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Existence and Hölder continuity of derivatives of single layer Φ-potentials

Dedicated to Professor Makoto Ohtsuka on the occasion of his 60th birthday

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Introduction

In the n-dimensional Euclidean space R^n, let S be a compact portion of a k-dimensional $(1 \leqq k \leqq n-1)$ Lipschitzian surface. Let Φ be a continuously differentiable function on the set $\{x \in R^n;\ 0<|x|<R_0\}$ for some positive number R_0 greater than the diameter of S. For a signed measure σ on S, we define the single layer Φ-potential of σ by

$$V_{\Phi}^{\sigma}(x) = \int_S \Phi(x-y)d\sigma(y),$$

provided the integral exists. Clearly, V_{Φ}^{σ} is continuously differentiable on $\Omega \backslash S$ for a neighborhood Ω of S, but, in general, not on S.

The purpose of this paper is to investigate the following problems under the conditions that S satisfies α_0-condition at $x^0 \in S$ (in the sense of [11]) and that $|(\partial\Phi/\partial x_i)(x)| \leqq C|x|^{-\lambda-1}$, $0<|x|<R_0$, for some λ with $0<\lambda<n$.

(I) Existence of limits of derivatives of V_{Φ}^{σ} along sets which are non-tangential to S at $x^0 \in S$;

(II) Hölder continuity of derivatives of V_{Φ}^{σ} on sets of the above type;

(III) Existence of derivatives of V_{Φ}^{σ} at $x^0 \in S$;

(IV) Hölder continuity of derivatives of V_{Φ}^{σ} on S.

In the case of the single layer Newtonian potentials V_1^f in R^3, i.e., in the case where $n=3$, $k=2$, $\Phi(x)=|x|^{-1}$ (hence $\lambda=1$) and $\sigma=fdS$ (dS: the surface element of S), many results on these problems have been obtained; see O. D. Kellogg [8], N. M. Günter [6] and M. Ohtsuka [9] and [11].

In case n and k $(1 \leqq k \leqq n-1)$ are arbitrary, S. Dümmel [3], and Dümmel and Siewert [4] have shown a few results concerning problem (I) for $\Phi(x)=|x|^{-\lambda}$; but in these papers, problems (II), (III) and (IV) are not discussed.

We shall extend these results to more general single layer Φ-potentials with conditions on σ and Φ suitable to respective problems; in particular when we consider normal derivatives $(d/dn)V_{\Phi}^{\sigma}$ we assume a local homogeneity condition for Φ (denoted by $(\Phi-4)$; see 1.3) and further, in case $\lambda=k-1$, a condition of the type (cf. $(\Phi-5)$ in 1.3)

$$|(d\Phi/dn)(x)| \leqq C|x^*|\,|x|^{-k-1},$$

where x^* is the projection of x to the space of normal directions to S at x^0. Note that $\Phi(x)=|x|^{-\lambda}$ satisfies both $(\Phi-4)$ and $(\Phi-5)$.

Basic notions and definitions for S, σ and Φ are given in §1. We assume that S contains the origin 0 and consider problems (I), (II) and (III) for $x^0=0$. In §2 we study the behavior of single layer Φ-potentials themselves (not their derivatives), and prove the existence of their limits along a set non-tangential at the origin or a non-tangential line terminating at the origin and their Hölder continuity on such a set. In §3 we are concerned with problems (I) and (II). We obtain in Theorem 3.1 the Hölder continuity of tangential derivatives of V_Φ^σ on a non-tangential set. The existence of limits of normal derivatives $(d/dn)V_\Phi^\sigma$ and that of functions of type $|x|^{\lambda-k+1}(d/dn)V_\Phi^\sigma(x)$ along non-tangential lines terminating at the origin are immediate consequences of the results in §2 (Theorems 3.2 and 3.2′). In Theorem 3.3 we obtain the Hölder continuity of directional derivatives of V_Φ^σ on a non-tangential line terminating at the origin. Note that normal derivatives, and hence directional derivatives, are Hölder continuous only on a line. In fact, limits of a normal derivative along lines depend on their directions. But, in case S is an $(n-1)$-dimensional surface, as in [9; Theorem 18], the Hölder continuity of directional derivatives on a non-tangential set can be proved (Corollary 3.1). In §4 we consider problem (III). We show in Theorem 4.1 the existence of a certain limit for V_Φ^σ which insures the existence of the tangential derivative of V_Φ^σ at the origin (Corollary 4.2). In Theorem 4.2 we give an answer to problem (III) for directional derivatives. §5 is devoted to problem (IV) in the case where σ has density f and $\lambda=k-1$ under the conditions that S satisfies uniform α_0-condition and f is Hölder continuous on S. We obtain in Theorem 5.1 the Hölder continuity of directional derivatives of V_Φ^σ on S and a generalization of a theorem of Liapunov (Theorems 5.2 and 5.2′).

