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Some notes on asymptotic values of meromorphic functions of smooth growth

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1. Introduction

Let $f(z)$ be a nonconstant meromorphic function in the complex plane $|z|<\infty$ and a be a value in the extended complex plane. We say that $f(z)$ has a as an asymptotic value when there exists a path Γ going from a finite point z_0 to ∞ in $|z|<\infty$ such that

$$f(z) \longrightarrow a, \text{ as } z \longrightarrow \infty \text{ along } \Gamma.$$

A few years ago, Hayman ([2]) gave a very interesting sufficient condition for a to be asymptotic, which is applicable to many cases. That is,

THEOREM A. *If*

$$\lim_{r\to\infty} \{T(r, f) - 2^{-1}r^{1/2}\int_r^\infty t^{-3/2}N(t, a)dt\} = \infty, \tag{1}$$

then a is an asymptotic value of $f(z)$.

Applying this theorem, he proved several interesting results. The following result is one of them.

PROPOSITION I. *If $f(z)$ has perfectly regular growth of order ρ, where $0<\rho<1/2$, that is,*

$$\lim_{r\to\infty} T(r, f)/r^\rho = c, \quad 0 < c < \infty \tag{2}$$

and if $\delta(a, f)>2\rho$, then a is asymptotic.

He asks whether this conclusion is sharp ([2], p. 144).

Recently, Yoshida ([5]) has generalized this result as follows.

PROPOSITION II. *Suppose that $f(z)$ satisfies*

$$\limsup_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) \leqq 1 \tag{3}$$

for any $x>1$, where ρ is the order of f $(0\leqq\rho<1/2)$, and that

$$\delta(a, f) > 2\rho. \tag{4}$$

Then, a is an asymptotic value of $f(z)$.

It is trivial that (2) implies (3). He also asks whether this is sharp ([5], p. 207).

In this paper, we shall improve Theorem A and then, using the improved result we shall show that neither Proposition I nor Proposition II is sharp. Besides, some notes on asymptotic values of meromorphic functions are given.

We will use the standard notation of Nevanlinna theory (See [1], [4]).

2. Lemmas

To begin with, we give a sufficient condition for a to be asymptotic in a somewhat stronger form than Theorem A.

LEMMA 1. *Let $f(z)$ be meromorphic and nonconstant in $|z|<\infty$. If*

$$\lim_{r\to\infty}\{T(r,f) - 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t,a)dt\} = \infty, \tag{5}$$

then a is an asymptotic value of $f(z)$.

PROOF. To prove this lemma, we first improve the inequalities of Lemma 5 and Theorem 8 in [2] and then carry out the rest of proof completely as in the case of Theorem A. We will use the same notation as in [2].

I. Improvement of Lemma 5 ([2], p. 138).

$$\log^+ d/|w| \leqq g(0,w) \leqq \begin{cases} \log(1+2d/|w|+2(d/|w|+(d/|w|)^2)^{1/2}) & (|w|>1) \\ \log(1+2d+2(d+d^2)^{1/2})-\log|w| & (|w|\leqq 1). \end{cases} \tag{6}$$

We have only to prove the second inequalities. From the following inequality in the proof of Lemma 5 ([2]):

$$|w| \leqq 4d|\xi|/(1-|\xi|)^2,$$

we have

$$|\xi|^{-1} \leqq 1+2d/|w| + 2(d/|w|+(d/|w|)^2)^{1/2}$$

so that

$$\log|\xi|^{-1} = g(0,w) \leqq \log(1+2d/|w|+2(d/|w|+(d/|w|)^2)^{1/2})$$

for any w in D. As the inequality

$$\log(1+2d+2(d+d^2)^{1/2}) - \log|w| \geqq \log(1+2d/|w|+2(d/|w|+(d/|w|)^2)^{1/2})$$

holds for $|w|\leqq 1$, we obtain (6).

II. Improvement of Theorem 8 ([2], p. 139).

In place of (5.1) in [2], we obtain the following

(7) $m(d, f) \leqq -\int_1^d t^{-1}n(t, \infty)dt + d^{1/2}\int_1^\infty t^{-1}(t+d)^{-1/2}n(t, \infty)dt + \log(M+1)$

for $d>1$, using (6).

