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Convergence of difference approximations for quasi-linear hyperbolic systems

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1. Introduction

From numerical aspects, difference schemes for solving initial value problems have been extensively investigated for a long time. In linear initial value problems, Lax obtained the remarkable result which states that the stability of a consistent difference scheme is a necessary and sufficient condition for the convergence if the problem is well posed [22]. This has become well known as Lax's Equivalence Theorem. The stability used there means conceptually the boundedness of numerical solutions given by the difference scheme. For numerical studies of engineering and physical problems, many authors have constructed several difference schemes. In particular, Peetre and Thomée [21] investigated these in the Sobolev spaces and gave several estimates for the rate of convergence, assuming that they are stable. For linear hyperbolic systems the theory of the Sobolev spaces of L_2-type has played an important role in establishing the existence and uniqueness of solution. Along such lines the L_2-stability of difference schemes has also been studied. Such a stability can be obtained in terms of the amplification matrix defined by the Fourier transform of the difference operator. Lax [13], Lax and Wendroff [14, 15], Kreiss [11], Yamaguti and Nogi [31], Lax and Nirenberg [16], Vaillancourt [28, 29, 30], Koshiba [10] and so on [20, 25, 26, 27] are relevant here.

Though a considerable portion of the progress in difference schemes for hyperbolic systems is confined to the linear theory, we must mention some results related to nonlinear theory of initial value problems. Strang [24] contributed to the establishment of convergence of difference approximations to smooth solutions in nonlinear problems. This will be briefly stated below. Modifying Lax's Equivalence Theorem, Kreth [12], von Dein [3] and Ansorge [1] discussed the convergence of difference approximations in an abstract setting and so, to our knowledge, their results seem less applicable.

Concerning weak solutions of quasi-linear hyperbolic equations of conservation laws, Le Roux [17] studied problems of the convergence for difference approximations.

In nonlinear initial value problems, Lax's Equivalence Theorem is not valid in general, so the convergence must be proved without the help of the stability of the nonlinear scheme. From this requirement, Strang [24] discussed the convergence of difference approximations for the following quasi-linear hyperbolic system:

$$(1.1)\quad \frac{\partial u}{\partial t}(t, x) = \sum_{j=1}^{n} A_j(t, x, u(t, x))\frac{\partial u}{\partial x_j}(t, x) + B(t, x, u(t, x)) \qquad (x \in \boldsymbol{R}^n;\ 0 \leqq t \leqq T),$$

$$(1.2)\quad u(0, x) = u_0(x),$$

where $A_j(t, x, \cdot)$ $(j=1,\dots, n)$ are $N \times N$ matrices, $B(t, x, \cdot)$ and $u(t, x)$ are N-vectors. From analytical point of views the existence and uniqueness of smooth solution of (1.1), (1.2) are proved by Fischer and Marsden [4, 5] and Kato [8, 9]. The approximating difference scheme for the above system is written in the following form:

$$(1.3)\quad v^{j+1}(x) = S_h(jk)v^j(x) \qquad (j = 0, 1,\dots, \nu-1;\ 0 \leqq \nu k \leqq T),$$

$$(1.4)\quad v^0(x) = u_0(x),$$

where k and h denote the time step and the space mesh width, respectively, and $S_h(t)$ is a difference operator derived from the discretization of (1.1). Strang obtained the following result: *Let Φ be*

$$\Phi(jk, x, v^j, k, h) = S_h(jk)v^j(x).$$

Suppose that A_j $(j=1,\dots, n)$, B, Φ and a solution u of (1.1) *have continuous derivatives up to order $m+[(n+1)/2]+r+2$*[1] *for some positive integer r and that the difference scheme* (1.3) *approximates* (1.1) *with accuracy of order m. Then if the first variation of Φ is l_2-stable, it holds that*

$$(1.5)\quad v^\nu(x) = u(t, x) + O(k^m) \qquad (x \in \boldsymbol{R}^n;\ t = \nu k \in [0, T]),$$

where k/h is kept constant as h varies.

His method of proof is as follows: He first constructed the expansion

$$(1.6)\quad w(t, x, k) = u(t, x) + \sum_{j=1}^{q} k^j w^{(j)}(t, x), \qquad q = m + [(n+1)/2],$$

so that (1.6) satisfies (1.3) with an error $o(k^{q+1})$. He found the fact that $w(\nu k, x, k) - v^\nu(x)$ is governed by the l_2-stability of the first variation of Φ. Then, estimating two quantities $w(\nu k, x, k)-v^\nu(x)$ and $w(\nu k, x, k)-u(\nu k, x)$, he obtained the rate of convergence (1.5). This method is simple, but the calculations are too complicated for us.

1) $[a]$ denotes the greatest integer not exceeding a.

Thus, in the same framework as Strang's we shall give a different method from his, so that we obtain improved results. Our method is to directly estimate $u(\nu k, x)-v^{\nu}(x)$ in the maximum norm by using the Sobolev-type theorem. The discussion is carried out in the way almost similar to the one used in [12], [3] and [1]. We assume that $S_h(t)v^j(x)$ is a function of $v^j(x+hp^{(1)}),\dots, v^j(x+hp^{(q)})$, t, x and h, where $p^{(i)}$'s are multi-indices of integers. Then we can divide the function $S_h(t)v^j(x)$ into two parts by the mean-value theorem, i.e.,

$$S_h(jk)v^j(x) = L_h(jk, v^j; h)v^j(x) + hG(jk; h)v^j(x), \tag{1.7}$$

where $L_h(jk, v^j; h)$ is the first variation with respect to $v^j(x)$ and $G(jk; h)v^j(x)$ is the remainder. Using the stability of L_h, we estimate $u(\nu k, x)-v^{\nu}(x)$ in the Sobolev spaces of L_2-type, and then, obtain a convergence theorem (Theorem 3.1), i.e.,

$$|u(\nu k, \cdot) - v^{\nu}(\cdot)|_r \leqq Ch^m,$$

where u is assumed to be differentiable up to order $m+[n/2]+r+2$, and $|\cdot|_r$ denotes the norm in the space C_B^r consisting of all functions which, together with all their derivatives up to order r, are continuous and bounded. We note that when $r=0$, this result leads to Strang's result (1.5).

We next show the stability of L_h by Lax-Nirenberg Theorem for difference operators, imposing some conditions on the amplification matrix l of L_h, and obtain a convergence theorem (Theorem 5.1), which states as follows: The difference approximations converge to the exact solution in the C_B^2-norm by the Sobolev-type theorem, if $I-l^*l\geqq 0$.

Since the original system (1.1) does not contain the second-order derivatives of u, it should seem needless to derive the C_B^2-convergence, but reasonable to derive the C_B^1-convergence. In order to prove such a convergence we prepare an inequality (Theorem 4.2) excluding the second-order derivatives of the amplification matrix. This inequality corresponds to the one in Lax-Nirenberg Theorem which serves to establish the energy inequality of linear difference scheme. Using it, we arrive at the C_B^1-convergence theorem (Theorem 5.2), where it is assumed that there exist two matrix functions $a(\geqq 0)$ and b satisfying $I-l^*l=b^*ab$.

2. Preliminaries

We denote by $\boldsymbol{C}^N$ the complex N-dimensional space. We abbreviate $\boldsymbol{C}^1=\boldsymbol{C}$ and denote the norm of z in $\boldsymbol{C}^N$ by $|z|$. We define the norm of an $N\times N$ matrix by its operator norm in $\boldsymbol{C}^N$, writing as $|a|$. Unless otherwise stated, we denote by u, v, w, f, φ, etc. N-vector functions. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ of non-negative integers we use the notations

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j} \ (j = 1, \dots, n), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

For any nonnegative integer m let C_B^m be the space consisting of all functions which, together with all their partial derivatives up to order m, are continuous and bounded on $\boldsymbol{R}^n$. Then C_B^m is a Banach space with the norm $|\cdot|_m$ given by

$$|\varphi|_m = \sum_{|\alpha| \leqq m} \sup_x |\partial_x^\alpha \varphi(x)|.$$

The space W^m stands for the Sobolev space of L_2-type with the scalar product $(\ ,\ )_m$ and the norm $\|\cdot\|_m$. For simplicity we write $(\ ,\ )$ and $\|\cdot\|$ instead of $(\ ,\ )_0$ and $\|\cdot\|_0$, respectively. In this paper we use the following Sobolev-type theorem [18, 19]:

THEOREM 2.1. (i) *Let* $m > n/2 + m'$ $(m' \geqq 0)$. *Then* $W^m \subset C_B^{m'}$, *i.e.,*

$$|f|_{m'} \leqq K_1(m', n, N) \|f\|_m \quad \textit{for all} \quad f(x) \in W^m, \tag{2.1}$$

where $K_1(m', n, N)$ *is a constant depending on* m', n *and* N.

(ii) *Let* $m > n/2$ *and let*

$$\sum_{j=1}^r |\alpha^{(j)}| \leqq m, \quad |\alpha^{(j)}| \geqq 1 \qquad (j = 1, \dots, r),$$

where $\alpha^{(j)}$ $(j = 1, \dots, r)$ *are multi-indices of nonnegative integers. Then*

$$\|\partial_x^{\alpha^{(1)}} f_1 \cdots \partial_x^{\alpha^{(r)}} f_r\| \leqq K_2(m, n, N) \prod_{j=1}^r \|f_j\|_m \tag{2.2}$$

for all $f_j(x) \in W^m$ $(j = 1, \dots, r)$, *where* $f_j(x)$ $(j = 1, \dots, r)$ *are N-vector functions, the operation* $\cdot$ *denotes the componentwise multiplication and* $K_2(m, n, N)$ *is a constant depending on* m, n *and* N *but not on* $\alpha^{(j)}$ $(j = 1, \dots, r)$.

