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Generalized J-integral and three dimensional fracture mechanics I

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1. Introduction

The principal objective of this work is to make a systematic study of a generalization of the Griffith theory in three dimensional fracture mechanics from mathematical viewpoint. We consider the situation where an elastic body containing a crack, in its non-deformed state, occupies a domain in R^3 of the form $\Omega = G - \Sigma$. Here we consider the crack as a discontinuity in the material in the form of a surface Σ, and we assume that G is a domain in R^3 with local Lipschitz property and Σ is a two dimensional manifold with boundary contained in G. This body is in a state of equilibrium under the influence of a load $\mathscr{L}$ consisting of a body force in Ω and a surface force on the boundary ∂G of G. By $\mathrm{I}(\mathscr{L}; \Sigma)$ we denote the potential energy of the elastic body containing the crack Σ under the load $\mathscr{L}$. The generalization of the Griffith theory can be expressed in terms of the concept of energy release rate as follows (cf. Palamiswamy and Knauss [19]). The crack extension process is considered to occur in a quasi-static manner, so that when we refer to time we use it as a parameter which indicates the sequence of events. We denote by $\Sigma(t)$ the surface obtained from Σ by extending it in the length of time $t\,(\geqq 0)$. Of course $\Sigma(t) \subset \Sigma(t')$ if $t < t'$, and $\Sigma = \Sigma(0) = \bigcap_{t \geqq 0} \Sigma(t)$. During crack extension let the load $\mathscr{L}$ be independent of t. If the crack extends from Σ to $\Sigma(t)$, the potential energy released by the increment $\Sigma(t) - \Sigma$ is given by

$$\mathrm{I}(\mathscr{L}; \Sigma) - \mathrm{I}(\mathscr{L}; \Sigma(t)).$$

Now we consider the limit

$$G(\mathscr{L}; \{\Sigma(t)\}) = \lim_{t \to 0} \frac{\mathrm{I}(\mathscr{L}; \Sigma) - \mathrm{I}(\mathscr{L}; \Sigma(t))}{|\Sigma(t) - \Sigma|} \tag{1.1}$$

where $|\Sigma(t) - \Sigma|$ denotes the surface measure of $\Sigma(t) - \Sigma$. If it exists, we call $G(\mathscr{L}; \{\Sigma(t)\})$ the energy release rate of the crack extension $\{\Sigma(t)\}$ under the load $\mathscr{L}$. This is expected to be a function of the "infinitesimal displacement" $d\{\Sigma(t)\}$ of the edge of the crack (see [19]). Then we may rewrite $G(\mathscr{L}; \{\Sigma(t)\})$ as $G(\mathscr{L}; d\{\Sigma(t)\})$.

Now the generalization of Griffith's energy balance can be expressed as follows (see [19]): of all crack extensions, there should be one $\{\Sigma(t)\}$ which makes $G(\mathscr{L};\ \cdot)$ an absolute maximum

$$G_{\max} = G(\mathscr{L};\ d\{\Sigma(t)\}).$$

Crack propagation in a brittle solid becomes possible when the energy release rate $G_{\max}$ reaches a critical value which depends on the material considered, and the crack will propagate in the direction determined by $d\{\Sigma(t)\}$.

Here the following questions arise:

(Q.1) *How to describe and measure the force which causes the crack extension?*
(Q.2) *How to define the infinitesimal displacement* $d\{\Sigma(t)\}$?
(Q.3) *Is there an absolute maximum* $G_{\max}$?

To our knowledge systematic studies of these questions have not appeared in the literature. The following result will be of great help in attacking these questions.

For a homogeneous elastic plate containing a crack which lies on the line $x_2=0$, it has been shown in Rice [21] that if the crack extends in the x_1-direction and the body force is zero, then the energy release rate is expressed as a path-independent integral

$$J = \int_C \{W\nu_1 - s\cdot(D_1 u)\}d\ell, \tag{1.2}$$

which is called the J-integral in fracture mechanics. Here u is the displacement vector, W the strain energy density, s the traction vector, C a closed curve surrounding the crack tip as illustrated in Figure 1, $d\ell$ the line element of C and ν_i the components of the unit outward normal to C. The work in [21] is intimately related to earlier investigations by Sanders [22] and Cherepanov [6].

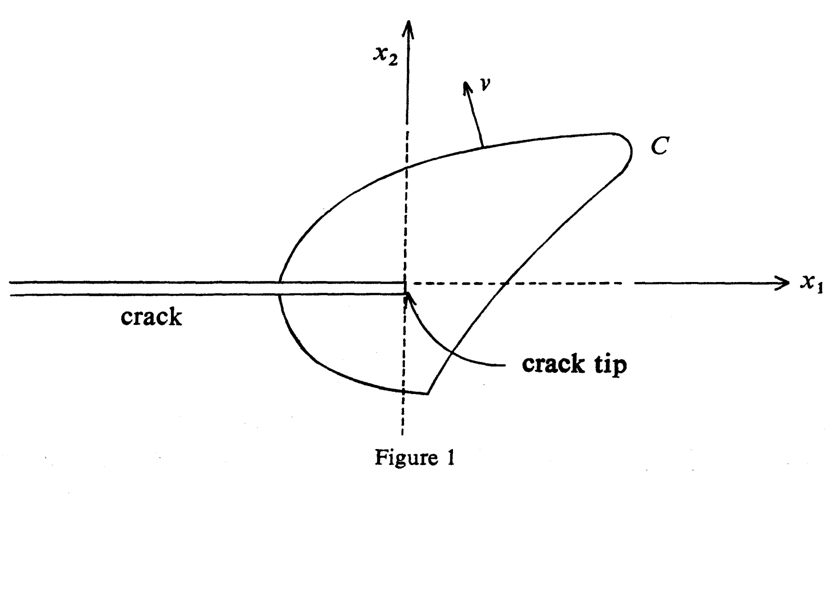

Figure 1

This representation of energy release rate indicates the following interesting fact: If the elastic field of the above plate under an arbitrary load is "regular" at the crack tip, then we see that $J=0$ by means of the divergence theorem (see Proposition 3.9) and hence the energy release rates are zero for all loads, which is contradictory to our experience. Hence there should exist a load $\mathscr{L}$ such that the elastic field of the above plate under $\mathscr{L}$ is "singular" at the crack tip. The meaning of the terms "regular" and "singular" will be clarified later (see Definition 3.1). A detailed mathematical investigation for the J-integral (1.2) is described in Ohtsuka [20].

The above considerations suggest that the crack extension force is described by the singularity of the elastic field at the crack tip and measurements of crack extension force can be made in terms of the J-integral. Thus our question (Q.1) can be reduced to the following problems:

(P.1) *Find a representation analogous to the J-integral of the energy release rate for three dimensional bodies under arbitrary loads.*
(P.2) *Show that this representation depends only on the singularity of elastic fields at the edge of the crack.*

It is difficult to calculate the energy release rate in general case. Hence, as a first step, we calculate it in the case of a linear elastic body containing a smooth crack which advances smoothly (see Definition 4.1).

The main result in this paper is the expression of energy release rate as a generalized J-integral (see section 3), which is an answer to (P.1) (see Theorem 4.5). Partial answer to (P.2) and (Q.2) are given in Corollary 4.6 and Theorem 5.5, respectively. Further discussions of these questions as well as (Q.3) will be given in a forthcoming paper.

Throughout this paper we use the following notations: For a domain A, ∂A is the boundary of A and $|A|$ the volume of A. For a surface S, ∂S is the boundary of S if S is a manifold with boundary, dS the surface element of S, $|S|$ the surface measure of S, and $\nu=(\nu_1, \nu_2, \nu_3)$ the unit outward normal to S if S is the boundary of a domain with local Lipschitz property or a two dimensional oriented smooth manifold. For an arbitrary open set Q, we denote the set $Q \cap \Omega$ by Q'.

2. Elastic bodies and energy release rate

In this section we shall discuss a linear elastic body containing a crack (not necessarily smooth) and a quasi-static problem which arises from consideration of a crack extension process.

First we define $\Omega = G - \Sigma$ more precisely. A domain Q is said to have local Lipschitz property if it is a bounded domain in R^3 such that, in a neighborhood of any point $x \in \partial Q$, ∂Q admits a representation as a surface $y_3 = \alpha(y_1, y_2)$, where α

is a Lipschitzian function and (y_1, y_2, y_3) is a Cartesian coordinate in R^3, and Q is locally located on one side of ∂Q. G is a domain in R^3 with local Lipschitz property and Σ a two dimensional manifold with boundary $\partial\Sigma$ in G which lies on the boundary $\partial\Xi$ of a domain Ξ with local Lipschitz property such that $\bar{\Xi} \subset G$.

The linear theory of an elastic body containing a crack is expressed as follows: Let $u=(u_i)$, $\varepsilon=(\varepsilon_{ij})$, $\sigma=(\sigma_{ij})$ denote the displacement vector, the strain tensor and the stress tensor, respectively. Then the strain-displacement and stress-strain relations of this elastic body are given by

$$\varepsilon_{ij}(x) = [\varepsilon_{ij}(u)](x) = (D_j u_i(x) + D_i u_j(x))/2,$$
$$\sigma_{ij}(x) = [\sigma_{ij}(u)](x) = a_{ijkl}(x)\varepsilon_{kl}(x) \quad \text{(Hooke's law)},$$

where a_{ijkl} denote the components of Hooke's tensor. We assume that a_{ijkl} belong to $C^\infty(\bar{\Omega})$ and satisfy the following property of symmetry

$$a_{ijkl} = a_{jilk} = a_{klij} \tag{2.1}$$

and of ellipticity

$$a_{ijkl}\xi_{kl}\xi_{ij} \geqq \alpha_0 \xi_{ij}\xi_{ij} \qquad \text{for all} \quad \xi_{ij} \neq 0 \tag{2.2}$$

with some positive constant α_0 independent of ξ_{ij}.

We consider the following circumstances: the elastic body cannot move along Γ_0 $(\subset \partial G)$, a surface force F is given on $\Gamma_1 = \partial G - \Gamma_0$, a body force f is given in Ω and the stress is free on Σ (see Figure 2). Then the displacement vector u satisfies the boundary value problem

$$\begin{cases} -D_j\sigma_{ij} = f_i & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_0, \\ \sigma_{ij}\nu_j = F_i \quad \text{on} \quad \Gamma_1, \\ \sigma^+_{ij}\nu_j = \sigma^-_{ij}\nu_j = 0 \quad \text{on} \quad \Sigma, \end{cases} \tag{2.3}$$

where f_i and F_i are the components of f and F, respectively. Here we assume that Γ_0 is measurable with respect to the surface element of ∂G and has positive measure, and that the stress tensor σ_{ij} has finite limits

$$\sigma^+_{ij}(x) = \lim_{y\to x} \sigma_{ij}(y), \quad \sigma^-_{ij}(x) = \lim_{z\to x} \sigma_{ij}(z)$$

for any $x \in \Sigma$, as y and z approach to x from $G - \bar{\Xi}$ and Ξ, respectively.