In fact, we have only to prove this inequality when

$$\int_1^\infty t^{-1}(t+d)^{-1/2}n(t, \infty)dt < \infty,$$

which is equivalent ot

$$\sum_{|b_\nu|>1} |b_\nu|^{-1/2} < \infty$$

as is easily seen. Then only one thing which is different from the proof of Theorem 8 ([2]) is the estimate of $g(0)$ when $f(0) \neq \infty$. That is, from (6)

$$\begin{aligned} g(0) &\leqq \sum_{|b_\nu|\leqq 1} \{\log(1+2d+2(d+d^2)^{1/2}) - \log|b_\nu|\} \\ &\quad + \sum_{|b_\nu|>1} \log(1+2d/|b_\nu|+2(d/|b_\nu|+(d/|b_\nu|)^2)^{1/2}) \\ &= \int_0^1 \{\log(1+2d+2(d+d^2)^{1/2}) - \log t\}dn(t, \infty) \\ &\quad + \int_1^\infty \log(1+2d/t+2(d/t+(d/t)^2)^{1/2})dn(t, \infty) \\ &= \int_0^1 t^{-1}n(t, \infty)dt + d^{1/2}\int_1^\infty t^{-1}(t+d)^{-1/2}n(t, \infty)dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} T(d, f) &= m(d, f) + N(d, f) \leqq \\ &\int_0^1 t^{-1}n(t, \infty)dt + d^{1/2}\int_1^\infty t^{-1}(t+d)^{-1/2}n(t, \infty)dt + \log(M+1), \end{aligned}$$

which reduces to (7).

III. Completion of the proof.

By integration by parts, we obtain

$$T(d, f) - 2^{-1}d^{1/2}\int_1^\infty (t+d)^{-3/2}N(t, \infty)dt \leqq \log(M+1) + N(1, \infty)/2$$

from (7). Using this inequality instead of (5.4) in [2], we carry out the proof as in §5.3 of [2], and obtain this lemma.

As a corollary of this lemma, similarly to Corollary 1([2]), we obtain the following.

LEMMA 2. *Suppose*

(8) $$\limsup_{r\to\infty} 2^{-1}T(r, f)^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}T(t, f)dt = K < \infty.$$

Then if $\delta(a, f) > 1 - K^{-1}$, *a is an asymptotic value of f.*

PROOF. As $\delta = \delta(a, f) > 1 - K^{-1}$, for every positive ε smaller than $(1 - K(1-\delta))/2(K+3)$, there exists a $t_0(>1)$ such that

$$N(t, a) < (1-\delta+\varepsilon)T(t, f) \qquad (t \geqq t_0).$$

Therefore,

$$\begin{aligned} 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt &= 2^{-1}r^{1/2}\int_1^{t_0}(t+r)^{-3/2}N(t, a)dt \\ &\quad + 2^{-1}r^{1/2}\int_{t_0}^\infty (t+r)^{-3/2}N(t, a)dt \\ &\leqq N(t_0, a) + 2^{-1}(1-\delta+\varepsilon)r^{1/2}\int_{t_0}^\infty (t+r)^{-3/2}T(t, f)dt. \end{aligned}$$

By the definition of K, since $N(t_0, a)$ is constant, there exists an r_0 such that, for any $r \geqq r_0$,

$$2^{-1}T(r, f)^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}T(t, f)dt < K + \varepsilon$$

and

$$N(t_0, a) < \varepsilon T(r, f),$$

so that we have for $r \geqq r_0$

$$2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt < \{K(1-\delta)+\varepsilon(K+3)\}\,T(r, f).$$

Therefore, we have for $r \geqq r_0$

$$T(r, f) - 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt > 2^{-1}(1-K(1-\delta))T(r, f),$$

which tends to ∞ as $r \to \infty$. This proves Lemma 2 by Lemma 1.

