0.00354
*33b	001101000011100		0.03288	0.00567	−0.00142
33c	010010010101100		0.03288	0.00496	−0.00213
*34a	101101000011100	0.18561E–50	0.03262	0.00446	−0.00127
34b	011010010101001		0.03262	−0.00414	−0.00159
35a	111101000011100	0.69404E–51	0.03230	0.00462	−0.00074
35b	111010010101100		0.03230	0.00342	−0.00194
36a	000101010011010	0.26215E–51	0.02932	−0.00116	0.00193
36b	000101001011100		0.02932	0.00231	−0.00154
37a	001101100011100	0.57023E–52	0.03189	0.00663	−0.00059
37b	001111000011010		0.03189	0.00243	0.00361
38a	101101100011100	0.16528E–52	0.03065	0.00421	0.00027
38b	011110010101001		0.03065	−0.00170	−0.00224
39	001010100111001	0.66896E–53	0.02806	−0.00337	−0.00112
40a	000101110011100	0.19486E–53	0.02930	0.00532	−0.00119
40b	000110110101001		0.02930	−0.00445	−0.00206
41a	001111100011001	0.40478E–54	0.03002	−0.00351	−0.00117
41b	010110011101010		0.03002	0.00000	0.00234
*42a	101001000011101	0.11309E–54	0.02728	−0.00322	−0.00182
42b	011100000011011		0.02728	−0.00434	−0.00070
43	111001000011101	0.41553E–55	0.02663	−0.00381	−0.00106
44a	011011010001101	0.20723E–55	0.02923	−0.00829	−0.00157
44b	101101001010110		0.02923	0.00650	−0.00336
45a	111000000111101	0.62218E–56	0.02417	0.00000	−0.00196
45b	111000000111110		0.02417	0.00294	0.00098
46	011110000011011	0.19513E–56	0.02604	−0.00304	−0.00188
47a	111010010101101	0.86870E–57	0.02337	0.00000	−0.00262
47b	111101000011110		0.02337	0.00393	0.00131
48a	011010000111011	0.29374E–57	0.02315	−0.00270	0.00039
48b	011100000111011		0.02315	−0.00193	0.00116
49a	111001000111101	0.11557E–57	0.02125	0.00046	−0.00015
49b	111100000111011		0.02125	0.00000	0.00030
50a	011111000011011	0.43914E–58	0.02097	−0.00076	0.00040
50b	011110010101011		0.02097	−0.00098	0.00018
51a	001111110001101	0.19350E–58	0.02021	−0.00079	−0.00089
51b	010111011001110		0.02021	0.00094	0.00084
51c	001111101010011		0.02021	−0.00174	0.00005
52	011110000111011	0.59747E–59	0.01984	0.00099	−0.00033
53	001110101110011	0.31083E–59	0.01940	0.00000	−0.00088
54a	011110101110011	0.12088E–59	0.01852	0.00000	0.00000
54b	101011110101101		0.01852	0.00000	0.00000
55a	111011110101101	0.65934E–60	0.01833	0.00000	0.00019
55b	111110101110011		0.01833	0.00028	−0.00009
56a	112011110101101	0.40968E–60	0.01811	−0.00033	0.00008
56b	112110101110011		0.01811	−0.00004	−0.00021

*This design is also optimal with respect to the trace criterion.
$\lambda' = (\lambda_{400}, \lambda_{040}, \lambda_{004}, \lambda_{310}, \lambda_{301}, \lambda_{130}, \lambda_{031}, \lambda_{103}, \lambda_{013}, \lambda_{220}, \lambda_{202}, \lambda_{022}, \lambda_{211}, \lambda_{121}, \lambda_{112})$.
$n_0\mathrm{E}-n_1 = n_0 10^{-n_1}$.

TABLE 2. (continued-1)

$V^{(00,20)}_{(0000)}$	$V^{(00,02)}_{(0000)}$	$V^{(00,11)}_{(0000)}$	$V^{(10,10)}_{(0000)}$	$V^{(10,10)}_{(1000)}$	$V^{(10,01)}_{(0000)}$	$V^{(10,01)}_{(0100)}$
−0.00113	−0.00161	0.00118	0.05315	0.00106	0.00184	0.00068
−0.00213	−0.00128	0.00151	0.06803	−0.00142	−0.00312	0.00151
−0.00567	−0.00009	0.00033	0.06250	−0.00347	−0.00496	0.00083
−0.00096	−0.00057	0.00068	0.06240	−0.00704	−0.00245	0.00218
−0.00255	−0.00004	−0.00015	0.05992	−0.00605	0.00328	−0.00251
−0.00089	−0.00114	0.00094	0.06232	−0.00712	−0.00273	0.00190
−0.00419	−0.00004	−0.00016	0.05827	−0.00771	−0.00408	0.00171
0.00116	−0.00116	−0.00000	0.05221	0.00459	0.00227	−0.00037
−0.00231	0.00000	0.00116	0.06597	−0.00347	−0.00231	0.00231
−0.00302	−0.00257	0.00291	0.06114	−0.00037	−0.00492	0.00104
−0.00218	−0.00285	0.00263	0.04679	−0.00083	−0.00013	0.00119
−0.00066	−0.00161	0.00169	0.05642	−0.00509	−0.00325	0.00270
−0.00632	0.00028	0.00020	0.05653	−0.00894	0.00321	−0.00142
0.00084	−0.00028	0.00084	0.05322	−0.00091	0.00065	−0.00078
0.00016	−0.00273	0.00251	0.04594	0.00487	−0.00285	0.00001
−0.00987	0.00061	0.00083	0.05353	−0.00113	0.00032	0.00199
−0.00234	−0.00208	0.00195	0.04261	−0.00041	−0.00050	0.00178
−0.00819	−0.00013	0.00000	0.04411	−0.00574	0.00000	0.00000
0.00384	0.00001	0.00058	0.03839	0.00556	−0.00151	0.00017
0.00186	0.00068	0.00124	0.04362	0.00195	−0.00326	0.00137
0.00450	−0.00082	0.00098	0.03786	0.00503	−0.00083	0.00086
0.01137	−0.00159	0.00063	0.04077	0.00836	−0.00105	0.00050
0.00022	0.00213	−0.00435	0.04234	0.00067	0.00157	−0.00306
0.00129	0.00069	0.00000	0.03307	0.00220	0.00000	0.00000
0.00188	0.00050	0.00020	0.04380	0.00136	0.00358	−0.00028
0.00130	0.00024	0.00260	0.03461	0.00313	−0.00132	0.00238
−0.00060	0.00060	0.00000	0.03063	−0.00024	0.00000	0.00000
0.00119	0.00000	0.00060	0.04097	−0.00008	0.00345	0.00005
0.00502	0.00013	−0.00026	0.03488	0.00505	−0.00190	0.00016
0.00193	0.00116	−0.00129	0.03797	0.00196	−0.00087	−0.00087
0.00182	0.00020	−0.00061	0.03223	0.00239	0.00126	−0.00080
0.00000	0.00081	0.00000	0.03601	0.00000	0.00000	0.00000
0.00168	−0.00031	0.00083	0.03186	0.00268	−0.00063	0.00078
−0.00152	0.00076	−0.00024	0.03242	−0.00098	−0.00044	−0.00044
−0.00015	−0.00018	0.00069	0.02918	−0.00000	−0.00061	0.00080
−0.00147	0.00026	0.00025	0.03242	−0.00098	0.00047	0.00047
0.00060	−0.00043	0.00044	0.03060	0.00141	−0.00108	0.00033
0.00099	0.00055	0.00033	0.02922	0.00144	0.00045	0.00045
0.00000	0.00088	0.00044	0.02778	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.02778	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.02778	0.00000	0.00000	0.00000
0.00028	−0.00028	0.00000	0.02778	0.00000	0.00000	0.00000
−0.00056	−0.00000	0.00028	0.02736	−0.00042	0.00014	0.00014
−0.00037	−0.00029	−0.00011	0.02730	−0.00048	−0.00015	−0.00015
−0.00091	−0.00011	0.00007	0.02691	−0.00087	−0.00002	−0.00002

TABLE 2. (continued-2)

$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$
−0.00170	−0.00170	−0.00125	−0.00241	0.00524	0.00177	0.00177
0.01616	−0.02551	−0.00605	−0.00142	−0.00269	−0.00269	0.01120
0.00298	−0.01786	0.00017	−0.00099	−0.01042	−0.00694	0.01042
0.02160	−0.02007	−0.00278	0.00185	−0.00656	−0.00656	0.00733
−0.01281	0.00803	−0.00033	0.00083	−0.01099	−0.00752	0.00984
0.02157	−0.02010	−0.00249	0.00214	−0.00669	−0.00669	0.00719
0.00904	−0.01179	0.00034	−0.00082	−0.01170	−0.00823	0.00913
−0.00459	0.01327	−0.00070	−0.00335	0.00496	0.00298	0.00298
0.01736	−0.02431	−0.00463	0.00000	−0.00174	−0.00174	0.01215
0.01554	−0.02315	−0.00198	−0.00297	0.00091	−0.00107	0.00587
0.00905	−0.00880	−0.00032	0.00100	0.00160	−0.00038	−0.00038
0.02015	−0.01854	−0.00010	−0.00110	−0.00147	−0.00346	0.00349
−0.00814	0.01270	−0.00130	0.00101	−0.01056	−0.00659	0.00829
−0.01331	0.00699	0.00444	−0.00233	−0.01046	−0.00761	0.01198
0.00389	−0.00275	−0.00219	−0.00303	−0.00083	0.00183	0.00552
0.00469	0.01843	0.00162	−0.00158	−0.00498	−0.00439	0.00610
−0.00402	0.00795	0.00114	0.00086	−0.00506	−0.00221	0.00349
0.00000	0.00000	0.00000	0.00000	0.00294	−0.00276	0.00080
−0.00563	0.00005	0.00111	0.00216	−0.00339	0.00166	−0.00087
−0.00977	0.01107	0.00109	−0.00123	0.00043	0.00043	0.00043
−0.00504	0.00064	0.00036	0.00141	−0.00303	0.00202	−0.00050
−0.01325	−0.01094	0.00126	0.00203	−0.00343	0.00119	−0.00112
0.01008	−0.01075	−0.00030	0.00202	−0.00302	−0.00302	−0.00302
0.00000	0.00000	0.00000	0.00000	−0.00243	0.00066	0.00220
0.01090	−0.01225	0.00161	−0.00224	0.00292	0.00138	−0.00055
−0.00015	0.00263	0.00131	−0.00054	−0.00262	0.00109	−0.00123
0.00000	0.00000	0.00000	0.00000	−0.00344	−0.00036	0.00119
0.01038	−0.00860	0.00091	−0.00109	0.00269	0.00115	−0.00147
−0.00383	−0.00228	−0.00038	0.00168	0.00101	0.00306	0.00024
−0.00711	0.00833	−0.00035	−0.00035	0.00371	−0.00041	0.00216
−0.00124	0.00108	−0.00082	0.00098	−0.00130	0.00076	0.00178
−0.00514	0.01029	0.00000	0.00000	0.00240	−0.00171	0.00086
0.00054	0.00159	0.00037	0.00089	−0.00051	0.00090	−0.00059
−0.00072	0.00456	−0.00018	−0.00018	0.00156	−0.00125	−0.00037
0.00021	0.00009	0.00028	−0.00051	−0.00122	0.00019	0.00002
0.00069	−0.00452	0.00012	0.00021	0.00158	−0.00123	−0.00041
−0.00081	0.00018	0.00026	0.00076	−0.00095	0.00045	−0.00107
0.00144	0.00144	0.00018	0.00018	0.00048	0.00048	0.00048
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00084	0.00084	0.00000	0.00000	−0.00042	−0.00042	−0.00042
−0.00094	−0.00094	−0.00001	−0.00001	−0.00016	−0.00016	−0.00016
0.00035	0.00035	−0.00015	−0.00015	−0.00072	−0.00072	−0.00072

TABLE 2. (continued-3)

$V_{(0000)}^{(01,01)}$	$V_{(0001)}^{(01,01)}$	$V_{(0000)}^{(01,20)}$	$V_{(0010)}^{(01,20)}$	$V_{(0000)}^{(01,02)}$	$V_{(0001)}^{(01,02)}$	$V_{(0001)}^{(01,11)}$
0.02310	−0.00120	−0.00850	0.00539	0.00300	−0.00742	0.00307
0.01814	−0.00038	−0.00404	0.00638	0.00035	−0.00080	0.00125
0.01998	0.00031	0.00468	0.01162	−0.00073	0.00043	0.00314
0.01806	−0.00046	−0.00468	0.00573	−0.00003	−0.00119	0.00171
0.01889	−0.00079	−0.00172	0.00522	−0.00084	0.00032	−0.00351
0.01715	−0.00137	−0.00479	0.00563	0.00093	−0.00023	0.00127
0.01850	−0.00117	−0.00354	0.00341	−0.00084	0.00032	0.00317
0.02200	−0.00181	−0.00426	0.00368	0.00293	−0.00699	0.00298
0.01742	0.00088	−0.00364	0.00827	−0.00066	0.00066	0.00083
0.01728	−0.00102	−0.00354	0.00738	0.00191	−0.00018	0.00091
0.02206	−0.00086	−0.00347	0.00049	0.00239	−0.00665	0.00235
0.01669	−0.00161	−0.00516	0.00575	0.00125	−0.00084	0.00175
0.01665	−0.00032	−0.00296	0.00399	−0.00209	0.00177	−0.00137
0.01731	0.00022	−0.00016	0.00732	0.00005	−0.00244	−0.00301
0.01679	−0.00104	−0.00018	0.00588	0.00188	−0.00024	0.00112
0.01426	0.00096	0.00158	0.00439	−0.00099	−0.00035	0.00097
0.01454	−0.00132	−0.00125	0.00345	0.00035	0.00002	0.00018
0.01404	0.00045	−0.00468	0.00088	0.00169	−0.00263	0.00000
0.01613	0.00014	0.00231	0.00042	−0.00196	0.00078	−0.00089
0.01438	0.00134	0.00311	−0.00131	−0.00083	0.00107	−0.00024
0.01524	−0.00075	0.00155	−0.00034	−0.00099	0.00174	−0.00136
0.01498	−0.00096	0.00016	−0.00061	−0.00104	0.00179	−0.00169
0.01446	0.00160	0.00192	−0.00348	−0.00173	−0.00044	0.00066
0.01579	0.00036	−0.00209	0.00486	−0.00280	0.00260	0.00000
0.01222	0.00064	0.00275	−0.00188	0.00083	−0.00148	0.00024
0.01320	0.00024	0.00176	−0.00102	−0.00122	0.00063	0.00120
0.01481	−0.00001	−0.00118	0.00160	−0.00252	0.00211	0.00000
0.01136	−0.00006	0.00156	−0.00168	0.00094	−0.00076	0.00050
0.01173	0.00042	0.00209	0.00055	0.00057	−0.00149	0.00015
0.01071	0.00145	0.00087	0.00087	−0.00010	−0.00010	−0.00058
0.01158	0.00027	−0.00010	0.00221	−0.00070	0.00110	−0.00168
0.01032	0.00106	0.00000	0.00000	−0.00025	−0.00025	0.00000
0.01045	−0.00021	0.00073	−0.00033	0.00022	−0.00031	0.00069
0.01026	0.00101	−0.00068	−0.00068	−0.00028	−0.00028	−0.00011
0.01076	0.00010	−0.00006	0.00006	−0.00010	0.00068	0.00053
0.00968	0.00042	−0.00074	−0.00074	0.00013	0.00013	0.00013
0.01029	−0.00038	0.00020	−0.00079	0.00017	−0.00033	0.00051
0.01004	0.00078	0.00045	0.00045	−0.00037	−0.00037	0.00015
0.01014	0.00088	−0.00000	−0.00000	−0.00088	−0.00088	−0.00044
0.00926	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00926	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00907	−0.00019	−0.00028	−0.00028	0.00028	0.00028	0.00000
0.00921	−0.00005	−0.00028	−0.00028	−0.00000	−0.00000	0.00014
0.00903	−0.00023	−0.00058	−0.00058	0.00028	0.00028	−0.00005
0.00916	−0.00010	−0.00046	−0.00046	−0.00005	−0.00005	0.00004

TABLE 2. (continued-4)

$V^{(01,11)}_{(0010)}$	$V^{(01,11)}_{(0011)}$	$V^{(20,20)}_{(0000)}$	$V^{(20,20)}_{(1000)}$	$V^{(20,20)}_{(2000)}$	$V^{(20,02)}_{(0000)}$	$V^{(20,02)}_{(0100)}$
0.00423	−0.00040	0.15689	0.00064	−0.09311	−0.00865	0.00177
0.00357	−0.00222	0.11894	0.00175	−0.05293	−0.00641	0.00661
0.00430	−0.00380	0.17219	0.01594	0.10969	−0.00581	0.00461
0.00402	−0.00176	0.11368	−0.00351	−0.05820	−0.00957	0.00345
−0.00467	0.00343	0.13475	−0.02150	0.07225	−0.00643	0.00398
0.00359	−0.00220	0.11366	−0.00352	−0.05821	−0.00946	0.00356
0.00433	−0.00377	0.12618	−0.03007	0.06368	−0.00642	0.00400
0.00364	−0.00099	0.09090	0.00608	−0.01625	−0.00360	0.00087
0.00215	0.00083	0.11954	0.00347	−0.05010	−0.00794	0.00694
0.00289	−0.00405	0.11702	0.00094	−0.05263	−0.00548	0.00543
0.00434	−0.00029	0.08435	−0.00047	−0.02279	−0.00004	0.00045
0.00373	−0.00321	0.11251	−0.00356	−0.05713	−0.00731	0.00361
−0.00137	0.00094	0.12752	−0.02873	0.06502	−0.00583	0.00459
−0.00586	0.00233	0.08757	0.00808	−0.02141	−0.00697	0.00008
0.00240	−0.00405	0.07254	0.00671	0.00339	−0.00574	0.00541
−0.00100	−0.00125	0.13939	−0.01221	0.08620	−0.00877	0.00526
−0.00267	0.00090	0.07901	−0.00048	−0.01747	0.00378	−0.00028
0.00000	0.00000	0.12193	−0.02807	0.07193	−0.00468	0.00088
−0.00258	0.00289	0.06601	0.00493	−0.02490	0.00117	−0.00088
−0.00361	−0.00151	0.09482	−0.00177	−0.03586	0.00047	−0.00080
−0.00305	0.00242	0.06536	0.00428	−0.02555	0.00200	−0.00005
−0.00324	0.00217	0.06615	0.01638	0.02911	−0.00319	−0.00126
0.00375	0.00220	0.08639	−0.00504	−0.03398	−0.00294	−0.00101
0.00000	0.00000	0.06555	0.00305	−0.02820	0.00731	0.00037
0.00178	−0.00092	0.09124	0.00096	−0.04765	0.00443	−0.00020
−0.00250	−0.00019	0.06257	−0.00202	−0.00410	0.00109	−0.00099
0.00000	0.00000	0.04973	−0.00027	0.01223	0.00027	0.00027
0.00205	−0.00089	0.07724	−0.00054	−0.03665	−0.00110	−0.00018
−0.00191	0.00092	0.05957	0.00633	−0.00524	0.00121	−0.00033
−0.00058	−0.00058	0.08272	0.00170	−0.03765	0.00035	0.00035
0.00038	0.00141	0.06024	0.00295	−0.02309	0.00335	0.00046
0.00000	0.00000	0.08076	−0.00026	−0.03961	0.00000	0.00000
−0.00072	0.00078	0.06124	−0.00205	−0.00284	0.00292	−0.00095
−0.00011	−0.00011	0.07278	−0.01029	0.03164	−0.00028	−0.00028
−0.00088	−0.00071	0.04759	−0.00022	0.01448	−0.00218	0.00061
0.00013	0.00013	0.07189	−0.01014	0.03283	−0.00135	−0.00005
−0.00090	0.00063	0.05601	−0.00351	−0.00054	0.00100	−0.00098
0.00015	0.00015	0.05352	0.00144	−0.00898	0.00018	0.00018
−0.00044	−0.00044	0.04861	0.00000	−0.00694	−0.00231	0.00000
0.00000	0.00000	0.04861	0.00000	−0.00694	−0.00231	0.00000
0.00000	0.00000	0.04861	0.00000	0.01389	−0.00231	0.00000
0.00000	0.00000	0.04819	−0.00042	0.01347	−0.00189	0.00042
0.00014	0.00014	0.04693	−0.00168	−0.00863	−0.00231	−0.00000
−0.00005	−0.00005	0.04633	−0.00228	0.01161	−0.00192	0.00039
0.00004	0.00004	0.04639	−0.00222	−0.00917	−0.00248	−0.00017

TABLE 2. (continued–5)