In order to give variational formulation of the problem (2.3), we consider the Sobolev space $W^{m,p}(Q)$ for an open set Q of R^3, $1 \leqq p < \infty$ and non-negative integer m. All functions considered in this paper are real valued. By $L^p(Q)$ we

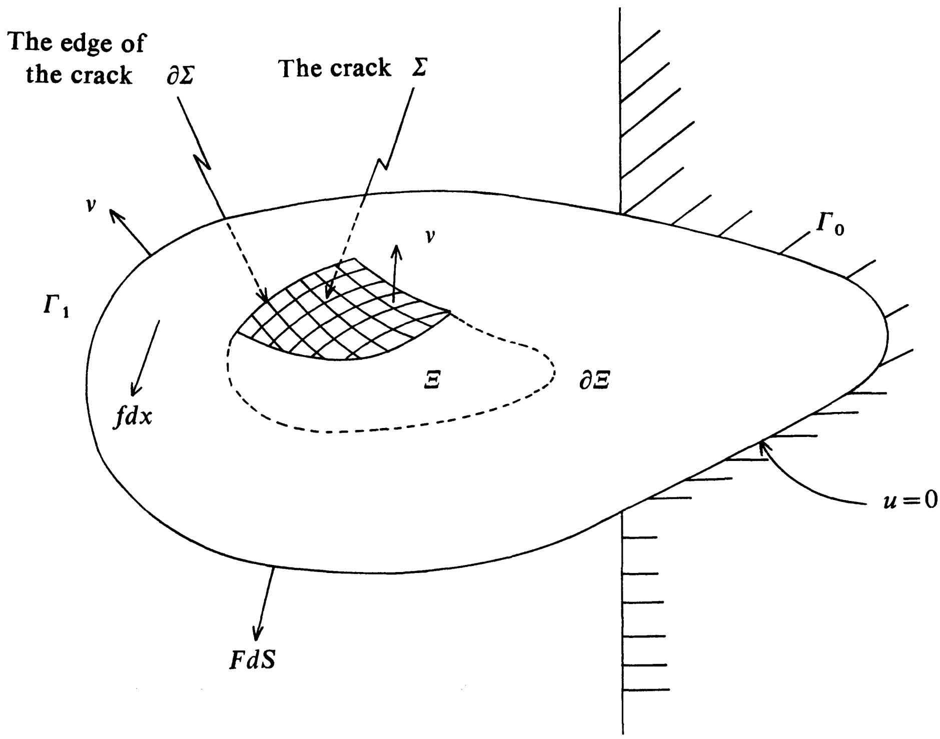

Figure 2

denote the classical Banach space consisting of p-integrable functions on Q with the norm

$$|v|_{L^p(Q)} = \left\{\int_Q |v(x)|^p dx\right\}^{1/p}.$$

$W^{m,p}(Q)$ is the space of all functions $v \in L^p(Q)$ such that

$$|v|_{m,p,Q} = \left\{\sum_{|\alpha| \leqq m} \int_Q |D^\alpha v|^p dx\right\}^{1/p} < \infty,$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $D^\alpha v$ mean distributional derivatives. $W^{m,p}(Q)$ is a Banach space equipped with the norm $|v|_{m,p,Q}$. Here we note that $W^{0,p}(Q) = L^p(Q)$. The case $p=2$ is special, since the space $W^{m,2}(Q)$ is a Hilbert space with respect to the scalar product

$$(v, w)_{m,Q} = \int_Q \sum_{|\alpha| \leqq m} D^\alpha v D^\alpha w \, dx$$

We set $|v|_{m,Q} = |v|_{m,2,Q}$ and $H^m(Q) = W^{m,2}(Q)$. For a surface S, $L^p(S)$ can be defined in terms of the surface element dS of S. For our later purposes we present the trace theorem, the density theorem and fundamental Green's formula. If a domain Q has local Lipschitz property, then $C^\infty(\bar{Q})$ is dense in $H^1(Q)$. For $v \in C^\infty(\bar{Q})$, we define

$$\gamma_Q v = \text{"}trace\ of\ v\ on\ \partial Q\text{"} = the\ restriction\ of\ v\ to\ \partial Q.$$

Then we have

TRACE THEOREM. *If Q has local Lipschitz property, then the mapping γ_Q is extended to a continuous linear operator from $H^1(Q)$ into $L^2(\partial Q)$.*

When there is no fear of confusion, we simply write v for $\gamma_Q v$. Although $C^\infty(\bar{\Omega})$ is not dense in $H^1(\Omega)$ for our domain Ω we can define the trace of $v \in H^1(\Omega)$ on $\partial\Omega = \partial G \cup \Sigma$ as follows: Since Ξ and $B = G - \bar{\Xi}$ have local Lipschitz property, and $v|_\Xi \in H^1(\Xi)$ and $v|_B \in H^1(B)$ for all $v \in H^1(\Omega)$, we obtain a trace operator

$$v \rightarrow (v^+, v^-, v) \in \{L^2(\Sigma)\}^2 \times L^2(\partial G), \tag{2.4}$$

where v^+ is the restriction of $\gamma_B(v|_B)$ to Σ, v^- the restriction of $\gamma_\Xi(v|_\Xi)$ to Σ and v the restriction of $\gamma_B(v|_B)$ to ∂G.

The density theorem is the following (see e.g. Adams [1]):

DENSITY THEOREM. *The subspace $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ for $1 \leqq p < \infty$.*

We now give well-known fundamental Green's formula, which is closely related to J-integrals. Let Q be a domain with local Lipschitz property. For any v, $w \in H^1(Q)$, we have

$$\int_Q v(D_i w)dx = -\int_Q (D_i v)w\,dx + \int_{\partial Q} vw\nu_i\,dS \tag{2.5}$$

for each $i = 1, 2, 3$ (see e.g. Nečas [17]). Since

$$aD_i(vw) = (aw)D_i v + vD_i(aw) - (D_i a)vw$$

for any v, $w \in H^1(Q)$, $a \in C^1(\bar{Q})$, (2.5) implies

LEMMA 2.1. *Let Q be a domain with local Lipschitz property. Then, for any v, $w \in H^1(Q)$, $a \in C^1(\bar{Q})$,*

$$\int_Q aD_i(vw)dx = -\int_Q (D_i a)vw\,dx + \int_{\partial Q} avw\nu_i dS \quad \textit{for each} \quad i = 1, 2, 3. \tag{2.6}$$

In general, Green's formulae (2.5) and (2.6) do not hold for Ω. We call Q $(\subset G)$ "regular relative to Ω" if Q is a domain with local Lipschitz property and the formula

$$(2.7)\qquad \int_{Q'} (D_i v) w dx = -\int_{Q'} v(D_i w) dx + \int_{\partial Q} v w \nu_i dS + \int_{\Sigma \cap Q} [\![vw]\!] \nu_i dS$$

hold for all $v, w \in H^1(\Omega)$, $i=1, 2, 3$, where $Q' = Q \cap \Omega$ and $[\![vw]\!]$ represents the discontinuity of vw across Σ, i.e.,

$$[\![vw]\!] = v^+ w^+ - v^- w^- \quad \text{(see (2.4))}.$$

If both $Q \cap \Xi$ and $Q \cap (G - \bar{\Xi})$ have local Lipschitz property, then by (2.5), it is easy to prove that Q is regular relative to Ω. If Q is regular relative to Ω, then, by an argument similar to Lemma 2.1,

$$(2.8)\quad \int_{Q'} a D_i(vw) dx = -\int_{Q'} (D_i a) v w dx + \int_{\partial Q} a v w \nu_i dx + \int_{\Sigma \cap Q} a [\![vw]\!] \nu_i dS$$

for $v, w \in H^1(\Omega)$, $a \in C^1(\bar{\Omega})$.

We shall now give the variational formulation of the problem (2.3). In what follows we shall use the notations

$$\boldsymbol{H}^m(Q) = \{H^m(Q)\}^3 \quad \text{and} \quad \boldsymbol{L}^2(Q) = \{L^2(Q)\}^3 \ (= \boldsymbol{H}^0(Q)),$$

which are equipped with the product norms

$$\|v\|_{m,Q} = \{\textstyle\sum_{j=1}^3 |v_j|_{m,Q}^2\}^{1/2}.$$

We define the space

$$V(\Omega) = \{v;\ v \in \boldsymbol{H}^1(\Omega),\ v=0 \quad \text{on} \quad \Gamma_0\},$$

which is a Hilbert space as a closed subspace of $\boldsymbol{H}^1(\Omega)$, and we consider the bilinear form

$$(2.9)\qquad a(v, w) = \int_\Omega \sigma_{ij}(v) \varepsilon_{ij}(w) dx \quad \text{for} \quad v, w \in V(\Omega),$$

which is symmetric by (2.1). Then the problem to find a displacement vector u satisfying (2.3) under a load $\mathscr{L} = (f, F) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Gamma_1)$ can be reformulated as follows:

(2.10) *Find $u \in V(\Omega)$ such that*

$$a(u, v) = \int_\Omega f \cdot v\, dx + \int_{\Gamma_1} F \cdot v\, dS \qquad \textit{for all} \quad v \in V(\Omega).$$

If the bilinear form $a(v, w)$ is coercive and bounded on $V(\Omega)$, then we can immediately conclude the unique solvability of the problem (2.10) by use of the Lax-Milgram lemma (see e.g. [17]). Related to the coercivity of $a(v, w)$ is Korn's inequality, that is, there exists a constant $c(\Omega) > 0$ such that

$$(2.11)\qquad \int_\Omega \varepsilon_{ij}(v) \varepsilon_{ij}(v) dx + \int_\Omega v_i v_i dx \geqq c(\Omega) \|v\|_{1,\Omega}$$

for all $v \in \boldsymbol{H}^1(\Omega)$. To obtain Korn's inequality, it is sufficient by Gobert [11] to show that Ω has cone property, that is, there exists a finite cone C such that each point x of Ω is the vertex of a cone C_x contained in Ω and congruent to C. It is easy to prove the following lemma (cf. Chenais [5]):

LEMMA 2.2. *Ω has cone property.*

Then we obtain the following compactness result.

LEMMA 2.3. *The imbedding $H^1(\Omega) \to L^2(\Omega)$ is compact.*

The proof of this lemma is found e.g. in [1].

By virtue of (2.2), (2.11) and Lemma 2.3, we can conclude the coercivity of $a(v, w)$ by an argument similar to that in Duvaut-Lions [7], Chapter 3, Theorem 3.3.

LEMMA 2.4. *There exists a positive constant $\alpha(\Omega)$ such that*

$$a(v, v) \geqq \alpha(\Omega) \|v\|_{1,\Omega} \quad \textit{for all} \quad v \in V(\Omega). \tag{2.12}$$

We then have

THEOREM 2.5. *For each load $\mathscr{L} = (f, F) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Gamma_1)$, there exists a unique solution $u \in V(\Omega)$ of the problem* (2.10). *Furthermore Green's operator*

$$T: \mathscr{L} = (f, F) \to u \tag{2.13}$$

is a bounded linear operator of $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Gamma_1)$ into $V(\Omega)$.

We now state the quasi-static problem which arises from a consideration of crack extension process. Let $\{\Sigma(t)\}_{t \in [0,1]}$ be a family of closed subsets of $\partial\Xi$. Then the problem we now consider is the following:

(2.14) *For a given load $\mathscr{L} = (f, F) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Gamma_1)$, we seek displacement vectors $v(t) \in V(\Omega(t))$, $t \in [0, 1]$, such that*

$$a_t(v(t), w) = \int_{\Omega(t)} f \cdot w \, dx + \int_{\Gamma_1} F \cdot w \, dS$$

for all $w \in V(\Omega(t))$, where $\Omega(t) = G - \Sigma(t)$,

$$V(\Omega(t)) = \{v;\ v \in \boldsymbol{H}^1(\Omega(t)),\ v = 0 \quad \textit{on} \quad \Gamma_0\},$$

$$a_t(w, w') = \int_{\Omega(t)} \sigma_{ij}(w) \varepsilon_{ij}(w') dx \qquad \textit{for} \quad w, w' \in V(\Omega(t)).$$

By virtue of Theorem 2.5 there exists a solution $v(t)$ of the problem (2.14) for each time t under an arbitrary load $\mathscr{L}$.

We shall compare the two potential energies

$$\mathrm{I}(\mathscr{L};\Sigma) = a_0(u, u)/2 - \int_\Omega f\cdot u\,dx - \int_{\Gamma_1} F\cdot u\,dS,$$

$$\mathrm{I}(\mathscr{L};\Sigma(t)) = a_t(v(t), v(t))/2 - \int_{\Omega(t)} f\cdot v(t)dx - \int_{\Gamma_1} F\cdot v(t)dS$$

of the elastic bodies each containing the crack Σ and $\Sigma(t)$, respectively, under the same load $\mathscr{L}$. Here $v(t)$ is the displacement vectors given above under the load $\mathscr{L}$ and $u=v(0)$. For simplicity, we put $a(w)=a(w, w)$ and $a_t(w)=a_t(w, w)$.

LEMMA 2.6. $\mathrm{I}(\mathscr{L};\Sigma) - \mathrm{I}(\mathscr{L};\Sigma(t)) = a_t(u-v(t))/2$.