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§1. Preliminaries

1.1 Basic notions

Let R^n be the n-dimensional Euclidean space with points $x=(x_1,\dots,x_n)$. The inner product of points $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ is defined by $\langle x, y\rangle=\sum_{i=1}^n x_iy_i$ and the distance of x, y by $|x-y|=\{\sum_{i=1}^n (x_i-y_i)^2\}^{1/2}$. We denote by $C\ell(E)$ the closure of a set E in R^n and by $B^{(n)}(x, r)$ the n-dimensional closed ball $\{y\in R^n;\ |y-x|\leqq r\}$. We write $e_1=(1, 0,\dots, 0),\dots, e_n=(0,\dots, 0, 1)$.

Let a be a unit vector in R^n and x^0 be a point in R^n. The open half line $\{x^0+\rho a;\ \rho>0\}$ is denoted by $L(x^0, a)$. For a set E in R^n, the contingent of E at x^0, denoted by contg (E, x^0), is the set of all half lines $L(x^0, a)$ for which there is a sequence $\{x^{(n)}\}$ in $E\backslash\{x^0\}$ satisfying $\lim_{n\to\infty} x^{(n)}=x^0$ and $\lim_{n\to\infty}(x^{(n)}-x^0)/|x^{(n)}-x^0|=a$.

LEMMA 1.1. *If E, F are sets such that $0\in C\ell(E\backslash\{0\})\cap C\ell(F\backslash\{0\})$ and* contg$(E, 0)\cap$ contg$(F, 0)=\emptyset$, *then there are positive numbers $C=C(E, F)$ and $r=r(E, F)$ such that*

$$|x| + |y| \leqq C|x-y|$$

for every $x\in E\cap B^{(n)}(0, r)$ and $y\in F$.

We can prove the lemma by the same argument as in the proof of [1; Proposition 0.1] and thus omit its proof.

Let $0<\alpha\leqq 1$. A function f defined on a set E is said to be *α-Hölder continuous* on E if there is a positive constant C such that

$$|f(x) - f(\tilde{x})| \leqq C|x - \tilde{x}|^{\alpha},$$

whenever x, $\tilde{x}\in E$. The smallest of such C is called the *Hölder constant* of f.

Let μ be a non-negative measure, let x^0 be a point in R^n and write $g(\rho)=\mu(B^{(n)}(x^0, \rho))$ for $\rho\geqq 0$. Then for any continuously differentiable function F on $(0, r]$ $(r>0)$ such that $\lim_{\rho\downarrow 0} F(\rho)g(\rho)=0$,

$$\int_{0<|x-x^0|\leqq r} F(|x-x^0|)d\mu(x) = F(r)g(r) - \int_0^r F'(\rho)g(\rho)d\rho, \tag{1.1}$$

provided at least one of the integrals exists. This formula will be often used in the sequel.

The letter C will be used to denote various positive constants independent of the variables in question.

1.2. The surface S

Let k be an integer such that $1\leqq k\leqq n-1$. For $x=(x_1,\dots, x_n)\in R^n$, let $x'=(x_1,\dots, x_k, 0,\dots, 0)$ and $x^*=x-x'=(0,\dots, 0, x_{k+1},\dots, x_n)$. We often regard x' as a point in R^k.