$V^{(20,02)}_{(0200)}$	$V^{(20,11)}_{(0000)}$	$V^{(20,11)}_{(0100)}$	$V^{(20,11)}_{(1000)}$	$V^{(20,11)}_{(1100)}$	$V^{(02,02)}_{(0000)}$	$V^{(02,02)}_{(0001)}$
0.01913	−0.00978	0.00064	0.00064	−0.00978	0.01688	0.00299
−0.00120	−0.00680	−0.00420	−0.00941	0.01403	0.01961	−0.00036
−0.01276	0.00298	−0.00744	−0.00744	0.02381	0.01329	−0.00060
−0.00436	−0.00306	−0.00045	−0.00566	0.01778	0.01771	−0.00225
−0.01338	−0.00516	0.00526	0.00526	−0.02599	0.01328	−0.00061
−0.00425	−0.00311	−0.00050	−0.00571	0.01772	0.01670	−0.00327
−0.01336	0.00354	−0.00688	−0.00688	0.02437	0.01328	−0.00061
0.01228	−0.00992	0.00050	0.00645	−0.00397	0.01638	0.00299
0.00099	−0.00818	−0.00223	−0.00818	0.01860	0.01609	−0.00077
−0.00449	−0.00256	−0.00256	−0.00851	0.01232	0.01392	−0.00206
0.00789	−0.00527	0.00515	−0.00080	−0.01122	0.01393	0.00142
−0.00631	−0.00023	−0.00023	−0.00619	0.01465	0.01318	−0.00280
−0.01277	−0.00449	0.00593	0.00593	−0.02532	0.01053	−0.00066
0.00158	0.01862	−0.00018	−0.00125	−0.02839	0.01014	0.00069
−0.00427	0.00049	−0.00461	−0.00328	0.01245	0.01392	−0.00206
−0.00848	−0.00708	0.00622	0.00843	−0.01994	0.00851	0.00014
0.00261	0.00554	0.00340	0.00233	−0.02064	0.01008	−0.00184
0.00643	0.00000	0.00000	0.00000	0.00000	0.01095	0.00045
−0.00640	0.00130	0.00414	−0.00154	0.00130	0.01110	−0.00060
0.00489	−0.00244	0.00040	0.00419	−0.01380	0.00837	0.00009
−0.00557	0.00090	0.00374	−0.00194	0.00090	0.01004	−0.00166
−0.00628	0.00050	0.00166	−0.00065	0.00050	0.01014	−0.00156
0.00787	0.00325	−0.00254	−0.00485	0.01019	0.00772	0.00065
−0.01005	0.00000	0.00000	0.00000	0.00000	0.00981	−0.00022
−0.00483	−0.00004	0.00111	−0.00352	0.00458	0.00888	0.00039
0.00387	−0.00265	0.00430	0.00152	−0.01237	0.00819	−0.00006
−0.00668	0.00000	0.00000	0.00000	0.00000	0.00899	−0.00027
0.00075	0.00253	0.00160	−0.00303	0.00299	0.00685	0.00006
−0.00188	0.00015	0.00054	−0.00101	−0.00757	0.00859	0.00036
0.00035	0.00272	−0.00384	0.00388	−0.00962	0.00659	0.00042
−0.00591	0.00242	0.00127	−0.00105	−0.00221	0.00657	−0.00031
0.00000	0.00403	−0.00253	0.00519	−0.00832	0.00653	0.00035
0.00213	−0.00276	0.00357	0.00251	−0.01199	0.00567	−0.00012
−0.00028	−0.00563	0.00136	0.00399	−0.00985	0.00652	0.00034
−0.00354	0.00088	−0.00036	−0.00023	−0.00147	0.00860	−0.00150
0.00125	0.00617	−0.00164	−0.00424	0.00878	0.00535	0.00004
0.00398	−0.00529	0.00314	0.00215	−0.01025	0.00555	−0.00008
0.00018	0.00222	0.00048	0.00048	−0.00820	0.00648	0.00031
0.00231	0.00000	0.00000	0.00000	−0.00694	0.00628	0.00088
0.00231	0.00000	0.00000	0.00000	−0.00694	0.00540	0.00000
−0.00463	0.00000	0.00000	0.00000	0.00000	0.00540	0.00000
−0.00421	0.00000	0.00000	0.00000	0.00000	0.00498	−0.00042
0.00231	0.00084	0.00084	0.00084	−0.00610	0.00540	0.00000
−0.00424	−0.00031	−0.00031	−0.00031	−0.00031	0.00498	−0.00042
0.00215	0.00052	0.00052	0.00052	−0.00643	0.00535	−0.00005

TABLE 2. (continued–6)

$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$	$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$
−0.00395	−0.00286	−0.00170	−0.00054	−0.01096	0.03883	0.01453
0.00514	−0.00038	−0.00067	−0.00009	−0.01658	0.04106	0.01676
−0.00523	0.00248	0.00017	0.00132	−0.00562	0.04167	0.01736
0.00325	0.00187	0.00158	0.00216	−0.01433	0.03840	0.01410
−0.00524	−0.00252	−0.00020	−0.00136	0.00559	0.04154	0.01723
0.00223	0.00233	0.00204	0.00262	−0.01387	0.03819	0.01389
−0.00524	0.00252	0.00020	0.00136	−0.00558	0.04123	0.01693
−0.00346	−0.00265	−0.00149	−0.00083	−0.01124	0.03869	0.01488
0.00783	−0.00496	0.00033	−0.00033	−0.01124	0.03435	0.01054
0.00742	−0.00417	0.00046	0.00245	−0.00913	0.03091	0.00710
−0.00415	−0.00145	−0.00029	0.00169	−0.00872	0.03634	0.01253
0.00668	−0.00322	0.00141	0.00339	−0.00818	0.02970	0.00589
−0.00259	0.00264	−0.00084	−0.00084	0.00032	0.03118	0.00737
−0.00197	−0.00682	0.00037	0.00073	0.00884	0.04246	0.02038
0.00742	−0.00421	0.00040	0.00242	−0.00918	0.03040	0.00765
0.00103	−0.00001	−0.00124	0.00048	0.00388	0.02737	0.00462
0.00553	−0.00379	0.00155	0.00190	−0.00433	0.02893	0.00685
−0.00695	0.00000	0.00000	0.00000	0.00000	0.02047	−0.00161
0.00352	−0.00014	0.00038	−0.00067	−0.00477	0.02423	0.00529
−0.00279	−0.00186	−0.00060	0.00108	−0.00228	0.02352	0.00458
0.00246	0.00037	0.00089	−0.00016	−0.00426	0.02398	0.00504
0.00294	0.00032	0.00070	−0.00007	−0.00431	0.02348	0.00497
0.00052	0.00077	−0.00116	−0.00194	−0.00155	0.02374	0.00522
−0.00370	0.00000	0.00000	0.00000	0.00000	0.02149	0.00452
−0.00501	0.00191	−0.00079	0.00075	−0.00195	0.01861	0.00163
−0.00292	−0.00136	−0.00021	0.00164	−0.00183	0.02133	0.00373
−0.00258	0.00000	0.00000	0.00000	0.00000	0.02107	0.00410
−0.00210	0.00176	−0.00071	0.00083	−0.00395	0.01970	0.00272
−0.00478	−0.00166	0.00116	−0.00089	0.00194	0.01820	0.00226
−0.00267	−0.00024	−0.00024	−0.00024	−0.00024	0.01735	0.00140
−0.00063	0.00170	0.00080	−0.00100	−0.00190	0.01983	0.00388
−0.00273	0.00000	0.00000	0.00000	0.00000	0.01648	0.00053
−0.00051	−0.00122	0.00079	0.00026	−0.00236	0.01951	0.00421
−0.00274	−0.00004	−0.00004	−0.00004	−0.00004	0.01631	0.00102
0.00459	−0.00127	0.00013	0.00092	−0.00231	0.01648	0.00119
−0.00064	−0.00006	0.00006	0.00010	−0.00209	0.01731	0.00201
0.00125	−0.00121	0.00069	0.00019	−0.00022	0.01966	0.00437
−0.00278	0.00006	0.00006	0.00006	0.00006	0.01550	0.00161
0.00011	0.00044	0.00044	0.00044	0.00276	0.01642	0.00254
−0.00077	−0.00000	0.00000	0.00000	0.00231	0.01620	0.00231
0.00154	0.00000	0.00000	0.00000	0.00000	0.01620	0.00231
0.00112	0.00000	0.00000	0.00000	0.00000	0.01620	0.00231
−0.00077	−0.00000	−0.00000	−0.00000	0.00231	0.01578	0.00189
0.00112	−0.00000	−0.00000	−0.00000	−0.00000	0.01615	0.00226
−0.00082	−0.00010	−0.00010	−0.00010	0.00221	0.01559	0.00170

TABLE 2. (continued–7)

$V^{(11,11)}_{(0111)}$	$V^{(11,11)}_{(1110)}$	$V^{(11,11)}_{(1011)}=V^{(11,11)}_{(1101)}$	$V^{(11,11)}_{(1111)}$
−0.00631	−0.00283	−0.00283	0.01453
0.00027	0.00027	0.00200	−0.00581
0.00000	−0.00347	0.00000	−0.01736
−0.00240	−0.00240	−0.00066	−0.00847
−0.00013	−0.00360	−0.00013	−0.01749
−0.00261	−0.00261	−0.00087	−0.00868
−0.00043	−0.00391	−0.00043	−0.01780
−0.00645	−0.00248	−0.00248	0.01488
0.00360	−0.00037	0.00360	0.00360
−0.00382	0.00015	0.00015	0.00015
−0.00880	−0.00483	−0.00483	0.01253
−0.00502	−0.00105	−0.00105	−0.00105
0.00192	−0.00602	−0.00007	−0.00751
−0.00142	−0.00070	0.00179	−0.02236
−0.00380	−0.00026	0.00048	0.00026
−0.00011	−0.00425	0.00033	−0.00395
−0.00383	−0.00312	−0.00063	−0.00116
−0.00161	−0.00303	0.00053	0.00267
0.00213	0.00213	−0.00039	−0.00608
0.00158	0.00284	−0.00031	0.00521
0.00188	0.00188	−0.00064	−0.00633
0.00149	0.00149	−0.00082	−0.00661
0.00175	0.00175	−0.00057	0.00753
0.00452	−0.00165	−0.00011	−0.00705
−0.00068	0.00241	−0.00068	−0.00184
−0.00009	0.00084	−0.00148	0.00373
0.00410	−0.00208	−0.00053	−0.00748
−0.00098	0.00211	−0.00098	−0.00005
−0.00102	0.00258	−0.00025	−0.00160
0.00121	0.00172	0.00044	0.00063
0.00311	−0.00101	0.00002	−0.00615
0.00034	0.00085	−0.00044	−0.00024
−0.00161	−0.00003	−0.00152	0.00316
0.00012	−0.00041	−0.00085	0.00075
0.00094	−0.00013	−0.00030	−0.00179
0.00003	−0.00055	−0.00096	0.00301
−0.00202	−0.00037	−0.00189	0.00573
0.00016	0.00016	0.00016	−0.00013
0.00022	0.00022	0.00022	0.00254
0.00000	0.00000	0.00000	0.00231
0.00000	0.00000	0.00000	−0.00463
0.00000	0.00000	0.00000	−0.00463
−0.00042	−0.00042	−0.00042	0.00189
−0.00005	−0.00005	−0.00005	−0.00468
−0.00062	−0.00062	−0.00062	0.00170

TABLE 3. Trace-optimal 3^4-BFF designs of resolution V

N	$\boldsymbol{\lambda}'$	$\mathrm{tr}(V_T)$	$V^{(00,00)}_{(0000)}$	$V^{(00,10)}_{(0000)}$	$V^{(00,01)}_{(0000)}$	$V^{(00,20)}_{(0000)}$	$V^{(00,20)}_{(0000)}$	$V^{(00,11)}_{(0000)}$	$V^{(10,10)}_{(0000)}$
57	001111110011101	0.61257	0.01945	−0.00079	−0.00146	0.00098	−0.00062	0.00079	0.02684
58	101111110011101	0.59595	0.01940	−0.00066	−0.00154	0.00066	−0.00066	0.00088	0.02649
59	102111110011101	0.58452	0.01912	−0.00106	−0.00166	0.00005	−0.00069	0.00074	0.02594
60	101011110111101	0.57665	0.01749	0.00000	−0.00051	0.00154	−0.00051	0.00000	0.02500
61	102011110111101	0.56377	0.01720	−0.00037	−0.00066	0.00086	−0.00054	−0.00012	0.02452
62	202011110111101	0.55218	0.01696	0.00000	−0.00078	0.00029	−0.00055	0.00000	0.02393
63	102111111011101	0.54812	0.01842	−0.00187	−0.00177	−0.00065	−0.00051	0.00081	0.02477
64	101111110111101	0.53183	0.01626	0.00110	−0.00076	0.00007	−0.00027	0.00000	0.02389
65	102111110111101	0.52048	0.01609	0.00076	−0.00086	−0.00039	−0.00030	−0.00010	0.02323
66	112010010111111	0.51163	0.01656	−0.00046	−0.00150	0.00158	0.00036	−0.00019	0.02275
67	212010010111111	0.50068	0.01612	0.00000	−0.00164	0.00084	0.00035	0.00000	0.02226
68	101111111111101	0.49163	0.01535	0.00000	−0.00094	−0.00103	−0.00009	0.00000	0.02249
69	111110010111111	0.48282	0.01597	0.00113	−0.00154	0.00108	0.00055	−0.00005	0.02140
70	112110010111111	0.47242	0.01562	0.00067	−0.00167	0.00046	0.00053	−0.00021	0.02078
71	111111011011111	0.46485	0.01467	−0.00079	−0.00090	0.00055	0.00003	0.00024	0.02089
72	011110011111111	0.45730	0.01535	−0.00036	−0.00158	0.00059	0.00075	−0.00024	0.01993
73	111110011111111	0.44417	0.01509	0.00000	−0.00171	0.00000	0.00069	0.00000	0.01944
74	112110011111111	0.43884	0.01498	−0.00014	−0.00177	−0.00024	0.00066	−0.00010	0.01925
75	212110011111111	0.43355	0.01488	0.00000	−0.00182	−0.00047	0.00063	0.00000	0.01905
76	222110011111111	0.42860	0.01377	0.00000	−0.00100	−0.00044	0.00013	0.00000	0.01905
77	111111011111111	0.42490	0.01337	0.00039	−0.00064	0.00000	0.00026	−0.00039	0.01910
78	112110111111111	0.41958	0.01321	−0.00057	−0.00069	−0.00030	0.00022	0.00027	0.01888
79	212111011111111	0.41433	0.01315	0.00046	−0.00074	−0.00047	0.00020	−0.00034	0.01866
80	101111111111111	0.41024	0.01260	0.00000	−0.00026	0.00000	0.00026	0.00000	0.01852
81	111111111111111	0.40741	0.01235	0.00000	0.00000	0.00000	0.00000	0.00000	0.01852

$\boldsymbol{\lambda}'=(\lambda_{400}, \lambda_{040}, \lambda_{004}, \lambda_{310}, \lambda_{301}, \lambda_{130}, \lambda_{031}, \lambda_{103}, \lambda_{013}, \lambda_{220}, \lambda_{202}, \lambda_{022}, \lambda_{211}, \lambda_{121}, \lambda_{112})$.

TABLE 3. (continued–1)

$V^{(10,10)}_{(1000)}$	$V^{(10,01)}_{(0000)}$	$V^{(10,01)}_{(0100)}$	$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$	$V^{(01,01)}_{(0000)}$
0.00077	−0.00084	0.00087	0.00007	0.00022	0.00026	−0.00049	−0.00292	0.00051	0.00071	0.01031
0.00043	−0.00062	0.00109	0.00092	0.00107	0.00038	−0.00038	−0.00316	0.00026	0.00046	0.01018
−0.00013	−0.00078	0.00093	0.00006	0.00021	0.00034	−0.00041	−0.00336	0.00006	0.00026	0.01013
0.00093	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00185	−0.00000	0.00093	0.00900
0.00044	−0.00019	−0.00019	−0.00088	−0.00088	−0.00003	−0.00003	−0.00201	−0.00016	0.00077	0.00893
−0.00014	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00221	−0.00036	0.00057	0.00887
−0.00099	−0.00114	0.00088	−0.00106	−0.00030	0.00043	−0.00008	−0.00333	−0.00038	0.00063	0.00988
−0.00003	0.00041	0.00016	0.00152	0.00113	−0.00016	−0.00028	−0.00193	−0.00039	0.00116	0.00863
−0.00069	0.00021	−0.00004	0.00060	0.00021	−0.00021	−0.00034	−0.00212	−0.00058	0.00097	0.00857
0.00184	−0.00015	−0.00015	−0.00077	−0.00077	−0.00001	−0.00001	−0.00128	0.00111	−0.00098	0.00887
0.00135	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00148	0.00091	−0.00118	0.00883
−0.00132	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00198	−0.00066	0.00132	0.00814
0.00130	0.00057	0.00010	0.00186	0.00092	0.00005	−0.00041	−0.00102	0.00056	−0.00037	0.00869
0.00068	0.00040	−0.00006	0.00104	0.00010	0.00003	−0.00044	−0.00123	0.00035	−0.00058	0.00864
0.00083	−0.00040	0.00012	−0.00089	−0.00012	0.00018	0.00044	−0.00148	0.00006	−0.00071	0.00799
0.00048	−0.00018	−0.00018	−0.00080	−0.00080	−0.00009	−0.00009	−0.00061	0.00032	0.00032	0.00847
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00093	−0.00000	−0.00000	0.00840
−0.00020	−0.00007	−0.00007	−0.00033	−0.00033	−0.00004	−0.00004	−0.00106	−0.00013	−0.00013	0.00838
−0.00040	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00119	−0.00026	−0.00026	0.00835
−0.00040	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00119	−0.00026	−0.00026	0.00774
0.00000	0.00019	−0.00039	0.00000	0.00000	−0.00019	0.00039	−0.00058	−0.00000	0.00000	0.00701
−0.00021	−0.00024	0.00033	−0.00034	−0.00034	0.00015	−0.00043	−0.00071	−0.00013	−0.00013	0.00700
−0.00043	0.00014	−0.00043	−0.00000	−0.00000	−0.00017	0.00041	−0.00084	−0.00026	−0.00026	0.00695
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00643
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00617

TABLE 3. (continued-2)

$V^{(01,01)}_{(0001)}$	$V^{(01,20)}_{(0000)}$	$V^{(01,20)}_{(0010)}$	$V^{(01,02)}_{(0000)}$	$V^{(01,02)}_{(0001)}$	$V^{(01,11)}_{(0001)}$	$V^{(01,11)}_{(0010)}$	$V^{(01,11)}_{(0011)}$	$V^{(20,20)}_{(0000)}$	$V^{(20,20)}_{(1000)}$	$V^{(20,20)}_{(2000)}$
−0.00032	0.00078	0.00093	−0.00043	0.00036	0.00043	−0.00078	−0.00058	0.03544	0.00329	0.00238
−0.00045	0.00026	0.00041	−0.00050	0.00028	0.00058	−0.00062	−0.00042	0.03340	0.00124	0.00033
−0.00050	0.00001	0.00016	−0.00052	0.00027	0.00053	−0.00068	−0.00048	0.03208	−0.00007	−0.00098
−0.00026	0.00077	0.00077	−0.00026	−0.00026	0.00000	0.00000	0.00000	0.03588	0.00289	0.00116
−0.00033	0.00043	0.00043	−0.00027	−0.00027	−0.00006	−0.00006	−0.00006	0.03426	0.00127	−0.00046
−0.00039	0.00015	0.00015	−0.00028	−0.00028	0.00000	0.00000	0.00000	0.03293	−0.00005	−0.00179
−0.00044	−0.00041	0.00035	−0.00061	0.00043	0.00049	−0.00119	−0.00018	0.03018	−0.00078	−0.00050
−0.00020	0.00016	0.00080	−0.00032	−0.00010	0.00013	0.00064	−0.00039	0.03315	0.00113	0.00036
−0.00026	−0.00012	0.00052	−0.00033	−0.00012	0.00007	0.00059	−0.00044	0.03188	−0.00014	−0.00091
0.00055	0.00079	−0.00041	−0.00096	−0.00002	−0.00006	−0.00006	−0.00006	0.03175	0.00156	−0.00086
0.00051	0.00056	−0.00065	−0.00096	−0.00003	0.00000	0.00000	0.00000	0.03052	0.00033	−0.00209
−0.00009	−0.00051	0.00103	−0.00043	0.00009	0.00000	0.00000	0.00000	0.03048	−0.00019	0.00039
0.00064	0.00054	−0.00013	−0.00106	0.00015	−0.00003	0.00044	−0.00023	0.03044	0.00132	−0.00001
0.00059	0.00032	−0.00035	−0.00107	0.00014	−0.00008	0.00038	−0.00028	0.02934	0.00023	−0.00111
−0.00000	0.00033	−0.00055	−0.00059	0.00036	0.00007	−0.00045	0.00043	0.02910	0.00001	0.00008
0.00075	0.00030	0.00030	−0.00117	0.00038	−0.00012	−0.00012	−0.00012	0.02911	0.00134	0.00134
0.00069	0.00000	0.00000	−0.00120	0.00034	0.00000	0.00000	0.00000	0.02778	0.00000	0.00000
0.00066	−0.00012	−0.00012	−0.00121	0.00033	−0.00005	−0.00005	−0.00005	0.02724	−0.00054	−0.00054
0.00063	−0.00024	−0.00024	−0.00123	0.00032	0.00000	0.00000	0.00000	0.02671	−0.00107	−0.00107
0.00002	−0.00026	−0.00026	−0.00085	0.00069	0.00000	0.00000	0.00000	0.02671	−0.00107	−0.00107
0.00026	0.00000	0.00000	−0.00045	0.00013	−0.00019	0.00039	0.00039	0.02778	0.00000	0.00000
0.00024	−0.00008	−0.00008	−0.00046	0.00012	0.00016	−0.00042	−0.00042	0.02723	−0.00054	−0.00054
0.00020	−0.00024	−0.00024	−0.00047	0.00010	−0.00022	0.00036	0.00036	0.02671	−0.00107	−0.00107
0.00026	0.00000	0.00000	−0.00026	−0.00026	0.00000	0.00000	0.00000	0.02778	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.02778	0.00000	0.00000

