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Subideals and serial subalgebras of Lie algebras

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The purpose of this note is to record several facts about the interplay between finiteness conditions on a Lie algebra, and the structure of its subideals and serial subalgebras (in the sense of [2] pp. 9, 258). Wielandt [16] has shown that a subgroup H of a finite group G is subnormal in G if and only if H is subnormal in $\langle H, g\rangle$ for every $g \in G$. This, and related criteria given by Wielandt in the same paper, have been extended to various classes of infinite groups by Hartley and Peng [7] and Whitehead [14, 15]. We obtain similar results for various classes of Lie algebras, though with somewhat different proofs owing to the unavailability of conjugacy arguments. In particular we prove an analogue of Wielandt's theorem for finite-dimensional Lie algebras over a field of characteristic zero. Chao and Stitzinger [3] prove a similar result for finite-dimensional soluble Lie algebras in arbitrary characteristic: their proof can be greatly simplified, and we do this in Theorem 2.

A generalization to locally finite Lie algebras leads to a criterion for a subalgebra of a locally finite Lie algebra to be serial, implying that a simple locally finite Lie algebra cannot have non-trivial serial subalgebras. (The group-theoretic analogue of this result appears to be unknown.) This is reminiscent of a theorem of Levič [9] and Amayo [1] on the nonexistence of ascendant subalgebras in arbitrary simple Lie algebras.

Notation for Lie algebras will follow Amayo and Stewart [2]. In particular '$\leq$', '$\lhd$', 'si', 'asc', and 'ser' denote the relations 'subalgebra', 'ideal', 'subideal', 'ascendant subalgebra', 'serial subalgebra' respectively (see [2] pp. 9, 10, 258). Triangular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. If L is a Lie algebra the *Fitting radical* $\nu(L)$ is the sum of the nilpotent ideals of L (equal to the nil radical in finite dimensions) and the *Hirsch-Plotkin radical* $\rho(L)$ is the unique maximal locally nilpotent ideal. If L is finite-dimensional we write $\sigma(L)$ for the soluble radical. In characteristic zero both $\nu(L)$ and $\sigma(L)$ are characteristic ideals (that is, invariant under derivations, see Jacobson [8] p. 74 and [2] p. 116). If L has finite dimension and the ground field has characteristic zero then $\nu(L)$ contains every nilpotent subideal of L ([2] p. 114). If $x, y \in L$ we write

$$[x, {}_n y] = [x, y, y, \dots, y] \qquad (n \text{ repetitions of } y)$$

with similar notation for subspaces. We put

$$L^{\omega} = \bigcap_{n=1}^{\infty} L^{n}.$$

If H si L then $H^{\omega} \lhd L$ (Schenkman [11], or [2] p. 10). This very useful fact does not work for groups. We write $H \dotplus K$ to denote the split extension of an ideal H by a subalgebra K (under suitably specified K-action on H).

1. Finite dimensions

Our results are most satisfactory over fields of characteristic zero:

THEOREM 1. *Let L be a finite-dimensional Lie algebra over a field of characteristic zero. Then a subalgebra H of L is a subideal of L if and only if H is a subideal of $\langle H, x\rangle$ for every $x \in L$.*

PROOF. If H si L then H si $\langle H, x\rangle$ for every $x \in L$. For the converse we argue for a contradiction, assuming L to be a counterexample of minimal dimension. We have H si $\langle H, x\rangle$ for all $x \in L$, but H not a subideal of L. By Schenkman [11] (or [2] Lemma 3.2 p. 10) we have $H^{\omega} \lhd \langle H, x\rangle$ for all $x \in L$, so that $H^{\omega} \lhd L$. If $H^{\omega} \neq 0$ then by minimality H/H^{ω} si L/H^{ω}, and H si L, a contradiction. Therefore $H^{\omega} = 0$ and H is nilpotent.

Therefore if $h \in H$ then $\langle h\rangle$ si H, by [2] Lemma 3.7 p. 12. Hence $\langle h\rangle$ si $\langle h, x\rangle$ for all $x \in L$. If the theorem were true for the case dim $H = 1$ it would follow that $\langle h\rangle$ si L for all $h \in H$, hence that $H \leq \nu(L)$. Therefore H would be a subideal. It follows that we may assume dim $H = 1$, so that $H = \langle h\rangle$ for some $h \in L$. For all $x \in L$ we have $[x, {}_n h] = 0$ for some $n > 0$, because $\langle h\rangle$ si $\langle h, x\rangle$. Since L has finite dimension, $[L, {}_n h] = 0$ for some $n > 0$.

Let $S = \sigma(L)$. If $S = L$ then every element h for which h^* is nilpotent lies in $\nu(L)$ by Mal'cev [10] (cf. [2] Theorem 4.2(a) p. 341). Therefore $H \leq \nu(L)$, so H si L, which again is a contradiction. Hence $S \neq L$. It follows that $S + H \neq L$, since $S + H$ is soluble. By minimality, we have H si $S + H$. If $S \neq 0$ then $(S+H)/S$ si L/S by minimality, which implies H si L. Therefore $S = 0$ and L is semisimple. By Jacobson [8] Theorem 8 p. 79 and Theorem 17(1) p. 100, there is an element $k \in L$ such that $\langle h, k\rangle$ is a 3-dimensional split simple Lie algebra. But this contradicts $\langle h\rangle$ si $\langle h, k\rangle$, and the theorem is proved.