Let S be a k-dimensional Lipschitz surface defined by

$$S = \{x\in R^n;\ x_{k+1} = \psi_{k+1}(x'),\dots, x_n = \psi_n(x'),\ |x'| \leqq r_0\}$$

for some $r_0>0$, where $\psi_{k+1},\dots, \psi_n$ are Lipschitz functions on $|x'|\leqq r_0$ such that $\psi_i(0)=0$, $i=k+1,\dots, n$. Let $\Psi(x')=(x', \psi_{k+1}(x'),\dots, \psi_n(x'))$ and assume that $|\Psi(x')|^2\leqq 2r_0^2$ for all x', $|x'|\leqq r_0$. Then the diameter of S does not exceed $3r_0$.

The k-dimensional Hausdorff measure m_k on S will be denoted by μ_S. It is expressed as $d\mu_S(\Psi(x'))=J_k\Psi(x')dx'$ with a bounded Borel measurable function $J_k\Psi$ on $|x'|\leqq r_0$ such that $J_k\Psi(x')\geqq 1$ dx'-a.e. (see e.g., [10; Theorem 3]), where $dx'=dx_1\cdots dx_k$ is the k-dimensional Lebesgue measure. In the sequel we denote by $S(0, r)$ the image of the k-dimensional closed ball $B^{(k)}(0, r)$ under Ψ for $0\leqq r\leqq r_0$.

Let $\alpha_0>0$. We say that S satisfies α_0-condition at the origin if the following condition holds:

$$\sum_{i,j}\left(\frac{\partial\psi_i}{\partial x_j}(x')\right)^2 \leqq K_1|x'|^{2\alpha_0} \quad dx'\text{-}a.e.$$

for some positive constant K_1. By Fubini's theorem and the absolute continuity of ψ_i the above condition implies that

$$(S\text{–}1) \qquad |\psi_i(x')| \leqq K_2|x'|^{1+\alpha_0}$$

for all x', $|x'|\leqq r_0$ and $i=k+1,\dots, n$, and

$$(S\text{–}2) \qquad 0\leqq J_k\Psi(x') - 1 \leqq K_3|x'|^{\alpha_0} \quad dx'\text{-}a.e.,$$

where K_2 and K_3 depend only on K_1. In this case it is easy to see that contg $(S, 0)=\{L(0, a);\ a^*=0\}$.

For $0<\varepsilon\leqq 1$, let $E(0, \varepsilon)=\{x\in B^{(n)}(0, r_0);\ |x^*|\geqq\varepsilon|x|\}$. The following lemma is a consequence of Lemma 1.1.

LEMMA 1.2. *Let $0<\varepsilon\leqq 1$. Assume that S satisfies α_0-condition at 0. Then there are positive numbers C and r depending only on K_2, α_0 and ε such that $S\cap E(0, \varepsilon/2)\cap B^{(n)}(0, r)=\{0\}$ and*

$$(1.2) \qquad |x| + |y| \leqq C|x-y|$$

for every $x\in E(0, \varepsilon)\cap B^{(n)}(0, r)$ and $y\in S$.

REMARK 1.1. If we replace the α_0-condition by the condition that $\lim_{x'\to 0}\psi_i(x')/|x'|=0$ $(i=k+1,\dots, n)$, then the assertion of the lemma is still valid.

Let σ be a signed measure on S. If there is a number A such that

$$(\sigma\text{–}1) \qquad \lim_{r\downarrow 0} r^{-k}|\sigma-A\mu_S|(S(0, r)) = 0,$$

then the origin is called a *Lebesgue point of* σ. Here we denote by $|\sigma|$ the total variation of σ. The origin is called a *Lebesgue point of order* α_1 (>0) if there are numbers A and $L_1>0$ such that

$$(\sigma\text{–}2) \qquad |\sigma-A\mu_S|(S(0, r)) \leqq L_1 r^{k+\alpha_1}, \quad 0\leqq r\leqq r_0$$

(cf. [4; p. 188]).