TABLE 3. (continued-3)

$V^{(20,02)}_{(0000)}$	$V^{(20,02)}_{(0100)}$	$V^{(20,02)}_{(0200)}$	$V^{(20,11)}_{(0000)}$	$V^{(20,11)}_{(0100)}$	$V^{(20,11)}_{(1000)}$	$V^{(20,11)}_{(1100)}$	$V^{(02,02)}_{(0000)}$	$V^{(02,02)}_{(0001)}$	$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$
−0.00036	0.00069	−0.00172	0.00062	−0.00059	−0.00029	−0.00149	0.00822	−0.00170	0.00421	−0.00123
−0.00064	0.00041	−0.00200	0.00122	0.00001	0.00032	−0.00089	0.00818	−0.00174	0.00417	−0.00115
−0.00070	0.00036	−0.00206	0.00092	−0.00029	0.00001	−0.00120	0.00817	−0.00174	0.00417	−0.00116
−0.00039	0.00019	−0.00270	0.00000	0.00000	0.00000	0.00000	0.00502	−0.00019	0.00116	0.00000
−0.00043	0.00014	−0.00275	−0.00029	−0.00029	−0.00029	−0.00029	0.00501	−0.00019	0.00116	−0.00001
−0.00048	0.00010	−0.00279	0.00000	0.00000	0.00000	0.00000	0.00501	−0.00020	0.00115	0.00000
−0.00101	0.00054	−0.00139	−0.00013	0.00025	−0.00038	−0.00000	0.00793	−0.00179	0.00432	−0.00164
−0.00041	0.00049	−0.00209	0.00116	−0.00039	0.00039	−0.00116	0.00486	−0.00024	0.00122	0.00039
−0.00049	0.00041	−0.00216	0.00090	−0.00065	0.00012	−0.00142	0.00486	−0.00025	0.00121	0.00037
0.00080	−0.00040	−0.00161	−0.00032	−0.00032	−0.00032	−0.00032	0.00546	0.00022	−0.00193	−0.00000
0.00079	−0.00042	−0.00162	0.00000	0.00000	0.00000	0.00000	0.00545	0.00022	−0.00193	0.00000
−0.00064	0.00071	−0.00141	0.00000	0.00000	0.00000	0.00000	0.00467	−0.00028	0.00133	0.00000
0.00050	−0.00017	−0.00084	0.00080	−0.00053	0.00040	−0.00093	0.00516	0.00019	−0.00168	0.00044
0.00046	−0.00021	−0.00087	0.00053	−0.00081	0.00012	−0.00121	0.00516	0.00019	−0.00168	0.00043
0.00011	−0.00033	0.00062	−0.00043	0.00088	0.00011	0.00004	0.00434	−0.00013	−0.00012	−0.00028
0.00015	0.00015	0.00015	−0.00053	−0.00053	−0.00053	−0.00053	0.00482	0.00019	−0.00136	−0.00006
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00480	0.00017	−0.00137	0.00000
−0.00006	−0.00006	−0.00006	−0.00022	−0.00022	−0.00022	−0.00022	0.00479	0.00016	−0.00138	−0.00002
−0.00012	−0.00012	−0.00012	0.00000	0.00000	0.00000	0.00000	0.00479	0.00016	−0.00138	0.00000
−0.00010	−0.00010	−0.00010	0.00000	0.00000	0.00000	0.00000	0.00456	−0.00007	−0.00162	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00373	0.00006	−0.00051	0.00019
−0.00008	−0.00008	−0.00008	−0.00020	−0.00020	−0.00020	−0.00020	0.00372	0.00005	−0.00052	−0.00022
−0.00012	−0.00012	−0.00012	0.00000	0.00000	0.00000	0.00000	0.00372	0.00005	−0.00053	0.00020
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00334	0.00026	0.00026	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00309	0.00000	0.00000	0.00000

TABLE 3. (continued-4)

$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$	$V^{(11,11)}_{(0111)}$	$V^{(11,11)}_{(1110)}$	$V^{(11,11)}_{(1011)}=V^{(11,11)}_{(1101)}$	$V^{(11,11)}_{(1111)}$
0.00018	0.00098	−0.00224	0.01521	0.00122	−0.00029	0.00042	0.00001	−0.00119
0.00026	0.00106	−0.00216	0.01503	0.00104	−0.00046	0.00024	−0.00016	−0.00137
0.00025	0.00105	−0.00217	0.01496	0.00097	−0.00054	0.00017	−0.00023	−0.00144
0.00000	0.00000	0.00000	0.01481	0.00185	−0.00116	0.00069	0.00023	−0.00370
−0.00001	−0.00001	−0.00001	0.01476	0.00180	−0.00121	0.00064	0.00018	−0.00376
0.00000	0.00000	0.00000	0.01470	0.00173	−0.00128	0.00058	0.00011	−0.00382
0.00042	0.00084	−0.00173	0.01354	0.00028	0.00015	−0.00090	−0.00006	−0.00039
−0.00013	0.00013	−0.00039	0.01312	0.00077	−0.00039	−0.00039	0.00039	−0.00231
−0.00014	0.00011	−0.00040	0.01306	0.00072	−0.00044	−0.00044	0.00033	−0.00237
−0.00000	−0.00000	−0.00000	0.01211	0.00046	−0.00013	0.00166	−0.00043	−0.00133
0.00000	0.00000	0.00000	0.01203	0.00038	−0.00022	0.00157	−0.00052	−0.00141
0.00000	0.00000	0.00000	0.01190	0.00000	0.00017	−0.00116	0.00050	−0.00132
−0.00023	0.00024	−0.00043	0.01108	0.00024	0.00049	0.00077	−0.00016	−0.00056
−0.00023	0.00023	−0.00043	0.01101	0.00017	0.00042	0.00070	−0.00023	−0.00063
0.00015	−0.00010	−0.00106	0.01117	0.00037	0.00060	0.00060	−0.00017	0.00068
−0.00006	−0.00006	−0.00006	0.01040	0.00021	0.00114	0.00021	0.00021	0.00021
0.00000	0.00000	0.00000	0.01019	0.00000	0.00093	−0.00000	0.00000	−0.00000
−0.00002	−0.00002	−0.00002	0.01010	−0.00009	0.00084	−0.00009	−0.00009	−0.00009
0.00000	0.00000	0.00000	0.01001	−0.00018	0.00075	−0.00018	−0.00018	−0.00018
0.00000	0.00000	0.00000	0.01001	−0.00018	0.00075	−0.00018	−0.00018	−0.00018
0.00019	−0.00039	−0.00039	0.00984	0.00000	0.00058	0.00000	0.00000	−0.00000
−0.00022	0.00036	0.00036	0.00976	−0.00008	0.00050	−0.00008	−0.00008	−0.00008
0.00020	−0.00037	−0.00037	0.00968	−0.00016	0.00042	−0.00016	−0.00016	−0.00016
0.00000	0.00000	0.00000	0.00926	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00926	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 4. Determinant-optimal 3^4-BFF designs of resolution V

N	λ'	det (V_T)	$V_{(0000)}^{(00,00)}$	$V_{(0000)}^{(00,10)}$	$V_{(0000)}^{(00,01)}$
57	010111011011110	0.19202E–60	0.01787	0.00031	−0.00033
58	011110111110011	0.10700E–60	0.01777	−0.00056	−0.00037
59	111111011011110	0.60360E–61	0.01741	0.00024	−0.00055
60a	101101101111101	0.31857E–61	0.01721	0.00000	0.00005
60b	011111010111011		0.01721	0.00008	−0.00003
61a	111111001111011	0.18235E–61	0.01665	−0.00000	0.00033
61b	111101101111101		0.01665	0.00000	0.00033
61c	111110110111011		0.01665	−0.00049	−0.00016
62a	101111111101101	0.10835E–61	0.01725	0.00000	−0.00016
62b	011111111110011		0.01725	−0.00024	0.00008
63	010111011111110	0.57380E–62	0.01637	0.00142	−0.00012
64	011110111111011	0.31979E–62	0.01626	−0.00169	−0.00017
65	111111011111110	0.18654E–62	0.01612	0.00141	−0.00030
66a	011111010011111	0.10465E–62	0.01640	0.00005	−0.00065
66b	101101101011111		0.01640	−0.00095	0.00035
67a	111111010011111	0.59918E–63	0.01581	0.00047	−0.00080
67b	111111010101111		0.01581	0.00144	0.00017
67c	111101101011111		0.01581	−0.00097	0.00063
*68a	101111111111101	0.34807E–63	0.01535	0.00000	−0.00094
68b	011111111111011		0.01535	−0.00141	0.00047
69a	010111011011111	0.20120E–63	0.01544	−0.00090	−0.00059
69b	001111110101111		0.01544	0.00044	0.00075
70a	011111011011111	0.11327E–63	0.01505	−0.00122	−0.00079
70b	101101111011111		0.01505	−0.00179	−0.00022
*71a	111111011011111	0.66048E–64	0.01467	−0.00079	−0.00090
71b	111101111011111		0.01467	−0.00174	0.00005
72a	000111111101111	0.38764E–64	0.01517	0.00000	0.00080
72b	000111111011111		0.01517	−0.00120	−0.00040
73a	010111111101111	0.22205E–64	0.01454	0.00000	0.00110
73b	001111111011111		0.01454	−0.00165	−0.00055
74a	011111111101111	0.13107E–64	0.01425	−0.00039	0.00095
74b	011111111011111		0.01425	−0.00162	−0.00028
75a	111111111101111	0.77673E–65	0.01399	0.00000	0.00082
75b	111111111011111		0.01399	−0.00123	−0.00041
76a	101111101111111	0.46213E–65	0.01378	0.00083	0.00023
76b	101111110111111		0.01352	0.00110	−0.00007
77a	111111101111111	0.26958E–65	0.01337	0.00077	0.00051
77b	111111110111111		0.01337	0.00116	0.00013
78	000111111111111	0.16172E–65	0.01323	0.00000	0.00000
79	001111111111111	0.95038E–66	0.01289	−0.00041	−0.00014
*80a	101111111111111	0.56099E–66	0.01260	0.00000	−0.00026
80b	011111111111111		0.01260	−0.00039	0.00013
*81	111111111111111	0.33244E–66	0.01235	0.00000	0.00000

*This design is also optimal with respect to the trace criterion.

$\lambda' = (\lambda_{400}, \lambda_{040}, \lambda_{004}, \lambda_{310}, \lambda_{301}, \lambda_{130}, \lambda_{031}, \lambda_{103}, \lambda_{013}, \lambda_{220}, \lambda_{202}, \lambda_{022}, \lambda_{211}, \lambda_{121}, \lambda_{112})$.

$n_0\mathrm{E} - n_1 = n_0 10^{-n_1}$.

TABLE 4. (continued-1)

$V^{(00,20)}_{(0000)}$	$V^{(00,02)}_{(0000)}$	$V^{(00,11)}_{(0000)}$	$V^{(10,10)}_{(0000)}$	$V^{(10,10)}_{(1000)}$	$V^{(10,01)}_{(0000)}$	$V^{(10,01)}_{(0100)}$
0.00070	−0.00021	0.00043	0.02677	0.00067	0.00015	0.00015
0.00028	−0.00028	−0.00056	0.02609	0.00000	−0.00028	−0.00028
−0.00047	−0.00032	0.00024	0.02580	−0.00029	0.00012	0.00012
−0.00090	0.00061	0.00000	0.02602	−0.00105	0.00000	0.00000
0.00114	−0.00007	−0.00068	0.02628	0.00066	−0.00009	−0.00057
−0.00098	0.00009	0.00007	0.02602	−0.00105	0.00000	0.00000
−0.00091	0.00016	0.00000	0.02602	−0.00105	0.00000	0.00000
−0.00014	−0.00019	0.00035	0.02598	0.00037	−0.00001	0.00047
−0.00237	0.00047	0.00000	0.02493	−0.00214	0.00000	0.00000
0.00047	−0.00047	−0.00095	0.02528	0.00000	−0.00012	−0.00071
−0.00015	0.00002	−0.00036	0.02546	0.00000	0.00058	−0.00019
−0.00059	−0.00005	0.00022	0.02477	−0.00069	−0.00071	0.00006
−0.00103	−0.00009	−0.00039	0.02426	−0.00121	0.00047	−0.00031
0.00277	0.00003	−0.00014	0.02319	0.00231	−0.00004	−0.00000
0.00054	0.00077	0.00060	0.02428	0.00052	−0.00040	0.00060
0.00177	−0.00009	0.00021	0.02289	0.00201	0.00007	0.00011
−0.00007	0.00052	−0.00041	0.02408	0.00020	0.00046	−0.00050
0.00055	0.00032	0.00061	0.02428	0.00052	−0.00040	0.00060
−0.00103	−0.00009	0.00000	0.02249	−0.00132	0.00000	0.00000
−0.00045	−0.00028	−0.00019	0.02395	−0.00052	−0.00049	−0.00027
0.00208	0.00017	0.00029	0.02164	0.00158	−0.00036	0.00015
0.00046	0.00071	−0.00025	0.02422	0.00045	0.00049	−0.00023
0.00126	0.00013	−0.00004	0.02138	0.00132	−0.00052	−0.00001
0.00055	0.00036	0.00020	0.02261	0.00039	−0.00093	0.00030
0.00055	0.00003	0.00024	0.02089	0.00083	−0.00040	0.00012
0.00055	0.00003	0.00024	0.02260	0.00038	−0.00097	0.00027
0.00000	0.00101	0.00000	0.02288	0.00000	0.00000	0.00000
0.00227	0.00025	0.00076	0.02116	0.00156	−0.00057	0.00052
0.00000	0.00055	0.00000	0.02288	0.00000	0.00000	0.00000
0.00123	0.00014	0.00041	0.02084	0.00123	−0.00068	0.00041
−0.00066	0.00048	−0.00019	0.02237	−0.00051	−0.00019	−0.00019
0.00120	−0.00014	0.00042	0.02084	0.00123	−0.00070	0.00039
−0.00123	0.00041	0.00000	0.02179	−0.00109	0.00000	0.00000
0.00062	−0.00021	0.00062	0.02026	0.00065	−0.00051	0.00058
0.00000	0.00060	0.00042	0.02084	0.00001	0.00034	0.00034
0.00110	0.00007	0.00000	0.02028	0.00118	0.00065	0.00008
0.00000	0.00026	0.00039	0.02083	0.00000	0.00039	0.00039
0.00116	−0.00013	0.00000	0.02025	0.00116	0.00058	0.00000
0.00132	0.00044	0.00000	0.01961	0.00109	0.00000	0.00000
0.00062	0.00034	−0.00021	0.01910	0.00058	−0.00017	−0.00017
0.00000	0.00026	0.00000	0.01852	0.00000	0.00000	0.00000
0.00058	0.00006	−0.00019	0.01910	0.00058	−0.00019	−0.00019
0.00000	0.00000	0.00000	0.01852	0.00000	0.00000	0.00000

TABLE 4. (continued-2)

$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$
0.00057	−0.00322	−0.00018	0.00025	−0.00091	−0.00007	0.00035
−0.00168	0.00210	−0.00000	−0.00042	−0.00126	−0.00042	0.00000
0.00100	−0.00278	−0.00003	0.00039	−0.00155	−0.00071	−0.00029
0.00000	0.00000	0.00000	0.00000	0.00019	−0.00052	0.00126
−0.00082	0.00323	−0.00042	0.00054	−0.00057	−0.00021	0.00046
−0.00013	0.00094	−0.00013	0.00094	0.00031	−0.00040	0.00031
0.00000	0.00000	0.00000	0.00000	0.00019	−0.00052	0.00126
−0.00009	0.00396	0.00034	−0.00063	−0.00081	−0.00045	0.00022
0.00000	0.00000	0.00000	0.00000	−0.00036	−0.00107	0.00071
−0.00160	0.00178	−0.00030	0.00012	−0.00053	−0.00018	0.00000
0.00162	−0.00301	−0.00052	0.00025	−0.00006	−0.00006	0.00109
−0.00276	0.00187	0.00035	−0.00042	−0.00041	−0.00041	0.00074
0.00193	−0.00270	−0.00042	0.00035	−0.00072	−0.00072	0.00043
0.00020	0.00005	0.00019	0.00000	−0.00130	0.00106	−0.00089
−0.00109	0.00026	0.00031	0.00006	0.00129	0.00017	−0.00058
0.00091	0.00076	0.00027	0.00008	−0.00154	0.00082	−0.00113
0.00208	0.00103	−0.00005	−0.00014	0.00128	0.00004	−0.00101
−0.00109	0.00026	0.00030	0.00005	0.00129	0.00017	−0.00058
0.00000	0.00000	0.00000	0.00000	−0.00198	−0.00066	0.00132
−0.00263	0.00167	−0.00002	0.00009	0.00029	−0.00037	0.00062
−0.00102	−0.00025	0.00010	0.00036	−0.00091	0.00063	−0.00014
0.00130	0.00022	−0.00022	−0.00058	0.00194	0.00040	−0.00068
−0.00169	−0.00091	0.00006	0.00032	−0.00118	0.00037	−0.00040
−0.00273	0.00190	0.00009	−0.00022	0.00029	0.00029	−0.00064
−0.00089	−0.00012	0.00018	0.00044	−0.00148	0.00006	−0.00071
−0.00273	0.00190	0.00013	−0.00017	0.00028	0.00028	−0.00064
0.00000	0.00000	0.00000	0.00000	0.00218	0.00000	0.00000
−0.00100	−0.00018	0.00025	−0.00002	−0.00033	0.00076	−0.00006
0.00000	0.00000	0.00000	0.00000	0.00218	0.00000	0.00000
−0.00174	−0.00093	0.00017	−0.00010	−0.00058	0.00051	−0.00031
−0.00087	−0.00087	−0.00010	−0.00010	0.00192	−0.00026	−0.00026
−0.00174	−0.00092	0.00019	−0.00008	−0.00058	0.00051	−0.00031
0.00000	0.00000	0.00000	0.00000	0.00163	−0.00054	−0.00054
−0.00087	−0.00005	0.00029	0.00002	−0.00087	0.00022	−0.00060
0.00174	−0.00174	0.00024	0.00024	0.00116	0.00000	0.00000
0.00176	0.00089	−0.00008	−0.00037	0.00000	0.00058	−0.00029
0.00174	−0.00174	0.00019	0.00019	0.00116	−0.00000	0.00000
0.00174	0.00087	−0.00000	−0.00029	0.00000	0.00058	−0.00029
0.00000	0.00000	0.00000	0.00000	0.00054	0.00054	0.00054
−0.00087	−0.00087	−0.00012	−0.00012	0.00029	0.00029	0.00029
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
−0.00087	−0.00087	−0.00010	−0.00010	0.00029	0.00029	0.00029
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 4. (continued-3)