3. Difference approximations and their convergence

3.1. The difference scheme

Put

$$J = [0, T], \qquad K = [0, k_0], \qquad H = [0, h_0], \tag{3.1}$$

where k_0 is a positive constant such that $k_0 \leqq T$ and $h_0 = k_0/\lambda$ for some constant λ. For the time step k and the space mesh width h we approximate the solution of (1.1), (1.2) by the following difference scheme of the form, keeping the ratio $\lambda = k/h$ constant as h varies:

$$v^{j+1}(x) = \Phi(jk, x, \{v^j(x + hp^{(1)})\}^T, \dots, \{v^j(x + hp^{(q)})\}^T, h) \tag{3.2}$$
$$(j = 0, 1, \dots, \nu - 1;\ k \in K;\ \nu k \leqq T),$$

$$v^0(x) = u_0(x), \tag{3.3}$$

where $p^{(j)}=(p_1^{(j)},\dots, p_n^{(j)})$ $(j=1,\dots, q)$ for integers $p_i^{(j)}$ $(i=1,\dots, n)$, $\Phi(t, x, y, \mu)$ is an N-vector function defined on $J \times \boldsymbol{R}^n \times (\boldsymbol{C}^N)^q \times H$ and $\{\cdot\}^T$ stands for the transpose of a vector. The space variable of $(\boldsymbol{C}^N)^q$ is denoted by y in the sense that

$$y = (y_{11},\dots, y_{1N}, y_{21},\dots, y_{2N},\dots, y_{q1},\dots, y_{qN}).$$

We assume that $\Phi(t, x, y, \mu)$ is continuously differentiable on $J\times\boldsymbol{R}^n\times(\boldsymbol{C}^N)^q\times H$ and that

$$\Phi(t, x, z,\dots, z, 0) = z \qquad \text{for all} \quad (t, x, z)\in J \times \boldsymbol{R}^n \times \boldsymbol{C}^N. \tag{3.4}$$

Then by the mean-value theorem we have

$$\begin{aligned}\Phi(t, x, v(x, h), h) &= v(x) + \Phi(t, x, v(x, h), h) - \Phi(t, x, v(x, 0), 0)\\ &= v(x) + \textstyle\sum_{j=1}^q l_j(t, x, v(x, h), h)\{v(x+hp^{(j)})-v(x)\}\\ &\quad + hg(t, x, v(x, h), h),\end{aligned}$$

where

$$\begin{aligned}l_j(t, x, y, \mu) &= \Big(\int_0^1 \partial_{y_{j1}}\Phi(t, x, \theta y+(1-\theta)y', \theta\mu)d\theta,\dots\\ &\qquad \dots, \int_0^1 \partial_{y_{jN}}\Phi(t, x, \theta y+(1-\theta)y', \theta\mu)d\theta\Big) \qquad (j = 1,\dots, q),\\ g(t, x, y, \mu) &= \int_0^1 \partial_\mu\Phi(t, x, \theta y+(1-\theta)y', \theta\mu)d\theta,\\ y' &= (y_{11},\dots, y_{1N}, y_{11},\dots, y_{1N},\dots, y_{11},\dots, y_{1N})\in(\boldsymbol{C}^N)^q,\\ \partial_{y_{ji}} &= \frac{\partial}{\partial y_{ji}} \qquad (j = 1,\dots, q;\ i = 1,\dots, N), \quad \partial_\mu = \frac{\partial}{\partial\mu}\end{aligned}$$

and

$$v(x, \mu) = (v^T(x+\mu p^{(1)}),\dots, v^T(x+\mu p^{(q)})) \qquad (\mu\in H). \tag{3.5}$$

Put

$$L_\eta(t, v; \mu) = I + \textstyle\sum_{j=1}^q l_j(t, x, v(x, \mu), \mu)(T_\eta^{p^{(j)}}-I) \quad (\eta\in H),$$

where $T_\eta^{p^{(j)}}$ is a translation operator defined by

$$T_\eta^{p^{(j)}}v(x) = v(x+\eta p^{(j)}) \qquad (j = 1,\dots, q).$$

Then it follows that

$$\Phi(t, x, v(x, h), h) = L_h(t, v; h)v(x) + hg(t, x, v(x, h), h) \tag{3.6}$$

and

(3.7) $$L_\eta(t, v; \mu)|_{\eta=0} = I \qquad \text{for all} \quad (t, \mu) \in J \times H.$$

It is clear that $L_h(t, v; h)$ is a linear operator for any fixed t, v and h.

From the above consideration, the difference scheme (3.2) may be represented by the form introduced in Section 1, i.e.,

(3.8) $$v^{j+1}(x) = S_h(jk)v^j(x) \qquad (j = 0, 1, \dots, \nu-1;\ k \in K;\ \nu k \leqq T),$$

(3.9) $$v^0(x) = u_0(x),$$

where $S_\mu(t)v(x)$ ($\mu \in H$) takes the form of

(3.10) $$S_\mu(t)v(x) = L_\mu(t, v; \mu)v(x) + \mu G(t; \mu)v(x)$$

by using

(3.11) $$L_\eta(t, v; \mu) = \textstyle\sum_{\sigma\in\Lambda} l_\sigma(t, x, v(x, \mu), \mu)T_\eta^\sigma, \qquad \sigma = (\sigma_1, \dots, \sigma_n)$$

and

(3.12) $$G(t; \mu)v(x) = g(t, x, v(x, \mu), \mu) \qquad (\mu, \eta \in H).$$

Here Λ is a finite set, σ_j $(j=1,\dots, n)$ are integers and $v(x, \mu)$ is given by (3.5). Furthermore, $N \times N$ matrix functions $l_\sigma(t, x, y, \mu)$ $(\sigma \in \Lambda)$ and an N-vector function $g(t, x, y, \mu)$ are defined on $J \times \boldsymbol{R}^n \times (\boldsymbol{C}^N)^q \times H$. In view of (3.7), we assume that

(3.13) $$\textstyle\sum_{\sigma\in\Lambda} l_\sigma(t, x, y, \mu) = I \qquad \text{for all} \quad (t, x, y, \mu) \in J \times \boldsymbol{R}^n \times (\boldsymbol{C}^N)^q \times H.$$

In the sequel we are concerned with the difference scheme of the form (3.8).

3.2. Accuracy and stability

To state the definitions of accuracy and stability we introduce two sets $W^m(d)$ and $U^{l,m}(d)$ for arbitrary integers l, m $(l \geqq 1;\ m \geqq 0)$ and an arbitrary number $d > 0$. $W^m(d)$ is given by

$$W^m(d) = \{\varphi(x) \in W^m:\ \|\varphi\|_m \leqq d\},$$

and $U^{l,m}(d)$ consists of all functions $u(t, x)$ with the properties:

1) $u(t, \cdot) \in \bigcap_{i=0}^{l} C^i(J;\ W^{m+l-i})$, where $C^i(J;\ W^j)$ denotes the space of all functions $u(t, \cdot)$ from J to W^j such that $u(t, \cdot)$ is i-times continuously differentiable with respect to t in the W^j-topology;

2) $\max_{0\leqq i\leqq l} \max_{0\leqq t\leqq T} \|\partial_t^i u(t, \cdot)\|_{m+l-i} \leqq d$, $\partial_t^i = \left(\dfrac{\partial}{\partial t}\right)^i$;

3) $u(t, \cdot)$ is the unique solution of (1.1) with the initial value $u(0, \cdot)$.

DEFINITION. Let d be a positive number and let m_1, m_2, m_3 $(m_1 \geqq 1;\ m_2 \geqq m_1 + 1;\ m_3 \geqq 0)$ be integers. Then we say that *the difference scheme* (3.8) *approximates* (1.1) *with accuracy of order* m_1 *in* $U^{m_2,m_3}(d)$ if the following conditions (a–1) and (a–2) are fulfilled:

(a–1) $S_h(t)$ is a mapping from W^{m_3} into itself for any fixed $(t, h) \in J \times H$;

(a–2) $U^{m_2, m_3}(d)$ is not empty and there exists a constant c_1 such that

$$\|u(t+k, \cdot) - S_h(t)u(t, \cdot)\|_{m_3} \leqq c_1 h^{m_1+1} \tag{3.14}$$

for all $u(t, x) \in U^{m_2, m_3}(d)$, $k \in K$ and $t, t+k \in J$.

For a given v we define the following difference scheme derived from (3.10) by

$$w^{j+1}(x) = L_h(jk, v; h)w^j(x) \qquad (j = 0, 1,\dots, \nu-1;\ k \in K;\ \nu k \leqq T), \tag{3.15}$$

$$w^0(x) = u_0(x). \tag{3.16}$$

From now on we call (3.15) the linear difference scheme of (3.10).

Definition. Let d and m be a positive number and a nonnegative integer, respectively. Then the linear difference scheme (3.15) is said to be *stable in W^m for all $w_j \in W^m(d)$ $(j=0, 1,\dots)$* if the operator $L_h(t, \cdot; h)$ satisfies the following conditions:

(s–1) When a function $v \in W^m(d)$ is given, $L_h(t, v; h)$ is a linear operator from W^m into itself for any fixed $(t, h) \in J \times H$;

(s–2) There exists a constant c_2 such that

$$\|L_h(t+(\nu-1)k, w_{\nu-1}; h)L_h(t+(\nu-2)k, w_{\nu-2}; h)\cdots L_h(t, w_0; h)u\|_m \leqq c_2\|u\|_m \tag{3.17}$$

for all $w_j(x) \in W^m(d)$ $(j=0, 1,\dots, \nu-1)$, $u(x) \in W^m$, $k \in K$ and $t, t+\nu k \in J$.

For all integers $\nu \geqq 0$ and $k \in K$ we define $\phi^\nu(t, v(x))$ by

$$\phi^{j+1}(t, v(x)) = S_h(t+jk)\phi^j(t, v(x)) \qquad (j = 0,1,\dots, \nu-1;\ \nu \geqq 1), \tag{3.18}$$

$$\phi^0(t, v(x)) = v(x). \tag{3.19}$$

3.3. Convergence theorem

We show the following convergence theorem which improves Strang's result [24].

Theorem 3.1. *Let the difference scheme* (3.8) *approximate* (1.1) *with accuracy of order m_1 in $U^{m_2, m_3}(d_1)$ $(d_1 > 0;\ m_1 \geqq 1;\ m_2 \geqq m_1 + 1;\ m_3 \geqq 0)$ and let*

$$\|\{L_h(t, w; h) - L_h(t, v; h)\}u\|_{m_3} \leqq c_3 h\|w - v\|_{m_3}, \tag{3.20}$$

$$\|G(t; h)w - G(t; h)v\|_{m_3} \leqq c_3\|w - v\|_{m_3} \tag{3.21}$$

for all $w(x), v(x) \in W^{m_3}(d_2)$, $u(x) \in W^{m_3+1}(d_2)$, $h \in H$ and $t \in J$, where d_2 $(d_2 > d_1)$ and c_3 are constants. Suppose that the linear difference scheme (3.15) *is stable* *in* W^{m_3} *for all* $w_j \in W^{m_3}(d_2)$ $(j=0, 1,\dots)$. *Then*

$$\phi^{\nu}(t, u(t, x)) \in W^{m_3}(d_2), \tag{3.22}$$

$$\|u(t+\nu k, \cdot) - \phi^{\nu}(t, u(t, \cdot))\|_{m_3} \leqq \{(1+c_4 h)^{\nu} - 1\} h^{m_1} \tag{3.23}$$

for all $u(t, x) \in U^{m_2, m_3}(d_1)$, $k \in [0, k_1]$ *and* t, $t+\nu k \in J$, *where*

$$c_4 = \max\{c_1, 2c_3, c_1 c_2, 2c_2 c_3\}, \tag{3.24}$$

$$k_1 = \min\{k_0, \lambda\{(d_2-d_1)\exp(-c_4 T/\lambda)\}^{1/m_1}\}, \tag{3.25}$$

c_1 *is a constant satisfying* (3.14) *with* $d=d_1$ *and* c_2 *is also a constant satisfying* (3.17) *with* $d=d_2$ *and* $m=m_3$.

REMARK 3.1. (i) Since

$$(1+c_4 h)^{\nu} \leqq \exp(c_4 T/\lambda)$$

for all integers $\nu \geqq 0$ and $k \in K$ such that $\nu k \leqq T$, it follows from (3.23) that

$$\|u(t+\nu k, \cdot) - \phi^{\nu}(t, u(t, \cdot))\|_{m_3} \leqq \{\exp(c_4 T/\lambda)\} h^{m_1} \tag{3.26}$$

for all $u(t, x) \in U^{m_2, m_3}(d_1)$, $k \in [0, k_1]$ and t, $t+\nu k \in J$.