PROOF. First we note that $u\in V(\Omega(t))$ since $\Omega(t)\subset\Omega$, and that the Lebesgue measure of $\Omega-\Omega(t)$ is zero. It follows from the symmetricity of the bilinear forms $a_0(w, w')$ and $a_t(w, w')$ that

$$\mathrm{I}(\mathscr{L};\Sigma) - \mathrm{I}(\mathscr{L};\Sigma(t)) = a_t(u-v(t))/2 + a_t(v(t), u-v(t))$$
$$- \int_{\Omega(t)} f\cdot(u-v(t))dx - \int_{\Gamma_1} F\cdot(u-v(t))dS.$$

Since $u-v(t)\in V(\Omega(t))$, we obtain

$$a_t(v(t), u-v(t)) = \int_{\Omega(t)} f\cdot(u-v(t))dx + \int_{\Gamma_1} F\cdot(u-v(t))dS.$$

Thus Lemma 2.6 follows.

REMARK 2.7. The strict inequality $\mathrm{I}(\mathscr{L};\Sigma)>\mathrm{I}(\mathscr{L};\Sigma(t))$ indicates that the elastic body containing the crack $\Sigma(t)$ is more stable than that containing the crack Σ. On the other hand, since

$$a_t(u-v(t)) \geqq \alpha(\Omega(t))\,\|u-v(t)\|_{1,\Omega(t)}$$

by Lemma 2.4, the equality $\mathrm{I}(\mathscr{L};\Sigma)=\mathrm{I}(\mathscr{L};\Sigma(t))$ implies that $v(t)=u$ in $\Omega(t)$. Therefore the equality shows that the elastic body has no discontinuity across $\Sigma(t)-\Sigma$, which means that the crack does not actually propagate.

3. Generalized J-integral

Before we calculate the energy release rate, let us give a brief summary of surface integrals of J-integral type. Earlier works which provide a three-dimensional version of the J-integral (1.2) are found in Eshelby [8] and Günther [12]. Eshelby [8], in a paper devoted to the continuum theory of lattice defects, deduced a surface integral representation

$$J_k(u) = \int_S \{W\nu_k - s\cdot(D_k u)\}dS \qquad (k=1, 2, 3) \tag{3.1}$$

for the force on elastic singularity or inhomogeneity of the portion enclosed by a surface S. Here W is the strain energy density, $s=(s_i)$ the traction vector, i.e., $s_i=\sigma_{ij}\nu_j$, and u the displacement vector as before. Günther [12], using Noether's theorem [18] on variational principles, obtained conservation laws for regular elastostatic fields appropriate to homogeneous solids. Here a linear elastic body is called homogeneous if all components of Hooke's tensor are constants. The meaning of the term the "elastic singularity" will be made precise later in terms of Sobolev spaces (see Definition 3.1). Let us consider a linear homogeneous elastic body whose elastic fields are regular and assume that the body force vanishes identically. Then the following conservation laws hold:

$$(3.2)\qquad J_k(u) = 0 \quad \text{for} \quad k = 1, 2, 3.$$

$$(3.3)\qquad M(u) = \int_S \{Wx_i\nu_i - s_j(D_iu_j)x_i - (s\cdot u)/2\}dS = 0.$$

If, moreover, the elastic body is isotropic, then also

$$(3.4)\qquad L_\alpha(u) = \int_S \varepsilon_{\alpha lk}\{Wx_k\nu_l - s_lu_k - s_p(D_lu_p)x_k\}dS = 0,$$

where $\varepsilon_{\alpha lk}$ are the components of the antisymmetric third order tensor such that $\varepsilon_{123}=+1$. Knowles and Sternberg [14] have shown that the conservation law (3.2) holds for more general class of materials, called hyperelastic materials, for which a strain energy density W is defined so that the stress tensor σ_{ij} is given by $\sigma_{ij}=\partial W/\partial\varepsilon_{ij}$. A hyperelastic material is homogeneous since W does not depend on x explicitly. They have also shown that the conservation law (3.4) holds if the material is isotropic and hyperelastic. Moreover the completeness of these three conservation laws for linear elasticity has been established in [14]. There have been many applications of two-dimensional versions of conservation laws to fracture mechanics (cf. Budiansky and Rice [2], Eshelby [9]). Here it should be noticed that, in general, the conservation laws do not hold for an elastic body containing a crack, because the elastic field is in general singular at the edge of the crack (cf. [20], [21]).

For linear (not necessarily homogeneous) elasticity, we consider a generalization of the surface integrals given in (3.1), (3.3), (3.4) (see Definition 3.3). As described later, we use this generalization to express the energy release rates for a class of smooth crack extensions (see Theorem 4.5).

We now turn our attention to the linear elastic body considered in section 2. First we define regular points and singular points of the elastic field.

DEFINITION 3.1. Let u be the displacement vector of the elastic body under a load $\mathscr{L}$, i.e., $u=T(\mathscr{L})$, and let β be a point of $\bar{\Omega}$. We call β a *regular point* of the elastic field under $\mathscr{L}$ if there exists an open neighborhood V_β of β such that

$u|_{V'_\beta} \in \boldsymbol{H}^2(V'_\beta)$. We call β a *singular point* of the elastic field under $\mathscr{L}$ if β is not a regular point.

We shall now show that singular points of the elastic field under an arbitrary load must belong to $\partial G \cup \partial\Sigma$, if the crack Σ is smooth, i.e., Σ is a 2-dimensional oriented C^∞-submanifold of R^3 with boundary. Since Σ is oriented, the unit outward normal to Σ is determined by the orientations of Σ and R^3. Moreover we can construct easily a domain Ξ with local Lipschitz property such that $\Sigma \subset \partial\Xi$ and the unit outward normal to $\partial\Xi$ equals that to Σ on Σ.

We may rewrite $D_j\sigma_{ij}(u)$ in the form

$$D_j\sigma_{ij}(u) = D_j(c_{ijkl}D_l u_k) \quad \text{with} \quad c_{ijkl} \in C^\infty(\bar{\Omega}) \tag{3.5}$$

(cf. Fichera [10]). By Lemma 2.4 the differential operator (3.5) satisfies the uniform ellipticity, that is, there exists a positive constant c such that

$$c_{ijkl}\xi_j\xi_l\eta_i\eta_k \geqq c|\xi|^2|\eta|^2$$

for any non-zero real vectors ξ, η (see e.g. [10]). Therefore a well-known regularity result for the elliptic partial differential system (see [10]) derives the following

PROPOSITION 3.2. *Let B be an open neighborhood of Σ in R^3 such that $\bar{B} \subset G$, and N an arbitrary open neighborhood of $\partial\Sigma$ in R^3 such that $\bar{N} \subset B$. If Σ is smooth, then the operator*

$$(f, F) \longrightarrow T(f, F)|_{(B-\bar{N})'}$$

is a bounded linear operator from $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Gamma_1)$ into $\boldsymbol{H}^2((B-\bar{N})')$.

From now on, we assume that Σ is smooth. For a domain A regular relative to Ω, we give a generalization $J_A(u;\mu)$ of the expressions $J_k(u)$, $M(u)$ and $L_\alpha(u)$ as a functional on the space $D(\Omega)$ of all displacement vectors u and the set $X(A)$ of all smooth vector fields defined on open neighborhoods of $\bar{A}$.

DEFINITION 3.3. Let A be a domain regular relative to Ω such that $\bar{A} \subset G$ and $\operatorname{dist}(\partial A, \partial\Sigma) > 0$. For each $u = T(f, F)$ and $\mu \in X(A)$, we define

$$J_A(u;\mu) = P_A(u;\mu) + R_A(u;\mu) \tag{3.6}$$

with

$$P_A(u;\mu) = \int_S \{W(\mu\cdot\nu) - s\cdot X_\mu(u)\}dS,$$

$$R_A(u;\mu) = -\int_{A'} \{(X_\mu(a_{ijkl})/2)\varepsilon_{kl}\varepsilon_{ij} - f\cdot X_\mu(u)\}dx + \int_{A'} \{\sigma_{ij}(D_j\mu^h)(D_h u_i) - W(\operatorname{div}\mu)\}dx,$$

where $W=\sigma_{ij}\varepsilon_{ij}/2$ (the strain energy density), $s_i=\sigma_{ij}\nu_j$, $S=\partial A$, $X_\mu=\mu^h D_h$ (μ^h the components of μ) and div $\mu=D_h\mu^h$. We call $J_A(u;\mu)$ the *generalized J-integral.*

First we shall show

PROPOSITION 3.4. *For any* $\mathscr{L}=(f, F)\in \boldsymbol{L}^2(\Omega)\times \boldsymbol{L}^2(\boldsymbol{\Gamma}_1)$ *and* $\mu\in X(A)$,

$$|J_A(T(\mathscr{L});\mu)| \leqq C(A)\,\|\mu\|_A\|\mathscr{L}\|^2, \tag{3.7}$$

where

$$\|\mathscr{L}\| = \{\|f\|_{0,\Omega}^2+\|F\|_{0,\Gamma_1}^2\}^{1/2}, \quad \|\mu\|_A = \sum_{h=1}^3 |\mu^h|_{C^2(\bar{A})}.$$

PROOF. Using the Schwarz inequality, we obtain

$$|R_A(u;\mu)| \leqq C\|\mu\|_A\|u\|_{1,A'}^2 \tag{3.8}$$

with some positive constant C, independent of A. Since $\bar{A}\subset G$ and dist $(\partial A, \partial\Sigma)>0$, we can take an open neighborhood N of ∂A such that $\bar{N}\subset G$ and $N\cap\partial\Sigma=\emptyset$. Using the Schwarz inequality and the trace theorem, we obtain

$$|P_A(u;\mu)| \leqq C(N, A)\,\|\mu\|_A\|u\|_{2,N'}^2.$$

Thus, by virtue of Theorem 2.5 and Proposition 3.2, (3.7) follows.

The connection between surface integrals in (3.1), (3.3), (3.4) and the generalized J-integral is given in the following.

THEOREM 3.5. *If all components a_{ijkl} of Hooke's tensor are constants and the body force f is zero, then for any domain A regular relative to Ω such that $\bar{A}\cap\Sigma=\emptyset$, and for $S=\partial A$, we have*

$$J_A(u;e_k) = J_k(u) \qquad \textit{for} \quad k = 1, 2, 3, \tag{3.9}$$

$$J_A(u;x) = M(u); \tag{3.10}$$

if, in addition, the elastic body is isotropic, then

$$J_A(u;q_\alpha) = L_\alpha(u) \qquad \textit{for} \quad \alpha = 1, 2, 3. \tag{3.11}$$

Here e_k is the unit base vector in the x_k-direction, $q_\alpha(x)=(\varepsilon_{\alpha 1k}x_k, \varepsilon_{\alpha 2k}x_k, \varepsilon_{\alpha 3k}x_k)$ and $x=(x_1, x_2, x_3)$.

PROOF. First we note that the first term of $R_A(u;\mu)$ in (3.6) vanishes, since a_{ijkl} are constants and $f=0$. Since $\bar{A}\subset G$ and $\bar{A}\cap\Sigma=\emptyset$, $u|_A\in \boldsymbol{H}^2(A)$ by Proposition 3.2.

(i) Proof of (3.9). By the fact that $x\to e_k$ is a constant vector field and $e_k\cdot\nu=\nu_k$, (3.9) is clear.

(ii) Proof of (3.10). Since $\sigma_{ij}(D_jx_h)D_hu_i=\sigma_{ij}D_ju_i$ and div $x=3$, it follows that

$$R_A(u;\, x) = -\int_A \sigma_{ij}(D_j u_i)/2\, dx.$$

Hence by (2.5)

$$R_A(u;\, x) = -\int_S (s\cdot u)/2\, dS,$$

which leads to (3.10).

(iii) Proof of (3.11). By a simple calculation

$$\sigma_{ij}D_j(\varepsilon_{\alpha hk}x_k)D_h u_i = \varepsilon_{\alpha hk}\sigma_{ih}D_i u_k \quad \text{(see [14])}.$$

Therefore, by (2.5)

$$R_A(u;\, q_\alpha) = \int_A \varepsilon_{\alpha hk}\sigma_{ih}D_i u_k\, dx = \varepsilon_{\alpha hk}\int_S s_h u_k\, dS,$$

which leads to (3.11).