$V^{(01,01)}_{(0000)}$	$V^{(01,01)}_{(0001)}$	$V^{(01,20)}_{(0000)}$	$V^{(01,20)}_{(0010)}$	$V^{(01,02)}_{(0000)}$	$V^{(01,02)}_{(0001)}$	$V^{(01,11)}_{(0001)}$
0.00910	−0.00016	0.00035	0.00035	−0.00011	−0.00011	0.00021
0.00907	−0.00019	0.00014	0.00014	−0.00014	−0.00014	−0.00028
0.00898	−0.00028	−0.00024	−0.00024	−0.00016	−0.00016	0.00012
0.00879	0.00041	−0.00116	0.00181	−0.00008	0.00048	0.00000
0.00870	−0.00016	0.00010	0.00018	0.00020	−0.00060	−0.00010
0.00866	0.00028	−0.00117	0.00082	0.00013	−0.00031	0.00001
0.00866	0.00028	−0.00115	0.00182	0.00014	0.00069	0.00000
0.00867	−0.00019	0.00014	0.00006	0.00017	−0.00063	0.00002
0.00847	0.00024	−0.00107	0.00047	−0.00020	0.00032	0.00000
0.00835	−0.00047	−0.00018	0.00012	0.00034	−0.00028	−0.00030
0.00836	0.00004	−0.00005	0.00098	0.00008	−0.00035	0.00000
0.00834	0.00002	−0.00026	0.00077	0.00004	−0.00039	−0.00006
0.00823	−0.00009	−0.00064	0.00039	0.00001	−0.00042	−0.00008
0.00824	−0.00003	0.00095	−0.00046	−0.00044	0.00025	−0.00004
0.00788	0.00057	0.00057	−0.00096	−0.00000	0.00028	0.00026
0.00820	−0.00006	0.00070	−0.00071	−0.00047	0.00022	0.00005
0.00781	0.00054	−0.00023	−0.00104	−0.00009	0.00019	−0.00022
0.00774	0.00043	0.00056	−0.00097	0.00022	0.00050	0.00025
0.00814	−0.00009	−0.00051	0.00103	−0.00043	0.00009	0.00000
0.00766	−0.00035	−0.00045	0.00077	0.00045	−0.00047	−0.00010
0.00812	0.00012	0.00094	0.00006	−0.00054	0.00041	0.00015
0.00726	0.00050	0.00032	−0.00056	0.00013	−0.00016	−0.00004
0.00802	0.00003	0.00053	−0.00034	−0.00056	0.00039	−0.00001
0.00761	0.00034	0.00032	−0.00071	−0.00017	0.00011	−0.00003
0.00799	−0.00000	0.00033	−0.00055	−0.00059	0.00036	0.00007
0.00742	0.00015	0.00032	−0.00071	0.00007	0.00035	−0.00006
0.00686	0.00069	0.00000	0.00000	0.00005	0.00005	0.00000
0.00744	0.00017	0.00076	−0.00006	−0.00028	−0.00001	0.00025
0.00672	0.00055	0.00000	0.00000	0.00027	0.00027	0.00000
0.00740	0.00014	0.00051	−0.00031	−0.00031	−0.00003	0.00017
0.00665	0.00048	−0.00033	−0.00033	0.00024	0.00024	−0.00010
0.00716	−0.00011	0.00054	−0.00027	−0.00005	0.00022	0.00016
0.00658	0.00041	−0.00062	−0.00062	0.00021	0.00021	0.00000
0.00709	−0.00017	0.00025	−0.00056	−0.00008	0.00019	0.00025
0.00663	0.00046	0.00000	0.00000	−0.00011	−0.00011	0.00017
0.00690	0.00015	0.00065	−0.00021	−0.00015	−0.00044	0.00000
0.00643	0.00026	0.00000	0.00000	0.00013	0.00013	0.00019
0.00662	−0.00013	0.00058	−0.00029	0.00013	−0.00016	0.00000
0.00654	0.00036	0.00054	0.00054	−0.00018	−0.00018	0.00000
0.00648	0.00031	0.00025	0.00025	−0.00022	−0.00022	−0.00008
0.00643	0.00026	0.00000	0.00000	−0.00026	−0.00026	0.00000
0.00624	0.00006	0.00029	0.00029	0.00003	0.00003	−0.00010
0.00617	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 4. (continued-4)

$V_{(0010)}^{(01,11)}$	$V_{(0011)}^{(01,11)}$	$V_{(0000)}^{(20,20)}$	$V_{(1000)}^{(20,20)}$	$V_{(2000)}^{(20,20)}$	$V_{(0000)}^{(20,02)}$	$V_{(0100)}^{(20,02)}$
0.00021	0.00021	0.04182	0.00173	0.00331	−0.00097	0.00040
−0.00028	−0.00028	0.03998	−0.00011	0.00147	−0.00126	0.00011
0.00012	0.00012	0.03840	−0.00169	−0.00011	−0.00134	0.00003
0.00000	0.00000	0.04446	−0.00018	−0.01358	−0.00024	0.00076
0.00025	0.00027	0.04256	0.00135	0.00181	0.00040	0.00025
0.00001	0.00100	0.04409	−0.00192	−0.00627	−0.00047	0.00014
0.00000	0.00000	0.04446	−0.00018	−0.01358	−0.00025	0.00075
−0.00033	−0.00035	0.04077	−0.00044	0.00002	−0.00019	−0.00004
0.00000	0.00000	0.04184	−0.00445	0.01175	−0.00220	0.00089
0.00006	0.00024	0.03971	−0.00018	0.00160	−0.00077	0.00018
0.00103	0.00013	0.04022	0.00125	0.00395	−0.00037	0.00053
−0.00109	−0.00019	0.03836	−0.00061	0.00209	−0.00066	0.00024
0.00095	0.00005	0.03704	−0.00193	0.00077	−0.00077	0.00013
−0.00008	0.00011	0.03314	0.00301	0.00204	0.00068	−0.00026
−0.00094	−0.00009	0.04165	0.00001	−0.01248	−0.00003	−0.00034
0.00001	0.00020	0.03143	0.00130	0.00034	0.00049	−0.00045
0.00094	0.00013	0.03915	−0.00106	0.00874	−0.00001	−0.00052
−0.00095	−0.00010	0.04165	0.00001	−0.01248	−0.00002	−0.00033
0.00000	0.00000	0.03048	−0.00019	0.00039	−0.00064	0.00071
−0.00076	0.00023	0.03823	−0.00063	0.00218	−0.00033	0.00027
−0.00037	0.00051	0.03212	0.00303	0.00310	0.00038	−0.00006
0.00047	−0.00040	0.04096	0.00076	0.01056	−0.00015	−0.00059
−0.00053	0.00035	0.03040	0.00131	0.00138	0.00029	−0.00014
−0.00106	−0.00024	0.03465	0.00001	−0.00547	0.00010	−0.00034
−0.00045	0.00043	0.02910	0.00001	0.00008	0.00011	−0.00033
−0.00108	−0.00026	0.03465	0.00001	−0.00547	0.00011	−0.00033
0.00000	0.00000	0.03889	0.00000	0.01111	0.00000	0.00000
−0.00084	−0.00002	0.03212	0.00304	0.00312	0.00056	0.00007
0.00000	0.00000	0.03889	0.00000	0.01111	0.00000	0.00000
−0.00092	−0.00010	0.03039	0.00131	0.00139	0.00036	−0.00013
−0.00010	−0.00010	0.03741	−0.00148	0.00964	−0.00016	−0.00016
−0.00093	−0.00011	0.03039	0.00130	0.00138	0.00833	−0.00016
0.00000	0.00000	0.03611	−0.00278	0.00833	−0.00031	−0.00031
−0.00084	−0.00002	0.02908	0.00000	0.00008	0.00018	−0.00031
0.00017	0.00017	0.03299	0.00000	−0.00521	0.00000	0.00000
0.00058	−0.00029	0.03040	0.00132	0.00002	0.00021	−0.00022
0.00019	0.00019	0.03299	0.00000	−0.00521	0.00000	0.00000
0.00058	−0.00029	0.03038	0.00130	0.00000	0.00029	−0.00014
0.00000	0.00000	0.03058	0.00280	0.00280	0.00039	0.00039
−0.00008	−0.00008	0.02908	0.00131	0.00131	0.00018	0.00018
0.00000	0.00000	0.02778	0.00000	0.00000	0.00000	0.00000
−0.00010	−0.00010	0.02908	0.00130	0.00130	0.00014	0.00014
0.00000	0.00000	0.02778	0.00000	0.00000	0.00000	0.00000

TABLE 4. (continued-5)

$V_{(0200)}^{(20,02)}$	$V_{(0000)}^{(20,11)}$	$V_{(0100)}^{(20,11)}$	$V_{(1000)}^{(20,11)}$	$V_{(1100)}^{(20,11)}$	$V_{(0000)}^{(02,02)}$	$V_{(0001)}^{(02,02)}$
0.00176	0.00110	0.00016	−0.00174	0.00426	0.00524	−0.00006
0.00147	−0.00168	−0.00074	0.00116	−0.00484	0.00519	−0.00011
0.00140	0.00100	0.00006	−0.00184	0.00416	0.00519	−0.00011
−0.00172	0.00000	0.00000	0.00000	0.00000	0.00494	−0.00002
0.00009	−0.00063	−0.00056	0.00158	−0.00529	0.00473	0.00015
0.00075	0.00010	0.00166	0.00060	−0.00478	0.00452	0.00017
−0.00173	0.00000	0.00000	0.00000	0.00000	0.00459	−0.00037
0.00011	−0.00005	0.00003	0.00217	−0.00470	0.00471	0.00013
−0.00297	0.00000	0.00000	0.00000	0.00000	0.00497	−0.00018
0.00113	−0.00178	−0.00071	0.00142	−0.00445	0.00425	−0.00018
0.00143	0.00108	−0.00085	−0.00162	0.00340	0.00449	0.00016
0.00114	−0.00165	0.00028	0.00105	−0.00397	0.00445	0.00012
0.00103	0.00116	−0.00077	−0.00154	0.00347	0.00444	0.00011
0.00019	−0.00018	−0.00020	−0.00005	−0.00145	0.00462	−0.00009
0.00074	−0.00053	0.00131	−0.00031	0.00014	0.00415	0.00030
−0.00001	0.00041	0.00039	0.00054	−0.00087	0.00460	−0.00011
−0.00103	0.00094	−0.00038	0.00031	−0.00101	0.00407	0.00019
0.00075	−0.00054	0.00131	−0.00031	0.00014	0.00380	−0.00005
−0.00141	0.00000	0.00000	0.00000	0.00000	0.00467	−0.00028
0.00088	−0.00176	0.00023	0.00122	−0.00374	0.00360	0.00006
0.00090	−0.00025	0.00106	0.00029	0.00021	0.00437	−0.00009
−0.00103	0.00043	−0.00089	−0.00011	−0.00143	0.00374	0.00051
0.00081	−0.00093	0.00038	−0.00039	−0.00047	0.00437	−0.00010
0.00062	−0.00135	0.00011	0.00088	0.00096	0.00402	0.00017
0.00062	−0.00043	0.00088	0.00011	0.00004	0.00434	−0.00013
0.00062	−0.00135	0.00011	0.00088	0.00096	0.00372	−0.00013
0.00000	0.00000	0.00000	0.00000	0.00000	0.00357	0.00048
0.00096	−0.00029	0.00101	0.00020	0.00011	0.00395	0.00010
0.00000	0.00000	0.00000	0.00000	0.00000	0.00322	0.00014
0.00077	−0.00087	0.00044	−0.00038	−0.00046	0.00393	0.00008
−0.00016	−0.00043	−0.00043	−0.00043	−0.00043	0.00321	0.00012
0.00073	−0.00087	0.00044	−0.00038	−0.00046	0.00366	−0.00019
−0.00031	0.00000	0.00000	0.00000	0.00000	0.00319	0.00010
0.00059	−0.00044	0.00087	0.00005	−0.00003	0.00364	−0.00021
0.00000	0.00087	0.00087	−0.00087	−0.00087	0.00345	0.00036
−0.00065	0.00087	−0.00043	0.00043	−0.00087	0.00353	0.00029
0.00000	0.00087	0.00087	−0.00087	−0.00087	0.00315	0.00006
−0.00058	0.00087	−0.00043	0.00043	−0.00087	0.00325	0.00002
0.00039	0.00000	0.00000	0.00000	0.00000	0.00340	0.00031
0.00018	−0.00044	−0.00044	−0.00044	−0.00044	0.00337	0.00028
0.00000	0.00000	0.00000	0.00000	0.00000	0.00334	0.00026
0.00014	−0.00043	−0.00043	−0.00043	−0.00043	0.00310	0.00002
0.00000	0.00000	0.00000	0.00000	0.00000	0.00309	0.00000

TABLE 4. (continued–6)

$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$	$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$
−0.00073	−0.00009	0.00001	0.00022	−0.00199	0.01533	0.00187
−0.00077	−0.00000	−0.00011	−0.00032	0.00189	0.01515	0.00168
−0.00078	−0.00003	0.00007	0.00028	−0.00193	0.01486	0.00139
0.00158	0.00000	0.00000	0.00000	0.00000	0.01327	0.00009
−0.00134	0.00021	0.00019	−0.00053	−0.00055	0.01390	0.00073
−0.00110	−0.00005	0.00052	−0.00054	0.00003	0.01344	0.00026
0.00123	0.00000	0.00000	0.00000	0.00000	0.01327	0.00009
−0.00137	−0.00028	−0.00025	0.00046	0.00048	0.01371	0.00053
0.00162	0.00000	0.00000	0.00000	0.00000	0.01300	−0.00018
0.00002	0.00036	0.00000	−0.00071	0.00125	0.01442	0.00125
−0.00108	0.00063	−0.00002	−0.00027	−0.00092	0.01305	0.00070
−0.00113	−0.00072	−0.00007	0.00018	0.00083	0.01287	0.00052
−0.00114	0.00068	0.00003	−0.00023	−0.00087	0.01268	0.00034
−0.00033	0.00007	−0.00014	0.00005	−0.00154	0.01251	0.00089
0.00092	−0.00052	−0.00004	0.00026	−0.00064	0.01180	0.00018
−0.00035	0.00014	−0.00007	0.00012	−0.00147	0.01231	0.00069
−0.00060	0.00065	0.00014	−0.00033	−0.00084	0.01180	0.00019
0.00057	−0.00051	−0.00003	0.00027	−0.00064	0.01180	0.00018
0.00133	0.00000	0.00000	0.00000	0.00000	0.01190	0.00000
−0.00040	−0.00038	0.00012	−0.00021	0.00028	0.01221	0.00031
−0.00009	−0.00032	0.00011	−0.00014	−0.00109	0.01164	0.00083
0.00037	0.00026	−0.00018	0.00008	−0.00036	0.01110	0.00030
−0.00009	−0.00036	0.00008	−0.00018	−0.00113	0.01137	0.00056
0.00080	−0.00063	−0.00014	0.00012	−0.00078	0.01117	0.00037
−0.00012	−0.00028	0.00015	−0.00010	−0.00106	0.01117	0.00037
0.00050	−0.00059	−0.00010	0.00015	−0.00075	0.01117	0.00037
0.00048	0.00000	0.00000	0.00000	0.00000	0.01035	0.00000
0.00072	−0.00047	0.00002	0.00029	−0.00060	0.01090	0.00056
0.00014	0.00000	0.00000	0.00000	0.00000	0.01035	0.00000
0.00070	−0.00053	−0.00004	0.00023	−0.00067	0.01071	0.00036
0.00012	−0.00005	−0.00005	−0.00005	−0.00005	0.01022	−0.00013
0.00044	−0.00052	−0.00003	0.00024	−0.00066	0.01071	0.00036
0.00010	0.00000	0.00000	0.00000	0.00000	0.01008	−0.00027
0.00042	−0.00047	0.00002	0.00029	−0.00061	0.01057	0.00022
0.00036	0.00012	0.00012	0.00012	0.00012	0.00984	0.00000
0.00015	0.00029	−0.00014	0.00014	−0.00029	0.01013	0.00029
0.00006	0.00010	0.00010	0.00010	0.00010	0.00984	−0.00000
−0.00013	0.00029	−0.00014	0.00014	−0.00029	0.01013	0.00029
0.00031	0.00000	0.00000	0.00000	0.00000	0.00953	0.00027
0.00028	−0.00006	−0.00006	−0.00006	−0.00006	0.00940	0.00015
0.00026	0.00000	0.00000	0.00000	0.00000	0.00926	0.00000
0.00002	−0.00005	−0.00005	−0.00005	−0.00005	0.00940	0.00014
0.00000	0.00000	0.00000	0.00000	0.00000	0.00926	0.00000

TABLE 4. (continued-7)

$V_{(0111)}^{(11,11)}$	$V_{(1110)}^{(11,11)}$	$V_{(1011)}^{(11,11)}=V_{(1101)}^{(11,11)}$	$V_{(1111)}^{(11,11)}$
−0.00076	0.00008	−0.00013	0.00250
−0.00095	−0.00011	−0.00032	0.00231
−0.00124	−0.00040	−0.00061	0.00202
0.00152	−0.00133	0.00045	−0.00151
−0.00113	0.00030	−0.00006	0.00031
−0.00007	−0.00078	−0.00007	0.00076
0.00152	−0.00133	0.00045	−0.00151
−0.00132	0.00010	−0.00025	0.00012
0.00125	−0.00160	0.00018	−0.00178
−0.00160	−0.00018	−0.00053	0.00231
−0.00046	−0.00046	0.00031	0.00109
−0.00063	−0.00063	0.00014	0.00091
−0.00082	−0.00082	−0.00005	0.00072
0.00024	0.00178	−0.00017	0.00015
0.00121	0.00065	−0.00025	0.00070
0.00003	0.00157	−0.00038	−0.00005
0.00123	0.00025	−0.00044	−0.00107
0.00121	0.00065	−0.00025	0.00070
0.00017	−0.00116	0.00050	−0.00132
−0.00126	−0.00060	0.00006	0.00097
0.00107	0.00107	0.00029	0.00114
0.00053	0.00053	−0.00024	−0.00078
0.00080	0.00080	0.00002	0.00087
0.00060	0.00060	−0.00017	0.00068
0.00060	0.00060	−0.00017	0.00068
0.00060	0.00060	−0.00017	0.00068
0.00109	0.00000	0.00000	−0.00000
0.00034	0.00088	0.00007	0.00096
0.00109	0.00000	0.00000	−0.00000
0.00015	0.00069	−0.00013	0.00077
0.00096	−0.00013	−0.00013	−0.00013
0.00014	0.00069	−0.00013	0.00077
0.00082	−0.00027	−0.00027	−0.00027
−0.00000	0.00054	−0.00027	0.00063
0.00058	0.00000	0.00000	0.00000
−0.00043	0.00072	−0.00014	−0.00058
0.00058	0.00000	0.00000	0.00000
−0.00043	0.00072	−0.00014	−0.00058
0.00027	0.00027	0.00027	0.00027
0.00015	0.00015	0.00015	0.00015
0.00000	0.00000	0.00000	0.00000
0.00014	0.00014	0.00014	0.00014
0.00000	0.00000	0.00000	0.00000

TABLE 5. Trace-optimal 3^5-BFF designs of resolution V

N	$\boldsymbol{\lambda}'$	$\mathrm{tr}(V_T)$	$V_{(0000)}^{(00,00)}$	$V_{(0000)}^{(00,10)}$	$V_{(0000)}^{(00,01)}$	$V_{(0000)}^{(00,20)}$	$V_{(0000)}^{(00,02)}$	$V_{(0000)}^{(00,11)}$	$V_{(0000)}^{(10,10)}$
51	100001001000110100000	1.70306	0.03406	0.00021	−0.00245	−0.00319	−0.00301	0.00195	0.04994
52	200001001000110100000	1.68527	0.03085	−0.00091	−0.00183	−0.00152	−0.00233	0.00140	0.04955
53	201001001000110100000	1.67447	0.03082	−0.00094	−0.00184	−0.00166	−0.00233	0.00134	0.04951
54	202100100011000001000	1.67015	0.03081	0.00096	−0.00185	−0.00172	−0.00233	−0.00132	0.04949
55	000100110011000001000	1.56073	0.03492	−0.00089	−0.00187	−0.00532	−0.00271	−0.00144	0.04795
56	010101010000100000010	1.27223	0.02987	0.00236	−0.00744	0.00391	−0.00452	0.00429	0.03094
57	101101010000100000010	1.21793	0.02670	−0.00069	−0.00479	0.00306	−0.00390	0.00254	0.02874
58	111101010000100000010	1.18994	0.02419	−0.00109	−0.00373	0.00169	−0.00231	0.00172	0.02868
59	121001000000100101000	1.16005	0.02160	−0.00285	−0.00318	0.00035	−0.00164	0.00129	0.02864
60	000111010000100000010	1.03823	0.02398	−0.00074	−0.00217	0.00216	−0.00197	0.00136	0.02784
61	100010111001000000010	1.01901	0.02321	0.00154	−0.00229	0.00116	−0.00187	−0.00116	0.02701
62	200010111001000000010	1.01225	0.02293	0.00183	−0.00233	0.00080	−0.00184	−0.00109	0.02671
63	210010111001000000010	1.00795	0.02253	0.00180	−0.00237	0.00059	−0.00153	−0.00101	0.02671
64	211000000110000011000	0.99482	0.01776	0.00080	−0.00027	−0.00051	−0.00082	−0.00097	0.02641
65	000111011000100000010	0.88409	0.02146	−0.00275	−0.00223	−0.00006	−0.00150	0.00107	0.02624
66	100110111001000000010	0.88070	0.02146	0.00277	−0.00224	−0.00009	−0.00150	−0.00106	0.02615
67	101001000000110101000	0.86984	0.02176	−0.00253	−0.00240	−0.00028	−0.00162	0.00126	0.02605
68	201000100011000101000	0.85505	0.02175	0.00255	−0.00241	−0.00037	−0.00162	−0.00124	0.02597
69	211000100010000001100	0.83767	0.01551	0.00048	−0.00083	0.00113	−0.00038	−0.00030	0.02539
70	000110110001001000010	0.78716	0.01554	−0.00005	−0.00082	−0.00101	−0.00042	−0.00002	0.02419

$\boldsymbol{\lambda}' = (\lambda_{500}, \lambda_{050}, \lambda_{005}, \lambda_{410}, \lambda_{401}, \lambda_{140}, \lambda_{041}, \lambda_{104}, \lambda_{014}, \lambda_{320}, \lambda_{302}, \lambda_{230}, \lambda_{032}, \lambda_{203}, \lambda_{023}, \lambda_{311}, \lambda_{131}, \lambda_{113}, \lambda_{221}, \lambda_{212}, \lambda_{122})$.