3. Smoothness conditions

Let f be nonconstant meromorphic in $|z| < \infty$. We discuss the smoothness of $T(r, f)$ in this section for later use. Yoshida ([5]) introduced two smoothness conditions for $T(r, f)$. That is,

(A) the *smoothness condition* (A) *of type* (ρ, c):

$$\limsup_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) \leqq c \qquad \text{for any} \quad x > 1;$$

(B) the *smoothness condition* (B) *of type* (ρ, c):

$$1 \leqq \liminf_{r\to\infty} r^{-\rho(r)}T(r, f) \leqq \limsup_{r\to\infty} r^{-\rho(r)}T(r, f) \leqq c,$$

where ρ is the order of f and $\rho(r)$ is a proximate order of $T(r, f)$.

In addition to these smoothness conditions, we consider the following condition which is useful in the sequel in this paper.

(C) For some $\rho > 0$,

$$\limsup_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) \leqq 1 \qquad \text{for any} \quad x > 0.$$

It is easily seen that (C) is stronger than (A) with $c=1$, but weaker than (B) with $c=1$.

REMARK 1. Each of the following conditions is equivalent to (C).

$$(\mathrm{C}_1) \qquad \lim_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) = 1 \qquad \text{for any} \quad x \geqq 1;$$

$$(\mathrm{C}_2) \qquad \lim_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) = 1 \qquad \text{for any} \quad x > 0.$$

In fact, using the relation

$$x^{-\rho}T(r, f)^{-1}T(xr, f) = y^{\rho}T(r', f)T(yr', f)^{-1},$$

where $xy=1$ and $yr'=r$, we can prove this remark easily.

LEMMA 3. *Let ρ be a positive number. Then, $T(r, f)$ satisfies the condition* (C) *if and only if, for any positive ε smaller than ρ, there is an r_0 such that*

$$(9) \quad (1-\varepsilon)(t/r)^{\rho-\varepsilon}T(r, f) \leqq T(t, f) \leqq (1+\varepsilon)(t/r)^{\rho+\varepsilon}T(r, f) \quad (r_0 \leqq r \leqq t).$$

PROOF. Suppose first that $T(r, f)$ satisfies (C) for ρ. Let a be a value for which $N(r, a)$ satisfies the relation

$$\lim_{r\to\infty} N(r, a)/T(r, f) = 1.$$

Then, $N(r, a)$ satisfies (C):

$$(10) \qquad \limsup_{r\to\infty} x^{-\rho}N(r, a)^{-1}N(xr, a) \leqq 1 \qquad (x > 0).$$

This implies

$$(11) \qquad \limsup_{r\to\infty} n(r, a)/N(r, a) \leqq \rho$$

by Lemma 5([5]). Next, for any $0<x<1$ and $r>0$,

$$n(r, a)\log x^{-1} \geqq \int_{xr}^{r} t^{-1}n(t, a)dt = N(r, a) - N(xr, a),$$

so that

$$n(r, a)/N(r, a) \geqq \{N(xr, a)/N(r, a) - 1\}/\log x.$$

This, together with (10) and Remark 1, gives

$$\liminf_{r\to\infty} n(r, a)/N(r, a) \geqq (x^{\rho}-1)/\log x,$$

and letting $x\to 1$, we have

(12) $$\liminf_{r\to\infty} n(r, a)/N(r, a) \geqq \rho.$$

Combining (11) with (12)

(13) $$\lim_{r\to\infty} n(r, a)/N(r, a) = \rho$$

(cf. Lemma 6([5])). Let ε be any positive number smaller than ρ. Then there exists an r_0 such that

$$(\rho-\varepsilon)\log(t/r) \leqq \log N(t, a)/N(r, a) = \int_r^t N(u, a)^{-1}u^{-1}n(u, a)du \leqq (\rho+\varepsilon)\log(t/r)$$

for $t\geqq r\geqq r_0$, that is,

(14) $$(t/r)^{\rho-\varepsilon} \leqq N(t, a)/N(r, a) \leqq (t/r)^{\rho+\varepsilon} \quad (t \geqq r \geqq r_0),$$

and such that

(15) $$(1-\varepsilon)N(t, a)/N(r, a) \leqq T(t, f)/T(r, f) \leqq (1+\varepsilon)N(t, a)/N(r, a) \quad (t \geqq r \geqq r_0).$$

From (14) and (15), we obtain (9).