1.3. The kernel Φ

LEMMA 1.3. *Let $\Phi \in C^1(B^{(n)}(0, 4r_0)\backslash\{0\})$, i.e., Φ be a continuously differentiable function on $B^{(n)}(0, 4r_0)\backslash\{0\}$. If there are positive numbers C and τ such that $\tau > 1$ and*

$$|D_i\Phi(x)| \leqq C|x|^{-\tau} \quad \text{for} \quad 0 < |x| \leqq 4r_0 \quad \text{and} \quad i = 1,\dots, n,$$

where $D_i = \partial/\partial x_i$, then there is a positive number C' depending only on C, r_0, τ and $\max_{|x|=4r_0}|\Phi(x)|$ such that

$$|\Phi(x)| \leqq C'|x|^{-\tau+1} \quad \text{for every } x,\ 0 < |x| \leqq 4r_0$$

and

$$|\Phi(x) - \Phi(\tilde{x})| \leqq C'|x - \tilde{x}|\,|x|^{-\tau+1}|\tilde{x}|^{-1} \qquad \text{for every } x \text{ and } \tilde{x},\ 0 < |x| \leqq |\tilde{x}| \leqq 4r_0.$$

The proof of this lemma is elementary.

Let Φ be a real valued continuous function on $B^{(n)}(0, 4r_0)\backslash\{0\}$. Let $0<\lambda<n$. In the sequel we shall consider the following conditions on Φ:

(Φ–1) $\quad |\Phi(x)| \leqq M_1|x|^{-\lambda}, \quad 0 < |x| \leqq 4r_0,$

(Φ–2) $\quad |\Phi(x)| \leqq M_2|x^*|\,|x|^{-\lambda-1}, \quad 0 < |x| \leqq 4r_0,$

(Φ–3) $\quad |\Phi(x) - \Phi(\tilde{x})| \leqq M_3|x - \tilde{x}|\,|x|^{-\lambda}|\tilde{x}|^{-1}, \quad 0 < |x| \leqq |\tilde{x}| \leqq 4r_0,$

(Φ–4) $\quad \Phi(hx) = h^{-\lambda}\Phi(x), \quad 0 < h \leqq 2$ and $0 < |x| \leqq 2r_0$;

in case $\Phi \in C^1(B^{(n)}(0, 4r_0)\backslash\{0\})$,

(Φ–5) $\quad |D_i\Phi(x)| \leqq M_4|x^*|\,|x|^{-\lambda-2}, \quad 0 < |x| \leqq 4r_0$ and $i = k + 1,\dots, n,$

(Φ–6) $\quad |D_i\Phi(x) - D_i\Phi(\tilde{x})| \leqq M_5|x - \tilde{x}|\,|x|^{-\lambda-1}|\tilde{x}|^{-1},$
$\qquad 0 < |x| \leqq |\tilde{x}| \leqq 4r_0$ and $i = 1,\dots, n$;

in case $\Phi \in C^2(B^{(n)}(0, 4r_0)\backslash\{0\})$, i.e., Φ is a 2-times continuously differentiable function on $B^{(n)}(0, 4r_0)\backslash\{0\}$,

(Φ–7) $\quad |D_iD_j\Phi(x) - D_iD_j\Phi(\tilde{x})| \leqq M_6|x - \tilde{x}|\,|x|^{-\lambda-2}|\tilde{x}|^{-1},$
$\qquad 0 < |x| \leqq |\tilde{x}| \leqq 4r_0$ and $i, j = 1,\dots, n.$

It is easy to see that (Φ–2) implies (Φ–1) with $M_1 = M_2$ and (Φ–3) implies (Φ–1) with $M_1 = 2M_3 + M_0r_0^{\lambda}$, where $M_0 = \max_{|x|=4r_0}|\Phi(x)|$. If Φ satisfies (Φ–6), then (Φ–3) holds for Φ by Lemma 1.3 and so does (Φ–1). Here the constants M_1 and M_3 depend only on M_0, M_5, r_0 and λ. If Φ satisfies (Φ–7), then (Φ–6) holds for Φ by Lemma 1.3 and so do (Φ–1) and (Φ–3). In this case the constant

M_5 depends only on M_0', M_6, r_0 and λ, where $M_0'=\max_{1\leq i\leq n}\max_{|x|=4r_0}|D_i\Phi(x)|$, and the constants M_1 and M_3 depend only on M_0, M_0', M_6, r_0 and λ.