TABLE 5. (continued-1)

$V^{(10,10)}_{(1000)}$	$V^{(10,01)}_{(0000)}$	$V^{(10,01)}_{(0100)}$	$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$	$V^{(01,01)}_{(0000)}$
−0.00660	−0.00693	0.00220	0.01037	−0.00898	−0.00084	0.00052	−0.00772	−0.00654	0.00423	0.01361
−0.00699	−0.00672	0.00242	0.01095	−0.00840	−0.00061	0.00076	−0.00791	−0.00674	0.00404	0.01349
−0.00704	−0.00674	0.00239	0.01076	−0.00860	−0.00061	0.00075	−0.00798	−0.00681	0.00396	0.01347
−0.00706	0.00675	−0.00238	−0.01068	0.00867	0.00062	−0.00075	−0.00801	−0.00684	0.00393	0.01347
−0.00564	0.00540	−0.00197	−0.00634	0.00786	0.00060	−0.00040	−0.00646	−0.00560	0.00331	0.01251
0.00162	−0.00306	−0.00152	−0.00109	0.00470	−0.00137	−0.00125	−0.00128	0.00027	0.00219	0.01441
−0.00058	−0.00123	0.00032	−0.00260	0.00319	−0.00038	−0.00025	−0.00264	−0.00110	0.00083	0.01201
−0.00064	−0.00106	0.00049	−0.00281	0.00297	−0.00013	0.00000	−0.00277	−0.00123	0.00070	0.01156
−0.00045	0.00109	0.00286	−0.00272	−0.00202	0.00120	0.00100	−0.00259	−0.00128	−0.00012	0.01396
0.00045	−0.00080	0.00113	0.00069	0.00185	0.00017	0.00004	−0.00377	−0.00068	0.00086	0.01017
−0.00039	0.00093	−0.00100	0.00035	−0.00081	−0.00027	−0.00014	−0.00397	−0.00089	0.00066	0.01015
−0.00068	0.00097	−0.00096	0.00071	−0.00044	−0.00031	−0.00018	−0.00405	−0.00096	0.00058	0.01015
−0.00068	0.00097	−0.00096	0.00070	−0.00046	−0.00029	−0.00016	−0.00404	−0.00096	0.00059	0.01014
0.00072	−0.00025	−0.00094	−0.00047	0.00335	−0.00019	−0.00008	−0.00140	−0.00140	−0.00082	0.00864
−0.00115	−0.00084	0.00107	−0.00104	0.00005	0.00056	0.00041	−0.00397	−0.00084	0.00061	0.01005
−0.00124	0.00088	−0.00103	0.00118	0.00009	−0.00054	−0.00039	−0.00402	−0.00089	0.00056	0.01003
−0.00121	−0.00114	0.00075	−0.00117	0.00049	0.00032	0.00019	−0.00403	−0.00067	0.00081	0.00999
−0.00129	0.00116	−0.00072	0.00150	−0.00016	−0.00033	−0.00019	−0.00410	−0.00074	0.00074	0.00998
0.00386	0.00002	−0.00021	−0.00075	−0.00029	−0.00105	−0.00035	−0.00267	0.00034	−0.00252	0.00824
−0.00185	0.00014	0.00014	0.00038	0.00038	0.00006	0.00006	−0.00311	−0.00138	0.00152	0.00803

TABLE 5. (continued-2)

$V_{(0001)}^{(01;01)}$	$V_{(0000)}^{(01;20)}$	$V_{(0010)}^{(01;20)}$	$V_{(0000)}^{(01;02)}$	$V_{(0001)}^{(01;02)}$	$V_{(0001)}^{(01;11)}$	$V_{(0010)}^{(01;11)}$	$V_{(0011)}^{(01;11)}$	$V_{(0000)}^{(20;20)}$	$V_{(1000)}^{(20;20)}$	$V_{(2000)}^{(20;20)}$
−0.00146	−0.00409	0.00303	0.00065	0.00015	0.00181	0.00138	−0.00162	0.06292	−0.00181	−0.00403
−0.00158	−0.00442	0.00271	0.00052	0.00002	0.00192	0.00149	−0.00151	0.06205	−0.00267	−0.00490
−0.00159	−0.00452	0.00261	0.00052	0.00002	0.00188	0.00145	−0.00155	0.06124	−0.00348	−0.00570
−0.00159	−0.00456	0.00257	0.00052	0.00001	−0.00186	−0.00143	0.00156	0.06092	−0.00380	−0.00603
−0.00150	−0.00223	0.00183	0.00031	0.00002	−0.00122	−0.00098	0.00091	0.05410	−0.00162	0.00515
0.00361	−0.00268	−0.00152	0.00237	0.00301	−0.00171	−0.00326	−0.00287	0.04926	0.00412	−0.00977
0.00120	−0.00243	−0.00127	0.00143	0.00208	−0.00026	−0.00181	−0.00142	0.04821	0.00307	−0.01081
0.00076	−0.00185	−0.00069	0.00076	0.00140	0.00008	−0.00146	−0.00108	0.04746	0.00232	−0.01156
0.00339	−0.00439	−0.00370	0.00157	0.00239	−0.00044	−0.00222	−0.00106	0.04956	−0.00044	0.01206
−0.00055	−0.00035	−0.00012	−0.00000	0.00059	0.00056	−0.00068	−0.00037	0.03549	0.00146	−0.00132
−0.00057	−0.00051	−0.00028	0.00001	0.00060	−0.00052	0.00071	0.00040	0.03418	0.00016	−0.00262
−0.00058	−0.00056	−0.00033	0.00002	0.00061	−0.00051	0.00072	0.00041	0.03373	−0.00030	−0.00308
−0.00058	−0.00059	−0.00036	0.00005	0.00064	−0.00051	0.00073	0.00042	0.03361	−0.00042	−0.00320
−0.00039	−0.00119	−0.00015	0.00051	−0.00030	0.00002	0.00002	0.00060	0.04312	−0.00080	−0.00306
−0.00052	−0.00080	0.00008	−0.00013	0.00070	0.00032	−0.00128	−0.00010	0.03016	−0.00107	−0.00104
−0.00054	−0.00087	0.00002	−0.00014	0.00069	−0.00029	0.00131	0.00012	0.02992	−0.00130	−0.00127
−0.00058	−0.00039	0.00036	−0.00016	0.00067	0.00025	−0.00141	−0.00023	0.02930	0.00065	0.00324
−0.00058	−0.00049	0.00026	−0.00016	0.00067	−0.00023	0.00143	0.00025	0.02799	−0.00066	0.00194
−0.00032	0.00045	−0.00094	−0.00047	0.00053	0.00033	0.00057	−0.00013	0.03097	0.00041	−0.00098
−0.00036	−0.00089	0.00085	−0.00017	0.00031	0.00005	0.00005	0.00005	0.03060	−0.00065	−0.00065

TABLE 5. (continued-3)

$V^{(20,02)}_{(0000)}$	$V^{(20,02)}_{(0100)}$	$V^{(20,02)}_{(0200)}$	$V^{(20,11)}_{(0000)}$	$V^{(20,11)}_{(0100)}$	$V^{(20,11)}_{(1000)}$	$V^{(20,11)}_{(1100)}$	$V^{(02,02)}_{(0000)}$	$V^{(02,02)}_{(0001)}$	$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$
−0.01175	0.00355	−0.00197	0.00199	−0.00375	−0.00551	0.00958	0.01502	−0.00325	0.00395	−0.00717
−0.01211	0.00320	−0.00232	0.00228	−0.00346	−0.00522	0.00987	0.01488	−0.00339	0.00380	−0.00705
−0.01213	0.00318	−0.00235	0.00197	−0.00377	−0.00553	0.00956	0.01488	−0.00339	0.00380	−0.00706
−0.01214	0.00317	−0.00236	−0.00184	0.00390	0.00566	−0.00944	0.01487	−0.00339	0.00380	0.00706
−0.01079	0.00388	−0.00228	−0.00619	0.00280	0.00510	−0.00674	0.01480	−0.00342	0.00382	0.00727
−0.00080	−0.00041	−0.00350	0.00412	−0.00051	0.00412	−0.00051	0.01126	−0.00061	0.00333	−0.00234
−0.00104	−0.00066	−0.00375	0.00373	−0.00090	0.00373	−0.00090	0.01135	−0.00053	0.00342	−0.00181
−0.00017	0.00021	−0.00287	0.00329	−0.00134	0.00329	−0.00134	0.01034	−0.00154	0.00241	−0.00130
−0.00491	−0.00051	−0.00306	0.00153	0.00153	0.00083	0.00083	0.01021	−0.00151	0.00298	−0.00250
−0.00083	0.00018	−0.00229	0.00431	0.00060	0.00153	−0.00218	0.01014	−0.00170	0.00228	−0.00097
−0.00070	0.00030	−0.00217	−0.00405	−0.00034	−0.00127	0.00244	0.01013	−0.00171	0.00227	0.00095
−0.00066	0.00035	−0.00212	−0.00395	−0.00025	−0.00118	0.00253	0.01013	−0.00171	0.00227	0.00094
−0.00049	0.00051	−0.00195	−0.00391	−0.00021	−0.00113	0.00257	0.00989	−0.00195	0.00203	0.00088
−0.00254	−0.00040	0.00174	−0.00075	−0.00046	0.00301	−0.00364	0.00488	−0.00001	−0.00026	0.00038
−0.00164	0.00039	−0.00105	0.00050	0.00054	−0.00010	−0.00006	0.00960	−0.00186	0.00250	−0.00222
−0.00167	0.00036	−0.00108	−0.00042	−0.00046	0.00018	0.00014	0.00960	−0.00186	0.00249	0.00223
−0.00163	0.00049	−0.00086	−0.00007	0.00010	0.00049	0.00066	0.00959	−0.00187	0.00249	−0.00235
−0.00162	0.00050	−0.00085	0.00036	0.00019	−0.00020	−0.00037	0.00959	−0.00187	0.00249	0.00234
0.00116	−0.00023	−0.00023	−0.00040	0.00007	−0.00040	0.00145	0.00557	0.00009	−0.00091	−0.00100
−0.00110	0.00064	−0.00110	0.00015	0.00015	0.00015	0.00015	0.00480	−0.00051	0.00074	0.00002

TABLE 5. (continued-4)

$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$	$V^{(11,11)}_{(0111)}$	$V^{(11,11)}_{(1110)}$	$V^{(11,11)}_{(1011)}=V^{(11,11)}_{(1101)}$	$V^{(11,11)}_{(1111)}$
0.00229	0.00283	−0.00391	0.02860	0.00724	−0.00360	−0.00279	0.00002	−0.00026
0.00241	0.00295	−0.00380	0.02850	0.00714	−0.00369	−0.00289	−0.00008	−0.00036
0.00240	0.00294	−0.00381	0.02838	0.00702	−0.00381	−0.00301	−0.00020	−0.00048
−0.00239	−0.00294	0.00381	0.02833	0.00697	−0.00386	−0.00306	−0.00025	−0.00053
−0.00241	−0.00300	0.00352	0.02672	0.00540	−0.00430	−0.00307	−0.00046	0.00064
−0.00105	−0.00002	−0.00337	0.02052	0.00509	0.00162	0.00316	0.00162	−0.00186
−0.00053	0.00050	−0.00284	0.01959	0.00416	0.00069	0.00223	0.00069	−0.00278
−0.00001	0.00102	−0.00232	0.01933	0.00390	0.00042	0.00197	0.00042	−0.00305
0.00020	0.00041	−0.00152	0.01638	0.00118	0.00250	−0.00113	0.00003	−0.00113
0.00037	0.00119	−0.00210	0.01840	0.00420	−0.00043	0.00173	0.00049	−0.00321
−0.00039	−0.00121	0.00208	0.01834	0.00415	−0.00048	0.00168	0.00044	−0.00326
−0.00040	−0.00122	0.00207	0.01832	0.00413	−0.00050	0.00166	0.00042	−0.00328
−0.00046	−0.00128	0.00201	0.01831	0.00411	−0.00052	0.00164	0.00041	−0.00329
0.00028	−0.00088	0.00134	0.01609	0.00220	−0.00175	0.00056	−0.00059	0.00124
0.00049	0.00074	−0.00118	0.01418	0.00085	0.00034	−0.00080	0.00005	−0.00075
−0.00048	−0.00073	0.00119	0.01416	0.00083	0.00032	−0.00082	0.00003	−0.00077
0.00037	0.00065	−0.00126	0.01395	0.00103	0.00012	−0.00051	0.00029	−0.00045
−0.00037	−0.00066	0.00126	0.01389	0.00097	0.00005	−0.00057	0.00023	−0.00051
0.00016	−0.00053	0.00201	0.01443	0.00216	0.00031	0.00301	0.00016	−0.00046
0.00002	0.00002	0.00002	0.01282	0.00067	0.00067	−0.00106	0.00067	−0.00164

TABLE 6. Determinant-optimal 3^5-BFF designs of resolution V

N	λ'	det (V_T)	$V_{(0000)}^{(00,00)}$	$V_{(0000)}^{(00,10)}$	$V_{(0000)}^{(00,01)}$
*51	100001001000110100000	0.19350E−86	0.03406	0.00021	−0.00245
*52	200001001000110100000	0.96752E−87	0.03085	−0.00091	−0.00183
53a	210001001000110100000	0.53825E−87	0.02977	−0.00087	−0.00201
53b	210000011001100100000		0.02977	0.00258	0.00144
53c	201001100000011100000		0.02977	−0.00087	−0.00201
54a	220000011001100100000	0.30709E−87	0.02975	0.00259	0.00146
54b	211001001000110100000		0.02975	−0.00090	−0.00203
54c	202001100000011100000		0.02975	−0.00090	−0.00203
55a	000100110001001100000	0.95051E−88	0.03492	0.00325	0.00049
*55b	000100110011000001000		0.03492	−0.00089	−0.00187
56a	100000111010000000100	0.46404E−90	0.02349	0.01029	0.00146
56b	100011010000001000100		0.03105	0.01016	0.00200
57a	110010000100000011000	0.90107E−91	0.02362	−0.00695	0.00390
57b	110000100001000101000		0.02362	0.00238	−0.00543
58a	111100000010000011000	0.22389E−91	0.02188	−0.00722	0.00027
58b	111010000100000011000		0.02188	−0.00402	0.00348
58c	111001000000100101000		0.02188	−0.00321	−0.00375
*59a	121001000000100101000	0.11849E−91	0.02160	−0.00285	−0.00318
59b	211010000100000011000		0.02160	−0.00335	0.00302
60a	000101110000001000010	0.18214E−92	0.02322	−0.00203	−0.00459
60b	000110110001000000001		0.02322	−0.00588	0.00331
61a	100001010010001000100	0.16994E−93	0.01685	0.00106	−0.00035
61b	100000110010001000100		0.01685	−0.00106	−0.00035
61c	100001010110000000001		0.01685	0.00106	−0.00035
61d	100000110110000000001		0.01685	−0.00106	−0.00035
61e	010100010011000000001		0.01685	0.00106	−0.00035
62a	110000000001100101000	0.32998E−94	0.01901	−0.00084	0.00027
62b	101000000110000011000		0.01901	0.00083	0.00028
63	111000000001100101000	0.88735E−95	0.01844	0.00000	0.00067
64	121000000001100101000	0.44367E−95	0.01776	0.00000	0.00053
65a	000100110010001000100	0.10224E−95	0.01604	0.00078	−0.00026
65b	000100110011000000001		0.01604	0.00000	0.00052
66	100011001000110000100	0.51687E−96	0.01593	0.00042	0.00048
67a	101001000000010100001	0.22224E−96	0.01656	0.00035	−0.00086
67b	110000010001000100001		0.01656	0.00146	0.00025
67c	110000001100000010010		0.01656	0.00111	0.00060
68	111010000000100001100	0.89610E−97	0.01649	−0.00112	0.00033
*69	211000100010000001100	0.46112E−97	0.01551	0.00048	−0.00083
70	000110101001010000001	0.62939E−98	0.01552	−0.00121	0.00040

* This design is also optimal with respect to the trace criterion.
$\lambda'=(\lambda_{500}, \lambda_{050}, \lambda_{005}, \lambda_{410}, \lambda_{401}, \lambda_{140}, \lambda_{041}, \lambda_{104}, \lambda_{014}, \lambda_{320}, \lambda_{302}, \lambda_{230}, \lambda_{032}, \lambda_{203}, \lambda_{023}, \lambda_{311}, \lambda_{131}, \lambda_{113}, \lambda_{221}, \lambda_{212}, \lambda_{122})$. $n_0\mathrm{E}-n_1=n_0 10^{-n_1}$.

TABLE 6. (continued–1)

$V_{(0000)}^{(00,20)}$	$V_{(0000)}^{(00,02)}$	$V_{(0000)}^{(00,11)}$	$V_{(0000)}^{(10,10)}$	$V_{(1000)}^{(10,10)}$	$V_{(0000)}^{(10,01)}$	$V_{(0100)}^{(10,01)}$
−0.00319	−0.00301	0.00195	0.04994	−0.00660	−0.00693	0.00220
−0.00152	−0.00233	0.00140	0.04955	−0.00699	−0.00672	0.00242
−0.00151	−0.00186	0.00126	0.04955	−0.00699	−0.00671	0.00242
−0.00644	−0.00021	−0.00039	0.05273	−0.00900	−0.00565	0.00176
−0.00151	−0.00186	0.00126	0.04955	−0.00699	−0.00671	0.00242
−0.00640	−0.00028	−0.00038	0.05273	−0.00900	−0.00566	0.00175
−0.00166	−0.00186	0.00120	0.04950	−0.00704	−0.00674	0.00240
−0.00166	−0.00186	0.00120	0.04950	−0.00704	−0.00674	0.00240
−0.00958	−0.00129	0.00002	0.04824	−0.00775	−0.00530	0.00127
−0.00532	−0.00271	−0.00144	0.04795	−0.00564	0.00540	−0.00197
−0.00347	−0.00204	−0.00344	0.04900	0.01968	0.00514	0.00360
−0.01659	0.00234	−0.00379	0.03858	0.00463	0.00064	0.00064
−0.01043	0.00008	0.00057	0.04346	0.00974	−0.00565	−0.00588
−0.00158	−0.00287	−0.00238	0.02947	0.00038	−0.00099	−0.00276
−0.00218	−0.00100	0.00254	0.04387	0.01548	−0.00315	−0.00160
−0.00660	0.00048	0.00106	0.03851	0.00479	−0.00493	−0.00516
0.00101	−0.00206	0.00148	0.02908	−0.00001	0.00179	0.00356
0.00035	−0.00164	0.00129	0.02864	−0.00045	0.00109	0.00286
−0.00554	0.00032	0.00067	0.03695	0.00323	−0.00386	−0.00409
0.00010	−0.00251	0.00240	0.02803	−0.00118	−0.00108	0.00036
−0.00923	0.00060	0.00071	0.03812	0.00460	−0.00229	−0.00229
−0.00087	−0.00003	−0.00048	0.03009	0.00231	−0.00106	−0.00106
0.00087	−0.00003	0.00048	0.03009	0.00231	0.00106	0.00106
0.00087	−0.00003	−0.00048	0.03009	0.00231	−0.00106	−0.00106
−0.00087	−0.00003	0.00048	0.03009	0.00231	0.00106	0.00106
0.00087	−0.00003	−0.00048	0.03009	0.00231	−0.00106	−0.00106
−0.00609	0.00052	−0.00032	0.02691	−0.00087	0.00059	0.00059
0.00013	−0.00155	−0.00175	0.02653	0.00084	−0.00046	−0.00115
−0.00405	0.00063	0.00000	0.02568	−0.00210	0.00000	0.00000
−0.00343	0.00015	0.00000	0.02568	−0.00210	0.00000	0.00000
−0.00090	−0.00010	0.00030	0.02621	−0.00099	0.00022	0.00080
0.00000	−0.00040	0.00000	0.02687	0.00141	0.00000	0.00000
−0.00043	−0.00042	0.00002	0.02524	−0.00023	0.00016	0.00016
0.00220	−0.00030	0.00043	0.02681	0.00528	−0.00049	−0.00025
−0.00078	0.00069	−0.00056	0.02656	0.00156	−0.00057	−0.00149
0.00052	0.00026	−0.00099	0.02510	0.00080	−0.00008	−0.00124
−0.00144	0.00076	0.00059	0.02494	−0.00006	0.00021	0.00114
0.00113	−0.00038	−0.00030	0.02539	0.00386	0.00002	−0.00021
−0.00137	−0.00039	0.00010	0.02425	−0.00114	0.00006	−0.00016