This proof fails in characteristic $p > 0$. For soluble algebras the result remains true, as is proved by Chao and Stitzinger [3]. Their method of proof can be simplified, as we now show in the case of interest here.

THEOREM 2. *Let L be a finite-dimensional soluble Lie algebra over any field. Then a subalgebra H of L is a subideal of L if and only if H is a subideal of $\langle H, x\rangle$ for every $x \in L$.*

PROOF. Assume L to be a counterexample of minimal dimension. Let B be a maximal subalgebra of L such that $H \leq B$. By minimality, H si B. Let C be the core in L of B. If $C \neq 0$ then $H+C/C$ si L/C by minimality. But $H+C \leq B$ so H si $H+C$, a contradiction. Therefore $C=0$. It follows easily (cf. [13]) that there is a minimal ideal M of L such that $L=M \dotplus B$. Now if

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = B$$

Then

$$H + M = H_0 + M \lhd \cdots \lhd H_n + M = L$$

so that

$$H + M \text{ si } L.$$

If $H+M \neq L$ then H si $H+M$, a contradiction. But if $H+M=L$ then $H=B$, so if $x \in L \backslash B$ we have H si $\langle H, x\rangle = \langle B, x\rangle = L$ since B is maximal. This is the final contradiction, and the theorem is proved.

(Note that this argument can be used instead of appealing to Mal'cev [10] in Theorem 1.)

As regards the other criteria in Wielandt [16]: only (e) does not involve conjugacy in its statement. However, the Lie analogue is false for finite-dimensional algebras in characteristic zero. It merely asserts that every element of H acts nilpotently on L. The split 3-dimensional simple algebra provides the required counterexample. It is of course true for finite-dimensional soluble algebras in characteristic zero by the result of Mal'cev [10] quoted above.

Theorem 1 would, of course, be immediate if it were the case that in a Lie algebra L every subalgebra H has a *subidealizer*, that is, a unique $S \geq H$ maximal with respect to H si S. However, as in finite groups, this is in general untrue. Examples are easy to come by: perhaps the most straightforward is the simple algebra of type A_2. If $\{\alpha, \beta\}$ is a system of simple roots (in the terminology of Jacobson [8] pp. 110, 120) then the subalgebra $H = \langle e_\alpha \rangle$ satisfies

$$H \lhd \langle e_\alpha, e_{\alpha+\beta}\rangle \lhd \langle e_\alpha, e_\beta, e_{\alpha+\beta}\rangle,$$

$$H \lhd \langle e_\alpha, e_{-\beta}\rangle \lhd \langle e_\alpha, e_{-\beta}, e_{-\alpha-\beta}\rangle,$$

whereas $\langle e_\alpha, e_\beta, e_{\alpha+\beta}, e_{-\beta}, e_{-\alpha-\beta}\rangle$ is the whole of A_2. This is simple, so $\langle e_\alpha \rangle$ cannot be a subideal.

It is a minor curiosity that in the simple subalgebra of type A_1, every subalgebra has a subidealizer. This is because the ideal relation is transitive in A_1 and its subalgebras, so the idealizer is the same as the subidealizer. For subalgebras H of a finite-dimensional soluble Lie algebra over a field of characteristic zero subidealizers exist, and may be characterized as the full inverse image in L of the Fitting null-component of H/H^ω acting on $I_L(H^\omega)/H^\omega$, using the methods of Theorem 1. For the soluble characteristic p case they also exist, as proved by Chao and Stitzinger [3] Theorem 6.

2. Infinite dimensions

For infinite-dimensional Lie algebras we do not expect a result like Theorem 1 in general. Thus let L be any locally nilpotent Lie algebra and $0 \neq x \in L$. Then $\langle x \rangle$ is a subideal of $\langle x, y \rangle$ for any $y \in L$ because $\langle x, y \rangle$ is nilpotent. But if L is not Baer (see [2] p. 119) then we can choose x so that $\langle x \rangle$ is not a subideal of L; and if L is not Gruenberg (see [2] p. 126) we can even ensure that $\langle x \rangle$ is not ascendant. By restricting attention to locally finite Lie algebras we can turn this situation to advantage: it is in a sense possible to trace any failure of Theorem 1 to a 'locally nilpotent situation'.