For a signed measure σ on S, we define the *single layer Φ-potential* of σ by

$$V_\Phi^\sigma(x) = V(\Phi, \sigma)(x) = \int_S \Phi(x-y)d\sigma(y),$$

whenever the integral exists. If f is a Borel measurable function on S with $\int_S |f| d\mu_S < \infty$ and $\sigma = f\mu_S$, then we denote V_Φ^σ by V_Φ^f; in particular, if $\Phi(x) = |x|^{-\lambda}$, we denote V_Φ^σ by V_λ^σ and V_Φ^f by V_λ^f.

LEMMA 1.4. *Let $0<r\leq r_0$. Assume that Φ satisfies (Φ–1). Then $(\partial/\partial x_i)\int_{|y|\leq r}\Phi(x-y)dy$ exists at 0 and equals*

$$-\int_{|y|=r}\Phi(-y)\langle\nu(y), e_i\rangle dm_{n-1}(y)$$

for $i=1,\dots, n$, where dy is the Lebesgue measure on R^n and $\nu(y)$ is the unit outer normal at y to the boundary $\partial B^{(n)}(0, r)$ of $B^{(n)}(0, r)$.

As in the proof of [7; Theorem 1.14], we can prove this lemma, and so we omit its proof.

§2. Hölder continuity of Φ-potentials on non-tangential sets

In this section we discuss the Hölder continuity of Φ-potentials on sets non-tangential to S at 0 and the existence of limits of functions of type $|x|^{\lambda-k}V_\Phi^\sigma(x)$ as $x\to 0$ along a non-tangential line.

2.1. Limits and Hölder continuity in general case

PROPOSITION 2.1. *Let S be a k-dimensional Lipschitz surface and let E be a set in $B^{(n)}(0, r_0)$ such that $0\in C\ell(E\setminus\{0\})$ and* $\operatorname{contg}(E, 0)\cap\operatorname{contg}(S, 0)=\emptyset$.

(i) *If Φ satisfies (Φ–1), then for a signed measure σ on S such that $V_\lambda^{|\sigma|}(0)<\infty$,*

$$\lim_{x\to 0, x\in E} V_\Phi^\sigma(x) = V_\Phi^\sigma(0).$$

(ii) *Assume that Φ satisfies (Φ–3) and a signed measure σ on S satisfies*

$$(\sigma\text{–}3) \qquad |\sigma|(B^{(n)}(0, r)) \leq L_2 r^\gamma \qquad \textit{for} \quad 0\leq r\leq r_0$$

with some $L_2>0$ and $\gamma>0$. If $\gamma>\lambda$, then $V_\Phi^\sigma(0)$ exists and V_Φ^σ is β-Hölder continuous on $E\cap B^{(n)}(0, r(E, S))$, where $\beta=\gamma-\lambda$, if $\gamma-\lambda<1$; $0<\beta<1$, if $\gamma-\lambda=1$; $\beta=1$, if $\gamma-\lambda>1$. The Hölder constant depends only on L_2, M_0, M_3, $C(E, S)$, r_0, β, γ and λ.

REMARK 2.1. This proposition still holds in case σ is not necessarily supported by S, if S is replaced by the support of σ.

PROOF OF (i) If $x \in E \cap B^{(n)}(0, r(E, S))\backslash\{0\}$, then $V_\Phi^\sigma(x)$ is well defined. By (Φ–1) and Lemma 1.1,

$$|\Phi(x-y)| \leqq M_1|x-y|^{-\lambda} \leqq C|y|^{-\lambda} \tag{2.1}$$

for every $x \in E \cap B^{(n)}(0, r(E, S))$ and every $y \in S$. Hence, assertion (i) follows from Lebesgue's dominated convergence theorem.

PROOF OF (ii). As stated in §1, (Φ–3) implies (Φ–1). Let $g(r)=|\sigma|(B^{(n)}(0, r))$. By using (1.1) and ($\Phi$–3), we have

$$V_\lambda^{|\sigma|}(0) = \int_S |y|^{-\lambda}d|\sigma|(y) \leqq C\left\{r_0^{-\lambda}g(r_0) + \int_0^{r_0} \rho^{-\lambda-1}g(\rho)d\rho\right\} \leqq Cr_0^{\gamma-\lambda} < \infty$$

for some constant $C>0$. Hence, $V_\Phi^\sigma(0)$ exists by (Φ–1).