(ii) Let $m_3 \geqq [n/2]+1$. Then, by using the Sobolev-type theorem (Theorem 2.1), we obtain from (3.26)

$$|u(t+\nu k, \cdot) - \phi^{\nu}(t, u(t, \cdot))|_{m_3-[n/2]-1} \leqq C h^{m_1} \tag{3.27}$$

for all $u(t, x) \in U^{m_2, m_3}(d_1)$, $k \in [0, k_1]$ and t, $t+\nu k \in J$, where C is a positive constant.

PROOF OF THEOREM 3.1. The proof will be done by induction on ν $(\nu \geqq 1)$. For $\nu=1$ it follows from (3.14) that

$$\|u(t+k, \cdot) - \phi^1(t, u(t, \cdot))\|_{m_3} \leqq c_1 h^{m_1+1} \leqq c_4 h^{m_1+1}. \tag{3.28}$$

Since (3.25) implies

$$c_4 h^{m_1+1} \leqq (c_4 T/\lambda)(d_2-d_1)\exp(-c_4 T/\lambda) \leqq d_2 - d_1,$$

we have

$$\|\phi^1(t, u(t, \cdot))\|_{m_3} \leqq \|u(t+k, \cdot)\|_{m_3} + d_2 - d_1 \leqq d_2. \tag{3.29}$$

Hence (3.22) and (3.23) for $\nu=1$ follow from (3.29) and (3.28).

Suppose that (3.22) and (3.23) hold for all j $(1 \leqq j \leqq \nu-1)$, where $\nu>1$. For simplicity we put

$$u_j = u(t+jk, x), \qquad w_j = \phi^j(t, u(t, x)),$$

$$L_j(v) = L_h(t+jk, v; h), \qquad G_j v = G(t+jk; h)v(x),$$

$$F_j(v)w = L_j(v)w + hG_jv \qquad (j = 0, 1,\dots, \nu).$$

Then

(3.30) $u_\nu - w_\nu$

$$= u_\nu - F_{\nu-1}(w_{\nu-1})u_{\nu-1} + \textstyle\sum_{i=1}^{\nu-1}\{\prod_{j=\nu-i}^{\nu-1}F_j(w_j)u_{\nu-i} - \prod_{j=\nu-i-1}^{\nu-1}F_j(w_j)u_{\nu-i-1}\}$$
$$= u_\nu - F_{\nu-1}(w_{\nu-1})u_{\nu-1} + \textstyle\sum_{i=1}^{\nu-1}\prod_{j=\nu-i}^{\nu-1}L_j(w_j)\{u_{\nu-i} - F_{\nu-i-1}(w_{\nu-i-1})u_{\nu-i-1}\}.$$

By the inductive hypothesis we have from (3.20) and (3.21)

(3.31) $\|u_j - F_{j-1}(w_{j-1})u_{j-1}\|_{m_3}$

$$\leqq \|u_j - F_{j-1}(u_{j-1})u_{j-1}\|_{m_3} + \|F_{j-1}(u_{j-1})u_{j-1} - F_{j-1}(w_{j-1})u_{j-1}\|_{m_3}$$
$$\leqq \|u_j - S_h(t+(j-1)k)u_{j-1}\|_{m_3} + \|\{L_{j-1}(u_{j-1}) - L_{j-1}(w_{j-1})\}u_{j-1}\|_{m_3}$$
$$+ h\|G_{j-1}u_{j-1} - G_{j-1}w_{j-1}\|_{m_3}$$
$$\leqq c_1h^{m_1+1} + 2c_3h\|u_{j-1} - w_{j-1}\|_{m_3} \leqq c_1h^{m_1+1} + 2c_3\{(1+c_4h)^{j-1} - 1\}h^{m_1+1}$$
$$(j = 1,\dots, \nu).$$

Applying the stability of the linear difference scheme (3.15) and using (3.31), we obtain from (3.30)

$\|u_\nu - w_\nu\|_{m_3}$

$$\leqq \|u_\nu - F_{\nu-1}(w_{\nu-1})u_{\nu-1}\|_{m_3} + c_2\textstyle\sum_{i=1}^{\nu-1}\|u_{\nu-i} - F_{\nu-i-1}(w_{\nu-i-1})u_{\nu-i-1}\|_{m_3}$$
$$\leqq c_1h^{m_1+1} + 2c_3\{(1+c_4h)^{\nu-1} - 1\}h^{m_1+1}$$
$$+ c_2\textstyle\sum_{i=1}^{\nu-1}\{c_1h^{m_1+1} + 2c_3\{(1+c_4h)^{\nu-i-1} - 1\}h^{m_1+1}\}$$
$$\leqq c_4\nu h^{m_1+1} + c_4\textstyle\sum_{j=0}^{\nu-1}\{(1+c_4h)^j - 1\}h^{m_1+1} \leqq \{(1+c_4h)^\nu - 1\}h^{m_1},$$

and by (3.25)

$$\|w_\nu\|_{m_3} \leqq \|u_\nu\|_{m_3} + \{(1+c_4h)^\nu - 1\}h^{m_1} \leqq d_1 + \{\exp(c_4T/\lambda)\}h^{m_1} \leqq d_2.$$

Hence our induction on ν is complete. This gives the proof.

REMARK 3.2. The assumption (3.13) is not used in the proof of Theorem 3.1. However, it will be used in Section 6.2.

4. Families of linear difference operators

4.1. Definitions

We regard $L_h(t, v; h)$ as a one parameter family of the linear difference operators depending on t, when v is given. In this section we show some properties of linear operators mapping L_2 into itself. Some results obtained here will be used to show the stability of the linear difference scheme (3.15).

For $N\times N$ matrices $a_\sigma(t, x, \mu)$ such that $\partial_x^\alpha a_\sigma(t, x, \mu)(|\alpha|\leqq r)$ and $\partial_\mu a_\sigma(t, x, \mu)$ are continuous and bounded on $J\times \boldsymbol{R}^n\times H$, we write

$$(4.1)\qquad A_h(t;\mu)=\textstyle\sum_\sigma a_\sigma(t, x, \mu)T_h^\sigma, \qquad \sigma=(\sigma_1,\dots,\sigma_n),$$

where the summation is taken over some finite set of σ. Here we define $\mathscr{A}^r$ $(r=1, 2)$ by the set of two parameter families of $A_h(t;\mu)$ $(h, \mu\in H)$ and also define $\mathscr{A}_0^r$ $(r=1, 2)$ by the set of one parameter families of $A_h(t; h)$ $(h\in H)$.

Then it is found that $\mathscr{A}^2\subset\mathscr{A}^1$, $\mathscr{A}_0^2\subset\mathscr{A}_0^1$ and that $A_h(t;\mu)$ is a bounded linear operator from L_2 into itself, if it belongs to $\mathscr{A}^1$. We have the following lemmas:

LEMMA 4.1. *Let $A_h(t;\mu)$ belong to $\mathscr{A}^1$. Then*

$$(4.2)\qquad \|\{A_h(t; h)-A_h(t; 0)\}u\|\leqq C_A h\|u\|$$

for all $u(x)\in L_2$, $h\in H$ and $t\in J$, where

$$(4.3)\qquad C_A=\textstyle\sum_\sigma \sup_{t,x,\mu}|\partial_\mu a_\sigma(t, x, \mu)|.$$

PROOF. The proof is obvious by the mean-value theorem.

LEMMA 4.2. *Let $A_h(t; h)$ and $B_h(t; h)$ belong to $\mathscr{A}_0^r$ $(r=1, 2)$. Then the sum $A_h(t; h)+B_h(t; h)$, the product $A_h(t; h)B_h(t; h)$, the operation $\rho A_h(t; h)$ $(\rho\in \boldsymbol{C})$ and the adjoint $A_h^*(t; h)$ also belong to $\mathscr{A}_0^r$ $(r=1, 2)$.*

PROOF. The statements for $A_h(t; h)+B_h(t; h)$, $A_h(t; h)B_h(t; h)$ and $\rho A_h(t; h)$ can be easily shown. Let $A_h(t;\mu)$ be written as (4.1). Then we have

$$(4.4)\qquad A_h^*(t; h)=\textstyle\sum_\sigma a_\sigma^*(t, x-h\sigma, h)T_h^{-\sigma},$$

which belongs to $\mathscr{A}_0^r$ $(r=1, 2)$, where a_σ^* denotes the conjugate transpose of the matrix a_σ.

Now we introduce the amplification matrix of $A_h(t; h)\in\mathscr{A}_0^1$, which will be used in the next subsection. Let $A_h(t;\mu)$ be of the form (4.1). We put

$$(4.5)\qquad a(t, x, \omega, \mu)=\textstyle\sum_\sigma a_\sigma(t, x, \mu)\exp(i\sigma\cdot\omega) \qquad (\omega\in \boldsymbol{R}^n).$$

Then

$$(4.6)\qquad A_h(t;\mu)u(x)=\mathscr{F}^{-1}a(t, x, h\xi, \mu)(\mathscr{F}u)(\xi) \qquad (\xi\in \boldsymbol{R}^n),$$

where $(\mathscr{F}u)(\xi)$ and $(\mathscr{F}^{-1}v)(x)$ denote the Fourier transform of $u(x)\in L_2$ and the inverse Fourier transform of $v(\xi)$, respectively. In the case when a_σ is independent of x, i.e., $a_\sigma(t, x, \mu)=a_\sigma(t, \mu)$, it follows that

(4.7) $$\|A_h(t; h)u(x)\| = \|a(t, h\xi, h)(\mathscr{F}u)(\xi)\|,$$

where

(4.8) $$a(t, \omega, \mu) = \sum_\sigma a_\sigma(t, \mu)\exp(i\sigma\cdot\omega) \qquad (\omega\in \boldsymbol{R}^n).$$

It is said, in view of (4.7), that $a(t, \omega, h)$ is the amplification matrix of $A_h(t; h)$. In the following we would say that $a(t, x, \omega, h)$ is the amplification matrix even if $a_\sigma(t, x, \mu)$ depends on x.

4.2. Lax-Nirenberg Theorem

Without proof we first state Lax-Nirenberg Theorem [16, 30] which plays an important role in establishing the stability of the linear difference scheme (3.15).

THEOREM 4.1 (*Lax-Nirenberg*). *Let $a(t, x, \omega, h)$ be the amplification matrix of $A_h(t; h)\in\mathscr{A}_0^2$, where $A_h(t; \mu)$ is of the form* (4.1). *Suppose that $a(t, x, \omega, 0)$ is hermitian and positive semi-definite*:

(4.9) $$a(t, x, \omega, 0) \geqq 0 \qquad \textit{for all} \quad (t, x, \omega)\in J\times \boldsymbol{R}^n \times \boldsymbol{R}^n.$$

Then there exists a constant K_A such that

(4.10) $$\mathrm{Re}(A_h(t; h)u, u) \geqq -K_A h\|u\|^2$$

for all $u(x)\in L_2$, $h\in H$ and $t\in J$, where

(4.11) $$K_A = 2^{-1}(|a|_{2,0} + c|a|_{0,2}) + C_A,$$

(4.12) $$|a|_{l,m} = \sum_\sigma[\sup_{|\alpha|\leqq l}\{\sup_{t,x}|\partial_x^\alpha a_\sigma(t, x, 0)|\}](1+\sigma_1^2+\cdots+\sigma_n^2)^{m/2} \qquad (l, m = 0, 1, 2),$$

c is a constant independent of $A_h(t; h)$ and C_A is given by (4.3).