Conversely, suppose that, for any postive ε smaller than ρ, (9) is satisfied for $t\geqq r\geqq r_0$. Let $x\geqq 1$ and $r\geqq r_0$. Put $t=xr(\geqq r)$. Then by (9) we have

$$(1-\varepsilon)x^{-\varepsilon} \leqq x^{-\rho}T(r, f)^{-1}T(xr, f) \leqq (1+\varepsilon)x^{\varepsilon}.$$

This reduces to the condition (C_1):

$$\lim_{r\to\infty} x^{-\rho}T(r, f)^{-1}T(xr, f) = 1,$$

which is equivalent to (C) by Remark 1.

COROLLARY 1. *If $T(r, f)$ satisfies* (C) *for $\rho\geqq 0$, then f has regular growth of order ρ.*

In fact, when $\rho=0$, it is well-known that f has order zero ([2]) and when $\rho>0$, we have this from (9) easily.

4. Results

In this section, we shall show that Propositions I and II are not sharp.

THEOREM 1. *Suppose that $f(z)$ is meromorphic and nonconstant in $|z|<\infty$*

and that $T(r,f)$ *satisfies* (C) *for some* ρ, *where* $0 \leqq \rho < 1/2$. *If*

$$\delta(a,f) > 1 - \pi^{1/2}/\Gamma(\rho+1)\Gamma(1/2-\rho),$$

then a *is asymptotic.*

PROOF. When $\rho=0$, (C) is equivalent to

$$\lim_{r\to\infty} T(2r,f)/T(r,f) = 1$$

and

$$1 - \pi^{1/2}/\Gamma(1)\Gamma(1/2) = 0.$$

Therefore, a is asymptotic by Corollary 2 ([2]).

Suppose now that ρ is positive. Let ε be any positive number smaller than $\min(\rho/2, 1/2-\rho)$. Then, by Lemma 3, $T(r,f)$ satisfies (9) since $T(r,f)$ satisfies (C). For any $r \geqq r_0$, we write

$$\begin{aligned} 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}T(t,f)dt &= 2^{-1}r^{1/2}\int_r^{r_0} (t+r)^{-3/2}T(t,f)dt \\ &+ 2^{-1}r^{1/2}\int_{r_0}^{r} (t+r)^{-3/2}T(t,f)dt + 2^{-1}r^{1/2}\int_r^\infty (t+r)^{-3/2}T(t,f)dt \\ &= I_1+I_2+I_3. \end{aligned}$$

We estimate I_1, I_2 and I_3 with the aid of (9).

$$I_1 = 2^{-1}r^{1/2}\int_1^{r_0}(t+r)^{-3/2}T(t,f)dt \leqq (1+r)^{-1/2}r^{1/2}T(r_0,f) < T(r_0,f).$$

$$\begin{aligned} I_2 &= 2^{-1}r^{1/2}\int_{r_0}^{r} (t+r)^{-3/2}T(t,f)dt \leqq r^{1/2}T(r,f)\int_{r_0}^{r} (t+r)^{-3/2}(t/r)^{\rho-\varepsilon}dt/2(1-\varepsilon) \\ &= T(r,f)\int_{r_0/r}^{1} (1+u)^{-3/2}u^{\rho-\varepsilon}du/2(1-\varepsilon) \leqq T(r,f)\int_0^1 (1+u)^{-3/2}u^{\rho-\varepsilon}du/2(1-\varepsilon). \end{aligned}$$

$$\begin{aligned} I_3 &= 2^{-1}r^{1/2}\int_r^\infty (t+r)^{-3/2}T(t,f)dt \leqq 2^{-1}(1+\varepsilon)r^{1/2}T(r,f)\int_r^\infty (t+r)^{-3/2}(t/r)^{\rho+\varepsilon}dt \\ &= 2^{-1}(1+\varepsilon)T(r,f)\int_1^\infty (1+u)^{-3/2}u^{\rho+\varepsilon}du. \end{aligned}$$

And so, as $(1+\varepsilon)<(1-\varepsilon)^{-1}$,

$$I_2+I_3 < T(r,f)\{\int_0^\infty(1+u)^{-3/2}u^{\rho+\varepsilon}du + \int_0^1 (1+u)^{-3/2}(u^{\rho-\varepsilon}-u^{\rho+\varepsilon})du\}/2(1-\varepsilon).$$