Now let $x, \tilde{x} \in E \cap B^{(n)}(0, r(E, S))$. Then

$$\begin{aligned}|V_\Phi^\sigma(x) - V_\Phi^\sigma(\tilde{x})| \leqq &\int_{S(0,|x-\tilde{x}|)} |\Phi(x-y)|d|\sigma|(y)\\ &+ \int_{S(0,|x-\tilde{x}|)} |\Phi(\tilde{x}-y)|d|\sigma|(y)\\ &+ \int_{S\backslash S(0,|x-\tilde{x}|)} |\Phi(x-y) - \Phi(\tilde{x}-y)|d|\sigma|(y).\end{aligned}$$

By (2.1),

$$\begin{aligned}\int_{S(0,|x-\tilde{x}|)} |\Phi(x-y)|d|\sigma|(y) &\leqq C\int_{S(0,|x-\tilde{x}|)} |y|^{-\lambda}d|\sigma|(y)\\ &\leqq C\left\{|x-\tilde{x}|^{\gamma-\lambda} + \int_0^{|x-\tilde{x}|} \rho^{\gamma-\lambda-1}d\rho\right\}\\ &\leqq C|x-\tilde{x}|^{\gamma-\lambda} \leqq C|x-\tilde{x}|^{\beta},\end{aligned}$$

since $\beta \leqq \gamma-\lambda$. Similarly,

$$\int_{S(0,|x-\tilde{x}|)} |\Phi(\tilde{x}-y)|d|\sigma|(y) \leqq C|x-\tilde{x}|^{\beta}.$$

By Lemma 1.1 and (Φ–3), we have for $y \in S$

$$|\Phi(x-y) - \Phi(\tilde{x}-y)| \leqq C|x-\tilde{x}|\,|y|^{-\lambda-1},$$

so that

$$\int_{S\backslash S(0,|x-\tilde{x}|)} |\Phi(x-y) - \Phi(\tilde{x}-y)|d|\sigma|(y)$$

$$\leqq C|x-\tilde{x}|\int_{S\setminus S(0,|x-\tilde{x}|)}|y|^{-\lambda-1}d|\sigma|(y)$$

$$\leqq C|x-\tilde{x}|\left\{r_0^{-\lambda}g(r_0)+\int_{|x-\tilde{x}|}^{r_0}\rho^{-\lambda-2}g(\rho)d\rho\right\}$$

$$\leqq C|x-\tilde{x}|\left\{r_0^{\gamma-\lambda}+\int_{|x-\tilde{x}|}^{r_0}\rho^{\gamma-\lambda-2}d\rho\right\}\leqq C|x-\tilde{x}|^{\beta}.$$

Thus combining above estimates we obtain

$$|V_{\Phi}^{\sigma}(x)-V_{\Phi}^{\sigma}(\tilde{x})|\leqq C|x-\tilde{x}|^{\beta},$$

where the constant C depends only on L_2, M_0, M_3, $C(E, S)$, r_0, β, γ and λ. Thus assertion (ii) is proved.

In the sequel we denote by ν_S the measure on S determined by $d\nu_S(\Psi(y'))=\{J_k\Psi(y')-1\}dy'$. By $(S-2)$ we obtain

COROLLARY 2.1. *Let* $0<\varepsilon\leqq 1$. *Assume that* Φ *satisfies* (Φ–3) *and* S *satisfies* α_0*-condition at* 0. *If* $k+\alpha_0>\lambda$, *then* $V(\Phi, \nu_S)$ *is* β*-Hölder continuous on* $E(0, \varepsilon)\cap B^{(n)}(0, r')$ *with* $r'>0$ *depending only on* K_2, α_0 *and* ε, *where* $\beta=k+\alpha_0-\lambda$, *if* $k+\alpha_0-\lambda<1$; $0<\beta<1$, *if* $k+\alpha_0-\lambda=1$; $\beta=1$, *if* $k+\alpha_0-\lambda>1$. *The Hölder constant depends only on* K_2, K_3, M_0, M_3, r_0, α_0, β, ε *and* λ.