The following proposition and Theorem 3.4 yield the conservation laws (3.2)–(3.4).

PROPOSITION 3.6. *If $\bar{A}\cap\Sigma=\emptyset$, then*

$$J_A(u;\, \mu) = 0 \quad \textit{for all} \quad u\in D(\Omega) \quad \textit{and} \quad \mu\in X(A).$$

PROOF. By Proposition 3.2, $u\,|_A\in \boldsymbol{H}^2(A)$. Hence, applying Lemma 2.1, we have

$$\int_A X_\mu(W)dx = -\int_A W(\operatorname{div}\mu)dx + \int_S W(\mu\cdot\nu)dS.$$

But

$$\int_A X_\mu(W)dx = \int_A \{X_\mu(a_{ijkl})/2\}\varepsilon_{kl}\varepsilon_{ij}\, dx$$
$$+\int_A \sigma_{ij}(u)D_j(X_\mu(u_i))dx - \int_A \sigma_{ij}(D_j\mu^h)(D_h u_i)dx.$$

From (2.5) it follows that

$$\int_A \sigma_{ij}(u)D_j(X_\mu(u_i))dx = -\int_A D_j(\sigma_{ij}(u))X_\mu(u_i)dx + \int_S s\cdot X_\mu(u)dS.$$

Collecting terms, we thus have

$$J_A(u;\, \mu) = 0 \quad \text{for all} \quad u\in D(\Omega) \quad \text{and} \quad \mu\in X(A).$$

Proposition 3.6 indicates that the generalized J-integral must vanish if the elastic field on A has no defects, i.e., $\bar{A}\cap\Sigma=\emptyset$. But, in general, $J_A(u;\, \mu)$ does not vanish and depends on the choice of A if $A \cap \Sigma \neq \emptyset$. Next we show that $J_A(u; \mu)$ is independent of the choice of A for some class of vector fields μ even if $A \cap \Sigma \neq \emptyset$.

PROPOSITION 3.7. *If the vector field μ is tangent to Σ, $J_A(u; \mu)$ takes the same value for all domains A regular relative to Ω such that $\partial\Sigma \subset A$ and $\mu \in X(A)$.*

PROOF. Assume that μ is defined on an neighborhood U of $\partial\Sigma$. Let A_1 and A_2 be two domains regular relative to Ω such that $\partial\Sigma \subset A_1$, $\bar{A}_1 \subset A_2$ and $\bar{A}_2 \subset U$. Letting $Q = A_2 - \bar{A}_1$, we have by (2.8),

$$(3.12)\quad \int_Q X_\mu(W)dx = \int_{Q'} W(\mu\cdot\nu)dS + \int_{\Sigma\cap Q} [\![\sigma_{ij}\varepsilon_{ij}]\!]\,(\mu\cdot\nu)\,dS - \int_{Q'} W(\operatorname{div}\mu)dx.$$

Here we used the fact that the elastic field is regular in Q' by Proposition 3.2, which also implies the applicability of (2.8). Since μ is tangent to Σ, the second term in the right-hand side of (3.12) vanishes. On the other hand, by symmetricity of the strain and stress tensors, we have

$$(3.13)\quad 2\int_{Q'} X_\mu(W)dx = \int_{Q'} X_\mu(a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}dx + 2\int_{Q'} \sigma_{ij}D_j(X_\mu(u_i))dx - 2\int_{Q'} \sigma_{ij}(D_j\mu^h)(D_hu_i)dx.$$

Applying (2.7), we obtain

$$(3.14)\quad \int_{Q'} \sigma_{ij}D_j(X_\mu(u_i))dx = -\int_{Q'} D_j\sigma_{ij}X_\mu(u_i)dx + \int_{\partial Q} \sigma_{ij}\nu_jX_\mu(u_i)dS + \int_{\Sigma\cap Q} [\![\sigma_{ij}\nu_jX_\mu(u_i)]\!]\,dS.$$

Since $\sigma^+_{ij}\nu_j = \sigma^-_{ij}\nu_j = 0$ on Σ, the last term in the right-hand side of (3.14) vanishes. Hence, noting that

$$-D_j\sigma_{ij} = f_i \quad \text{in} \quad \Omega,$$

we obtain

$$(3.15)\quad \int_{Q'} X_\mu(W)dx = \int_{Q'} \{(X_\mu(a_{ijkl})/2)\varepsilon_{kl}\varepsilon_{ij} + f\cdot X_\mu(u)\}dx + \int_{\partial Q} s\cdot X_\mu(u)dS - \int_{Q'} \sigma_{ij}(D_j\mu^h)(D_hu_i)dx.$$

Combining (3.12) and (3.15), we obtain the equation $J_{A_1}(u; \mu) - J_{A_2}(u; \mu) = 0$. This completes the proof of Proposition 3.7.

By the proposition, we may omit the subscript A of $J_A(u; \mu)$ if the vector field μ, defined on a neighborhood U of $\partial\Sigma$, is tangent to Σ, $\partial\Sigma\subset A$ and $\bar{A}\subset U$. The following proposition shows that we can neglect the volume integral part $R_A(u;\mu)$ if $|A|$ is sufficiently small.

PROPOSITION 3.8. *If a vector field μ is tangent to Σ, then*

$$J(u;\mu)=\lim_{|A|\to 0}\int_{\partial A}\{W(\mu\cdot\nu)-s\cdot X_\mu(u)\}dS.$$

PROOF. By virtue of Theorem 2.5, $u\in\boldsymbol{H}^1(\Omega)$. Hence $\|u\|_{1,A}\to 0$ as $|A|\to 0$. Since $f\in\boldsymbol{L}^2(\Omega)$, $\|f\|_{0,A}\to 0$ as $|A|\to 0$. Hence by (3.8)

$$R_A(u;\mu)\to 0 \quad \text{as} \quad |A|\to 0.$$

Thus we complete the proof of Proposition 3.8.

By an argument similar to the proof of Proposition 3.7, we can prove the following

PROPOSITION 3.9. *If the elastic field is regular at the edge $\partial\Sigma$ of the crack, then $J(u;\mu)=0$ for all μ tangent to Σ.*

4. Calculation of energy release rate

Next we shall show that the energy release rates are expressed as the generalized J-integral for the following class of crack extensions $\{\Sigma(t)\}$.

DEFINITION 4.1. A family $\{\Sigma(t)\}_{t\in[0,1]}$ of 2-dimensional C^∞-submanifolds of R^3 with boundary is called a *smooth crack extension* of Σ if it satisfies the following conditions (4.1–4):

(4.1) There exists a 2-dimensional oriented C^∞-submanifold Π of R^3 with boundary such that $\Pi\subset G$ and

$$\Sigma(t)\subset\Pi^\circ\ (=\Pi-\partial\Pi) \qquad \text{for all} \quad t\in[0,1].$$

(4.2) $$\Sigma(0)=\Sigma\subset\Sigma(t)\subset\Sigma(t') \quad \text{if} \quad 0<t<t'.$$

(4.3) For each $t\in[0,1]$, there exists a C^∞-diffeomorphism

$$\phi_t\colon \partial\Sigma\longrightarrow\partial\Sigma(t)$$

such that the map $\phi_t\colon\partial\Sigma\times[0,1]\to\Pi$ is of class C^∞.

(4.4) The limit $\lim_{t\to 0}t^{-1}|\Sigma(t)-\Sigma|$ exists and is non zero.

We now introduce a curvilinear coordinate system (U,p) in a region U near the edge $\partial\Sigma$ of the crack.

LEMMA 4.2. *There exists a C^∞-diffeomorphism p from $\partial\Sigma\times I^2$ onto an* *open neighborhood* U *of* $\partial\Sigma$ *in* R^3 *such that*

(4.5) $$p(x, 0, 0) = x \quad \textit{whenever } x \textit{ is in } \partial\Sigma.$$

(4.6) $$U \cap \Pi = p(\partial\Sigma\times \mathrm{I}\times\{0\}),\ U \cap \Sigma = p(\partial\Sigma\times \mathrm{I}'\times\{0\}),$$

where $\mathrm{I}=(-1, 1)$ *and* $\mathrm{I}'=(-1, 0]$.

We shall call the pair (U, p) a *product neighborhood* of $\partial\Sigma$.

PROOF. Since we assumed that $\partial\Sigma\subset\Pi^\circ$, there exists a neighborhood V of $\partial\Sigma$ in Π, $V\subset\Pi^\circ$, and a C^∞-diffeomorphism p_0 from $\partial\Sigma\times\mathrm{I}$ onto V such that $p_0(\partial\Sigma\times\mathrm{I}')=U\cap\Sigma$ and $p_0(x, 0)=x$ whenever x is in $\partial\Sigma$. Here we used the well-known result on the existence of the product neighborhood (see e.g. Munkres [16]). Since Π is oriented and $V\subset\Pi^\circ$, there exists a diffeomorphism p_1 from $V\times\mathrm{I}$ onto an open neighborhood U of $\partial\Sigma$ in R^3 such that $p_1(x, 0)=x$ whenever x is in V (see [16]). Let us now set

$$p(\xi, \eta, \lambda) = p_1(p_0(\xi, \eta), \lambda) \quad \text{for} \quad (\xi, \eta, \lambda)\in\partial\Sigma\times\mathrm{I}^2.$$

Then it is clear that $p\colon \partial\Sigma\times\mathrm{I}^2\to U$ satisfies the assertions in Lemma 4.2.

In terms of a product neighborhood of $\partial\Sigma$, each edge $\partial\Sigma(t)$ of newly created crack is represented by the graph of a smooth function h_t defined on $\partial\Sigma$.

LEMMA 4.3. *Let* $\{\Sigma(t)\}_{t\in[0,1]}$ *be a smooth crack extension and* (U, p) *a product neighborhood of* $\partial\Sigma$. *Then there exist a positive number* t_0 *and a family* $\{h_t\}_{t\in[0,t_0]}$ *of smooth functions defined on* $\partial\Sigma$ *such that*

(4.7) $$0\leqq h_t\leqq 1 \quad \textit{for all } t\in[0, t_0] \textit{ and } h_0(x)=0 \textit{ for all } x\in\partial\Sigma,$$

(4.8) $$\textit{the map } h_t\colon \partial\Sigma\times[0, t_0]\to[0, 1) \textit{ is of class } C^\infty,$$

(4.9) $$\partial\Sigma(t)\cap U = \{x;\ x=p(\xi, h_t(\xi), 0),\ \xi\in\partial\Sigma\},$$
$$\Sigma(t)\cap U = \{x;\ x=p(\xi, \eta, 0),\ \xi\in\partial\Sigma,\ -1<\eta\leqq h_t(\xi)\}.$$

Before proving this lemma, we prepare some geometrical concepts. Let M be an m-dimensional C^∞-submanifold (with boundary) of R^3, and consider the Riemannian metric on M induced by the imbedding into R^3. We denote the tangent space to M at x by T_xM and the tangent bundle $\cup_x T_xM$ (disjoint union) by TM; for each $x\in M$, T_xM is identified with the m-dimensional subspace of R^3. Let M and N be two C^∞-submanifolds of R^3 and $f\colon M\to N$ a C^1-map. By df_x we denote the differential of f at x, which is a linear map from T_xM to $T_{f(x)}N$.