To do this we need a theorem due, for groups, to Hartley [6]. The proof can be modified to work for Lie algebras (see [2] p. 258). The result is:

LEMMA 3. *Let L be a locally finite Lie algebra. Then a subalgebra H is serial in L if and only if $H \cap F$ is serial in F for every finite-dimensional subalgebra F of L.*

This immediately gives a generalization of Theorem 1:

THEOREM 4. *If L is a locally finite Lie algebra over a field of characteristic zero and H is a subalgebra, then H ser L if and only if H ser $\langle H, x \rangle$ for every $x \in L$.*

PROOF. One implication is trivial. Suppose now that $H \operatorname{ser} \langle H, x \rangle$ for every $x \in L$. Let F be any finite-dimensional subalgebra of L, and let $x \in F$. Then $H \operatorname{ser} \langle H, x \rangle$, so

$$H \cap F \operatorname{ser} \langle H, x \rangle \cap F \geq \langle H \cap F, x \rangle$$

so $H \cap F \operatorname{ser} \langle H \cap F, x \rangle$. The latter has finite dimension, so $H \cap F \operatorname{si} \langle H \cap F, x \rangle$ for all $x \in F$. By Theorem 1, $H \cap F \operatorname{si} F$. This being true for all F we may appeal to Lemma 3 to deduce that $H \operatorname{ser} L$.

A similar argument works in characteristic $p > 0$ provided we assume L locally soluble.

Lemma 3 gives us strong control over the serial subalgebras of a locally finite Lie algebra when coupled with Schenkman's result mentioned in the introduction. In fact Schenkman's result generalizes. Amayo [1] has obtained such a generalization for ascendant subalgebras. We must replace L^ω by the *locally* *nilpotent residual* $\lambda_{L\mathfrak{N}}(L)$, defined to be the intersection of the ideals $I \lhd L$ for which L/I is locally nilpotent.

THEOREM 5. *If L is a locally finite Lie algebra and H ser L then $\lambda_{L\mathfrak{N}}(H) \lhd L$.*

PROOF. Define a subalgebra K of L by

$$K = \Sigma_F (F \cap H)^{\omega},$$

where the sum is over all $F \leq L$ with $\dim F < \infty$. In fact K is an ideal of L. For let $x \in L$, $k \in (F \cap H)^{\omega}$. We can put $G = \langle F, x \rangle$ which is finite-dimensional, and then $k \in (F \cap H)^{\omega} \leq (G \cap H)^{\omega} \lhd G$, since $G \cap H$ si G and Schenkman's result applies. Hence $[k, x] \in K$, so $K \lhd L$. It remains to note that $K = \lambda_{L\mathfrak{N}}(H)$. Clearly H/K is locally nilpotent, so $K \geq \lambda_{L\mathfrak{N}}(H)$; but on the other hand each $(F \cap H)^{\omega}$ is contained in $\lambda_{L\mathfrak{N}}(H) \cap F$, because $(H \cap F)/(\lambda_{L\mathfrak{N}}(H) \cap F)$ is nilpotent; so $K \leq \lambda_{L\mathfrak{N}}(H)$. This proves the theorem.

COROLLARY 6. *Let L be locally finite over a field of characteristic zero. Then H ser L if and only if $H/\lambda_{L\mathfrak{N}}(H) \leq \rho(L/\lambda_{L\mathfrak{N}}(H))$.*

PROOF. One implication follows because every subalgebra of a locally nilpotent Lie algebra is serial (use Lemma 3 again). The other follows from the fact that in a locally finite Lie algebra of characteristic zero, the Hirsch-Plotkin radical contains every locally nilpotent serial subalgebra ([2] p. 261).

COROLLARY 7. *Let L be locally finite over a field of characteristic zero, and suppose that whenever U and V are ideals of L with $V \leq U$ and U/V locally nilpotent, it follows that U/V has finite dimension. If H ser $\langle H, x \rangle$ for every $x \in L$ then H si L.*

PROOF. Let $K = \lambda_{L\mathfrak{N}}(H) \lhd L$. Then $H/K \leq \rho(L/K)$ and this has finite dimension by hypothesis, so is nilpotent, and it follows that H si L.

Corollary 7 applies in particular if L (in addition to being locally finite) satisfies either of the chain conditions Min-$\lhd^2$ or Max-$\lhd^2$ (see [2] pp. 174, 177), and also if L is *semisimple* (defined by insisting that for U, V as in Corollary 7 we have $U = V$). This reduces to the usual concept in finite dimensions. In particular a simple locally finite Lie algebra is semisimple. Thus we have proved:

THEOREM 8. *Over any field of characteristic zero, a locally finite simple Lie algebra can have no non-trivial serial subalgebras.*

Amayo [1] has proved that a simple Lie algebra can never have a non-trivial ascendant subalgebra (see also Levič [9]). This is in contrast to a result of Hall [5] for groups. In fact for groups one must distinguish between simple groups, strictly simple groups (with no non-trivial ascendant subgroups) and absolutely simple groups (no non-trivial serial subgroups). Chehata [4] constructs a non-absolutely simple group. It is not known if a non-absolutely simple Lie algebra can exist.

Theorem 4 can be extended to locally finite groups by the same method. Theorem 5 is false for locally finite groups, because it is false even for finite groups. The group-theoretic analogue of Theorem 8 appears to be open.

Many of the results of Whitehead [14, 15] extend to Lie algebras, with no essential changes in their proofs.

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