We next show the inequality, which plays the same role as the one (4.10), excluding the second-order derivatives of the amplification matrix. For this purpose we introduce Condition A to be imposed on the amplification matrix, and Property B, which a partition of unity $\{\psi_\alpha^2(x)\}$ possesses, to be used in establishing the above inequality.

CONDITION A. When $h=0$, the amplification matrix $a(t, x, \omega, h)$ of $A_h(t; h)\in\mathscr{A}_0^1$ is written as

(4.13) $$a(t, x, \omega, 0) = g(t, x, \omega)\sum_j|s_j(\omega)|^2,$$

where

(4.14) $$g(t, x, \omega) = \sum_m g_m(t, x)e_m(\omega),$$

(4.15) $$s_j(\omega) = \sum_r q_{jr} \exp(ir\cdot\omega), \qquad r = (r_1,\dots, r_n),$$

(4.16) $$e_m(\omega)\sum_j |s_j(\omega)|^2 = \sum_l e_{ml} \exp(il\cdot\omega), \quad l = (l_1,\dots, l_n),$$

the summations are taken over respective finite sets of indices, q_{jr} and e_{ml} are constants, $\mathrm{ess\cdot sup}_\omega |e_m(\omega)|$ is finite and $\partial_x^\alpha g_m(t, x)$ ($|\alpha|\leqq 1$) are continuous and bounded on $J\times \boldsymbol{R}^n$.

PROPERTY B. (1) $\psi_\alpha(x)\geqq 0$, $\psi_\alpha(x)\in C^\infty$, $\mathrm{supp}\,\psi_\alpha(x)\subset V_\alpha=\{x\in \boldsymbol{R}^n: |x-x^{(\alpha)}| <\delta\}$ for all α, where δ is a positive constant and

$$x^{(\alpha)} = (\delta\alpha_1/\sqrt{n},\dots, \delta\alpha_n/\sqrt{n}),\ \alpha = (\alpha_1,\dots, \alpha_n),\ \alpha_i = 0, \pm 1,\dots\ (i = 1,\dots, n);$$

(2) $\sum_\alpha \psi_\alpha^2(x) = 1$;

(3) There exists a constant $K(\delta)$ such that

$$\sum_{|\beta|\leqq 1} |\partial_x^\beta \psi_\alpha(x)| \leqq K(\delta) \qquad \text{for all} \quad x\in \boldsymbol{R}^n.$$

Let $M(x)$ denote the number of α such that $\psi_\alpha(x)\neq 0$ for each $x\in \boldsymbol{R}^n$. Since $M(x)$ is bounded on $\boldsymbol{R}^n$ by (1) of Property B, we put $M=\sup_x M(x)$. Then we have

THEOREM 4.2. *Let the amplification matrix $a(t, x, \omega, h)$ of $A_h(t; h)\in \mathscr{A}_0^1$ satisfy Condition* A. *Suppose that $g(t, x, \omega)$ is hermitian and that there exists a positive constant ε such that*

(4.17) $$g(t, x, \omega) \geqq \varepsilon I$$

for all $(t, x)\in J\times \boldsymbol{R}^n$ and for almost all $\omega\in \boldsymbol{R}^n$. Then

(4.18) $$\mathrm{Re}(A_h(t, h)u, u) \geqq \sum_\alpha \sum_j (\varepsilon-\delta|g|_{1,\infty}) \|A_{jh}\psi_\alpha u\|^2 - 2|g|_{1,\infty} h \sum_\alpha \sum_j C_{1j}\|A_{jh}\psi_\alpha u\|^2 - C_2 h\|u\|^2,$$

for all $u(x)\in L_2$, $h\in H$ and $t\in J$, where

(4.19) $$A_{jh} = \sum_r q_{jr} T_h^r,$$

(4.20) $$|g|_{1,\infty} = \sum_m \{\textstyle\sum_{i=1}^n \sup_{t,x} |\partial_{x_i} g_m(t, x)|\}\, \mathrm{ess\cdot sup}_\omega |e_m(\omega)|,$$

(4.21) $$C_{1j} = \sum_r (\textstyle\sum_{i=1}^n |r_i|) \max(|q_{jr}|, 1),$$

(4.22) $$C_2 = |g|_{1,\infty} \sum_j C_{1j} + C_A + MK^2(\delta)\, |a|_{0,2} h_0,$$

and C_A is given by (4.3).

This theorem will be proved in Section 6.1.

5. Convergence theorems

5.1. Condition C

In this section we are concerned with the difference scheme (3.8)–(3.12) under the assumption that (3.13) holds. We introduce Condition C imposed on the difference scheme.

CONDITION C. (1) $l_\sigma(t, x, y, \mu)$ $(\sigma \in \Lambda)$, together with all their first-order derivatives, are continuous and bounded on $J \times \boldsymbol{R}^n \times (\boldsymbol{C}_d^N)^q \times H$ for any fixed number $d>0$, where $\boldsymbol{C}_d^N = \{z \in \boldsymbol{C}^N : |z| \leqq d\}$;

(2) $\partial_x^\alpha \partial_y^\beta l_\sigma(t, x, y, \mu)$ $(\sigma \in \Lambda)$ and $\partial_x^\alpha \partial_y^\beta g(t, x, y, \mu)$ $(|\alpha|+|\beta| \leqq m)$ are continuous and bounded on $J \times \boldsymbol{R}^n \times (\boldsymbol{C}_d^N)^q \times H$ for any fixed number $d>0$, where $m \geqq 2$.

Under Condition C, $L_h(t, w; h)$ belongs to $\mathscr{A}_0^r$ $(r=1, 2)$ for a given $w(x) \in W^s$ $(s>n/2+r)$. We have the following lemma.

LEMMA 5.1. *Let Condition C be satisfied, where $m>n/2+1$. Then:*

(i) *For every $d>0$ there exists a constant $c_3(d)$ satisfying*

$$\|\{L_h(t, w; h) - L_h(t, v; h)\}u\|_{m-1} \leqq c_3(d)h\|w-v\|_{m-1} \tag{5.1}$$

and

$$\|G(t; h)w - G(t; h)v\|_{m-1} \leqq c_3(d)\,\|w-v\|_{m-1} \tag{5.2}$$

for all $w(x), v(x) \in W^{m-1}(d)$, $u(x) \in W^m(d)$, $h \in H$ and $t \in J$.

(ii) *For a positive number d and a constant M_0 let*

$$\|L_h(t, w; h)u\| \leqq (1+M_0 h)\,\|u\| \tag{5.3}$$

for all $w(x) \in W^m(d)$, $u(x) \in L_2$, $h \in H$ and $t \in J$. Then there exists a constant M_1 such that

$$\|L_h(t, w; h)u\|_m \leqq (1+M_1 h)\,\|u\|_m \tag{5.4}$$

for all $w(x) \in W^m(d)$, $u(x) \in W^m$, $h \in H$ and $t \in J$. Moreover, the linear difference scheme (3.15) *is stable in W^m for all $w_j \in W^m(d)$ $(j=0, 1,\dots)$.*

The proof will be given in Section 6.2.

Under Condition C let $l(t, x, w(x), \omega, h)$ be the amplification matrix of $L_h(t, w; h)$ and let

$$\tilde{l}(t, x, z, \omega) = \textstyle\sum_{\sigma \in \Lambda} l_\sigma(t, x, z,\dots, z, 0)\exp(i\sigma\cdot\omega). \tag{5.5}$$

Then it follows that

$$(5.6) \qquad l(t, x, w(x), \omega, h) = \sum_{\sigma\in\Lambda} l_\sigma(t, x, w(x, h), h)\exp(i\sigma\cdot\omega)$$

and

$$(5.7) \qquad l(t, x, w(x), \omega, 0) = \tilde{l}(t, x, w^T(x), \omega)$$

for all $w(x)\in W^m$ $(m>n/2+1)$ and $(t, x, \omega)\in J\times \boldsymbol{R}^n\times \boldsymbol{R}^n$, where $w(x, \mu)$ is defined as $v(x, \mu)$ (see (3.5)).

5.2. Convergence theorems

In this subsection we state two convergence theorems (Theorems 5.1 and 5.2) in terms of the amplification matrix.

THEOREM 5.1. *Assume that Condition* C *is satisfied, where* $m\geqq[n/2]+4$, *and that the difference scheme* (3.8) *approximates* (1.1) *with accuracy of order* m_1 *in* $U^{m_2,m_3}(d_1)$, *where*

$$(5.8) \qquad d_1>0, \quad m_1\geqq 1, \quad m_2\geqq m_1+1, \quad m_3=[n/2]+3.$$

Suppose that there exists a constant d_2 $(d_2>d_1)$ *such that*

$$(5.9) \qquad I-l^*(t, x, w(x), \omega, 0)l(t, x, w(x), \omega, 0)\geqq 0$$

for all $w(x)\in W^{m_3}(d_2)$ *and* $(t, x, \omega)\in J\times \boldsymbol{R}^n\times \boldsymbol{R}^n$. *Then for sufficiently small* $k_1>0$ *there exists a constant* c_5 *such that*

$$(5.10) \qquad \|u(t+\nu k, \cdot)-\phi^\nu(t, u(t, \cdot))\|_{m_3}\leqq c_5 h^{m_1}$$

for all $u(t, x)\in U^{m_2,m_3}(d_1)$, $k\in[0, k_1]$ *and* $t, t+\nu k\in J$. *Moreover, it holds that*

$$(5.11) \qquad |u(t+\nu k, \cdot)-\phi^\nu(t, u(t, \cdot))|_2\leqq Ch^{m_1}$$

for all $u(t, x)\in U^{m_2,m_3}(d_1)$, $k\in[0, k_1]$ *and* $t, t+\nu k\in J$, *where* C *is some constant.*

REMARK 5.1. As a sufficient condition under which (5.9) holds, we obtain

$$(5.12) \qquad I-\tilde{l}^*(t, x, z, \omega)\tilde{l}(t, x, z, \omega)\geqq 0$$

for all $(t, x, z, \omega)\in J\times \boldsymbol{R}^n\times \boldsymbol{C}^N_{d_3}\times \boldsymbol{R}^n$, where $d_3=K_1(0, n, N)d_2$. The expression (5.12) can be shown by the Sobolev-type theorem.