Let us denote the space of all C^1-maps of M to N by $F^1(M, N)$, which is topologized as follows: Given a C^1-map $f\colon M\to N$ and a positive continuous function δ on M, let $W(f, \delta)$ be the set of all C^1-maps $g\colon M\to N$ such that

$$|f(x)-g(x)| \leqq \delta(x), \quad |df_x(v)-dg_x(v)| \leqq \delta(x)\,|v|$$

for each $x \in M$ and each $v \in T_xM$, where $|\cdot|$ stands for the Euclidian distance in R^3. The sets $W(f, \delta)$ form a basis for what is called the fine C^1-topology on $F^1(M, N)$. Then one can prove (see e.g. [16]):

PROPOSITION 4.4. *Let M and N be two manifolds and $f\colon M \to N$ a C^1-map. If f is an imbedding, there exists a fine C^1-neighborhood of f consisting only of imbeddings. If f is a C^1-diffeomorphism, there exists a fine C^1-neighborhood of f such that if g is in this neighborhood and carries ∂M into ∂N, then g is a diffeomorphism.*

PROOF OF LEMMA 4.3. By assumption (4.3) there exists a positive number $\tau_0 \leqq 1$ such that $\partial\Sigma(t) \subset U$ for all $t \leqq \tau_0$. Now we put

$$((\theta_t^1(\xi), \theta_t^2(\xi), 0) = p^{-1} \circ \phi_t(\xi); \ \xi \in \partial\Sigma, \quad t \in [0, \tau_0]. \tag{4.10}$$

Then from (4.2), (4.3) and Lemma 4.2 it follows that

(4.11) $\quad 0 \leqq \theta_t^2 < 1$ for all $t \in [0, \tau_0]$, $\theta_0^2(\xi) = 0$ for all $\xi \in \partial\Sigma$,

(4.12) $\quad$ the maps $\theta_t^1\colon \partial\Sigma \times [0, \tau_0] \to \partial\Sigma$ and $\theta_t^2\colon \partial\Sigma \times [0, \tau_0] \to [0, 1)$ are of class C^∞.

Applying Proposition 4.4, we can take a positive number τ such that there exists an inverse $(\theta_t^1)^{-1}$ of θ_t^1 for each $t \in [0, \tau]$. Here we used the fact that $\theta_0^1(\xi) = \xi$ for all $\xi \in \partial\Sigma$ and the boundary of $\partial\Sigma$ is empty. Next we shall show that

(4.13) the map $(\theta_t^1)^{-1}\colon \partial\Sigma \times [0, t_0] \to \partial\Sigma$ is of class C^∞ for some positive number t_0 $(\leqq \tau)$.

Let $\{(\alpha_i, V_i)\}_{i=1,2,\dots,m}$ be a local coordinate system of $\partial\Sigma$. Choose a positive number t_0 and a refinement $\{W_i\}$ of the covering $\{V_i\}$ of $\partial\Sigma$ such that $\theta_t^1(\xi) \in V_i$ for all $t \leqq t_0$ whenever ξ is in W_i. Setting $\psi_{it}(\omega) = \alpha_i \circ (\theta_t^1)^{-1} \circ \alpha_i^{-1}(\omega)$, $\omega \in \alpha_i(W_i)$, and using the implicit function theorem, we obtain the following ordinary differential equations depending on the parameter t,

$$\frac{d}{d\omega}\psi_{it}(\omega) = f_{it}(\omega), \tag{4.14}$$

where $f_{it}(\omega) = \left(\dfrac{d}{d\omega}(\alpha_i \circ \theta_t^1 \circ \alpha_i^{-1})(\omega)\right)^{-1}$, $\omega \in \alpha_i(W_i)$. Since the maps $f_{it}\colon \alpha_i(W_i) \times [0, t_0] \to R^1$ are of class C^∞, the assertion (4.13) follows from (4.14) and the well-known result on ordinary differential equations (see e.g. Hartman [13]).

Let us now put

$$h_t(\xi) = \theta_t^2((\theta_t^1)^{-1}(\xi)) \qquad \text{for} \quad \xi \in \partial\Sigma, \qquad t \in [0, t_0].$$

Then (4.7) follows from (4.11), (4.8) follows from (4.12) and (4.13), and (4.9) is clear from (4.3), (4.5) and (4.10).

Lemma 4.3 gives a smooth vector field $\tau(x)$ on U as follows: For each $x \in U$, there exists a positive number c_x such that

$$\kappa_t(x) = p(\xi(x), \eta(x) + h_t(\xi(x)), \lambda(x))$$

belongs to U if $t \in [0, c_x)$, where $(\xi(x), \eta(x), \lambda(x)) = p^{-1}(x)$. By virtue of Lemma 4.2 and Lemma 4.3, the parametrized path $t \to \kappa_t(x)$, $t \in [0, c_x)$, is of class C^∞ and $\kappa_0(x) = x$ for all $x \in U$. The vector field τ on U is defined by

$$\tau(x) = \left(\frac{d}{dt}\kappa_t(x)\right)_{t=0} \in T_x R^3 = R^3; \qquad x \in U. \tag{4.15}$$

The main result in this paper is the following theorem.

THEOREM 4.5. *For a given load* $\mathscr{L}$ *and a given smooth crack extension* $\{\Sigma(t)\}$, *the energy release rate*

$$G(\mathscr{L}; \{\Sigma(t)\}) = \lim_{t \to 0} \frac{\mathrm{I}(\mathscr{L}; \Sigma) - \mathrm{I}(\mathscr{L}; \Sigma(t))}{|\Sigma(t) - \Sigma|},$$

given in (1.1), *is expressed in the form*

$$G(\mathscr{L}; \{\Sigma(t)\}) = k^{-1} J(T(\mathscr{L}); \tau) \tag{4.16}$$

where $k = \lim_{t \to 0} t^{-1}|\Sigma(t) - \Sigma|$ *and* τ *is the vector field given in* (4.15) *obtained from* $\{\Sigma(t)\}$.

From Proposition 3.9 the following result follows.

COROLLARY 4.6. *If, for a given load* $\mathscr{L}$, *the elastic field is regular at the edge* $\partial\Sigma$ *of the crack* (*see Definition* 3.1), *then* $G(\mathscr{L}; \{\Sigma(t)\}) = 0$ *for any smooth crack extension* $\{\Sigma(t)\}$.

REMARK 4.7. Although τ may depend on the choice of a product neighborhood (U, p) of $\partial\Sigma$, $J(u; \tau)$ is independent of the choice of (U, p), because $G(\mathscr{L}; \{\Sigma(t)\})$ depends only on $\{\Sigma(t)\}$.

Before proving Theorem 4.5 we prepare some auxiliary results. Hereafter we fix a smooth crack extension $\{\Sigma(t)\}$ and assume that $\{\Sigma(t)\}$ is expressed as in Lemma 4.3.

The following maps ρ_t play basic roles in our calculation. For the construction of ρ_t, we take a function $\beta \in C_0^\infty(R^3)$ such that

$$\operatorname{supp} \beta \subset U \quad \text{and} \quad \beta = 1 \quad \text{on} \quad Q, \tag{4.17}$$

where Q is an open neighborhood of $\partial\Sigma$ in R^3 such that

(4.18) $$\partial\Sigma(t) \subset Q \quad \text{for any} \quad t \in [0, t_0] \quad \text{and} \quad \bar{Q} \subset U.$$

We now put

$$\rho_t(x) = \begin{cases} p(\xi(x), \eta(x) - \beta(x)h_t(\xi(x)), \lambda(x)), & \text{for} \quad x \in U, \\ x, \quad \text{for} \quad x \in R^3 - U, \end{cases}$$

where $(\xi(x), \eta(x), \lambda(x)) = p^{-1}(x)$.

Our construction of ρ_t yields the following

LEMMA 4.8. *There exists a positive number $\theta \leqq t_0$ such that the family $\{\rho_t\}_{t\in[0,\theta]}$ of maps satisfies the following*:

(4.19) *The map ρ_t: $R^3 \to R^3$ is a C^∞-diffeomorphism for each $t \in [0, \theta]$.*

(4.20) $\rho_t(\Sigma(t)) = \Sigma$ *for any $t \in [0, \theta]$.*

(4.21) *The map ρ_t: $R^3 \times [0, \theta] \to R^3$ is of class C^∞.*

PROOF. First we note that $\rho_t(x) = x$ for all $x \in R^3 - \operatorname{supp}\beta$ and $\operatorname{supp}\beta \subset U$. Hence, from Lemma 4.2 and Lemma 4.3, it follows that the map ρ_t: $R^3 \times [0, t_0] \to R^3$ is of class C^∞. Moreover $\rho_0(x) = x$ for all $x \in R^3$ by (4.7) and each ρ_t reduces to the identity except on the bounded set $\operatorname{supp}\beta$. Therefore, using Proposition 4.4, we can take a positive number θ such that ρ_t, $t \in [0, \theta]$, are diffeomorphisms from R^3 onto R^3. By (4.9) and the construction of ρ_t, the set $(p^{-1} \circ \rho_t)(\Sigma(t) \cap U)$ is expressed as

$$\{(\xi, \eta, 0);\ \xi \in \partial\Sigma,\ -1 - k_t(\xi, \eta) \leqq \eta \leqq h_t(\xi) - k_t(\xi, \eta)\},$$

where $k_t(\xi, \eta) = \beta(p(\xi, \eta, 0))h_t(\xi)$. From this it follows that $(p^{-1} \circ \rho_t)(\Sigma(t) \cap U) = p^{-1}(\Sigma \cap U)$ if $t \in [0, \theta]$. This completes the proof of Lemma 4.8.

For a function v defined on Ω, we define a function v_t^* on $\Omega(t)$ by

$$v_t^*(x) = v(\rho_t(x)).$$

For simplicity we put

$$D^{m,p}(\Omega) = W^{m,p}(\Omega) \cap C^\infty(\Omega) \quad \text{and} \quad D^m(\Omega) = H^m(\Omega) \cap C^\infty(\Omega),$$

which are dense in $W^{m,p}(\Omega)$ and $H^m(\Omega)$, respectively, by the density theorem. Since the mappings ρ_t reduce to the identity except on the bounded set $\operatorname{supp}\beta$, Lemma 4.8 leads to the estimates

(4.22) $$C^{-1}|v|_{m,p,\Omega} \leqq |v_t^*|_{m,p,\Omega(t)} \leqq C|v|_{m,p,\Omega}$$

for all $v \in D^{m,p}(\Omega)$, $m \geqq 0$ and $1 \leqq p < \infty$, where C is a positive constant independent of t. In what follows we denote by C various constants independent of t. The density theorem implies the following

LEMMA 4.9. *The pull-backs* $v \to v_t^*$ *are extended by continuity to linear homeomorphisms from* $W^{m,p}(\Omega)$ *onto* $W^{m,p}(\Omega(t))$ *for any* $m \geqq 0$ *and* $1 \leqq p < \infty$, *and also from* $V(\Omega)$ *onto* $V(\Omega(t))$. *Furthermore* v_t^* *satisfy the estimates* (4.22).

Let $v(t)$ be the solution of the quasi-static problem (2.14) under a load $\mathscr{L}$ and let $u = v(0)$. Then u_t^* gives a fairly good approximation of $v(t)$ as the following lemma shows.

LEMMA 4.10. *There exists a positive constant* t_1 *such that*

$$\|u_t^* - v(t)\|_{1,\Omega(t)} \leqq Ct\|\mathscr{L}\| \qquad \text{for all} \quad t \in [0, t_1]. \tag{4.23}$$

PROOF. 1) Let us denote the components of the matrix $D\rho_t(x) - I$ by $b_{ijt}(x)$; $i, j = 1, 2, 3$, where $D\rho_t(x)$ is the Jacobian matrix of ρ_t at x. For $w \in \{D^1(\Omega)\}^3$,

$$\varepsilon_{ij}(w_t^*) = [\varepsilon_{ij}(w)]_t^* + v_{ijt}(w) \text{ with } v_{ijt}(w) = \{(D_k w_i)_t^* b_{kjt} + (D_k w_j)_t^* b_{kit}\}/2. \tag{4.24}$$

Hence, for any $v, w \in \{D^1(\Omega)\}^3$,

$$\int_{\Omega(t)} \sigma_{ij}(v_t^*)\varepsilon_{ij}(w_t^*)dx = \int_{\Omega(t)} [a_{ijkl}]_t^*[\varepsilon_{kl}(v)]_t^*[\varepsilon_{ij}(w)]_t^*|\det(D\rho_t)|dx + l_t(v, w),$$

where

$$\begin{aligned} l_t(v, w) = &\int_{\Omega(t)} a_{ijkl}\{[\varepsilon_{kl}(v)]_t^* v_{ijt}(w) + v_{klt}(v)[\varepsilon_{ij}(w)]_t^* + v_{klt}(v)v_{ijt}(w)\}dx \\ &+ \int_{\Omega(t)} [a_{ijkl}]_t^*[\varepsilon_{kl}(v)]_t^*[\varepsilon_{ij}(w)]_t^*(1 - |\det(D\rho_t)|)dx \\ &+ \int_{\Omega(t)} (a_{ijkl} - [a_{ijkl}]_t^*)[\varepsilon_{kl}(v)]_t^*[\varepsilon_{ij}(w)]_t^* dx. \end{aligned}$$

By the change of variables $y = \rho_t(x)$,

$$\int_{\Omega(t)} [a_{ijkl}]_t^*[\varepsilon_{kl}(v)]_t^*[\varepsilon_{ij}(w)]_t^*|\det(D\rho_t)|dx = a_0(v, w).$$

We thus have for $v, w \in \{D^1(\Omega)\}^3$ the formula

$$a_t(v_t^*, w_t^*) = a_0(v, w) + l_t(v, w), \tag{4.25}$$

where $l_t(v, w)$ satisfies the estimate

$$|l_t(v, w)| \leqq Ct\|v\|_{1,\Omega}\|w\|_{1,\Omega} \tag{4.26}$$

by virtue of Lemma 4.9, the Schwarz inequality, the fact that $a_{ijkl} \in C^\infty(\bar{\Omega})$ and the following inequalities:

(4.27) $\qquad \sup_{x\in R^3}|\rho_t(x)-x| \leqq Ct, \quad \max_{i,j}\sup_{x\in R^3}|b_{ijt}(x)| \leqq Ct.$

The inequalities (4.27) follow from Lemma 4.8 and the fact that ρ_t reduce to the identity except on a bounded set. According to the density theorem, (4.25) and (4.26) hold for all v, $w\in \boldsymbol{H}^1(\Omega)$.