2.2. Lemmas

For $x\in B^{(n)}(0, r_0)\setminus\{0\}$, let

$$Y(x)=Y(x;\Phi)=\int_{|y'|\leqq r_0}\{\Phi(x-\Psi(y'))-\Phi(x-y')\}dy'$$

and

$$\tilde{V}(x)=\tilde{V}(x;\Phi)=\int_{|y'|\leqq r_0}\Phi(x-y')dy'.$$

LEMMA 2.1. *Let* $0<\varepsilon\leqq 1$. *Assume that* S *satisfies* α_0*-condition at* 0.

(i) *If* (Φ–3) *is valid for* Φ *and if* $k+\alpha_0>\lambda$, *then* $Y(0)$ *exists, and*

$$\lim_{x\to 0, x\in E(0,\varepsilon)}Y(x)=Y(0).$$

(ii) *If* $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$ *and it satisfies* (Φ–6) *and if* $k+\alpha_0>\lambda$, *then* Y *is* β*-Hölder continuous on* $E(0, \varepsilon)\cap B^{(n)}(0, r')$ *with* $r'>0$ *depending only on* K_2, α_0 *and* ε, *where* β *is as in Corollary* 2.1. *The Hölder constant depends only on* K_2, M_0, M_5, r_0, α_0, β, ε *and* λ.

PROOF OF (i). By Lemma 1.2, (S–1) and (Φ–3), we have

$$|\Phi(x-\Psi(y'))-\Phi(x-y')|\leqq C|y'|^{\alpha_0-\lambda} \tag{2.2}$$

for every $x \in E(0, \varepsilon) \cap B^{(n)}(0, r')$ and $y \in S$ with $r' = r(E(0, \varepsilon), S)$, and

$$\int_{|y'| \leqq r_0} |y'|^{\alpha_0 - \lambda} dy' < \infty,$$

since $k + \alpha_0 > \lambda$. Thus $Y(0)$ exists, and Lebesgue's dominated convergence theorem implies

$$\lim_{x \to 0, x \in E(0,\varepsilon)} Y(x) = Y(0).$$

PROOF OF (ii). As is seen in §1, (Φ–1) and (Φ–3) are valid. For $x, \tilde{x} \in E(0, \varepsilon) \cap B^{(n)}(0, r')$,

$$\begin{aligned} |Y(x) - Y(\tilde{x})| &\leqq \int_{|y'| \leqq |x - \tilde{x}|} |\Phi(x - \Psi(y')) - \Phi(x - y')| dy' \\ &+ \int_{|y'| \leqq |x - \tilde{x}|} |\Phi(\tilde{x} - \Psi(y')) - \Phi(\tilde{x} - y')| dy' \\ &+ \int_{|x - \tilde{x}| \leqq |y'| \leqq r_0} |G(x, \tilde{x}, \Psi(y')) - G(x, \tilde{x}, y')| dy' \\ &= I_1(x, \tilde{x}) + I_2(x, \tilde{x}) + I_3(x, \tilde{x}), \end{aligned}$$

where $G(x, \tilde{x}, y) = \Phi(x - y) - \Phi(\tilde{x} - y)$. By (2.2) we have

$$I_1(x, \tilde{x}) \leqq C \int_{|y'| \leqq |x - \tilde{x}|} |y'|^{\alpha_0 - \lambda} dy' = C|x - \tilde{x}|^{k + \alpha_0 - \lambda} \leqq C|x - \tilde{x}|^{\beta},$$

since $k + \alpha_0 - \lambda \geqq \beta$. Similarly,

$$I_2(x, \tilde{x}) \leqq C|x - \tilde{x}|^{\beta}.$$

Applying the mean value theorem, by Lemma 1.2, (S–1) and (Φ–6), we have

$$|G(x, \tilde{x}, \Psi(y')) - G(x, \tilde{x}, y')| \leqq C|x - \tilde{x}|\,|y'|^{\alpha_0 - \lambda - 1}$$

for every y', $|y'| \leqq r_0$, so that

$$I_3(x, \tilde{x}) \leqq C|x - \tilde{x}| \int_{|x - \tilde{x}| \leqq |y'| \leqq r_0} |y'|^{\alpha_0 - \lambda - 1} dy' \leqq C|x - \tilde{x}|^{\beta}.$$

Therefore we obtain

$$|Y(x) - Y(\tilde{x})| \leqq C|x - \tilde{x}|^{\beta},$$

where the constant C depends only on the values described in the lemma.