TABLE 6. (continued-2)

$V^{(10,20)}_{(0000)}$	$V^{(10,20)}_{(1000)}$	$V^{(10,02)}_{(0000)}$	$V^{(10,02)}_{(0100)}$	$V^{(10,11)}_{(0100)}$	$V^{(10,11)}_{(1000)}$	$V^{(10,11)}_{(1100)}$
0.01037	−0.00898	−0.00084	0.00052	−0.00772	−0.00654	0.00423
0.01095	−0.00840	−0.00061	0.00076	−0.00791	−0.00674	0.00404
0.01095	−0.00840	−0.00063	0.00074	−0.00790	−0.00673	0.00404
0.00955	−0.00804	−0.00037	0.00015	−0.00641	−0.00764	0.00440
0.01095	−0.00840	−0.00063	0.00074	−0.00790	−0.00673	0.00404
0.00954	−0.00805	−0.00034	0.00017	−0.00641	−0.00764	0.00439
0.01076	−0.00860	−0.00063	0.00074	−0.00798	−0.00680	0.00397
0.01076	−0.00860	−0.00063	0.00074	−0.00798	−0.00680	0.00397
0.00690	−0.00582	−0.00090	−0.00012	−0.00413	−0.00493	0.00330
−0.00634	0.00786	0.00060	−0.00040	−0.00646	−0.00560	0.00331
−0.00521	−0.01215	−0.00273	−0.00402	−0.00834	−0.00680	−0.00603
−0.00926	−0.01389	0.00129	0.00231	−0.00154	−0.00463	−0.00309
0.01311	0.01705	−0.00107	−0.00081	−0.00053	−0.00385	−0.00277
0.00478	0.00408	−0.00134	−0.00113	−0.00239	−0.00108	0.00008
0.00203	0.00689	0.00035	0.00032	−0.00875	−0.00674	−0.00759
0.00666	0.01060	−0.00174	−0.00149	−0.00136	−0.00468	−0.00360
−0.00354	−0.00285	0.00173	0.00152	−0.00282	−0.00151	−0.00035
−0.00272	−0.00202	0.00120	0.00100	−0.00259	−0.00128	−0.00012
0.00418	0.00811	−0.00138	−0.00113	−0.00045	−0.00377	−0.00269
−0.00298	0.00205	0.00008	−0.00000	−0.00230	−0.00038	0.00081
0.00787	0.01218	−0.00082	−0.00082	0.00135	−0.00248	−0.00104
−0.00000	−0.00694	−0.00096	0.00135	−0.00231	−0.00231	−0.00000
−0.00000	−0.00694	0.00096	−0.00135	−0.00231	−0.00231	−0.00000
0.00000	0.00694	−0.00096	0.00135	−0.00231	−0.00231	−0.00000
0.00000	0.00694	0.00096	−0.00135	−0.00231	−0.00231	−0.00000
0.00000	0.00694	−0.00096	0.00135	−0.00231	−0.00231	−0.00000
0.00301	0.00301	0.00017	0.00017	−0.00087	−0.00087	0.00145
−0.00010	0.00372	0.00002	0.00013	−0.00147	−0.00147	−0.00089
0.00000	0.00000	0.00000	0.00000	−0.00134	−0.00134	0.00097
0.00000	0.00000	0.00000	0.00000	−0.00134	−0.00134	0.00097
0.00197	−0.00295	0.00064	−0.00100	−0.00143	−0.00085	0.00021
−0.00042	0.00536	0.00000	0.00000	−0.00054	−0.00169	0.00024
0.00209	−0.00370	0.00006	0.00006	−0.00061	−0.00177	0.00016
0.00125	0.00079	0.00164	0.00095	−0.00330	−0.00029	−0.00314
0.00570	0.00431	−0.00000	−0.00078	0.00208	0.00023	−0.00147
0.00182	−0.00049	−0.00077	−0.00039	0.00077	−0.00039	−0.00116
−0.00251	−0.00112	−0.00033	0.00044	0.00195	0.00010	−0.00160
−0.00075	−0.00029	−0.00105	−0.00035	−0.00267	0.00034	−0.00252
−0.00209	0.00301	0.00026	0.00015	−0.00029	−0.00116	0.00032

TABLE 6. (continued–3)

$V^{(01,01)}_{(0000)}$	$V^{(01,01)}_{(0001)}$	$V^{(01,20)}_{(0000)}$	$V^{(01,20)}_{(0010)}$	$V^{(01,02)}_{(0000)}$	$V^{(01,02)}_{(0001)}$	$V^{(01,11)}_{(0001)}$
0.01361	−0.00146	−0.00409	0.00303	0.00065	0.00015	0.00181
0.01349	−0.00158	−0.00442	0.00271	0.00052	0.00002	0.00192
0.01346	−0.00161	−0.00441	0.00271	0.00060	0.00010	0.00189
0.01240	−0.00094	−0.00502	0.00165	0.00060	−0.00002	0.00205
0.01346	−0.00161	−0.00441	0.00271	0.00060	0.00010	0.00189
0.01237	−0.00096	−0.00506	0.00161	0.00066	0.00004	0.00205
0.01344	−0.00162	−0.00451	0.00261	0.00060	0.00009	0.00186
0.01344	−0.00162	−0.00451	0.00261	0.00060	0.00009	0.00186
0.01241	−0.00080	−0.00388	0.00208	0.00076	0.00010	0.00170
0.01251	−0.00150	−0.00223	0.00183	0.00031	0.00002	−0.00122
0.01102	0.00021	−0.00174	−0.00174	−0.00050	−0.00102	−0.00172
0.00970	0.00044	−0.00193	−0.00193	0.00101	−0.00054	−0.00006
0.01204	0.00301	−0.00732	−0.00662	0.00151	0.00074	0.00150
0.01670	0.00613	−0.00320	−0.00251	0.00318	0.00401	0.00168
0.01015	−0.00066	−0.00084	−0.00084	0.00030	−0.00021	0.00118
0.01193	0.00290	−0.00638	−0.00569	0.00160	0.00083	0.00162
0.01508	0.00451	−0.00570	−0.00501	0.00240	0.00322	−0.00081
0.01396	0.00339	−0.00439	−0.00370	0.00157	0.00239	−0.00044
0.01119	0.00217	−0.00468	−0.00398	0.00136	0.00059	0.00100
0.01311	0.00241	−0.00147	0.00045	0.00139	0.00224	−0.00102
0.00975	0.00049	−0.00359	−0.00359	0.00120	−0.00034	0.00028
0.00932	0.00006	0.00087	0.00087	0.00061	−0.00093	0.00048
0.00932	0.00006	−0.00087	−0.00087	0.00061	−0.00093	−0.00048
0.00932	0.00006	−0.00087	−0.00087	0.00061	−0.00093	0.00048
0.00932	0.00006	0.00087	0.00087	0.00061	−0.00093	−0.00048
0.00932	0.00006	−0.00087	−0.00087	0.00061	−0.00093	0.00048
0.00919	0.00086	−0.00046	0.00093	0.00053	0.00099	0.00023
0.00932	0.00029	−0.00159	−0.00055	−0.00025	−0.00106	−0.00022
0.00891	0.00058	−0.00189	−0.00050	0.00045	0.00092	0.00000
0.00889	0.00055	−0.00177	−0.00038	0.00036	0.00082	0.00000
0.00888	0.00020	−0.00170	0.00032	0.00023	−0.00064	−0.00021
0.00866	−0.00060	0.00000	0.00000	0.00082	−0.00073	0.00000
0.00865	−0.00061	−0.00017	−0.00017	0.00081	−0.00073	0.00001
0.00850	−0.00006	0.00062	−0.00076	−0.00079	0.00022	0.00009
0.00858	0.00118	−0.00257	−0.00326	0.00011	0.00050	−0.00033
0.00907	0.00143	0.00149	−0.00175	0.00082	0.00121	0.00046
0.00851	0.00110	−0.00187	−0.00256	0.00004	0.00043	0.00030
0.00824	−0.00032	0.00045	−0.00094	−0.00047	0.00053	0.00033
0.00812	−0.00048	−0.00146	0.00061	0.00063	−0.00058	−0.00004

TABLE 6. (continued-4)

$V^{(01,11)}_{(0010)}$	$V^{(01,11)}_{(0011)}$	$V^{(20,20)}_{(0000)}$	$V^{(20,20)}_{(1000)}$	$V^{(20,20)}_{(2000)}$	$V^{(20,02)}_{(0000)}$	$V^{(20,02)}_{(0100)}$
0.00138	−0.00162	0.06292	−0.00181	−0.00403	−0.01175	0.00355
0.00149	−0.00151	0.06205	−0.00267	−0.00490	−0.01211	0.00320
0.00146	−0.00153	0.06205	−0.00267	−0.00490	−0.01211	0.00320
0.00242	−0.00165	0.15002	−0.03387	0.03224	−0.01460	0.00423
0.00146	−0.00153	0.06205	−0.00267	−0.00490	−0.01211	0.00320
0.00242	−0.00166	0.14997	−0.03392	0.03219	−0.01450	0.00432
0.00142	−0.00157	0.06124	−0.00348	−0.00570	−0.01214	0.00317
0.00142	−0.00157	0.06124	−0.00348	−0.00570	−0.01214	0.00317
0.00201	−0.00151	0.14676	−0.03299	0.03725	−0.01408	0.00497
−0.00098	0.00091	0.05410	−0.00162	0.00515	−0.01079	0.00388
−0.00018	−0.00172	0.06250	−0.00000	−0.00000	0.00608	0.00029
−0.00006	−0.00006	0.10069	0.00347	0.03125	−0.00386	−0.00309
0.00173	0.00220	0.09710	0.00196	0.03182	−0.00276	−0.00122
0.00345	0.00230	0.05497	0.00497	0.01747	−0.00468	−0.00029
−0.00036	0.00118	0.05421	−0.00360	0.00109	0.00150	−0.00134
0.00185	0.00232	0.08869	−0.00645	0.02341	−0.00364	−0.00210
−0.00258	−0.00143	0.05111	0.00111	0.01361	−0.00589	−0.00149
−0.00222	−0.00106	0.04956	−0.00044	0.01206	−0.00491	−0.00051
0.00123	0.00169	0.08476	−0.01038	0.01948	−0.00307	−0.00153
−0.00294	−0.00182	0.04091	0.00117	−0.00731	−0.00142	0.00045
0.00028	0.00028	0.09433	−0.00265	0.02536	−0.00128	−0.00128
0.00048	0.00048	0.05208	0.00347	−0.00347	−0.00174	−0.00058
−0.00048	−0.00048	0.05208	0.00347	−0.00347	0.00174	0.00058
0.00048	0.00048	0.05208	0.00347	−0.00347	0.00174	0.00058
−0.00048	−0.00048	0.05208	0.00347	−0.00347	−0.00174	−0.00058
0.00048	0.00048	0.05208	0.00347	−0.00347	0.00174	0.00058
0.00023	0.00023	0.05497	0.00497	0.01747	−0.00350	0.00066
−0.00022	0.00036	0.04451	0.00059	−0.00167	−0.00224	−0.00010
0.00000	0.00000	0.04762	−0.00238	0.01012	−0.00391	0.00025
0.00000	0.00000	0.04707	−0.00293	0.00957	−0.00348	0.00069
0.00037	−0.00088	0.04063	−0.00089	−0.00075	−0.00116	0.00004
0.00000	0.00000	0.05047	0.00244	−0.00392	0.00000	0.00000
0.00001	0.00001	0.04878	0.00075	−0.00562	−0.00006	−0.00006
−0.00014	0.00055	0.03222	0.00167	0.00028	0.00100	−0.00039
0.00106	0.00021	0.04959	0.00740	0.01522	−0.00062	−0.00149
0.00208	0.00100	0.05753	0.00406	−0.02024	−0.00093	−0.00069
−0.00109	−0.00024	0.04328	0.00110	0.00891	0.00004	−0.00083
0.00057	−0.00013	0.03097	0.00041	−0.00098	0.00116	−0.00023
−0.00091	0.00043	0.03982	−0.00168	−0.00152	−0.00111	−0.00008

TABLE 6. (continued-5)

$V_{(0200)}^{(20,02)}$	$V_{(0000)}^{(20,11)}$	$V_{(0100)}^{(20,11)}$	$V_{(1000)}^{(20,11)}$	$V_{(1100)}^{(20,11)}$	$V_{(0000)}^{(02,02)}$	$V_{(0001)}^{(02,02)}$
−0.00197	0.00199	−0.00375	−0.00551	0.00958	0.01502	−0.00325
−0.00232	0.00228	−0.00346	−0.00522	0.00987	0.01488	−0.00339
−0.00233	0.00228	−0.00346	−0.00522	0.00987	0.01467	−0.00360
−0.00472	0.01321	−0.00790	−0.00809	0.01247	0.00656	−0.00082
−0.00233	0.00228	−0.00346	−0.00522	0.00987	0.01467	−0.00360
−0.00463	0.01320	−0.00791	−0.00809	0.01246	0.00639	−0.00098
−0.00235	0.00197	−0.00377	−0.00553	0.00956	0.01467	−0.00360
−0.00235	0.00197	−0.00377	−0.00553	0.00956	0.01467	−0.00360
−0.00376	0.01669	−0.00765	−0.00746	0.00987	0.00670	−0.00066
−0.00228	−0.00619	0.00280	0.00510	−0.00674	0.01480	−0.00342
0.00145	0.00651	−0.00043	−0.00391	0.00998	0.00748	0.00062
−0.00231	0.01100	0.00174	−0.00058	0.01100	0.00742	0.00107
0.00033	−0.01052	0.00013	0.00314	−0.00705	0.00525	0.00045
−0.00283	−0.00061	−0.00061	0.00008	0.00008	0.01121	−0.00051
0.00277	−0.00964	0.00095	0.00338	−0.00687	0.00556	−0.00046
−0.00055	−0.01160	−0.00095	0.00206	−0.00813	0.00516	0.00036
−0.00404	0.00196	0.00196	0.00126	0.00126	0.01084	−0.00088
−0.00306	0.00153	0.00153	0.00083	0.00083	0.01021	−0.00151
0.00002	−0.01016	0.00049	0.00350	−0.00668	0.00508	0.00027
−0.00115	−0.00146	−0.00087	0.00110	0.00169	0.01017	−0.00130
−0.00128	−0.00765	0.00168	0.00456	−0.00694	0.00414	0.00028
0.00058	−0.00174	0.00058	−0.00289	0.00637	0.00682	0.00064
−0.00058	−0.00174	0.00058	−0.00289	0.00637	0.00682	0.00064
−0.00058	0.00174	−0.00058	0.00289	−0.00637	0.00682	0.00064
0.00058	0.00174	−0.00058	0.00289	−0.00637	0.00682	0.00064
−0.00058	0.00174	−0.00058	0.00289	−0.00637	0.00682	0.00064
−0.00212	0.00116	0.00116	0.00116	0.00116	0.00542	−0.00013
0.00204	−0.00131	−0.00102	0.00245	−0.00421	0.00574	0.00086
−0.00252	0.00000	0.00000	0.00000	0.00000	0.00540	−0.00016
−0.00209	0.00000	0.00000	0.00000	0.00000	0.00506	−0.00049
0.00125	0.00172	−0.00035	−0.00180	0.00307	0.00587	0.00049
0.00000	0.00167	−0.00046	0.00359	−0.00547	0.00400	0.00014
−0.00006	−0.00159	0.00053	−0.00352	0.00555	0.00400	0.00014
−0.00039	0.00087	0.00041	0.00087	−0.00098	0.00596	0.00048
−0.00236	0.00134	−0.00023	0.00012	−0.00144	0.00510	0.00057
0.00093	0.00047	−0.00254	0.00116	−0.00046	0.00442	0.00041
−0.00169	−0.00108	0.00048	0.00013	0.00169	0.00503	0.00050
−0.00023	−0.00040	0.00007	−0.00040	0.00145	0.00557	0.00009
0.00095	−0.00123	0.00089	0.00219	−0.00264	0.00382	0.00012

TABLE 6. (continued-6)

$V^{(02,02)}_{(0002)}$	$V^{(02,11)}_{(0000)}$	$V^{(02,11)}_{(0001)}$	$V^{(02,11)}_{(0010)}$	$V^{(02,11)}_{(0011)}$	$V^{(11,11)}_{(0110)}$	$V^{(11,11)}_{(1001)}$
0.00395	−0.00717	0.00229	0.00283	−0.00391	0.02860	0.00724
0.00380	−0.00705	0.00241	0.00295	−0.00380	0.02850	0.00714
0.00360	−0.00699	0.00247	0.00301	−0.00373	0.02848	0.00712
0.00107	−0.00168	0.00079	0.00089	−0.00127	0.02599	0.00463
0.00360	−0.00699	0.00247	0.00301	−0.00373	0.02848	0.00712
0.00090	−0.00167	0.00080	0.00090	−0.00126	0.02599	0.00463
0.00360	−0.00699	0.00246	0.00301	−0.00374	0.02836	0.00700
0.00360	−0.00699	0.00246	0.00301	−0.00374	0.02836	0.00700
0.00124	−0.00158	0.00072	0.00084	−0.00149	0.02343	0.00210
0.00382	0.00727	−0.00241	−0.00300	0.00352	0.02672	0.00540
−0.00084	0.00172	0.00005	0.00018	0.00313	0.02353	0.00810
−0.00219	0.00064	0.00039	−0.00141	−0.00167	0.01861	0.00318
0.00028	0.00117	0.00091	−0.00011	0.00194	0.01890	0.00370
0.00398	0.00319	0.00049	0.00029	0.00221	0.01697	0.00177
0.00045	−0.00171	0.00071	−0.00007	0.00004	0.02389	0.00869
0.00018	0.00106	0.00080	−0.00023	0.00183	0.01876	0.00356
0.00360	−0.00277	−0.00007	0.00014	−0.00179	0.01650	0.00130
0.00298	−0.00250	0.00020	0.00041	−0.00152	0.01638	0.00118
0.00010	0.00085	0.00059	−0.00044	0.00162	0.01822	0.00302
0.00305	−0.00304	−0.00032	−0.00003	−0.00194	0.01607	0.00194
−0.00049	0.00010	0.00010	0.00010	0.00010	0.01620	0.00208
−0.00244	0.00096	0.00058	−0.00058	−0.00096	0.01591	0.00203
−0.00244	−0.00096	−0.00058	0.00058	0.00096	0.01591	0.00203
−0.00244	0.00096	0.00058	−0.00058	−0.00096	0.01591	0.00203
−0.00244	−0.00096	−0.00058	0.00058	0.00096	0.01591	0.00203
−0.00244	0.00096	0.00058	−0.00058	−0.00096	0.01591	0.00203
0.00125	0.00006	0.00006	0.00006	0.00006	0.01534	0.00145
0.00061	0.00073	0.00063	−0.00053	0.00169	0.01660	0.00271
0.00123	0.00000	0.00000	0.00000	0.00000	0.01515	0.00126
0.00090	0.00000	0.00000	0.00000	0.00000	0.01515	0.00126
−0.00181	0.00035	−0.00072	0.00111	0.00003	0.01370	0.00039
−0.00063	0.00000	0.00000	0.00000	0.00000	0.01486	0.00155
−0.00063	0.00000	0.00000	0.00000	0.00000	0.01486	0.00155
−0.00053	0.00049	−0.00067	0.00003	−0.00252	0.01515	0.00288
−0.00088	0.00173	0.00067	−0.00030	−0.00136	0.01353	0.00126
0.00087	0.00123	0.00069	0.00031	0.00116	0.01323	0.00096
−0.00095	−0.00175	−0.00069	0.00027	0.00133	0.01352	0.00125
−0.00091	−0.00100	0.00016	−0.00053	0.00201	0.01443	0.00216
−0.00049	−0.00050	0.00017	−0.00027	0.00040	0.01272	0.00057

TABLE 6. (continued-7)