PROOF OF THEOREM 5.1. In view of Theorem 3.1 and Remark 3.1, it suffices to show that the assumptions of Theorem 3.1 are satisfied. By the assertion (i) of Lemma 5.1 the inequalities (3.20) and (3.21) hold with $c_3=c_3(d_2)$. To prove the stability of the linear difference scheme (3.15) we now show that there exists a constant M_0 such that the inequality (5.3) holds, where $d=d_2$ and $m=m_3$. Give a function $w(x)\in W^{m_3}(d_2)$. It follows from Lemma 4.2 that

$$A_h(t, w; h) = I - L_h^*(t, w; h)L_h(t, w; h)$$

belongs to $\mathscr{A}_0^2$. Denoting by $a(t, x, w(x), \omega, h)$ the amplification matrix of $A_h(t, w; h)$, we find that

$$a(t, x, w(x), \omega, 0) = I - l^*(t, x, w(x), \omega, 0)l(t, x, w(x), \omega, 0).$$

Since (5.9) leads to $a(t, x, w(x), \omega, 0) \geqq 0$, Theorem 4.1 yields

$$\text{(5.13)} \qquad \operatorname{Re}(A_h(t, w; h)u, u) \geqq -K_A h \|u\|^2$$

for all $u(x) \in L_2$, $h \in H$, $t \in J$, and hence

$$\text{(5.14)} \qquad \|L_h(t, w; h)u\| \leqq (1 + K_A h)\|u\|$$

for all $u(x) \in L_2$, $h \in H$, $t \in J$.

Applying the Sobolev-type theorem

$$\text{(5.15)} \quad |w|_2 \leqq K_1(2, n, N)\|w\|_{m_3} \leqq K_1(2, n, N)d_2 \qquad \text{for all} \quad w(x) \in W^{m_3}(d_2)$$

and Condition C to the representations of C_A ((4.3)) and K_A ((4.11)), we obtain a constant M_0 independent of $w(x)$ such that

$$K_A \leqq M_0 \qquad \text{for all} \quad w(x) \in W^{m_3}(d_2).$$

Therefore (5.3) follows from (5.14), and the linear difference scheme (3.15) is stable in W^{m_3} for all $w_j \in W^{m_3}(d_2)$ $(j=0, 1,\dots)$ by the assertion (ii) of Lemma 5.1. Thus the assumptions of Theorem 3.1 are satisfied. This completes the proof.

According to this theorem, the difference approximations $v^\nu(x)$ defined by (3.8) and (3.9) converge to the exact solution in the C_B^2-norm. However it should seem needless to derive the C_B^2-convergence, because the original system (1.1) does not contain the second-order derivatives of u. We shall derive the C_B^1-convergence. For this purpose we introduce the conditions.

CONDITION D. $b(t, x, z, \omega)$ is an $N \times N$ matrix function and can be written as

$$\text{(5.16)} \qquad b(t, x, z, \omega) = \textstyle\sum_\sigma b_\sigma(t, x, z)\exp(i\sigma\cdot\omega), \quad \sigma = (\sigma_1,\dots, \sigma_n),$$

where the summation is taken over a finite set of σ and $\partial_x^\alpha \partial_z^\beta b_\sigma(t, x, z)$ $(|\alpha| + |\beta| \leqq 1)$ are continuous and bounded on $J \times \boldsymbol{R}^n \times \boldsymbol{C}_d^N$ for any fixed number $d > 0$.

CONDITION E. (1) $a(t, x, z, \omega)$ is an $N \times N$ matrix function satisfying Condition D;

(2) $a(t, x, z, \omega)$ is written as

$$\text{(5.17)} \qquad a(t, x, z, \omega) = g(t, x, z, \omega)\textstyle\sum_j |s_j(\omega)|^2,$$

where

$$g(t, x, z, \omega) = \sum_m g_m(t, x, z) e_m(\omega), \tag{5.18}$$

$$s_j(\omega) = \sum_r q_{jr} \exp(ir \cdot \omega), \qquad r = (r_1, \dots, r_n), \tag{5.19}$$

$$e_m(\omega) \sum_j |s_j(\omega)|^2 = \sum_l e_{ml} \exp(il \cdot \omega), \quad l = (l_1, \dots, l_n), \tag{5.20}$$

the summations are taken over respective finite sets of indices, q_{jr} and e_{ml} are constants, $\text{ess}\cdot\sup_\omega |e_m(\omega)|$ is finite and $\partial_x^\alpha \partial_z^\beta g_m(t, x, z)$ $(|\alpha|+|\beta| \leqq 1)$ are continuous and bounded on $J \times \boldsymbol{R}^n \times \boldsymbol{C}_d^N$ for any fixed number $d>0$.

Now we state the following convergence theorem.

THEOREM 5.2. *Assume that Condition* C *is satisfied, where* $m \geqq [n/2]+3$, *and that the difference scheme* (3.8) *approximates* (1.1) *with accuracy of order* m_1 *in* $U^{m_2,m_3}(d_1)$, *where*

$$d_1 > 0, \quad m_1 \geqq 1, \quad m_2 \geqq m_1 + 1, \quad m_3 = [n/2] + 2. \tag{5.21}$$

Let $I - l^* l$ *be written as*

$$\begin{aligned} I - l^*(t, x, w(x), \omega, 0) l(t, x, w(x), \omega, 0) \\ = b^*(t, x, w^T(x), \omega) a(t, x, w^T(x), \omega) b(t, x, w^T(x), \omega), \end{aligned} \tag{5.22}$$

where $b(t, x, z, \omega)$ *and* $a(t, x, z, \omega)$ *satisfy Conditions* D *and* E, *respectively. Suppose that* $g(t, x, z, \omega)$ *is hermitian and that there exist constants* d_2 $(d_2 > d_1)$ *and* ε $(\varepsilon > 0)$ *such that*

$$g(t, x, w^T(x), \omega) \geqq \varepsilon I \tag{5.23}$$

for all $w(x) \in W^{m_3}(d_2)$, $(t, x) \in J \times \boldsymbol{R}^n$ *and for almost all* $\omega \in \boldsymbol{R}^n$. *Then for sufficiently small* $k_1 > 0$ *there exists a constant* c_6 *such that*

$$\|u(t+\nu k, \cdot) - \phi^\nu(t, u(t, \cdot))\|_{m_3} \leqq c_6 h^{m_1} \tag{5.24}$$

for all $u(t, x) \in U^{m_2,m_3}(d_1)$, $k \in [0, k_1]$ *and* $t, t+\nu k \in J$. *Moreover, it holds that*

$$|u(t+\nu k, \cdot) - \phi^\nu(t, u(t, \cdot))|_1 \leqq C h^{m_1} \tag{5.25}$$

for all $u(t, x) \in U^{m_2,m_3}(d_1)$, $k \in [0, k_1]$ *and* $t, t+\nu k \in J$, *where* C *is some constant.*

REMARK 5.2. As a sufficient condition under which (5.23) holds, we have

$$g(t, x, z, \omega) \geqq \varepsilon I \tag{5.26}$$

for all $(t, x, z) \in J \times \boldsymbol{R}^n \times \boldsymbol{C}_{d_3}^N$ and for almost all $\omega \in \boldsymbol{R}^n$, where $d_3 = K_1(0, n, N) d_2$.

PROOF OF THEOREM 5.2. Since $a(t, x, z, \omega)$ satisfies (1) of Condition E, we may write it as

$$a(t, x, z, \omega) = \textstyle\sum_\alpha a_\alpha(t, x, z)\exp(i\alpha\cdot\omega), \quad \alpha = (\alpha_1,\dots, \alpha_n).$$

For each $w(x)\in W^{m_3}(d_2)$ we introduce the one parameter families of linear operators $A_h(t, w)$ and $B_h(t, w)$ belonging to $\mathscr{A}_0^1$, i.e.,

$$A_h(t, w)u = \textstyle\sum_\alpha a_\alpha(t, x, w^T(x))T_h^\alpha u$$

and

$$B_h(t, w)u = \textstyle\sum_\sigma b_\sigma(t, x, w^T(x))T_h^\sigma u$$

for all $u(x)\in L_2$, $h\in H$ and $t\in J$.

Now we shall prove the theorem by using the properties of $A_h(t, w)$ and $B_h(t, w)$. To this end it suffices to show that there exists a constant M_0 such that the inequality (5.3) holds, when $d=d_2$ and $m=m_3$.

We first prove the following inequality:

$$\mathrm{Re}(A_h(t, w)u, u) \geqq -K_A h\|u\|^2 \tag{5.27}$$

for all $u(x)\in L_2$, $h\in H$, $t\in J$ and $w(x)\in W^{m_3}(d_2)$, where K_A is a constant independent of $w(x)$. By Condition E the amplification matrix of $A_h(t, w)$, $a(t, x, w^T(x), \omega)$, satisfies Condition A for each $w(x)\in W^{m_3}(d_2)$. Since $g(t, x, w^T(x), \omega)\geqq\varepsilon I$ holds for $w(x)\in W^{m_3}(d_2)$, Theorem 4.2 yields

$$\begin{aligned}\mathrm{Re}(A_h(t, w)u, u) \geqq{}& \textstyle\sum_\alpha\sum_j(\varepsilon-\delta|g|_{1,\infty})\|\Lambda_{jh}\psi_\alpha u\|^2\\ &-2|g|_{1,\infty}h\textstyle\sum_\alpha\sum_j C_{1j}\|\Lambda_{jh}\psi_\alpha u\|^2 - C_2 h\|u\|^2\end{aligned}\tag{5.28}$$

for all $u(x)\in L_2$, $h\in H$ and $t\in J$, where $|g|_{1,\infty}$, C_{1j} and C_2 are given by (4.20), (4.21) and (4.22), respectively. The constant C_A in (4.22) vanishes, because $a(t, x, z, \omega)$ is independent of h. Applying the Sobolev-type theorem (Theorem 2.1) and Condition E to $|g|_{1,\infty}$ depending on $w(x)$, we obtain a constant M_2 independent of $w(x)\in W^{m_3}(d_2)$ such that

$$|g|_{1,\infty}\leqq M_2 \qquad \text{for all}\quad w(x)\in W^{m_3}(d_2). \tag{5.29}$$

In a similar way, we obtain a constant M_3 such that

$$|a|_{0,2}\leqq M_3 \qquad \text{for all}\quad w(x)\in W^{m_3}(d_2). \tag{5.30}$$

Since $\sum_\alpha\psi_\alpha^2=1$, we have

$$\textstyle\sum_\alpha\|\Lambda_{jh}\psi_\alpha u\|^2 \leqq (\sum_r|q_{jr}|)^2\|u\|^2. \tag{5.31}$$

Choose δ small so that $\delta<\varepsilon/M_2$ and put

(5.32) $$K_A = 3M_2 \sum_j C_{1j} \max\{(\sum_r |q_{jr}|)^2, 1\} + MM_3K^2(\delta)h_0.$$

Then by (5.29)–(5.32), (5.28) is reduced to

$$\operatorname{Re}(A_h(t, w)u, u) \geqq -2M_2h \sum_j C_{1j} \sum_\alpha \|\Lambda_{jh}\psi_\alpha u\|^2 - C_2h\|u\|^2 \geqq -K_Ah\|u\|^2.$$

Thus we obtain (5.27).