Here,

$$\begin{aligned} S_1 &= 2^{-1}\int_0^\infty (1+u)^{-3/2}u^{\rho+\varepsilon}du = B(\rho+1+\varepsilon,\ 1/2-\rho-\varepsilon)/2 \\ &= \pi^{-1/2}\Gamma(\rho+1+\varepsilon)\Gamma(1/2-\rho-\varepsilon), \end{aligned}$$

which tends to $\Gamma(\rho+1)\Gamma(1/2-\rho)/\pi^{1/2}$ as $\varepsilon\to 0$ and

$$0 \leqq S_2 = 2^{-1}\int_0^1 (1+u)^{-3/2}(u^{\rho-\varepsilon}-u^{\rho+\varepsilon})du$$
$$\leqq \max_{[0,1]}(u^{\rho-\varepsilon}-u^{\rho+\varepsilon}) \leqq \max_{[0,1]}(u^{\rho-2\varepsilon}-1)$$

tends to zero as $\varepsilon\to 0$. From these estimates, we have

$$\limsup_{r\to\infty} 2^{-1}T(r,f)^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}T(t,f)dt \leqq \Gamma(\rho+1)\Gamma(1/2-\rho)/\pi^{1/2}.$$

Applying Lemma 2, we obtain the conclusion.

REMARK 2. $2\rho > 1-\pi^{1/2}/\Gamma(\rho+1)\Gamma(1/2-\rho)$ $(0<\rho<1/2)$.
In fact,

$$(1-2\rho)^{-1} = 2^{-1}\int_0^\infty (1+u)^{\rho-3/2}du > 2^{-1}\int_0^\infty u^\rho(1+u)^{-3/2}du$$
$$= \Gamma(\rho+1)\Gamma(1/2-\rho)/\pi^{1/2}.$$

This shows that Proposition I is not sharp.

REMARK 3. Suppose that $T(r,f)$ satisfies

$$c_1(t/r)^{\rho-\varepsilon}T(r,f) \leqq T(t,f) \leqq c_2(t/r)^{\rho+\varepsilon}T(r,f) \qquad (t \geqq r \geqq r_0(\varepsilon))$$

for every sufficiently small positive ε, where $0<\rho<1/2$ and $0<c_1<1$, $c_2>1$ are constants. Then, if

$$\delta(a,f) > 1 - \pi^{1/2}/c\Gamma(\rho+1)\Gamma(1/2-\rho), \qquad c = \max(c_1^{-1}, c_2),$$

a is asymptotic.

We can prove this as in the same way as Theorem 1.

THEOREM 2. *Suppose that $f(z)$ is a nonconstant meromorphic function of order ρ in $|z|<\infty$ for which $T(r,f)$ satisfies* (A) *with $c=1$, where $0\leqq\rho<1/2$ and that*

$$\delta(a,f) > (K-2^{-1/2})/(K+1-2^{-1/2}) \qquad (>0),$$

where $K=2^{-1}\int_1^\infty u^\rho(1+u)^{-3/2}du$. Then, a is asymptotic.

PROOF. Let ε be any positive number smaller than $1/2-\rho$. Then there exists an r_0 such that

$$(16) \qquad T(t,f) \leqq (1+\varepsilon)(t/r)^{\rho+\varepsilon}T(r,f) \qquad (t \geqq r \geqq r_0)$$

(see [5], Proof of Theorem 3) and

$$N(r, a) < (1+\varepsilon-\delta)T(r, f) \qquad (r \geqq r_0;\ \delta = \delta(a, f)). \tag{17}$$