$V_{(0111)}^{(11,11)}$	$V_{(1110)}^{(11,11)}$	$V_{(1011)}^{(11,11)}=V_{(1101)}^{(11,11)}$	$V_{(1111)}^{(11,11)}$
−0.00360	−0.00279	0.00002	−0.00026
−0.00369	−0.00289	−0.00008	−0.00036
−0.00371	−0.00291	−0.00010	−0.00038
−0.00123	−0.00333	0.00093	−0.00277
−0.00371	−0.00291	−0.00010	−0.00038
−0.00123	−0.00333	0.00093	−0.00277
−0.00383	−0.00303	−0.00022	−0.00050
−0.00383	−0.00303	−0.00022	−0.00050
−0.00128	−0.00392	0.00063	−0.00085
−0.00430	−0.00307	−0.00046	0.00064
−0.00116	0.00270	−0.00000	0.00231
0.00087	0.00087	0.00010	0.00048
−0.00024	0.00092	−0.00032	0.00261
0.00308	−0.00054	0.00061	−0.00054
−0.00028	0.00219	0.00030	0.00580
−0.00038	0.00078	−0.00046	0.00248
0.00261	−0.00101	0.00015	−0.00101
0.00250	−0.00113	0.00003	−0.00113
−0.00091	0.00025	−0.00099	0.00194
0.00219	−0.00008	0.00094	−0.00007
−0.00152	0.00016	−0.00080	−0.00020
−0.00106	0.00125	0.00010	−0.00068
−0.00106	0.00125	0.00010	−0.00068
−0.00106	0.00125	0.00010	−0.00068
−0.00106	0.00125	0.00010	−0.00068
−0.00106	0.00125	0.00010	−0.00068
0.00299	−0.00164	0.00068	−0.00241
−0.00124	0.00107	−0.00009	0.00175
0.00281	−0.00182	0.00049	−0.00259
0.00281	−0.00182	0.00049	−0.00259
−0.00056	−0.00018	−0.00008	0.00061
−0.00205	0.00123	−0.00012	−0.00064
−0.00205	0.00123	−0.00012	−0.00064
0.00103	0.00373	0.00087	0.00026
0.00236	0.00020	−0.00022	−0.00171
0.00235	0.00181	0.00057	0.00158
0.00235	0.00019	−0.00023	−0.00172
0.00031	0.00301	0.00016	−0.00046
−0.00144	−0.00057	−0.00013	0.00032

TABLE 7. Trace-optimal 3^6-BTO designs

N	$\boldsymbol{\lambda}^{*\prime}$	$\mathrm{tr}(V_T)$	$V^{*(00,00)}_{(0000)}$	$V^{*(00,10)}_{(0000)}$	$V^{*(00,01)}_{(0000)}$	$V^{*(00,20)}_{(0000)}$	$V^{*(00,11)}_{(0000)}$	$V^{*(00,30)}_{(0000)}$
78	1000011000100000100010000000	3.14404	0.01891	−0.00430	−0.00118	0.00092	0.00072	0.00050
79	1010011000100000100010000000	3.04537	0.01875	−0.00416	−0.00124	0.00019	0.00061	−0.00012
80	2010011000100000100010000000	3.03471	0.01847	−0.00378	−0.00128	−0.00011	0.00062	0.00006
81	2020011000100000100010000000	3.02551	0.01845	−0.00376	−0.00129	−0.00019	0.00061	−0.00000
82	2120011000100000100010000000	3.02414	0.01742	−0.00368	−0.00107	−0.00019	0.00062	−0.00003
83	0001011000100000100010000000	2.74266	0.01733	−0.00231	−0.00130	−0.00012	0.00059	0.00121
84	1000111000000100100010000000	2.66620	0.02236	−0.00413	−0.00085	0.00104	0.00059	0.00317
85	1010111000000100100010000000	2.58420	0.02095	−0.00364	−0.00117	−0.00062	0.00019	0.00136
86	2010111000000100100010000000	2.57677	0.02092	−0.00371	−0.00117	−0.00056	0.00020	0.00129
87	1000010000001100101000000000	2.34797	0.01585	0.00242	−0.00060	0.00269	0.00090	0.00160
88	1010010000001100101000000000	2.10582	0.01473	−0.00041	−0.00123	−0.00042	0.00051	−0.00025
89	2010010000001100101000000000	2.09693	0.01471	−0.00043	−0.00122	−0.00033	0.00050	−0.00033
90	2020010000001100101000000000	2.08793	0.01467	−0.00052	−0.00124	−0.00045	0.00049	−0.00039
91	2120010000001100101000000000	2.08649	0.01389	−0.00039	−0.00104	−0.00043	0.00043	−0.00034
92	0001001000110000100010000000	1.96316	0.01491	0.00053	−0.00141	−0.00155	−0.00058	−0.00055
93	1000101000110000100010000000	1.92325	0.01468	0.00052	−0.00144	−0.00188	−0.00050	−0.00080
94	1010101000110000100010000000	1.85174	0.01418	0.00083	−0.00130	−0.00088	−0.00040	0.00019
95	2010101000110000100010000000	1.84609	0.01417	0.00080	−0.00130	−0.00085	−0.00040	0.00016
96	2020101000110000100010000000	1.83830	0.01411	0.00083	−0.00128	−0.00072	−0.00039	0.00026
97	2120101000110000100010000000	1.83686	0.01334	0.00069	−0.00107	−0.00070	−0.00033	0.00022
98	0001101000110000100010000000	1.76105	0.01470	0.00057	−0.00146	−0.00196	−0.00053	−0.00087
99	0001011100100000000000000100	1.64431	0.01629	−0.00061	−0.00128	0.00138	0.00014	0.00220
100	1001011100100000000000000100	1.63439	0.01615	−0.00037	−0.00126	0.00150	0.00005	0.00204

$\boldsymbol{\lambda}^{*\prime}=(\lambda^*_{600}, \lambda^*_{060}, \lambda^*_{006}, \lambda^*_{510}, \lambda^*_{501}, \lambda^*_{150}, \lambda^*_{051}, \lambda^*_{105}, \lambda^*_{015}, \lambda^*_{420}, \lambda^*_{402}, \lambda^*_{240}, \lambda^*_{042}, \lambda^*_{204}, \lambda^*_{024}, \lambda^*_{330}, \lambda^*_{303}, \lambda^*_{033}, \lambda^*_{411}, \lambda^*_{141}, \lambda^*_{114}, \lambda^*_{321}, \lambda^*_{312}, \lambda^*_{231}, \lambda^*_{132}, \lambda^*_{213}, \lambda^*_{123}, \lambda^*_{222})$.

TABLE 7. (continued–1)

$V^{*(10,10)}_{(0000)}$	$V^{*(10,10)}_{(1000)}$	$V^{*(10,01)}_{(0000)}$	$V^{*(10,01)}_{(0100)}$	$V^{*(10,20)}_{(0000)}$	$V^{*(10,20)}_{(1000)}$	$V^{*(10,11)}_{(0100)}$	$V^{*(10,11)}_{(1000)}$	$V^{*(10,11)}_{(1100)}$	$V^{*(10,30)}_{(0000)}$	$V^{*(10,30)}_{(1000)}$
0.05105	−0.00188	−0.00189	0.00013	−0.00532	0.00044	−0.00930	−0.00072	0.00049	−0.00139	0.00134
0.05093	−0.00200	−0.00183	0.00019	−0.00469	0.00106	−0.00921	−0.00063	0.00058	−0.00086	0.00187
0.05042	−0.00251	−0.00178	0.00024	−0.00429	0.00147	−0.00923	−0.00064	0.00057	−0.00111	0.00162
0.05040	−0.00253	−0.00177	0.00025	−0.00421	0.00155	−0.00922	−0.00063	0.00058	−0.00105	0.00168
0.05039	−0.00253	−0.00179	0.00023	−0.00421	0.00155	−0.00922	−0.00063	0.00058	−0.00104	0.00168
0.04847	−0.00432	−0.00002	−0.00006	−0.00377	0.00167	−0.00911	−0.00128	0.00091	−0.00228	0.00092
0.05581	−0.00350	−0.00155	0.00014	−0.00632	0.00154	−0.00896	−0.00118	0.00061	−0.00229	0.00131
0.05564	−0.00368	−0.00144	0.00025	−0.00573	0.00212	−0.00882	−0.00104	0.00075	−0.00166	0.00194
0.05546	−0.00385	−0.00144	0.00026	−0.00557	0.00229	−0.00881	−0.00103	0.00076	−0.00184	0.00176
0.05097	0.00203	−0.01138	0.00473	0.01118	0.00697	−0.00684	−0.00506	0.00399	0.00263	0.00723
0.04377	−0.00517	−0.01298	0.00313	0.00328	−0.00092	−0.00782	−0.00604	0.00301	−0.00209	0.00251
0.04377	−0.00517	−0.01297	0.00314	0.00333	−0.00088	−0.00783	−0.00604	0.00301	−0.00213	0.00247
0.04351	−0.00543	−0.01303	0.00308	0.00303	−0.00118	−0.00786	−0.00608	0.00297	−0.00228	0.00232
0.04348	−0.00546	−0.01307	0.00304	0.00302	−0.00118	−0.00785	−0.00607	0.00298	−0.00229	0.00231
0.03708	−0.00460	0.00826	−0.00193	−0.00361	0.00275	−0.00606	−0.00287	0.00182	0.00036	0.00052
0.04360	−0.00529	0.01318	−0.00300	−0.00262	0.00205	−0.00778	−0.00600	0.00296	−0.00118	0.00292
0.04341	−0.00548	0.01310	−0.00309	−0.00324	0.00143	−0.00784	−0.00607	0.00290	−0.00179	0.00231
0.04331	−0.00558	0.01312	−0.00306	−0.00312	0.00154	−0.00785	−0.00607	0.00289	−0.00192	0.00218
0.04330	−0.00559	0.01312	−0.00307	−0.00318	0.00148	−0.00785	−0.00608	0.00289	−0.00196	0.00213
0.04327	−0.00562	0.01315	−0.00303	−0.00318	0.00149	−0.00784	−0.00607	0.00290	−0.00197	0.00213
0.03601	−0.00440	0.00684	−0.00164	−0.00210	0.00212	−0.00607	−0.00252	0.00172	−0.00053	0.00155
0.04134	0.00201	0.00383	−0.00085	−0.00442	0.00536	−0.00852	−0.00470	0.00102	0.00441	−0.00456
0.04093	0.00160	0.00379	−0.00088	−0.00461	0.00516	−0.00836	−0.00454	0.00118	0.00470	−0.00428

TABLE 7. (continued-2)

$V^{*(01,01)}_{(0000)}$	$V^{*(01,01)}_{(0001)}$	$V^{*(01,20)}_{(0000)}$	$V^{*(01,20)}_{(0010)}$	$V^{*(01,11)}_{(0001)}$	$V^{*(01,11)}_{(0010)}$	$V^{*(01,11)}_{(0011)}$	$V^{*(01,30)}_{(0000)}$	$V^{*(01,30)}_{(0010)}$	$V^{*(20,20)}_{(0000)}$	$V^{*(20,20)}_{(1000)}$
0.04683	−0.00873	−0.00067	0.00084	−0.00006	−0.01521	0.00398	−0.00061	0.00090	0.03490	0.00299
0.04681	−0.00875	−0.00095	0.00056	−0.00010	−0.01525	0.00394	−0.00085	0.00067	0.03162	−0.00029
0.04680	−0.00876	−0.00100	0.00052	−0.00010	−0.01525	0.00394	−0.00082	0.00069	0.03129	−0.00061
0.04680	−0.00876	−0.00102	0.00049	−0.00010	−0.01526	0.00394	−0.00084	0.00067	0.03096	−0.00094
0.04675	−0.00881	−0.00102	0.00049	−0.00010	−0.01526	0.00394	−0.00084	0.00068	0.03096	−0.00094
0.02042	−0.00345	−0.00401	0.00238	0.00030	−0.00329	0.00088	0.00306	−0.00275	0.03323	0.00207
0.04651	−0.00863	−0.00068	0.00093	0.00024	−0.01516	0.00385	−0.00006	0.00109	0.02848	0.00012
0.04644	−0.00871	−0.00106	0.00055	0.00016	−0.01525	0.00376	−0.00047	0.00068	0.02652	−0.00184
0.04644	−0.00871	−0.00106	0.00055	0.00015	−0.01525	0.00376	−0.00046	0.00068	0.02636	−0.00200
0.01851	−0.00259	−0.00078	0.00322	0.00330	0.00434	−0.00179	0.00132	0.00102	0.03425	0.00975
0.01815	−0.00295	−0.00254	0.00146	0.00308	0.00412	−0.00201	0.00028	−0.00003	0.02559	0.00108
0.01814	−0.00295	−0.00258	0.00142	0.00309	0.00413	−0.00201	0.00031	0.00000	0.02530	0.00080
0.01813	−0.00297	−0.00265	0.00135	0.00308	0.00412	−0.00201	0.00027	−0.00004	0.02495	0.00045
0.01807	−0.00302	−0.00265	0.00134	0.00309	0.00413	−0.00200	0.00026	−0.00005	0.02495	0.00045
0.01416	−0.00211	−0.00341	0.00234	−0.00164	−0.00153	0.00105	0.00197	−0.00133	0.02721	0.00334
0.01807	−0.00290	−0.00185	0.00143	−0.00314	−0.00417	0.00210	−0.00041	0.00069	0.02157	0.00128
0.01804	−0.00294	−0.00212	0.00116	−0.00317	−0.00420	0.00207	−0.00068	0.00042	0.01958	−0.00071
0.01803	−0.00294	−0.00215	0.00113	−0.00317	−0.00420	0.00207	−0.00064	0.00045	0.01946	−0.00083
0.01803	−0.00295	−0.00219	0.00109	−0.00317	−0.00420	0.00207	−0.00068	0.00042	0.01920	−0.00109
0.01797	−0.00300	−0.00219	0.00109	−0.00319	−0.00421	0.00205	−0.00066	0.00043	0.01920	−0.00109
0.01224	−0.00174	−0.00159	0.00129	−0.00162	−0.00105	0.00094	0.00061	−0.00012	0.02163	0.00136
0.01037	−0.00142	−0.00182	0.00083	−0.00096	−0.00284	0.00097	0.00229	−0.00241	0.02334	0.00265
0.01037	−0.00143	−0.00184	0.00081	−0.00095	−0.00283	0.00099	0.00231	−0.00239	0.02324	0.00256

TABLE 7. (continued-3)

$V^{*(20,20)}_{(2000)}$	$V^{*(20,11)}_{(0000)}$	$V^{*(20,11)}_{(0100)}$	$V^{*(20,11)}_{(1000)}$	$V^{*(20,11)}_{(1100)}$	$V^{*(20,20)}_{(0000)}$	$V^{*(20,20)}_{(1000)}$	$V^{*(20,20)}_{(2000)}$	$V^{*(11,11)}_{(0110)}$	$V^{*(11,11)}_{(1001)}$	$V^{*(11,11)}_{(0111)}$
0.00234	−0.01183	0.00423	0.00309	−0.00168	0.00275	−0.00018	0.01252	0.04049	0.01397	−0.00684
−0.00094	−0.01231	0.00375	0.00262	−0.00216	−0.00001	−0.00294	0.00976	0.04042	0.01390	−0.00690
−0.00127	−0.01230	0.00376	0.00263	−0.00214	0.00019	−0.00274	0.00996	0.04042	0.01390	−0.00690
−0.00160	−0.01234	0.00372	0.00258	−0.00219	−0.00007	−0.00299	0.00971	0.04041	0.01390	−0.00691
−0.00160	−0.01234	0.00372	0.00258	−0.00219	−0.00007	−0.00299	0.00971	0.04041	0.01390	−0.00691
0.00217	−0.00998	0.00376	0.00441	−0.00268	0.00449	0.00044	0.01201	0.03485	0.01255	−0.00535
0.00301	0.00168	0.00093	0.00087	0.00011	−0.00279	0.00520	−0.00243	0.03162	0.00526	−0.00500
0.00105	0.00122	0.00047	0.00041	−0.00035	−0.00492	0.00307	−0.00457	0.03151	0.00515	−0.00510
0.00089	0.00121	0.00046	0.00040	−0.00036	−0.00474	0.00325	−0.00439	0.03151	0.00515	−0.00511
0.00685	0.00449	−0.00007	0.00003	0.00216	0.00773	0.00635	−0.00198	0.02401	0.00321	−0.00297
−0.00181	0.00341	−0.00114	−0.00105	0.00109	0.00256	0.00118	−0.00715	0.02388	0.00307	−0.00310
−0.00210	0.00343	−0.00112	−0.00103	0.00110	0.00281	0.00143	−0.00690	0.02388	0.00307	−0.00310
−0.00245	0.00339	−0.00116	−0.00107	0.00106	0.00263	0.00125	−0.00708	0.02387	0.00307	−0.00311
−0.00245	0.00339	−0.00116	−0.00107	0.00106	0.00263	0.00124	−0.00708	0.02387	0.00306	−0.00311
0.00108	−0.00174	0.00152	0.00184	−0.00158	0.00178	0.00184	0.00886	0.02132	0.00079	−0.00305
0.00259	−0.00201	0.00175	0.00158	−0.00135	0.00400	0.00075	0.00444	0.02366	0.00285	−0.00317
0.00061	−0.00221	0.00155	0.00138	−0.00155	0.00203	−0.00123	0.00246	0.02364	0.00283	−0.00319
0.00049	−0.00220	0.00156	0.00138	−0.00155	0.00217	−0.00109	0.00260	0.02364	0.00283	−0.00319
0.00022	−0.00223	0.00153	0.00136	−0.00157	0.00195	−0.00131	0.00238	0.02364	0.00283	−0.00320
0.00022	−0.00223	0.00153	0.00136	−0.00157	0.00195	−0.00131	0.00238	0.02363	0.00283	−0.00320
0.00270	−0.00223	0.00177	0.00151	−0.00117	0.00438	0.00123	0.00502	0.02118	0.00074	−0.00303
−0.00275	0.00342	0.00044	0.00217	−0.00314	−0.00448	0.00124	0.00418	0.01439	0.00336	−0.00006
−0.00285	0.00350	0.00051	0.00224	−0.00306	−0.00434	0.00138	0.00432	0.01433	0.00329	−0.00012

TABLE 7. (continued-4)

$V^{*(11,11)}_{(1110)}$	$V^{*(11,11)}_{(1011)} = V^{*(11,11)}_{(1101)}$	$V^{*(11,11)}_{(1111)}$	$V^{*(11,30)}_{(0000)}$	$V^{*(11,30)}_{(0010)}$	$V^{*(11,30)}_{(1000)}$	$V^{*(11,30)}_{(1010)}$	$V^{*(30,30)}_{(0000)}$	$V^{*(30,30)}_{(1000)}$	$V^{*(30,30)}_{(2000)}$	$V^{*(30,30)}_{(3000)}$
−0.00027	−0.00292	0.00102	−0.00408	0.00402	0.00478	−0.00795	0.03999	0.00354	−0.00166	0.00877
−0.00034	−0.00299	0.00095	−0.00448	0.00363	0.00438	−0.00834	0.03767	0.00122	−0.00398	0.00645
−0.00034	−0.00299	0.00095	−0.00449	0.00362	0.00438	−0.00835	0.03755	0.00110	−0.00410	0.00633
−0.00034	−0.00300	0.00094	−0.00452	0.00359	0.00434	−0.00838	0.03735	0.00091	−0.00429	0.00613
−0.00034	−0.00300	0.00094	−0.00452	0.00359	0.00434	−0.00838	0.03735	0.00090	−0.00429	0.00613
−0.00553	−0.00270	0.00289	−0.00585	0.00573	0.00382	−0.00544	0.03841	0.00365	0.00014	0.01226
0.00170	−0.00093	−0.00019	−0.00172	0.00236	0.00259	−0.00374	0.03271	0.00119	0.00093	0.01628
0.00159	−0.00104	−0.00030	−0.00222	0.00186	0.00209	−0.00425	0.03038	−0.00113	−0.00140	0.01396
0.00159	−0.00104	−0.00030	−0.00221	0.00187	0.00210	−0.00423	0.03019	−0.00133	−0.00159	0.01377
−0.00161	0.00120	−0.00072	0.00013	0.00211	0.00223	−0.00389	0.03406	0.00519	−0.00025	0.00213
−0.00174	0.00107	−0.00086	−0.00051	0.00147	0.00159	−0.00453	0.03097	0.00210	−0.00334	−0.00096
−0.00174	0.00106	−0.00086	−0.00053	0.00145	0.00158	−0.00454	0.03074	0.00187	−0.00356	−0.00118
−0.00174	0.00106	−0.00086	−0.00055	0.00143	0.00156	−0.00456	0.03065	0.00178	−0.00365	−0.00127
−0.00175	0.00106	−0.00087	−0.00055	0.00144	0.00156	−0.00456	0.03065	0.00178	−0.00366	−0.00128
−0.00310	0.00055	0.00040	−0.00249	0.00215	0.00142	−0.00204	0.02965	0.00349	0.00077	0.00586
−0.00178	0.00101	−0.00075	−0.00122	0.00164	0.00185	−0.00340	0.02526	0.00150	0.00117	0.00866
−0.00180	0.00099	−0.00077	−0.00142	0.00144	0.00165	−0.00360	0.02330	−0.00046	−0.00079	0.00670
−0.00180	0.00099	−0.00077	−0.00143	0.00143	0.00164	−0.00360	0.02314	−0.00062	−0.00095	0.00654
−0.00180	0.00099	−0.00077	−0.00145	0.00141	0.00162	−0.00362	0.02296	−0.00080	−0.00113	0.00636
−0.00181	0.00098	−0.00078	−0.00144	0.00141	0.00163	−0.00362	0.02296	−0.00080	−0.00113	0.00635
−0.00323	0.00054	0.00042	−0.00200	0.00198	0.00177	−0.00235	0.02485	0.00157	0.00172	0.00969
0.00281	0.00049	−0.00161	−0.00281	0.00029	−0.00107	0.00435	0.02680	0.00281	−0.00173	−0.00766
0.00274	0.00042	−0.00167	−0.00292	0.00017	−0.00118	0.00423	0.02660	0.00261	−0.00193	−0.00786