2) Next we shall show that there exist positive constants α (independent of t) and t_1 such that

(4.28) $$a_t(v, v) \geqq \alpha\|v\|_{1,\Omega(t)}^2$$

for all $v\in V(\Omega(t))$ and $t\in[0, t_1]$. From Lemma 2.4, (4.25) and (4.26) it follows that

$$a_t(v_t^*, v_t^*) \geqq a_0(v, v) - Ct\|v\|_{1,\Omega}^2 \geqq \alpha(\Omega)\|v\|_{1,\Omega}^2 - Ct\|v\|_{1,\Omega}^2$$

for all $v\in V(\Omega)$ and $t\in[0, \theta]$. We now take a number t_1 to satisfy $Ct_1\leqq\alpha(\Omega)/2$ and then we obtain

$$a_t(v_t^*, v_t^*) \geqq (\alpha(\Omega)/2)\|v\|_{1,\Omega}^2$$

for all $v\in V(\Omega)$ and $t\in[0, t_1]$. The desired estimate (4.28) follows from Lemma 4.9.

3) For $\psi\in V(\Omega)$, by (4.25),

$$a_t(v_t^*, \psi_t^*) = a_t(u, \psi) + l_t(u, \psi) = \int_\Omega f\cdot\psi\, dx + \int_{\Gamma_1} F\cdot\psi\, dS + l_t(u, \psi)$$
$$= \int_{\Omega(t)} f\cdot\psi_t^* dx + \int_{\Gamma_1} F\cdot\psi_t^* dS + \int_{\Omega(t)} f\cdot(\psi-\psi_t^*)dx + l_t(u, \psi),$$

since u is the solution of the problem (2.10), $\psi=\psi_t^*$ on Γ, and the Lebesgue measure of $\Omega-\Omega(t)$ is zero. Then we have

$$a_t(u_t^*-v(t), \psi_t^*) = \int_{\Omega(t)} f\cdot(\psi-\psi_t^*)dx + l_t(u, \psi).$$

Hence by virtue of (4.26), we have

(4.29) $\qquad |a_t(u_t^*-v(t), \psi_t^*)| \leqq \{\|f\|_{0,\Omega}\|\psi-\psi_t^*\|_{0,\Omega(t)} + Ct\|u\|_{1,\Omega}\|\psi\|_{1,\Omega}\}$

for all $\psi\in V(\Omega)$. We shall now prove

(4.30) $\qquad \|\psi-\psi_t^*\|_{0,\Omega(t)} \leqq Ct\|\psi\|_{1,\Omega} \qquad$ for all $\psi\in\boldsymbol{H}^1(\Omega)$.

It is sufficient to prove our assertion for $\psi\in\{D^1(\Omega)\}^3$. For each $t\in[0, t_1]$ and $x\in\Omega(t)$, the path $C_t=\{\rho_s(x);\ s\in[0, t]\}$ is contained in Ω, whence $s\to\psi(\rho_s(x))$ is a C^1-map from $[0, t]$ to R^2. Using the chain rule, the Schwarz inequality, Fubini's theorem and Lemma 4.9, we then have

$$\begin{aligned}\int_{\Omega(t)} |\psi_t^*(x)-\psi(x)|^2 dx &\leqq C\int_{\Omega(t)} \left|\int_0^t \frac{d}{ds}(\psi(\rho_s(x)))ds\right|^2 dx \\ &\leqq Ct\int_{\Omega(t)}\int_0^t \left|\frac{d}{ds}\psi(\rho_s(x))\right|^2 dsdx \\ &\leqq Ct\int_0^t \|\psi_t^*\|_{1,\Omega(s)}^2 ds \leqq Ct^2\|\psi\|_{1,\Omega}^2,\end{aligned}$$

which leads to (4.30). By virtue of Theorem 2.5, (4.29) and (4.30), it follows that

$$|a_t(u_t^* - v(t),\ \psi_t^*)| \leqq Ct\|\mathscr{L}\|\ \|\psi_t^*\|_{1,\Omega(t)}. \tag{4.31}$$

Taking for ψ_t^* in (4.31) the function $u_t^* - v(t)$ and using (4.28), we now have

$$\alpha\|u_t^* - v(t)\|_{1,\Omega(t)}^2 \leqq a_t(u_t^* - v(t),\ u_t^* - v(t)) \leqq Ct\|\mathscr{L}\|\ \|u_t^* - v(t)\|_{1,\Omega(t)}.$$

Hence

$$\|u_t^* - v(t)\|_{1,\Omega(t)} \leqq Ct\|\mathscr{L}\|.$$

This completes the proof of Lemma 4.10.

LEMMA 4.11. *If $f \in W^{1,1}(\Omega)$, then*

$$\lim_{t\to 0} t^{-1}\int_{\Omega(t)} \{f_t^* - f\}dx = -\int_\Omega X_\tau(f)\beta\, dx, \tag{4.32}$$

where τ and β are those given in (4.15) *and* (4.17), *respectively.*

PROOF. Since $D^{1,1}(\Omega)$ is dense in $W^{1,1}(\Omega)$, there exists a sequence $\{f_j\}$ of functions in $D^{1,1}(\Omega)$ such that $f_j \to f$ in $W^{1,1}(\Omega)$ as $j\to\infty$. By the same argument as in the proof of Lemma 4.10 we obtain

$$\begin{aligned}&\left|\int_{\Omega(t)} \{(f_j)_t^* - f_j\} - \{f_t^* - f\}dx\right| \\ &\quad \leq \int_{\Omega(t)}\int_0^t \left|\frac{d}{d\zeta}[f_j - f]_\zeta^*(x)\right| d\zeta dx \leqq Ct\|f_j - f\|_{1,1,\Omega},\end{aligned}$$

from which it follows that

$$t^{-1}\int_{\Omega(t)} \{(f_j)_t^* - f_j\}dx \longrightarrow t^{-1}\int_{\Omega(t)} \{f_t^* - f\}dx$$

uniformly with respect to t as $j\to\infty$. Hence

$$Q(f) = \lim_{j\to\infty} Q(f_j),$$

where $Q(f)$ stands for the left-hand side of (4.32). Therefore it is sufficient to prove (4.32) for $f\in D^{1,1}(\Omega)$. For some fixed number $\theta>0$, we have

$$Q(f) = \lim_{t\to 0} t^{-1}\int_{\Omega(\theta)} \{f_t^* - f\}dx,$$

since $\Omega(t)\supset\Omega(\theta)$ by (4.2) and the Lebesgue measures of $\Omega(t)-\Omega(\theta)$ are zero for all $t\in[0, \theta]$. For an arbitrary domain Ω_0 strictly contained in $\Omega(\theta)$, we obtain by the Lebesgue dominated convergence theorem

$$\lim_{t\to 0} t^{-1}\int_{\Omega_0} \{f_t^* - f\}dx = \int_{\Omega_0} \lim_{t\to 0} t^{-1}\{f_t^* - f\}dx$$

for all $f\in D^{1,1}(\Omega)$, since $f\in C^\infty(\bar{\Omega}_0)$. By an argument analogous to the above we obtain for $t\in[0, \theta]$

$$\left|t^{-1}\int_{\Omega(\theta)-\Omega_0} \{f_t^* - f\}dx\right|$$

$$\leqq t^{-1}\left|\int_{\Omega(\theta)-\Omega_0}\int_0^t \frac{d}{d\zeta} f_t^* d\zeta dx\right| \leqq C\|f\|_{1,1,M(\Omega_0)},$$

where $M(\Omega_0)=\bigcup_{t\in[0,\theta]}\rho_t(\Omega(\theta)-\Omega_0)$. We can see that $|M(\Omega_0)|\to 0$ if $\Omega_0\to\Omega(\theta)$. Thus we can deduce that

$$Q(f) = \int_{\Omega(\theta)} \lim_{t\to 0} t^{-1}(f_t^* - f)dx. \tag{4.33}$$

Setting $p^{-1}(x)=(\xi(x), \eta(x), \lambda(x))$, we have

$$\tau(x) = \left(\frac{d}{dt} h_t(\xi(x))\Big|_{t=0}\right)\frac{\partial p}{\partial\eta}(\xi(x), \eta(x), \lambda(x)),$$

$$\frac{d}{dt}\rho_t(x)\Big|_{t=0} = -\beta(x)\left(\frac{d}{dt} h_t(\xi(x))\Big|_{t=0}\right)\frac{\partial p}{\partial\eta}(\xi(x), \eta(x), \lambda(x)).$$

Since $f\in D^{1,1}(\Omega)$, we obtain

$$\lim_{t\to 0} t^{-1}(f(\rho_t(x))-f(x)) = \frac{d}{dt} f(\rho_t(x))\Big|_{t=0} = \left(\frac{d}{dt}\rho_t^j(x)\Big|_{t=0}\right)D_j f(x)$$

$$= -\beta(x)\left(\frac{d}{dt} h_t(\rho(x))\Big|_{t=0}\right)\frac{\partial p_j}{\partial\eta}(\xi(x), \eta(x), \lambda(x))D_j f(x)$$

$$= -\beta(x)\tau^j(x)D_j f(x) = -X_\tau(f)(x)\beta(x).$$

Therefore (4.32) follows from (4.33).

In the same way we can prove

LEMMA 4.12. *Let τ and β be as in Lemma 4.10.*
(a) *If $f\in H^1(\Omega)$ and $g\in L^2(\Omega)$, then*

(4.34) $$\lim_{t\to 0} t^{-1}\int_{\Omega(t)} g(f_t^* - f)dx = -\int_{\Omega} gX_\tau(f)\beta\, dx.$$

(b) *If* $f \in C^\infty(\bar{\Omega})$ *and* $g \in L^1(\Omega)$, *then*

(4.35) $$\lim_{t\to 0} t^{-1}\int_{\Omega(t)} g_t^*(f_t^* - f)dx = -\int_{\Omega} gX_\tau(f)\beta\, dx.$$

We are now in a position to prove Theorem 4.5.