Putting

$$P_h(t, w; h) = I - L_h^*(t, w; h)L_h(t, w; h)$$

and

$$Q_h(t, w; h) = B_h^*(t, w)A_h(t, w)B_h(t, w),$$

we next show that there exists a constant K_L such that

(5.33) $$\|\{P_h(t, w; h) - Q_h(t, w; h)\}u\| \leqq K_Lh\|u\|$$

for all $u(x) \in L_2$, $h \in H$, $t \in J$ and $w(x) \in W^{m_3}(d_2)$. Since $P_h(t, w; h)$ and $Q_h(t, w; h)$ belong to $\mathscr{A}_0^1$ for each $w(x) \in W^{m_3}(d_2)$, we have by Lemma 4.1

(5.34) $$\|\{P_h(t, w; h) - P_h(t, w; 0)\}u\| \leqq C_Ph\|u\|$$

and

(5.35) $$\|\{Q_h(t, w; h) - Q_h(t, w; 0)\}u\| \leqq C_Qh\|u\|$$

for all $u(x) \in L_2$, $h \in H$, $t \in J$ and for each $w(x) \in W^{m_3}(d_2)$, where C_P and C_Q are defined as C_A (see (4.3)).

Let us denote by $p(t, x, w, \omega, h)$ and $q(t, x, w, \omega, h)$ the amplification matrices of $P_h(t, w; h)$ and $Q_h(t, w; h)$, respectively. Since (5.22) implies

$$p(t, x, w, \omega, 0) = q(t, x, w, \omega, 0),$$

we have

$$P_h(t, w; 0) = Q_h(t, w; 0).$$

Hence, by (5.34) and (5.35)

(5.36) $$\|\{P_h(t, w; h) - Q_h(t, w; h)\}u\| \leqq (C_P + C_Q)h\|u\|$$

for all $u(x) \in L_2$, $h \in H$ and $t \in J$. In the same way as the one used in the proof of (5.29) we know that there exists a constant M_4 satisfying

$$C_P + C_Q \leqq M_4 \qquad \text{for all} \quad w(x) \in W^{m_3}(d_2).$$

Thus (5.33) follows from (5.36) where $K_L = M_4$.

Finally we show that the inequality (5.3) holds for some constant M_0. By (5.27) and (5.33) we obtain

$$(5.37)\quad \begin{aligned} \mathrm{Re}(P_h(t, w; h)u, u) &\geqq \mathrm{Re}(Q_h(t, w; h)u, u) - K_L h\|u\|^2 \\ &\geqq \mathrm{Re}(A_h(t, w)B_h(t, w)u, B_h(t, w)u) - K_L h\|u\|^2 \\ &\geqq -K_A h\|B_h(t, w)u\|^2 - K_L h\|u\|^2. \end{aligned}$$

Since Condition D implies the existence of a constant M_5 satisfying

$$\textstyle\sum_\sigma \sup_{t,x} |b_\sigma(t, x, w^T(x))| \leqq M_5 \qquad \text{for all} \quad w(x) \in W^{m_3}(d_2),$$

it follows that

$$\|B_h(t, w)u\| \leqq M_5\|u\|$$

for all $u(x) \in L_2$, $h \in H$, $t \in J$ and $w(x) \in W^{m_3}(d_2)$. Hence, from (5.37) we have (5.3) with $M_0 = K_A M_5 + K_L$, $d = d_2$ and $m = m_3$. Thus the linear difference scheme (3.15) is stable in W^{m_3} for all $w_j \in W^{m_3}(d_2)$ $(j = 0, 1,\dots)$ by the assertion (ii) of Lemma 5.1. This completes the proof.

5.3. Examples

In this subsection we introduce two difference schemes proposed by Gourlay and Morris [7], which approximate the following symmetric hyperbolic system:

$$(5.38)\qquad \frac{\partial u}{\partial t}(t, x) = \textstyle\sum_{j=1}^n A_j(u^T(t, x)) \dfrac{\partial u}{\partial x_j}(t, x)$$

with the initial condition (1.2), and seek some restrictions on λ under which the inequalities (5.9) and (5.23) hold. For simplicity we assume that $A_j(z)$ $(j = 1,\dots, n)$ are sufficiently smooth. The approximating difference schemes are written as (3.8) and (3.9) with the difference operators S_{jh} $(j = 1, 2)$ given by

$$(5.39)\qquad S_{1h}v(x) = C_h v(x) + (\lambda/2) \textstyle\sum_{j=1}^n A_j(v^T(x))(T_{jh} - T_{jh}^{-1})v(x)$$

and

$$(5.40)\qquad S_{2h}v(x) = v(x) + (\lambda/2) \textstyle\sum_{j=1}^n A_j(\tilde{v}^T(x, h))(T_{jh} - T_{jh}^{-1})\tilde{v}(x, h),$$

where

$$C_h = (2n)^{-1} \textstyle\sum_{j=1}^n (T_{jh} + T_{jh}^{-1}), \quad T_{jh}^{\pm 1}v(x) = v(x \pm he_j),$$

$$\tilde{v}(x, h) = C_h v(x) + (\lambda/4) \textstyle\sum_{j=1}^n A_j(v^T(x))(T_{jh} - T_{jh}^{-1})v(x)$$

and e_j is the unit vector in $\boldsymbol{R}^n$ whose j-th entry is unity. It is clear that the operators S_{jh} $(j = 1, 2)$ give the difference schemes with accuracy of order j $(j = 1, 2)$ in $U^{j+1,m}(d)$ $(m \geqq [n/2] + 1)$. We put

$$(5.41)\qquad L_{1h}(v) = C_h + (\lambda/2) \textstyle\sum_{j=1}^n A_j(v^T(x))(T_{jh} - T_{jh}^{-1})$$

and

(5.42) $$L_{2h}(v;\mu) = I + (\lambda/2)\sum_{j=1}^{n} A_j(\tilde{v}^T(x,\mu))(T_{jh} - T_{jh}^{-1})C_h + (\lambda^2/8)\sum_{j=1}^{n} A_j(\tilde{v}^T(x,\mu))\sum_{i=1}^{n} A_i(v^T(x+\mu e_j))T_{jh}(T_{ih} - T_{ih}^{-1}) - (\lambda^2/8)\sum_{j=1}^{n} A_j(\tilde{v}^T(x,\mu))\sum_{i=1}^{n} A_i(v^T(x-\mu e_j))T_{jh}^{-1}(T_{ih} - T_{ih}^{-1}).$$

Then $L_{1h}(v)$ and $L_{2h}(v; h)$ belong to $\mathscr{A}_0^1$ for a given $v(x)\in W^m$ $(m\geqq[n/2]+2)$ and satisfy the equality (3.13). Furthermore,

$$S_{1h}v(x) = L_{1h}(v)v(x), \qquad S_{2h}v(x) = L_{2h}(v;h)v(x).$$

Denoting by $l_1(w(x),\omega)$ and $l_2(w(x),\omega,h)(w(x)\in W^m, m\geqq[n/2]+2)$ the amplification matrices of $L_{1h}(w)$ and $L_{2h}(w; h)$, respectively, we have

(5.43) $$l_1(w(x),\omega) = c(\omega)I + i\lambda A(w(x),\sin\omega)$$

and

(5.44) $$l_2(w(x),\omega,0) = I + i\lambda A(w(x),\sin\omega)c(\omega) - (\lambda^2/2)A^2(w(x),\sin\omega),$$

where

$$c(\omega) = n^{-1}\sum_{j=1}^{n}\cos\omega_j, \quad \sin\omega = (\sin\omega_1,\dots,\sin\omega_n)$$

and

$$A(w(x),\omega) = \sum_{j=1}^{n} A_j(w^T(x))\omega_j.$$

Following Yamaguti and Nogi [31], we obtain

(5.45) $$I - l_1^*(w(x),\omega)l_1(w(x),\omega) = b_1^*a_1b_1$$

and

(5.46) $$I - l_2^*(w(x),\omega,0)l_2(w(x),\omega,0) = b_2^*a_2b_2,$$

where

(5.47) $$a_1(w^T(x),\omega) = \{n^{-1}I - \lambda^2A_z^2(w(x),\sin\omega)\}\sum_{j=1}^{n}\sin^2\omega_j + n^{-2}(\textstyle\sum_{j>k}c_{jk}^2(\omega))I,$$

(5.48) $$a_2(w^T(x),\omega) = \{n^{-1}I - (\lambda^2/4)A_z^2(w(x),\sin\omega)\}\sum_{j=1}^{n}\sin^2\omega_j + n^{-2}(\textstyle\sum_{j>k}c_{jk}^2(\omega))I,$$

$$A_z(w(x),\omega) = \sum_{j=1}^{n} A_j(w^T(x))\omega_j/|\omega|, \quad c_{jk}(\omega) = \cos\omega_j - \cos\omega_k,$$

$$b_1 = I, \quad b_2(w^T(x),\omega) = \lambda A(w(x),\sin\omega).$$

Here we put

$$f(\omega) = \sum_{j=1}^{n}\sin^2\omega_j + \sum_{j>k}c_{jk}^2(\omega)$$

and

(5.49) $$g_j(w^T(x), \omega) = a_j(w^T(x), \omega)/f(\omega) \qquad (j = 1, 2).$$

Let us seek some restrictions on λ under which the inequalities (5.9) and (5.23) hold. For any fixed number $d>0$ we put

$$\rho_d = \sup_{\omega\neq 0, w(x)} \rho(A_z(w(x), \omega)),$$

where the supremum with respect to $w(x)$ is taken over $W^{[n/2]+2}(d)$ and $\rho(A_z)$ denotes the spectral radius of A_z. We choose λ satisfying $\lambda\rho_d \leqq 1/\sqrt{n}$, so that $a_1 \geqq 0$ and

$$I - l_1^*(w(x), \omega)l_1(w(x), \omega) \geqq 0 \qquad \text{for all } w(x) \in W^{[n/2]+2}(d).$$

Hence, if $\lambda\rho_{d_2} \leqq 1/\sqrt{n}$, then the inequality (5.9) holds for each $d_2>0$, where $l=l_1$.

For each λ satisfying $\lambda\rho_d \leqq 2/\sqrt{n}$ we similarly know $a_2 \geqq 0$, and therefore obtain (5.9) with $l=l_2$ for each $d_2>0$, if $\lambda\rho_{d_2} \leqq 2/\sqrt{n}$.

It is known that a_j $(j=1, 2)$ satisfy Condition E by (5.47), (5.48) and (5.49) and that (5.22) holds by (5.45) and (5.46), where $l=l_j$, $b=b_j$ and $a=a_j$ $(j=1, 2)$. For each λ such that $\lambda\rho_d < 1/\sqrt{n}$ we have

$$g_1(w^T(x), \omega) \geqq \min\{(n^{-1} - \lambda^2\rho_d^2), n^{-2}\}I > 0$$

for all $w(x) \in W^{[n/2]+2}(d)$ and for almost all $\omega \in \boldsymbol{R}^n$. Hence, if $\lambda\rho_{d_2} < 1/\sqrt{n}$, then the inequality (5.23) holds for each $d_2>0$, where $g=g_1$. Similarly we can show (5.23) with $g=g_2$ for each $d_2>0$, if $\lambda\rho_{d_2} < 2/\sqrt{n}$.