TABLE 8. Determinant-optimal 3^6-BTO designs

N	$\boldsymbol{\lambda}^{*\prime}$	$\det(V_T)$	$V^{*(00,00)}_{(0000)}$	$V^{*(00,10)}_{(0000)}$	$V^{*(00,01)}_{(0000)}$	$V^{*(00,20)}_{(0000)}$	$V^{*(00,11)}_{(0000)}$	$V^{*(00,30)}_{(0000)}$
78	1001001000100000100010000000	0.10820E–126	0.08691	−0.03381	−0.01137	0.00028	0.01065	0.00544
79	1011010000100000100010000000	0.81858E–128	0.04039	0.01011	−0.00781	−0.00010	−0.00467	0.00218
80	1111010000100000100010000000	0.52517E–128	0.02885	0.00571	−0.00476	−0.00028	−0.00277	0.00140
81	2021010000100000100010000000	0.30381E–128	0.03902	0.00913	−0.00736	0.00020	−0.00418	0.00259
82	2121010000100000100010000000	0.19917E–128	0.02839	0.00526	−0.00459	−0.00011	−0.00253	0.00166
*83	0001011000100000100010000000	0.31562E–130	0.01733	−0.00231	−0.00130	−0.00012	0.00059	0.00121
84	1001011000100000100010000000	0.16611E–130	0.01711	−0.00214	−0.00130	−0.00033	0.00044	0.00074
85	1011011000100000100010000000	0.27945E–131	0.01705	−0.00210	−0.00134	−0.00076	0.00038	0.00035
86	2011011000100000100010000000	0.20459E–131	0.01700	−0.00206	−0.00133	−0.00072	0.00034	0.00031
*87	1000010000001100101000000000	0.43287E–133	0.01585	0.00242	−0.00060	0.00269	0.00090	0.00160
*88	1010010000001100101000000000	0.30381E–134	0.01473	−0.00041	−0.00123	−0.00042	0.00051	−0.00025
*89	2010010000001100101000000000	0.16690E–134	0.01471	−0.00043	−0.00122	−0.00033	0.00050	−0.00033
*90	2020010000001100101000000000	0.86797E–135	0.01467	−0.00052	−0.00124	−0.00045	0.00049	−0.00039
*91	2120010000001100101000000000	0.77243E–135	0.01389	−0.00039	−0.00104	−0.00043	0.00043	−0.00034
*92	0001001000110000100010000000	0.18027E–136	0.01491	0.00053	−0.00141	−0.00155	−0.00058	−0.00055
93	1010010000000100111000000000	0.65216E–137	0.01659	−0.00086	−0.00182	−0.00121	0.00043	−0.00041
94	1001001100100000000000000100	0.39055E–141	0.06558	−0.03749	−0.01124	−0.00256	0.00869	0.00513
95	1011001100100000000000000100	0.13995E–141	0.06213	−0.03816	−0.01106	−0.00416	0.00857	0.00369
96	2011001100100000000000000100	0.71014E–142	0.04772	−0.02578	−0.00805	−0.00193	0.00554	0.00222
97	2021001100100000000000000100	0.43398E–142	0.04729	−0.02601	−0.00805	−0.00222	0.00555	0.00200
98	2121001100100000000000000100	0.28608E–142	0.03364	−0.01803	−0.00510	−0.00159	0.00380	0.00149
99	0001101100100000000000000100	0.11067E–143	0.15104	−0.09722	−0.02527	−0.00822	0.02257	0.01979
100	1001101100100000000000000100	0.26412E–144	0.06124	−0.03246	−0.00982	−0.00121	0.00725	0.00540

* This design is also optimal with respect to the trace criterion.

$\boldsymbol{\lambda}^{*\prime}=(\lambda^*_{600}, \lambda^*_{060}, \lambda^*_{006}, \lambda^*_{510}, \lambda^*_{501}, \lambda^*_{150}, \lambda^*_{051}, \lambda^*_{105}, \lambda^*_{015}, \lambda^*_{420}, \lambda^*_{402}, \lambda^*_{240}, \lambda^*_{042}, \lambda^*_{204}, \lambda^*_{024}, \lambda^*_{330}, \lambda^*_{303}, \lambda^*_{033}, \lambda^*_{411}, \lambda^*_{141}, \lambda^*_{114}, \lambda^*_{321}, \lambda^*_{312}, \lambda^*_{231}, \lambda^*_{132}, \lambda^*_{213}, \lambda^*_{123}, \lambda^*_{222})$.

$n_0\text{E}-n_1=n_0 10^{-n_1}$.

TABLE 8. (continued-1)

$V^{*(10,10)}_{(0000)}$	$V^{*(10,10)}_{(1000)}$	$V^{*(10,01)}_{(0000)}$	$V^{*(10,01)}_{(0100)}$	$V^{*(10,20)}_{(0000)}$	$V^{*(10,20)}_{(1000)}$	$V^{*(10,11)}_{(0100)}$	$V^{*(10,11)}_{(1000)}$	$V^{*(10,11)}_{(1100)}$	$V^{*(10,30)}_{(0000)}$	$V^{*(10,30)}_{(1000)}$
0.13084	−0.00371	−0.02949	0.01131	−0.00200	0.00061	−0.03009	0.00087	−0.00115	−0.01244	0.00753
0.11601	−0.01036	0.02508	−0.00911	−0.01393	0.00794	−0.02574	−0.00778	0.00298	0.00037	0.00141
0.11434	−0.01204	0.02624	−0.00795	−0.01400	0.00787	−0.02502	−0.00706	0.00370	0.00007	0.00111
0.11528	−0.01109	0.02539	−0.00880	−0.01382	0.00805	−0.02537	−0.00741	0.00335	0.00063	0.00167
0.11387	−0.01250	0.02640	−0.00779	−0.01393	0.00794	−0.02477	−0.00682	0.00395	0.00030	0.00133
0.04847	−0.00432	−0.00002	−0.00006	−0.00377	0.00167	−0.00911	−0.00128	0.00091	−0.00228	0.00092
0.04834	−0.00445	−0.00003	−0.00006	−0.00361	0.00183	−0.00899	−0.00116	0.00103	−0.00192	0.00128
0.04832	−0.00448	−0.00000	−0.00004	−0.00333	0.00211	−0.00895	−0.00112	0.00107	−0.00167	0.00153
0.04828	−0.00451	−0.00001	−0.00005	−0.00336	0.00208	−0.00893	−0.00109	0.00110	−0.00164	0.00157
0.05097	0.00203	−0.01138	0.00473	0.01118	0.00697	−0.00684	−0.00506	0.00399	0.00263	0.00723
0.04377	−0.00517	−0.01298	0.00313	0.00328	−0.00092	−0.00782	−0.00604	0.00301	−0.00209	0.00251
0.04377	−0.00517	−0.01297	0.00314	0.00333	−0.00088	−0.00783	−0.00604	0.00301	−0.00213	0.00247
0.04351	−0.00543	−0.01303	0.00308	0.00303	−0.00118	−0.00786	−0.00608	0.00297	−0.00228	0.00232
0.04348	−0.00546	−0.01307	0.00304	0.00302	−0.00118	−0.00785	−0.00607	0.00298	−0.00229	0.00231
0.03708	−0.00460	0.00826	−0.00193	−0.00361	0.00275	−0.00606	−0.00287	0.00182	0.00036	0.00052
0.05722	−0.00916	−0.00786	0.00193	0.00230	−0.00114	−0.01122	−0.00534	0.00392	−0.00415	0.00430
0.08875	0.02548	0.00782	0.00731	−0.00483	0.00983	−0.02098	−0.01120	−0.00374	0.00406	−0.00829
0.08862	0.02535	0.00786	0.00734	−0.00514	0.00952	−0.02100	−0.01123	−0.00377	0.00378	−0.00857
0.07798	0.01471	0.00527	0.00476	−0.00706	0.00760	−0.01840	−0.00863	−0.00117	0.00505	−0.00730
0.07785	0.01458	0.00527	0.00475	−0.00722	0.00744	−0.01840	−0.00862	−0.00116	0.00493	−0.00742
0.07319	0.00992	0.00355	0.00303	−0.00758	0.00708	−0.01737	−0.00760	−0.00014	0.00522	−0.00712
0.11733	0.06882	0.01387	0.01749	0.00436	0.01048	−0.02795	−0.01650	−0.01474	−0.01091	−0.01471
0.07063	0.02211	0.00272	0.00635	−0.00070	0.00542	−0.01690	−0.00545	−0.00368	−0.00053	−0.00433

TABLE 8. (continued–2)

$V^{*(01,01)}_{(0000)}$	$V^{*(01,01)}_{(0001)}$	$V^{*(01,20)}_{(0000)}$	$V^{*(01,20)}_{(0010)}$	$V^{*(01,11)}_{(0001)}$	$V^{*(01,11)}_{(0010)}$	$V^{*(01,11)}_{(0011)}$	$V^{*(01,30)}_{(0000)}$	$V^{*(01,30)}_{(0010)}$	$V^{*(20,20)}_{(0000)}$	$V^{*(20,20)}_{(1000)}$
0.03887	−0.00540	−0.00504	0.00277	0.00705	−0.00809	−0.00182	0.00657	−0.00761	0.03316	0.00191
0.03541	−0.00432	−0.00953	0.00450	−0.00490	−0.00380	0.00435	0.00283	−0.00398	0.03499	0.00017
0.03461	−0.00513	−0.00948	0.00454	−0.00540	−0.00430	0.00385	0.00304	−0.00377	0.03499	0.00017
0.03526	−0.00447	−0.00964	0.00439	−0.00506	−0.00395	0.00419	0.00269	−0.00411	0.03468	−0.00014
0.03454	−0.00519	−0.00956	0.00447	−0.00549	−0.00438	0.00377	0.00294	−0.00387	0.03467	−0.00015
0.02042	−0.00345	−0.00401	0.00238	0.00030	−0.00329	0.00088	0.00306	−0.00275	0.03323	0.00207
0.02042	−0.00345	−0.00401	0.00239	0.00030	−0.00329	0.00088	0.00306	−0.00274	0.03303	0.00188
0.02040	−0.00347	−0.00426	0.00214	0.00027	−0.00332	0.00084	0.00284	−0.00297	0.03008	−0.00107
0.02040	−0.00348	−0.00427	0.00213	0.00028	−0.00332	0.00085	0.00285	−0.00296	0.03004	−0.00111
0.01851	−0.00259	−0.00078	0.00322	0.00330	0.00434	−0.00179	0.00132	0.00102	0.03425	0.00975
0.01815	−0.00295	−0.00254	0.00146	0.00308	0.00412	−0.00201	0.00028	−0.00003	0.02559	0.00108
0.01814	−0.00295	−0.00258	0.00142	0.00309	0.00413	−0.00201	0.00031	0.00000	0.02530	0.00080
0.01813	−0.00297	−0.00265	0.00135	0.00308	0.00412	−0.00201	0.00027	−0.00004	0.02495	0.00045
0.01807	−0.00302	−0.00265	0.00134	0.00309	0.00413	−0.00200	0.00026	−0.00005	0.02495	0.00045
0.01416	−0.00211	−0.00341	0.00234	−0.00164	−0.00153	0.00105	0.00197	−0.00133	0.02721	0.00334
0.01273	−0.00172	−0.00349	0.00207	0.00163	0.00098	−0.00077	−0.00143	0.00168	0.02445	0.00123
0.01299	0.00047	−0.00045	0.00135	−0.00163	−0.00455	−0.00103	0.00140	−0.00272	0.02491	0.00322
0.01298	0.00046	−0.00037	0.00143	−0.00162	−0.00454	−0.00102	0.00147	−0.00265	0.02416	0.00248
0.01235	−0.00017	−0.00084	0.00096	−0.00099	−0.00391	−0.00039	0.00178	−0.00234	0.02381	0.00213
0.01235	−0.00017	−0.00084	0.00096	−0.00099	−0.00391	−0.00039	0.00178	−0.00234	0.02362	0.00194
0.01171	−0.00080	−0.00097	0.00083	−0.00061	−0.00353	−0.00001	0.00189	−0.00223	0.02359	0.00191
0.01421	0.00286	0.00191	0.00131	−0.00312	−0.00556	−0.00365	−0.00237	−0.00409	0.01844	0.00170
0.01155	0.00020	0.00071	0.00011	−0.00048	−0.00293	−0.00101	0.00011	−0.00161	0.01789	0.00116

TABLE 8. (continued-3)

$V^{*(20,20)}_{(2000)}$	$V^{*(20,11)}_{(0000)}$	$V^{*(20,11)}_{(0100)}$	$V^{*(20,11)}_{(1000)}$	$V^{*(20,11)}_{(1100)}$	$V^{*(20,30)}_{(0000)}$	$V^{*(20,30)}_{(1000)}$	$V^{*(20,30)}_{(2000)}$	$V^{*(11,11)}_{(0110)}$	$V^{*(11,11)}_{(1001)}$	$V^{*(11,11)}_{(0111)}$
0.00191	−0.01038	0.00322	0.00467	−0.00257	0.00351	0.00004	0.01219	0.04038	0.01154	−0.00003
−0.00340	−0.00662	0.00521	0.00359	−0.00541	0.00095	−0.00262	0.00943	0.03983	0.01614	−0.00136
−0.00340	−0.00659	0.00524	0.00362	−0.00538	0.00094	−0.00263	0.00942	0.03952	0.01583	−0.00167
−0.00371	−0.00668	0.00515	0.00354	−0.00547	0.00079	−0.00278	0.00927	0.03965	0.01596	−0.00155
−0.00372	−0.00663	0.00520	0.00358	−0.00542	0.00076	−0.00281	0.00924	0.03939	0.01570	−0.00180
0.00217	−0.00998	0.00376	0.00441	−0.00268	0.00449	0.00044	0.01201	0.03485	0.01255	−0.00535
0.00197	−0.01013	0.00362	0.00426	−0.00283	0.00405	−0.00001	0.01157	0.03474	0.01244	−0.00546
−0.00097	−0.01054	0.00321	0.00385	−0.00324	0.00138	−0.00267	0.00890	0.03468	0.01238	−0.00552
−0.00102	−0.01051	0.00323	0.00388	−0.00321	0.00142	−0.00264	0.00893	0.03466	0.01236	−0.00554
0.00685	0.00449	−0.00007	0.00003	0.00216	0.00773	0.00635	−0.00198	0.02401	0.00321	−0.00297
−0.00181	0.00341	−0.00114	−0.00105	0.00109	0.00256	0.00118	−0.00715	0.02388	0.00307	−0.00310
−0.00210	0.00343	−0.00112	−0.00103	0.00110	0.00281	0.00143	−0.00690	0.02388	0.00307	−0.00310
−0.00245	0.00339	−0.00116	−0.00107	0.00106	0.00263	0.00125	−0.00708	0.02387	0.00307	−0.00311
−0.00245	0.00339	−0.00116	−0.00107	0.00106	0.00263	0.00124	−0.00708	0.02387	0.00306	−0.00311
0.00108	−0.00174	0.00152	0.00184	−0.00158	0.00178	0.00184	0.00886	0.02132	0.00079	−0.00305
−0.00189	0.00171	−0.00045	−0.00100	0.00116	0.00259	0.00102	−0.00688	0.02083	0.00020	−0.00127
−0.00318	0.00408	0.00073	0.00125	−0.00441	−0.00528	0.00112	0.00475	0.01782	0.00531	0.00325
−0.00393	0.00402	0.00068	0.00119	−0.00447	−0.00595	0.00046	0.00408	0.01782	0.00530	0.00324
−0.00427	0.00449	0.00114	0.00166	−0.00400	−0.00572	0.00069	0.00431	0.01718	0.00467	0.00261
−0.00447	0.00449	0.00115	0.00166	−0.00399	−0.00587	0.00054	0.00416	0.01718	0.00467	0.00261
−0.00449	0.00457	0.00123	0.00174	−0.00391	−0.00584	0.00056	0.00419	0.01696	0.00444	0.00238
0.00024	−0.00042	−0.00046	−0.00092	−0.00328	−0.00154	−0.00008	−0.00140	0.01773	0.00540	0.00535
−0.00030	0.00078	0.00073	0.00027	−0.00209	−0.00042	0.00104	−0.00028	0.01511	0.00279	0.00274

TABLE 8. (continued–4)

$V^{*(11,11)}_{(1110)}$	$V^{*(11,11)}_{(1011)} = V^{*(11,11)}_{(1101)}$	$V^{*(11,11)}_{(1111)}$	$V^{*(11,30)}_{(0000)}$	$V^{*(11,30)}_{(0010)}$	$V^{*(11,30)}_{(1000)}$	$V^{*(11,30)}_{(1010)}$	$V^{*(30,30)}_{(0000)}$	$V^{*(30,30)}_{(1000)}$	$V^{*(30,30)}_{(2000)}$	$V^{*(30,30)}_{(3000)}$
−0.00437	−0.00273	0.00383	−0.00399	0.00846	0.00180	−0.00659	0.04289	0.00469	−0.00225	0.00643
−0.00530	−0.00128	0.00212	−0.00735	0.00448	0.00287	−0.00614	0.03524	0.00042	−0.00315	0.00890
−0.00561	−0.00160	0.00181	−0.00722	0.00461	0.00300	−0.00601	0.03519	0.00037	−0.00320	0.00885
−0.00548	−0.00147	0.00193	−0.00748	0.00435	0.00274	−0.00627	0.03510	0.00028	−0.00329	0.00876
−0.00574	−0.00172	0.00168	−0.00733	0.00450	0.00288	−0.00612	0.03502	0.00020	−0.00338	0.00868
−0.00553	−0.00270	0.00289	−0.00585	0.00573	0.00382	−0.00544	0.03841	0.00365	0.00014	0.01226
−0.00564	−0.00281	0.00277	−0.00618	0.00540	0.00349	−0.00577	0.03742	0.00266	−0.00085	0.01127
−0.00570	−0.00287	0.00272	−0.00655	0.00503	0.00311	−0.00614	0.03500	0.00025	−0.00326	0.00886
−0.00572	−0.00289	0.00270	−0.00658	0.00500	0.00309	−0.00617	0.03497	0.00021	−0.00329	0.00882
−0.00161	0.00120	−0.00072	0.00013	0.00211	0.00223	−0.00389	0.03406	0.00519	−0.00025	0.00213
−0.00174	0.00107	−0.00086	−0.00051	0.00147	0.00159	−0.00453	0.03097	0.00210	−0.00334	−0.00096
−0.00174	0.00106	−0.00086	−0.00053	0.00145	0.00158	−0.00454	0.03074	0.00187	−0.00356	−0.00118
−0.00174	0.00106	−0.00086	−0.00055	0.00143	0.00156	−0.00456	0.03065	0.00178	−0.00365	−0.00127
−0.00175	0.00106	−0.00087	−0.00055	0.00144	0.00156	−0.00456	0.03065	0.00178	−0.00366	−0.00128
−0.00310	0.00055	0.00040	−0.00249	0.00215	0.00142	−0.00204	0.02965	0.00349	0.00077	0.00586
−0.00291	0.00148	−0.00089	−0.00029	0.00210	0.00060	−0.00407	0.03154	0.00229	−0.00357	−0.00144
0.00376	0.00188	−0.00078	−0.00305	0.00029	−0.00022	0.00544	0.02750	0.00304	−0.00197	−0.00838
0.00376	0.00187	−0.00079	−0.00310	0.00024	−0.00027	0.00539	0.02691	0.00245	−0.00257	−0.00897
0.00312	0.00124	−0.00142	−0.00341	−0.00007	−0.00058	0.00508	0.02676	0.00230	−0.00272	−0.00912
0.00312	0.00124	−0.00142	−0.00341	−0.00006	−0.00058	0.00508	0.02664	0.00218	−0.00283	−0.00924
0.00290	0.00101	−0.00165	−0.00347	−0.00013	−0.00064	0.00502	0.02662	0.00216	−0.00285	−0.00926
0.00476	0.00352	0.00241	0.00231	0.00235	0.00281	0.00518	0.02237	0.00286	0.00279	0.00133
0.00215	0.00091	−0.00020	−0.00015	−0.00010	0.00036	0.00272	0.02007	0.00055	0.00048	−0.00097

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Maritime Safety Academy,
Kure, Hiroshima