Now, for $r \geqq r_0$, by (17) and (16)

$$\begin{aligned}
2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt &\leqq N(r_0, a)+2^{-1}(1+\varepsilon-\delta)r^{1/2}\int_{r_0}^\infty (t+r)^{-3/2}T(t, f)dt \\
&\leqq N(r_0, a)+(1+\varepsilon-\delta)T(r, f)(1-2^{-1/2}+2^{-1}(1+\varepsilon)r^{1/2}\int_r^\infty (t+r)^{-3/2}(t/r)^{\rho+\varepsilon}dt) \\
&= N(r_0, a)+(1+\varepsilon-\delta)(1-2^{-1/2}+(1+\varepsilon)K(\varepsilon))T(r, f),
\end{aligned}$$

where

$$K(\varepsilon) = 2^{-1}\int_1^\infty (1+u)^{-3/2}u^{\rho+\varepsilon}du,$$

so that

$$\begin{aligned}
&T(r, f)-2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt \\
&\quad \geqq T(r, f)\{\delta(1-2^{-1/2}+(1+\varepsilon)K(\varepsilon)) \\
&\qquad - ((1+\varepsilon)^2K(\varepsilon)+\varepsilon(1-2^{-1/2})-2^{-1/2})\} - N(r_0, a)
\end{aligned}$$

Since $K(\varepsilon)\to K$ as $\varepsilon\to 0$, for sufficiently small $\varepsilon>0$,

$$\delta(1-2^{-1/2}+(1+\varepsilon)K(\varepsilon)) - ((1+\varepsilon)^2K(\varepsilon) + (1-2^{-1/2})-2^{-1/2}) > 0$$

by the hypothesis. Therefore, letting r tend to ∞, we obtain

$$\lim_{r\to\infty}\{T(r, f) - 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t, a)dt\} = \infty.$$

This shows that a is asymptotic by Lemma 1.

REMARK 4. $2\rho > (K-2^{-1/2})/(K+1-2^{-1/2}) \qquad (0 < \rho < 1/2)$.
In fact

$$(1-2\rho)^{-1} = 2^{-1}\int_1^\infty u^{\rho-3/2}du$$

and

$$K = 2^{-1}\int_1^\infty (1+v)^{-3/2}v^\rho dv = 2^{-3/2}\int_1^\infty u^{-3/2}(2u-1)^\rho du < 2^{\rho-3/2}\int_1^\infty u^{\rho-3/2}du.$$

And so,

$$(1-2\rho)^{-1}-K > (1-2^{\rho-1/2})2^{-1}\int_1^\infty u^{\rho-3/2}du = (1-2^{\rho-1/2})/(1-2\rho) > 1-2^{-1/2}.$$

This shows that Proposition II is not sharp.

REMARK 5. i) If $T(r, f)$ satisfies the inequality

(18) $$T(t, f) \leqq c(t/r)^{\rho+\varepsilon} T(r, f) \qquad (t \geqq r \geqq r_0(\varepsilon))$$

for any sufficiently samll positive ε, where $0 \leqq \rho < 1/2$ and c is constant, and if

$$\delta(a, f) > (cK - 2^{-1/2})/(cK + 1 - 2^{-1/2})$$

(K is the value given in Theorem 2), then a is asymptotic.

ii) If $T(r, f)$ satisfies (A) for $0 \leqq \rho < 1/2$ and if $\delta(a, f) = 1$, then a is asymptotic.

We can prove i) easily applying the method used in the proof of Theorem 2. As for ii), we note that if $T(r, f)$ satisfies (A), then for any μ greater than ρ, there exist an r_0 and a constatn k such that

$$T(t, f) \leqq k(t/r)^{\mu} T(r, f) \qquad (t \geqq r \geqq r_0)$$

(cf. Remark 1 ([5])). Making use of this inequality instead of (18), we obtain ii) as in the case of i).

Let $f(z)$ be nonconstant meromorphic of order ρ in $|z| < \infty$ and

$$T_0(r, f) = \int_0^r t^{-1} A(t) dt$$

be the Ahlfors-Shimizu characteristic of f (See [1]). Then,

(19) $$|T(r, f) - T_0(r, f)| < O(1).$$

THEOREM 3. *Suppose that $f(z)$ is a transcendental meromorphic function in $|z| < \infty$ satisfying*

$$\int_1^{\infty} t^{-3/2} T(t, f) dt < \infty$$

and that

(20) $$\liminf_{r \to \infty} m(r, a)/A(r) > 2.$$

Then, a is an asymptotic value of $f(z)$.