PROOF OF THEOREM 4.5. 1) By virtue of Lemma 2.6 and the symmetricity of $a_t(v, w)$, we have

$$\begin{aligned} 2\{\mathrm{I}(\mathscr{L}; \Sigma) - \mathrm{I}(\mathscr{L}; \Sigma(t))\} &= a_t(u - v(t)) = a_t(u - u_t^* + u_t^* - v(t)) \\ &= a_t(u) - a_t(u_t^*) + 2a_t(v(t), u_t^* - u) + a_t(u_t^* - v(t)) \\ &= a_t(u) - a_t(u_t^*) + a_t(u_t^* - v(t)) + 2\int_{\Omega(t)} f\cdot(u_t^* - u)dx, \end{aligned}$$

since $v(t)$ is the solution of the problem (2.14), $u_t^* - u \in V(\Omega(t))$ and $u_t^* - u = 0$ on Γ. Hence

(4.36) $$kG(\mathscr{L}; \{\Sigma(t)\}) = \lim_{t\to 0} t^{-1}\{(a_t(u) - a_t(u_t^*))/2\} + \lim_{t\to 0} t^{-1}\int_{\Omega(t)} f\cdot(u_t^* - u)dx + \lim_{t\to 0} t^{-1} a_t(u_t^* - v(t))/2.$$

By Lemma 4.10,

$$|a_t(u_t^* - v(t))| \leqq C\|u_t^* - v(t)\|_{1,\Omega(t)}^2 \leqq Ct^2\|\mathscr{L}\|^2.$$

Hence the last term in the right-hand side of (4.36) vanishes. Thus from Lemma 4.12 we can deduce that

(4.37) $$kG(\mathscr{L}; \{\Sigma(t)\}) = \lim_{t\to 0} t^{-1}\mathrm{I}_t - \int_{\Omega} f\cdot X_\tau(u)\beta\, dx,$$

where $2\mathrm{I}_t = a_t(u) - a_t(u_t^*)$. By symmetricity of a_{ijkl}, $[\varepsilon_{ij}(u)]_t^*$ and $v_{ijt}(v)$ (see (2.1) and (4.24)), I_t is written in the form

(4.38) $$2\mathrm{I}_t = \mathrm{I}_{1t} + \mathrm{I}_{2t} + \mathrm{I}_{3t} + \mathrm{I}_{4t},$$

where

$$\mathrm{I}_{1t} = 2\int_{\Omega(t)} (W - W_t^*)dx,$$

$$\mathrm{I}_{2t} = -2\int_{\Omega(t)} a_{ijkl}[\varepsilon_{kl}(u)]_t^* v_{ijt}(u)dx,$$

$$I_{3t} = \int_{\Omega(t)} [(a_{ijkl})_t^* - a_{ijkl}]\,[\varepsilon_{kl}(u)]_t^*[\varepsilon_{ij}(u)]_t^* dx,$$

$$I_{4t} = -\int_{\Omega(t)} a_{ijkl} v_{klt}(u) v_{ijt}(u) dx.$$

Since, by (4.22) and (4.27),

$$|v_{ijt}(u)|_{1,\Omega(t)} \leqq Ct\|u\|_{1,\Omega} \qquad \text{for all} \quad i, j = 1, 2, 3,$$

we have

(4.39) $$\lim_{t\to 0} t^{-1} I_{4t} = 0,$$

and by Lemma 4.12, (b), we obtain

(4.40) $$\lim_{t\to 0} t^{-1} I_{3t} = -\int_{\Omega} X_\tau(a_{ijkl})\varepsilon_{kl}(u)\varepsilon_{ij}(u)\beta\, dx.$$

2) Singularity at the edge $\partial\Sigma$ of the crack gives rise to difficulties in calculating the limits $\lim_{t\to 0} t^{-1} I_{1t}$ and $\lim_{t\to 0} t^{-1} I_{2t}$. So we introduce cut-off functions ζ, $\varphi \in C_0^\infty(R^3)$ such that

(4.41) $0 \leqq \zeta,\ \varphi \leqq 1$, $\beta = 1$ on $\operatorname{supp}(1-\zeta)$, $\zeta = 1$ near ∂Q, $\zeta = 0$ near $\partial\Sigma$;

$\varphi = 1$ on $\operatorname{supp}\zeta$ and $\varphi = 0$ near $\partial\Sigma$.

We put

$$I_{1t,1} = \int_{\Omega(t)} \{(1-\zeta)W - ((1-\zeta)W)_t^*\} dx.$$

Then, by the change of variables $y = \rho_t(x)$, we may rewrite it as

$$I_{1t,1} = \int_{\Omega} (1-\zeta)W(1-J_t) dx,$$

where $J_t = |\det(D\rho_t)|^{-1}$. Since the map $J_t\colon R^3 \times [0, t_1] \to R^1$ is of class C^∞ by (4.21) and $W \in L^1(\Omega)$, we obtain

(4.42) $$\lim_{t\to 0} t^{-1} I_{1t,1} = -\int_{\Omega} (1-\zeta)WJ' dx,$$

where $J' = \dfrac{d}{dt} J_t|_{t=0}$. Since $\zeta W \in W^{1,1}(\Omega)$ by Proposition 3.2, we obtain by Lemma 4.11

$$\lim_{t\to 0} t^{-1}\int_{\Omega(t)} \{\zeta W - (\zeta W_t^*)\} dx = \int_{\Omega} X_\tau(\zeta W)\beta\, dx.$$

Thus we have

(4.43) $$\lim_{t\to 0} t^{-1}(I_{1t}/2) = \int_{\Omega} X_\tau(\zeta W)\beta\, dx - \int_{\Omega} (1-\zeta)WJ' dx.$$

From the symmetricity of a_{ijkl} and $[\varepsilon_{ij}(u)]_t^*$, it follows that

$$\mathrm{I}_{2t}/2 = -\int_{\Omega(t)} a_{ijkl}[\varepsilon_{kl}(u)]_t^* D_j(\rho_t^* - x_h)(D_h u_i)_t^h dx.$$

Hence

$$\lim_{t\to 0} t^{-1}(\mathrm{I}_{2t}/2) = \int_\Omega a_{ijkl}\varepsilon_{kl}(u) D_j(\beta\tau^h) D_h u_i\, dx$$

$$= \int_\Omega \sigma_{ij}(D_j\beta) X_\tau(u_i) dx + \int_\Omega \sigma_{ij}(D_j\tau^h)(D_h u_i)\beta\, dx.$$

Since $D_j\beta = 0$ on $\mathrm{supp}\,(1-\zeta)$ and $\varphi = 1$ on $\mathrm{supp}\,\zeta$, we have

$$\int_\Omega \sigma_{ij}(D_j\beta)(X_\tau(u_i))\, dx = \int_\Omega \varphi\sigma_{ij}(D_j\beta)(X_\tau(\zeta u_i)) dx.$$

Thus, since $\varphi\sigma_{ij}$, $\beta X_\tau(\zeta u_i) \in H^1(\Omega)$ by Proposition 3.2, (2.8) implies

$$(4.44)\quad \int_\Omega \sigma_{ij}(D_j\beta)(X_\tau(u_i)) dx = -\int_\Omega D_j[\varphi\sigma_{ij} X_\tau(\zeta u_i)]\beta\, dx + \int_\Sigma [\![\varphi\sigma_{ij}\nu_j X_\tau(\zeta u_i)]\!]\, dS.$$

The last term of (4.44) vanishes by the assumption that $\sigma_{ij}^+\nu_j = \sigma_{ij}^-\nu_j = 0$ on Σ. Thus we have

$$(4.45)\quad \lim_{t\to 0} t^{-1}(\mathrm{I}_{2t}/2) = -\int_\Omega D_j[\varphi\sigma_{ij} X_\tau(\zeta u_i)]\beta\, dx - \int_\Omega \sigma_{ij}(D_j\tau^h)(D_h u_i)\beta\, dx.$$

3) Without loss of generality we may assume that the set Q given in (4.18) is a domain regular relative to Ω. Let us take a sequence $\{\beta_j\}$ of functions $C_0^\infty(R^3)$ such that $\beta_j = 1$ on Q for all j and $\beta_j \to \chi_Q$ a.e. as $j \to \infty$, where χ_Q is the characteristic function of Q; namely, $\chi_Q(x) = 1$ for $x \in Q$ and $\chi_Q(x) = 0$ for $x \notin Q$. We replace β in the integrals (4.37), (4.40), (4.43) and (4.45) by β_j and let $j \to \infty$. Here we notice that $\beta_j = 1$ on $\mathrm{supp}\,(1-\zeta)$ for all j, whence the integral (4.42) is independent of j. The passage to the limit in the integrals (4.37), (4.40), (4.43) and (4.45) is easily done using the Lebesgue dominated convergence theorem. Hence we find in the limit

$$(4.46)\quad kG(\mathscr{L}; \{\Sigma(t)\}) = \int_{Q'} X_\tau(\zeta W) dx - \int_{Q'} D_j\{\varphi\sigma_{ij} X_\tau(\zeta u_i)\} dx$$

$$- \int_{Q'} \{(X_\tau(a_{ijkl})/2)\varepsilon_{kl}\varepsilon_{ij} + f\cdot X_\tau(u)\} dx$$

$$- \int_{Q'} \sigma_{ij}(D_j\tau^h)(D_h u_i) dx - \int_{Q'} (1-\zeta) W J' dx.$$

Since $\zeta W = (\varphi\sigma_{ij})(\zeta\varepsilon_{ij})/2$ and $\varphi\sigma_{ij}$, $\zeta\varepsilon_{ij}$, $X_\tau(u_i) \in H^1(Q')$, we can apply (2.8) and obtain

$$(4.47)\quad \int_{Q'} X_\tau(\zeta W)dx = \int_{\partial Q} W(\tau\cdot\nu)dS + \int_{\Sigma\cap Q} [\![\zeta W(\tau\cdot\nu)]\!]\,dS - \int_{Q'} \zeta W(\operatorname{div}\tau)dx$$
$$= \int_{\partial Q} W(\tau\cdot\nu)dS - \int_{Q'} W(\operatorname{div}\tau)\,dx + \int_{Q'} (1-\zeta)W(\operatorname{div}\tau)dx,$$

$$\int_{Q'} D_j[\varphi\sigma_{ij}X_\tau(\zeta u_i)]dx = \int_{\partial Q} \sigma_{ij}\nu_j X_\tau(u_i)dS + \int_{\Sigma\cap Q} [\![\varphi\sigma_{ij}\nu_j X_\tau(\zeta u_i)]\!]\,dS$$
$$= \int_{\partial Q} \sigma_{ij}\nu_j X_\tau(u_i)dS.$$

Here we used the fact that $\tau\cdot\nu=0$ on Σ and $\sigma^+_{ij}\nu_j=\sigma^-_{ij}\nu_j=0$ on Σ. Therefore (4.46) is written in the form

$$(4.49)\qquad kG(\mathscr{L};\{\Sigma(t)\}) = J_Q(u;\tau) + \int_{Q'} (1-\zeta)\{W(\operatorname{div}\tau) - WJ'\}dx.$$

Let us now take a sequence $\{\zeta_j\}$ of functions of $C_0^\infty(R^3)$ such that $\zeta_j(x)\to 1$ a.e. in Q. Replacing ζ in (4.49) by ζ_j and letting $j\to\infty$, we conclude that

$$kG(\mathscr{L};\{\Sigma(t)\}) = J_Q(u;\tau).$$

This completes the proof of Theorem 4.5 in view of Proposition 3.7.

5. Infinitesimal crack extension

Using Theorem 4.5, we wish to investigate the infinitesimal crack extension (see Introduction, (Q.2)) for a smooth crack extension. By $\Lambda(\Sigma;\Pi)$ we denote the set of all smooth crack extensions which are located on a 2-dimensional oriented C^∞-submanifold Π as in (4.1). Let $(x,\gamma)\to\exp_x(\gamma)$ denote the exponential map from $T\Pi$ into Π (see e.g. Milnor [15]), which has the following properties:

$$(5.1)\qquad \exp_x(0) = x \quad\text{and}\quad \frac{d}{dt}\exp_x(t\gamma)|_{t=0} = \gamma$$

and the path

$$t \longrightarrow \exp_x(t\gamma), \qquad 0 \leqq t < q',$$

is the geodesic on Π starting from x in the γ direction for $x\in\Pi^\circ$ for some $q'>0$. By $\zeta(x)\in T_x\Pi$, $x\in\partial\Sigma$, we denote the unit outward normal to $\partial\Sigma$ at x relative to Π.