6. Proofs

6.1. Proof of Theorem 4.2

We prepare the following two lemmas:

LEMMA 6.1. *For all $u(x) \in L_2$, $h \in H$ and $t \in J$ and for all j*

(6.1) $$|(A_h(t; 0)u, u) - \textstyle\sum_\alpha (A_h(t; 0)\psi_\alpha u, \psi_\alpha u)| \leqq |a|_{0,2} C_3 h \|u\|^2,$$

(6.2) $$A_h(t, 0) = G_h(t) \textstyle\sum_j \Lambda_{jh}^* \Lambda_{jh},$$

(6.3) $$\|\{G_h(t)\Lambda_{jh}^* - \Lambda_{jh}^* G_h(t)\}u\| \leqq |g|_{1,\infty} C_{1j} h \|u\|,$$

where

$$C_3 = MK^2(\delta)h_0, \; G_h(t) = \textstyle\sum_m g_m(t, x)E_{mh}, \; E_{mh} = \mathscr{F}^{-1}e_m(h\xi)\mathscr{F}.$$

LEMMA 6.2. *For all $u(x) \in L_2$, $h \in H$ and $t \in J$ and for all j and α*

(6.4) $$|(\{G_h(t) - G_{\alpha h}(t)\}\Lambda_{jh}\psi_\alpha u, \Lambda_{jh}\psi_\alpha u)| \leqq |g|_{1,\infty}(\delta + C_{1j}h)\|\Lambda_{jh}\psi_\alpha u\|^2,$$

where

$$G_{\alpha h}(t) = \sum_m g_m(t, x^{(\alpha)})E_{mh}.$$

By these lemmas we have

$$\begin{aligned}
(6.5)\quad \operatorname{Re}(A_h(t; h)u, u) &\geqq \operatorname{Re}(A_h(t; 0)u, u) - C_A h\|u\|^2 \\
&\geqq \sum_\alpha \operatorname{Re}(A_h(t; 0)\psi_\alpha u, \psi_\alpha u) - (C_A + |a|_{0,2}C_3)h\|u\|^2 \\
&= \sum_\alpha \sum_j \operatorname{Re}(G_h(t)\Lambda_{jh}^*\Lambda_{jh}\psi_\alpha u, \psi_\alpha u) - (C_A + |a|_{0,2}C_3)h\|u\|^2 \\
&\geqq \sum_\alpha \sum_j \{\operatorname{Re}(G_h(t)\Lambda_{jh}\psi_\alpha u, \Lambda_{jh}\psi_\alpha u) - |g|_{1,\infty}C_{1j}h\|\Lambda_{jh}\psi_\alpha u\| \, \|\psi_\alpha u\|\} \\
&\quad - (C_A + |a|_{0,2}C_3)h\|u\|^2 \\
&\geqq \sum_\alpha \sum_j \{\operatorname{Re}(G_h(t)\Lambda_{jh}\psi_\alpha u, \Lambda_{jh}\psi_\alpha u) - |g|_{1,\infty}C_{1j}h\|\Lambda_{jh}\psi_\alpha u\|^2\} \\
&\quad - (|g|_{1,\infty}\sum_j C_{1j} + C_A + |a|_{0,2}C_3)h\|u\|^2 \\
&\geqq \sum_\alpha \sum_j \{\operatorname{Re}(G_{\alpha h}(t)\Lambda_{jh}\psi_\alpha u \; \Lambda_{jh}\psi_\alpha u) - |g|_{1,\infty}(\delta + 2C_{1j}h)\, \|\Lambda_{jh}\psi_\alpha u\|^2\} \\
&\quad - (|g|_{1,\infty}\sum_j C_{1j} + C_A + |a|_{0,2}C_3)h\|u\|^2.
\end{aligned}$$

Since $G_{\alpha h}(t)$ is an operator independent of x, the assumption (4.17) yields

$$(G_{\alpha h}(t)\Lambda_{jh}\psi_\alpha u, \Lambda_{jh}\psi_\alpha u) \geqq \varepsilon\|\Lambda_{jh}\psi_\alpha u\|^2.$$

Hence, combining the above with (6.5), we have (4.18).

PROOF OF LEMMA 6.1. Putting

$$F = (A_h(t; 0)u, u) - \sum_\alpha (A_h(t; 0)\psi_\alpha u, \psi_\alpha u),$$

we have, by (2) of Property B,

$$\begin{aligned}
(6.6)\quad F &= \int \sum_\sigma \{a_\sigma(t, x, 0)u(x+h\sigma)\}^*(1 - \sum_\alpha \psi_\alpha(x+h\sigma)\psi_\alpha(x))u(x)dx \\
&= 2^{-1}\int \sum_\sigma \{a_\sigma(t, x, 0)u(x+h\sigma)\}^* \sum_\alpha (\psi_\alpha(x+h\sigma) - \psi_\alpha(x))^2 u(x)dx.
\end{aligned}$$

Since

$$\sum_\alpha (\psi_\alpha(x+h\sigma) - \psi_\alpha(x))^2 \leqq 2MK^2(\delta)(\textstyle\sum_{i=1}^n \sigma_i^2)h^2,$$

it holds from (6.6) that

$$\begin{aligned}
|F| &\leqq 2^{-1}\sum_\sigma \sup_{t,x}|a_\sigma(t, x, 0)| \, \|u\| 2MK^2(\delta)(\textstyle\sum_{i=1}^n \sigma_i^2)h^2\|u\| \\
&= MK^2(\delta)h^2 \sum_\sigma \sup_{t,x}|a_\sigma(t, x, 0)| (\textstyle\sum_{i=1}^n \sigma_i^2)\, \|u\|^2 \\
&\leqq MK^2(\delta)h_0 h|a|_{0,2}\|u\|^2,
\end{aligned}$$

which yields the desired inequality (6.1).

It is clear that Λ_{jh} and E_{mh} are both bounded linear operators from L_2 into itself. By (4.13), (4.14) and (4.16) we have

$$\sum_\sigma a_\sigma(t, x, 0)\exp(i\sigma\cdot\omega) = \sum_m g_m(t, x)\sum_l e_{ml}\exp(il\cdot\omega)$$

and

$$E_{mh}\sum_j \Lambda_{jh}^*\Lambda_{jh} = \sum_l e_{ml}T_h^l.$$

Then $A_h(t; 0)$ is represented as

$$\begin{aligned} A_h(t; 0) &= \sum_m g_m(t, x)\sum_l e_{ml}T_h^l \\ &= \sum_m g_m(t, x)E_{mh}\sum_j \Lambda_{jh}^*\Lambda_{jh} = G_h(t)\sum_j \Lambda_{jh}^*\Lambda_{jh}, \end{aligned}$$

which is (6.2).

Finally (6.3) is proved by the fact that

$$\begin{aligned} &\|\{G_h(t)\Lambda_{jh}^* - \Lambda_{jh}^*G_h(t)\}u\| \leqq \sum_m \|\{g_m(t, x)\Lambda_{jh}^* - \Lambda_{jh}^*g_m(t, x)\}E_{mh}u\| \\ &\quad \leqq \sum_m\sum_r |q_{jr}|\,\|\{g_m(t, x) - g_m(t, x-hr)\}T_h^{-r}E_{mh}u\| \\ &\quad \leqq \sum_m\sum_r |q_{jr}|h(\textstyle\sum_{i=1}^n |r_i|)\sum_{i=1}^n \sup_{t,x}|\partial_{x_i}g_m(t, x)|\,\|E_{mh}u\| \leqq |g|_{1,\infty}C_{1j}h\|u\|. \end{aligned}$$

Here we used the relation $E_{mh}\Lambda_{jh}^* = \Lambda_{jh}^*E_{mh}$.

Proof of Lemma 6.2. By the mean-value theorem we have

$$\begin{aligned} &(\{G_h(t) - G_{\alpha h}(t)\}\Lambda_{jh}\psi_\alpha u,\ \Lambda_{jh}\psi_\alpha u) = \sum_m(\{g_m(t, x) - g_m(t, x^{(\alpha)})\}E_{mh}\Lambda_{jh}\psi_\alpha u,\ \Lambda_{jh}\psi_\alpha u) \\ &\quad = \sum_m\sum_{i=1}^n\left(\int_0^1 \partial_{x_i}g_m(t, x^{(\alpha)}+\theta(x-x^{(\alpha)}))d\theta E_{mh}\Lambda_{jh}\psi_\alpha u,\ (x_i - x_i^{(\alpha)})\Lambda_{jh}\psi_\alpha u\right). \end{aligned}$$

Since Λ_{jh} is written as a finite sum of translation operators T_h^r, it follows that

$$\operatorname{supp}(\Lambda_{jh}\psi_\alpha u) \subset W_\alpha = \{x \in \boldsymbol{R}^n\colon |x - x^{(\alpha)}| < \delta + \max_r(\textstyle\sum_{i=1}^n |r_i|)h\}.$$

Hence we obtain

$$|(\{G_h(t) - G_{\alpha h}(t)\}\Lambda_{jh}\psi_\alpha u,\ \Lambda_{jh}\psi_\alpha u)| \leqq |g|_{1,\infty}\|\Lambda_{jh}\psi_\alpha u\|(\delta + C_{1j}h)\|\Lambda_{jh}\psi_\alpha u\|,$$

which is (6.4).

6.2. Proof of Lemma 5.1

We show (i). From (3.11) and (3.13) it follows that

(6.7) $\quad \{L_h(t, w; h) - L_h(t, v; h)\}u(x) = \sum_{\sigma\in\Lambda} J_\sigma(t, w, v; h)(T_h^\sigma - I)u(x),$

where

$$J_\sigma(t, w, v; h) = l_\sigma(t, x, w(x, h), h) - l_\sigma(t, x, v(x, h), h) \qquad (\sigma \in \Lambda).$$

Carrying out the differentiations, we obtain

$$(6.8)\quad \partial_x^s\{J_\sigma(t, w, v; h)(T_h^\sigma - I)u(x)\}$$
$$= \textstyle\sum_{a+b=s} c(a, b)\{\partial_x^a J_\sigma(t, w, v; h)\}\{\partial_x^b(T_h^\sigma - I)u(x)\}$$
$$= \textstyle\sum_{a+b=s} c(a, b)\sum'^{a}_{\alpha,\beta} C_{\alpha\beta}[\sum_{i=1}^{q}\sum_{j=1}^{N}\int_0^1 \partial_x^\alpha\partial_y^\gamma E_{ij\sigma}(\theta, w, v)d\theta$$
$$\times \{w_j(x+hp^{(i)}) - v_j(x+hp^{(i)})\}\{\partial_x^\delta F_\beta(w)\}\partial_x^b(T_h^\sigma - I)u(x)$$
$$+ \partial_x^\alpha\partial_y^\gamma l_\sigma(t, x, v(x, h), h)\{\partial_x^\delta(F_\beta(w) - F_\beta(v))\}\partial_x^b(T_h^\sigma - I)u(x)\}],$$

where

$$s = (s_1,\dots, s_n)\ (|s| \leqq m-1),\ a = (a_1,\dots, a_n),\ b = (b_1,\dots, b_n),\ \alpha = (\alpha_1,\dots, \alpha_n),$$
$$\beta = (\beta^{(1)},\dots, \beta^{(n)}),\ \beta^{(k)} = (\beta_{11}^{(k)},\dots, \beta_{1N}^{(k)}, \beta_{21}^{(k)},\dots, \beta_{qN}^{(k)}) \qquad (k = 1,\dots, n),$$
$$\gamma = \textstyle\sum_{k=1}^{n}\beta^{(k)},\ \sum'^{a}_{\alpha,\beta} = \sum_{\alpha_1+|\beta^{(1)}|\leqq a_1}\cdots\sum_{\alpha_n+|\beta^{(n)}|\leqq a_n},\ c(a, b) = (a+b)!/(a!b!),$$
$$\delta = (\delta_1,\dots, \delta_n),\ \delta_k = a_k - \alpha_k - |\beta^{(k)}| \qquad (k = 1,\dots, n),$$
$$w(x) = (w_1(x),\dots, w_N(x))^T,\ v(x) = (v_1(x),\dots, v_N(x))^T,$$
$$E_{ij\sigma}(\theta, w, v) = \partial_{y_{ij}} l_\sigma(t, x, \theta w(x, h)+(1-\theta)v(x, h), h)$$
$$(\sigma\in\Lambda;\ i = 1,\dots, q;\ j = 1,\dots, N),$$

$F_\beta(w)$ is the product of the powers of $\partial_{x_k} w_j(x+hp^{(i)})$ with exponent $\beta_{ij}^{(k)}$ for $i=1,\dots, q$, $j=1,\dots, N$, $k=1,\dots, n$, $F_\beta(v)$ is defined similarly, and $C_{\alpha\beta}$ is a constant depending on α and β.