PROOF. By (19) and the first fundamental theorem of Nevanlinna, we have

$$T(r, f) - 2^{-1} r^{1/2} \int_r^{\infty} t^{-3/2} N(t, a) dt$$
$$\geqq 2^{-1} r^{1/2} \int_r^{\infty} t^{-3/2} m(t, a) dt + T_0(r, f) - 2^{-1} r^{1/2} \int_r^{\infty} t^{-3/2} T_0(t, f) dt - O(1)$$

$$= 2^{-1}r^{1/2}\int_r^\infty t^{-3/2}m(t,\ a)dt - r^{1/2}\int_r^\infty t^{-3/2}A(t)dt - O(1)$$

$$= 2^{-1}r^{1/2}\int_r^\infty t^{-3/2}(m(t,\ a)-2A(t))dt - O(1).$$

Thus the condition of Theorem A is satisfied under the condition (20). This yields that a is asymptotic.

REMARK 6. More generally, we can conclude the following easily from the proof. That is, suppose that $f(z)$ is a nonconstant meromorphic function satisfying

$$\int_1^\infty t^{-3/2}T(t, f)dt < \infty.$$

If

$$\lim_{r\to\infty}\{m(r,\ a) - 2A(r)\} = \infty,$$

then a is asymptotic.

REMARK 7. 1) This theorem contains an improvement of Corollary 2([2]), Propositions I and II when f is transcendental.

To see this, we first note that, if $T(r, f)$ satisfies (A) with $c=1$, then $T_0(r, f)$ also satisfies (A) with $c=1$ by (19), and in this case, as Lemma 5 ([5]) we have

$$\limsup_{r\to\infty} A(r)/T_0(r, f) \leqq \rho. \tag{21}$$

Similarly, when $T(r, f)$ satisfies (C), $T_0(r, f)$ also does and we have

$$\lim_{r\to\infty} A(r)/T_0(r, f) = \rho \tag{22}$$

as in the proof of Lemma 3.

Now, first suppose that

$$\lim_{r\to\infty} T(2r, f)/T(r, f) = 1, \tag{23}$$

which is equivalent to (A) with $\rho=0$ and $c=1$ (see Remark 3([5])). Then, if a is deficient,

$$m(r,\ a)/A(r) = \{m(r,\ a)/T(r, f)\}\{T(r, f)/T_0(r, f)\}\{T_0(r, f)/A(r)\}$$
$$\longrightarrow \infty \qquad (r \longrightarrow \infty)$$

by (19) and (21). Thus, a is asymptotic by Theorem 3. This shows that Theorem 3 is an improvement of Corollary 2 ([2]) when f is transcendental.

Secondly, suppose that $T(r, f)$ satisfies (A) with $c=1$ and that $\delta(a, f)>2\rho$, where $0<\rho<1/2$. Then, by (19) and (21), we have

$$\liminf_{r\to\infty} m(r, a)/A(r)$$
$$\geqq \liminf_{r\to\infty} m(r, a)/T(r, f) \liminf_{r\to\infty} T(r, f)/T_0(r, f) \liminf_{r\to\infty} T_0(r, f)/A(r)$$
$$\geqq \delta(a, f)/\rho > 2,$$

which shows that a is asymptotic by Theorem 3. That is, Theorem 3 is stronger than Propositions I and II when f is transcendental.

2) Suppose that f is a transcendental meoromorphic function in $|z|<\infty$ satisfying

$$\limsup_{r\to\infty} T(r, f)/(\log r)^\alpha = A \qquad (\alpha > 1)$$

and

$$\liminf_{r\to\infty} m(r, a)/(\log r)^{\alpha-1} > 2^{\alpha+1}A.$$

Then, a is asymptotic (cf. [2], p. 143).

In fact, as in [2], p. 143,

$$A(r) \leqq T_0(r^2, f)(\log r)^{-1} \leqq (2^\alpha A + o(1))(\log r)^{\alpha-1} \qquad (r \longrightarrow \infty)$$

and

$$\liminf_{r\to\infty} m(r, a)/A(r)$$
$$\geqq \liminf_{r\to\infty} m(r, a)/(\log r)^{\alpha-1} \liminf_{r\to\infty} (\log r)^{\alpha-1}/A(r) > 2,$$

which shows that a is asymptotic by Theorem 3.