LEMMA 5.1 (see [15]). *There exists a positive number q_0 and an open neighborhood V of $\partial\Sigma$ in Π such that the map $(x,t)\to\exp_x(t\zeta(x))$ is a diffeomorphism from $\partial\Sigma\times(-q_0,q_0)$ onto V. Moreover we have*

$$V \cap \Sigma = \{\exp_x(t\zeta(x));\ x \in \partial\Sigma,\ -q_0 < t \leqq 0\}.$$

We put

$$\mathrm{Exp}_x(tv(x)) = x + tv(x) \qquad \text{for} \quad x \in V,$$

where v is the outward normal to Π which is determined by the orientations of Π and R^3. Hereafter we denote $\exp_x(t\zeta(x))$ and $\mathrm{Exp}_y(tv(y))$ simply by $\exp(x, t)$ and $\mathrm{Exp}(y, t)$, respectively. A result analogous to Lemma 5.1 holds for the map $(x, t) \rightarrow \mathrm{Exp}(x, t)$ from $\partial\Sigma \times (-q_1, q_1)$ into R^3 with some $q_1 > 0$. Thus we can derive the following

LEMMA 5.2. *There exist a positive number q and an open neighborhood U of $\partial\Sigma$ in R^3 such that (U, p) is a product neighborhood of $\partial\Sigma$ (see Lemma 4.2), where*

$$p(\xi, \eta, \lambda) = \mathrm{Exp}(\exp(\xi, q\eta), q\lambda) \qquad \textit{for} \quad (\xi, \eta, \lambda) \in \partial\Sigma \times \mathrm{I}^2.$$

If $\{\Sigma(t)\} \in \Lambda(\Sigma; \Pi)$, then there exists a family $\{\phi_t\}_{t\in[0,1]}$ of smooth maps from $\partial\Sigma$ into Π, which satisfy (4.3).

LEMMA 5.3. *Let $\{h_t\}_{t\in[0,t_0]}$ be the family of smooth functions on $\partial\Sigma$ which is defined as in the proof of Lemma 4.3 in terms of $\{\phi_t\}_{t\in[0,1]}$ and the product neighborhood (U, p) given in Lemma 5.2. Then*

$$q\left(\frac{d}{dt} h_t(x)\Big|_{t=0}\right) = \left\langle \frac{d}{dt}\phi_t(x)\Big|_{t=0}, \zeta(x)\right\rangle \tag{5.2}$$

for all $x \in \partial\Sigma$, where $\langle\cdot, \cdot\rangle$ denotes the inner product in R^3.

PROOF. We denote by p_r the projection from $\partial\Sigma \times \mathrm{I}^2$ onto $\partial\Sigma$. Set $k_t(x) = p_r \circ p^{-1} \circ \phi_t(x)$ for $x \in \partial\Sigma$. Since $k_t(x) \in \partial\Sigma$ and $k_0(x) = x$ for all $x \in \partial\Sigma$,

$$r_x = \frac{d}{dt} k_t(x)\,|_{t=0} \in T_x(\partial\Sigma).$$

By (4.10) we may rewrite $\phi_t(x)$ as

$$\phi_t(x) = \exp(k_t(x), qh_t(k_t(x))).$$

Using the fact that $h_0(x) = 0$ for all $x \in \partial\Sigma$, we then have by (5.1)

$$\begin{aligned}\frac{d}{dt}\phi_t(x)\Big|_{t=0} &= q\left(\frac{d}{dt}h_t(x)\Big|_{t=0}\right)\frac{d}{dt}\exp(x, \eta)\Big|_{\eta=0} + r_x \\ &= q\left(\frac{d}{dt}h_t(x)\Big|_{t=0}\right)\zeta(x) + r_x.\end{aligned}$$

Since $r_x \in T_x(\partial\Sigma)$, we then have

$$\left\langle \frac{d}{dt} \phi_t(x) \Big|_{t=0}, \zeta(x) \right\rangle = q\left(\frac{d}{dt} h_t(x) \Big|_{t=0} \right).$$

This completes the proof of Lemma 5.3.

Furthermore we have

LEMMA 5.4.

$$\lim_{t\to 0} t^{-1}|\Sigma(t)-\Sigma| = \int_{\partial\Sigma} \left\langle \frac{d}{dt} \phi_t(\xi)\Big|_{t=0}, \zeta(\xi) \right\rangle d\ell(\xi), \tag{5.3}$$

where $d\ell$ is the line element on $\partial\Sigma$.

PROOF. Let θ be a positive number such that

$$\partial\Sigma(t) \subset U \quad \text{if} \quad 0 \leqq t \leqq \theta.$$

We set

$$K = \cup_{t\in[0,\theta]} \partial\Sigma(t).$$

Let $\{(V_i, \alpha_i)\}_{i=1,2,\dots,m}$ be a local coordinate system on $\partial\Sigma$, $U_i = V_i \times \mathrm{I}^2$ and $\{\gamma_i\}$ a partition of unity subordinate to the covering $\{p(U_i)\}$ of K. To simplify the notation, we put

$$h_{ti}(\omega) = h_t(\alpha_i^{-1}(\omega)),\ \tilde{V}_i = \alpha_i(V_i),\ p_i(\omega, \eta) = p(\alpha_i^{-1}(\omega), \eta, 0),$$

$$\tilde{\gamma}_i = \gamma_i \circ p_i \quad \text{and} \quad \Phi(x) = \frac{d}{dt} \phi_t(x)\Big|_{t=0}.$$

Then we have (cf. Cesari [3])

$$|\Sigma(t) - \Sigma| = \sum_{i=1}^m \int_{\tilde{V}_i} \int_0^{qh_{ti}(\omega)} \tilde{\gamma}_i(\omega, \eta) J_i(\omega, \eta) d\omega d\eta.$$

Here

$$J_i^2(\omega, \eta) = \sum_{k\neq l} \left\{ \frac{\partial}{\partial\eta} p_i^k(\omega, \eta) \frac{\partial}{\partial\omega} p_i^l(\omega, \eta) - \frac{\partial}{\partial\omega} p_i^k(\omega, \eta) \frac{\partial}{\partial\eta} p_i^l(\omega, \eta) \right\},$$

where $p_i^k(\omega, \eta)$ are the components of $p_i(\omega, \eta)$. For each i, we consider the function

$$g_{ti}(\omega) = \int_0^{ph_{ti}(\omega)} \tilde{\gamma}_i(\omega, \eta) J_i(\omega, \eta) d\eta$$

with the parameter t. We then have

$$\lim_{t\to 0} t^{-1} g_{ti}(\omega) = q\left(\frac{d}{dt} h_{ti}(\omega) \Big|_{t=0} \right) \tilde{\gamma}_i(\omega, 0) J_i(\omega, 0).$$

Hence using the Lebesgue dominated convergence theorem and Lemma 5.3, we deduce that

$$\lim_{t\to 0} t^{-1}|\Sigma(t)-\Sigma| = \sum_{i=1}^{m}\int_{\mathcal{V}_i}\tilde{\gamma}_i(\omega, 0)\langle\Phi(\alpha_i^{-1}(\omega)), \zeta(\alpha_i^{-1}(\omega))\rangle J_i(\omega, 0)d\omega.$$

From (5.1) it follows that

$$\left(\frac{\partial}{\partial\omega}p_i(\omega, \eta)\Big|_{\eta=0}\right)\cdot\left(\frac{\partial}{\partial\eta}p_i(\omega, \eta)\Big|_{\eta=0}\right) = 0,$$

$$\left|\frac{\partial}{\partial\eta}p_i(\omega, \eta)|_{\eta=0}\right| = 1,$$

so that we obtain

$$J_i^2(\omega, 0) = \sum_{k=1}^{3}\left\{\frac{\partial}{\partial\omega}p_i^k(\omega, 0)\right\}^2.$$

Hence

$$\int_{\partial\Sigma}f(\xi)d\ell(\xi) = \sum_{i=1}^{m}\int_{\mathcal{V}_i}\tilde{\gamma}_i(\omega, 0)f(\alpha_i^{-1}(\omega))J_i(\omega, 0)d\omega$$

holds for any function f defined on $\partial\Sigma$ (see [3]). This completes the proof of Lemma 5.4.

We can now assert the following theorem as a consequence of Theorem 4.5, Lemma 5.3 and Lemma 5.4.

THEOREM 5.5. *For each* $(\mathscr{L}, \{\Sigma(t)\})\in[\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Gamma_1)]\times\Lambda(\Sigma; \Pi)$,

$$G(\mathscr{L}; \{\Sigma(t)\}) = \left\{\int_{\partial\Sigma}\delta\{\Sigma(t)\}(\xi)d\ell(\xi)\right\}^{-1}J(T(\mathscr{L}), \tau), \tag{5.4}$$

where

$$\delta\{\Sigma(t)\}(\xi) = \left\langle\frac{d}{dt}\phi_t(\xi)|_{t=0}, \zeta(\xi)\right\rangle \tag{5.5}$$

and

$$\tau(x) = \delta\{\Sigma(t)\}(\xi(x))\frac{\partial}{\partial\eta}p(\xi(x), \eta(x), \lambda(x)), \tag{5.6}$$

if $\{\Sigma(t)\}$ *is given as in Definition* 4.1.

REMARK 5.6. From Lemma 5.3, the quantity $\delta\{\Sigma(t)\}$ is independent of the choice of the representation $\{\phi_t\}$ of the smooth crack extension $\{\Sigma(t)\}$.

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References

[1] R. A. Adams, Sobolev spaces, Academic Press, New York London, 1975.
[2] B. Budiansky and J. R. Rice, Conservation laws and energy release rates, Trans. ASME J. Appl. Mech. **40** (1972), 201–203.
[3] L. Cesari, Surface area, Ann. of Math. Studies, no. 35, Princeton, New Jersey, 1956.
[4] F. H. K. Chen and R. T. Shield, Conservation laws in elasticity of the J-integral type, J. Appl. Math. Phys. **28** (1977), 1–22.
[5] D. Chenais, On the existence of a solution in a domain identificantion problem, J. Math. Anal. Appl. **52** (1975), 189–219.
[6] G. P. Cherepanov, Crack propagation in continuous media, J. Appl. Math. Mech. **31** (1967), 503–512.
[7] G. Duvaut and J. L. Lions, Inequalities in mechanics and physics, Springer-Verlag, Berlin·Heidelberg·New York, 1972.
[8] J. D. Eshelby, The continuum theory of lattice defects, Solid State Physics **3** (1956), 79–144.
[9] J. D. Eshelby, The calculation of energy release rates, Prospect of Fracture Mechanics, Ed. by G. C. Sih, H. C. Van Elst and D. Brock, Noordhoff International Publishing, 1974, 69–84.
[10] G. Fichera, Linear elliptic differential systems and eigenvalue problems, Lecture Notes in Mathematics 8, Springer-Verlag, Berlin·Heidelberg·New York, 1965.
[11] J. Gobert, Une inégalité fondamentale de la théorie de l'élasticité, Bull. Soc. Royale Sciences Liège, **31** (1962), 182–191.
[12] W. Günther, Über einige Randintegrale der Elastomechanik, Abh. Braunschw Wiss. Ges. **14** (1962), 53–72.
[13] P. Hartman, Ordinary differential equations, John Wiley & Sons, New York, 1964.
[14] J. K. Knowles and E. Sternberg, On a class of conservation laws in linearized and finite elastostatics, Arch. Rational Mech. Anal. **44** (1971), 187–211.
[15] J. Milnor, Morse theory, Ann. of Math. Studies, no. 51, Princeton, New Jersey, 1963.
[16] J. R. Munkres, Elementary differential topology, Ann. of Math. Studies, no. 54, Princeton, New Jersey, 1963.
[17] J. Nečas, Les méthods directes en théorie des équations elliptiques, Masson et C^{ie}, Paris, 1967.
[18] E. Noether, Invariante Variationsprobleme, Göttingen Nachrichten, Mathematisch-Physikalische Klasse **2** (1918), 235–257.
[19] K. Palaniswamy and W. G. Knauss, On the problem of crack extension in brittle solids under general loading, Mechanics Today Vol. 4, Ed. by S. Nemat-Nasser, 1978, 87–148.
[20] K. Ohtsuka, J-integral in two dimensional fracture mechanics, RIMS Kokyuroku 386, Kyoto Univ., 231–248, 1980.

[21] J. R. Rice, Mathematical analysis in the mechanics of fracture, "Fracture" An Advance Treaties Vol. 2, Ed. by H. Liebowitz, Academic Press, New York·London, 1968, 191–311.

[22] J. L. Sanders, On the Griffith-Irwin fracture theory, J. Appl. Mech. **27** (1960), 352.

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