Since

$$(6.9)\qquad \|\{(T_h^\sigma - I)u\}\|_{m-1} \leqq |\sigma|h\|u\|_m \leqq |\sigma|hd \qquad (\sigma\in\Lambda)$$

for all $u(x)\in W^m(d)$, the Sobolev-type theorem yields

$$(6.10)\qquad \|(w_j - v_j)\{\partial_x^\delta F_\beta(w)\}\partial_x^b (T_h^\sigma - I)u\|$$
$$\leqq K_1(0, n, N)\|w-v\|_{m-1}K_3(m, n, N)(1+d)^{m-1}|\sigma|hd \qquad (j=1,\dots, N)$$

and

$$(6.11)\qquad \|\{\partial_x^\delta(F_\beta(w) - F_\beta(v))\}\partial_x^b(T_h^\sigma - I)u\|$$
$$\leqq (m-1)K_3(m, n, N)\|w-v\|_{m-1}(1+d)^{m-2}|\sigma|hd$$

for all $w(x), v(x)\in W^{m-1}(d)$ and $u(x)\in W^m(d)$, where $K_3(m, n, N)$ is a constant independent of β, δ, σ and b.

By Condition C we have

$$(6.12)\qquad |\partial_x^\alpha\partial_y^\gamma E_{ij\sigma}(\theta, w, v)| \leqq c_7(d') \qquad (\sigma\in\Lambda;\ 0\leqq\theta\leqq 1)$$

and

(6.13) $$|\partial_x^\alpha \partial_y^\gamma l_\sigma(t, x, v(x, h), h)| \leqq c_7(d') \qquad (\sigma \in \Lambda)$$

for all $w(x), v(x) \in W^m(d)$, where $d' = K_1(0, n, N)d$ and $c_7(d')$ is a constant depending on d'. Put

$$C(d) = (\sum_{\sigma\in\Lambda} |\sigma|)\{\sum_{|s|\leqq m-1} \sum_{a+b=s} c(a, b) \sum'^{a}_{\alpha,\beta} C_{\alpha\beta}\}$$
$$\times \{NqK_1(0, n, N)(1+d)+m-1\}K_3(m, n, N)d(1+d)^{m-2}c_7(d').$$

Then by (6.8) and (6.10)–(6.13) it follows from (6.7) that

$$\|\{L_h(t, w; h) - L_h(t, v; h)\}u\|_{m-1} \leqq C(d)h\|w-v\|_{m-1},$$

and (5.1) follows. Similarly it can be shown that (5.2) holds.

We show (ii). From (3.11) it follows that

(6.14) $$\|L_h(t, w; h)u\|_m^2 = \sum_{|s|\leqq m} \|\partial_x^s\{L_h(t, w; h)u\}\|^2$$
$$= \sum_{|s|\leqq m} \|\sum_{a+b=s,|a|\geqq 1} c(a, b)V(a, b, w, u) + L_h(t, w; h)\partial_x^s u\|^2,$$

where

$$V(a, b, w, u) = \sum_{\sigma\in\Lambda} (\partial_x^a\{l_\sigma(t, x, w(x, h), h)\})\partial_x^b T_h^\sigma u.$$

By (5.3) we have

(6.15) $$\|L_h(t, w; h)\partial_x^s u\| \leqq (1+M_0 h)\|\partial_x^s u\| \qquad (|s| \leqq m).$$

We estimate the L_2-norm of $V(a, b, w, u)$. From the assumption (3.13) we obtain

$$\sum_{\sigma\in\Lambda} \partial_x^a\{l_\sigma(t, x, w(x, h), h)\} = 0 \qquad (|a| \geqq 1),$$

so that

$$V(a, b, w, u) = \sum_{\sigma\in\Lambda} (\partial_x^a\{l_\sigma(t, x, w(x, h), h)\})\partial_x^b(T_h^\sigma - I)u \quad (|a| \geqq 1;\ |b| \leqq m-1).$$

Hence by the Sobolev-type theorem, Condition C and (6.9) we have

(6.16) $$\|V(a, b, w, u)\| \leqq (\sum_{\sigma\in\Lambda} |\sigma|)M_2(d, m, n, N)(1+d)^m h\|u\|_m$$

for all $w(x) \in W^m(d)$ and $u(x) \in W^m$, where $M_2(d, m, n, N)$ is a constant independent of a and b. Put

(6.17) $$M_1 = 2M_0 + M_0^2 h_0 + M_3^2 h_0 + 2M_3(1+M_0 h_0),$$

where

$$M_3 = \sum_{|s|\leqq m} \sum_{a+b=s,|a|\geqq 1} c(a, b)(\sum_{\sigma\in\Lambda} |\sigma|)M_2(d, m, n, N)(1+d)^m.$$

Then (5.4) holds by (6.14)–(6.17).

By (5.4) we have

$$\|L_h(t+(\nu-1)k,\ w_{\nu-1},\ h)L_h(t+(\nu-2)k,\ w_{\nu-2},\ h)\cdots L_h(t,\ w_0,\ h)u\|_m \leqq (1+M_1h)^\nu\|u\|_m \leqq \{\exp(M_1T/\lambda)\}\ \|u\|_m$$

for all $w_j(x) \in W^m(d)$ $(j=0, 1,\dots, \nu-1)$, $u(x) \in W^m$, $k \in K$ and t, $t+\nu k \in J$. Hence the linear difference scheme (3.15) is stable in W^m for all $w_j \in W^m(d)$ $(j=0, 1,\dots)$. This completes the proof.

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References

[1] R. Ansorge, *Differenzenapproximationen partieller Anfangswertaufgaben,* B. G. Teubner, Stuttgart, 1978.

[2] S. Z. Burstein and A. A. Mirin, *Third order difference methods for hyperbolic equations,* J. Computational Phys., **5** (1970), 547–571.

[3] H. von Dein, *Konvergenzbedingungen bei der numerischen Lösung nichtlinearer Anfangswertaufgaben mittels Differenzenverfahren,* ISNM, vol. **31** (editors: J. Albrecht and L. Collatz) Birkhaeuser, Basel und Stuttgart, 1976.

[4] A. Fischer and J. Marsden, *The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, I,* Comm. Math. Phys., **28** (1972), 1–38.

[5] A. Fischer and J. Marsden, *General relativity, partial differential equations, and dynamical systems,* Proc. Symp. Pure Math., **23** (1973), 309–327.

[6] D. Gottlieb, *Strang-type difference schemes for multidimensional problems,* SIAM J. Numer. Anal., **9** (1972), 650–661.

[7] A. R. Gourlay and J. Ll. Morris, *Finite-difference methods for nonlinear hyperbolic systems,* Math. Comp., **22** (1968), 28–39.

[8] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations,* Springer Lecture Notes, **448** (1975), 25–70.

[9] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems,* Arch. Rational Mech. Anal., **58** (1975), 181–205.

[10] Z. Koshiba, *On the general form of Yamaguti-Nogi-Vaillancourt's stability theorem,* Publ. RIMS, Kyoto Univ., **15** (1979), 289–313.

[11] H. O. Kreiss, *On difference approximations of the dissipative type for hyperbolic differential equations,* Comm. Pure Appl. Math., **17** (1964), 335–353.

[12] H. Kreth, *Ein Äquivalenzsatz bei der numerischen Lösung quasilinearer Anfangswertaufgaben,* Springer Lecture Notes, **395** (1974), 33–55.

[13] P. D. Lax, *On the stability of difference approximations to solutions of hyperbolic equations with variable coefficients,* Comm. Pure Appl. Math., **14** (1961), 497–520.

[14] P. D. Lax and B. Wendroff, *On the stability of difference schemes,* ibid., **15** (1962), 363–371.

[15] P. D. Lax and B. Wendroff, *Difference schemes for hyperbolic equations with high order of accuracy,* ibid., **17** (1964), 381–398.

[16] P. D. Lax and L. Nirenberg, *On stability for difference schemes; a sharp form of Gårding's inequality,* ibid., **19** (1966), 473–492.

[17] A. Y. Le Roux, *A numerical conception of entropy for quasilinear equations,* Math. Comp., **31** (1977), 848–872.

[18] S. Mizohata, *The Theory of Partial Differential Equations,* Cambridge Univ. Press, Cambridge, 1973.

[19] L. Nirenberg, *On elliptic partial differential equations,* Ann. Scuola Norm. Sup. Pisa, **13** (1959), 115–162.

[20] B. Parlett, *Accuracy and dissipation in difference schemes,* Comm. Pure Appl. Math., **19** (1966), 111–123.

[21] J. Peetre and V. Thomée, *On the rate of convergence for discrete initial-value problems,* Math. Scand., **21** (1967), 159–176.

[22] R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems,* Interscience, New York, 1967.

[23] G. Strang, *Accurate partial difference methods I: Linear Cauchy problems,* Arch. Rational Mech. Anal., **12** (1963), 392–402.

[24] G. Strang, *Accurate partial difference methods II. Non-linear problems,* Numer. Math., **6** (1964), 37–46.

[25] V. Thomée, *Stability theory for partial difference operators,* SIAM Rev., **11** (1969), 152–195.

[26] H. Shintani and K. Tomoeda, *Stability of difference schemes for nonsymmetric linear hyperbolic systems with variable coefficients,* Hiroshima Math. J., **7** (1977), 309–378.

[27] K. Tomoeda, *Stability of difference schemes for nonsymmetric linear hyperbolic systems,* ibid., **7** (1977), 787–812.

[28] R. Vaillancourt, *A strong form of Yamaguti and Nogi's stability theorem for Friedrichs' scheme,* Publ. RIMS, Kyoto Univ., **5** (1969), 113–117.

[29] R. Vaillancourt, *On the stability of Friedrichs' scheme and the modified Lax-Wendroff scheme,* Math. Comp., **24** (1970), 767–770.

[30] R. Vaillancourt, *A simple proof of Lax-Nirenberg theorems,* Comm. Pure Appl. Math., **23** (1970), 151–163.

[31] M. Yamaguti and T. Nogi, *An algebra of pseudo difference schemes and its application,* Publ. RIMS, Kyoto Univ., Ser. A, **3** (1967), 151–166.

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