3) We can improve the condition (20) for functions satisfying (A) with $c=1$ or (C) by making use of Theorem 2 and (21) or Theorem 1 and (22) respectively.

5. Miscellaneous notes

Suppose that $f(z)$ is meromorphic and nonconstant in $|z|<\infty$.

1. *Suppose*

$$\limsup_{r\to\infty} T(r, f)/(\log r)^2 = A < \infty.$$

If

$$\liminf_{r\to\infty} m(r, a)/\log r > 8A \log(2^{1/2}+1), \tag{24}$$

then a is asymptotic.

In fact, as is known,

$$n(r, a) \leqq (4A + o(1)) \log r \qquad (r \longrightarrow \infty)$$

in this case. Hence, for any sufficiently large r,

$$\begin{aligned} T(r,f) &- 2^{-1}r^{1/2}\int_1^\infty (t+r)^{-3/2}N(t,a)dt \\ &\geqq m(r,a) - r^{1/2}\int_r^\infty t^{-1}(t+r)^{-1/2}n(t,a)dt - O(1) \\ &\geqq m(r,a) - (4A+o(1))r^{1/2}\int_r^\infty t^{-1}(t+r)^{-1/2}\log t dt - O(1) \\ &= m(r,a) - (4A+o(1))2\log(2^{1/2}+1)\log r - O(1). \end{aligned}$$

Thus, the condition of Lemma 1 is satisfied under (24). Since $\log(2^{1/2}+1)<1$, this is somewhat better than a result given in [2], p. 143.

2. *If*
$\limsup_{r\to\infty} N(r,a)/r^\rho < \pi^{1/2}/\Gamma(\rho+1)\Gamma(1/2-\rho)\liminf_{r\to\infty} T(r,f)/r^\rho$
for some ρ *such that* $0<\rho<1/2$, *then* a *is asymptotic.*

This is an improvement of Corollary 4 in [2] since

$$1 - 2\rho < \pi^{1/2}/\Gamma(\rho+1)\Gamma(1/2-\rho)$$

(Remark 2). We can prove this as in [2] by using Lemma 1 instead of Theorem A.

3. Finally, we show that "$\delta(a,f)>1-c^{-1}$" *is not sharp for* a *to be asymptotic when* f *satisfies* (*B*) *with* $\rho=0$ *and* $c>1$ (see [5], p. 207).

To begin with, we show that if f satisfies the condition (A) with $\rho=0$, i.e., if

$$\limsup_{r\to\infty} T(xr,f)/T(r,f) \leqq c \tag{25}$$

for any $x>1$, then the order of f is equal to zero and we may take $c=1$.

In fact, let b be a value for which $N(r,b)$ satisfies

$$\lim_{r\to\infty} N(r,b)/T(r,f) = 1.$$

Then $N(r,b)$ also satisfies (25) for $x>1$. Since

$$n(r,b)\log x \leqq \int_r^{xr} t^{-1}n(t,b)dt = N(xr,b) - N(r,b),$$

we have

$$\limsup_{r\to\infty} n(r,b)/N(r,b) \leqq (c-1)/\log x \longrightarrow 0 \qquad (x\longrightarrow\infty).$$

That is,

$$\lim_{r\to\infty} n(r,b)/N(r,b) = 0,$$

so that, for any positive ε, there is an r_0 such that

$$T(t, f) \leqq (1+\varepsilon)(t/r)^{\varepsilon} T(r, f) \qquad (t \geqq r \geqq r_0)$$

as in [2]. Let x be any number larger than 1 and put $t = xr (r \geqq r_0)$. Then we have

$$T(xr, f)/T(r, f) \leqq (1+\varepsilon)x^{\varepsilon},$$

which yields

(26) $$\limsup_{r\to\infty} T(xr, f)/T(r, f) \leqq 1.$$

This shows that we may take $c=1$.

In this case, by Corollary 2([2]), if $\delta(a, f) > 0$, then a is asymptotic. Since (B) implies (A) with the same c, the constant $1-c^{-1}$ is not sharp.

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