Let $\rho > 0$ and w be a unit vector in R^n. For a Borel measurable function F defined on $B^{(n)}(0, r_0)$, write

$$p(\rho; w, F) = \int_{|y'| = 1} F(r_0(1 + \rho)^{-1}(-\rho y' + w)) dm_{k-1}(y'),$$

provided the integral exists.

LEMMA 2.2. *Let* $0<\varepsilon\leqq 1$. *Assume that* Φ *satisfies* (Φ–2) *with* $\lambda=k$.

(i) *There is a positive number* C *depending only on* M_2, r_0 *and* ε *such that* $|\tilde{V}(x)|\leqq C$ *for every* $x\in E(0, \varepsilon)\backslash\{0\}$.

(ii) *If* Φ *satisfies* (Φ–4) *with* $\lambda=k$, *and if* a *is a unit vector such that* $a^*\neq 0$, *then*

$$\lim_{x\to 0, x\in L(0,a)} \tilde{V}(x) = r_0^k\int_0^\infty \rho^{k-1}(1+\rho)^{-k}p(\rho;\, a^*/|a^*|,\, \Phi)d\rho.$$

(iii) *If* Φ *satisfies* (Φ–3) *and* (Φ–4) *with* $\lambda=k$, *then there exists a positive number* C *depending only on* M_2, M_3, r_0 *and* ε *such that*

$$|\tilde{V}(x)-\tilde{V}(\tilde{x})| \leqq C\{|x-\tilde{x}| + |(x^*/|x^*|) - (\tilde{x}^*/|\tilde{x}^*|)|\}$$

for all x, $\tilde{x}\in E(0, \varepsilon)\backslash\{0\}$.

PROOF. For $x\in E(0, \varepsilon)\cap B^{(n)}(0, r_0/4)\backslash\{0\}$, let $d=r_0-|x'|$ and $F=\{y';\, |y'|\leqq r_0,\, |y'-x'|>d\}$. Then we write

$$\tilde{V}(x) = \int_0^d \rho^{k-1}d\rho\int_{|y'|=1} \Phi(-\rho y'+x^*)dm_{k-1}(y') + \int_F \Phi(x-y')dy', \tag{2 3}$$

Since $|\Phi(-\rho y'+x^*)|\leqq M_2|x^*|\{\rho^2+|x^*|^2\}^{-(k+1)/2}$ by (Φ–2) with $\lambda=k$, the absolute value of the first integral on the right of (2.3) is dominated by

$$C|x^*|\int_0^d \rho^{k-1}(\rho+|x^*|)^{-k-1}d\rho \leqq C\int_0^\infty \rho^{k-1}(1+\rho)^{-k-1}d\rho < \infty.$$

Since $d\geqq 3r_0/4$, we have

$$\begin{aligned}|\Phi(x-y')| &\leqq M_2|x^*|\,|x-y'|^{-k-1}\\ &\leqq M_2 \min\{|x^*|(4/3r_0)^{k+1},\, (4/3r_0)^k\}\end{aligned} \tag{2.4}$$

for all $y'\in F$. Thus the second term on the right of (2.3) is dominated by $C(4/3r_0)^k r_0^k=C(4/3)^k$ in absolute value. Hence, assertion (i) is obtained.

Next, we prove (ii). It follows from (Φ–2) with $\lambda=k$ that

$$|p(\rho;\, x^*/|x^*|,\, \Phi)| \leqq Cr_0^{-k}(1+\rho)^{-1} \tag{2.5}$$

for every $x\in E(0, \varepsilon)\cap B^{(n)}(0, r_0/4)\backslash\{0\}$. By ($\Phi$–4) with $\lambda=k$, we can write

$$\tilde{V}(x) = r_0^k\int_0^{d/|x^*|} \rho^{k-1}(1+\rho)^{-k}p(\rho;\, x^*/|x^*|,\, \Phi)d\rho + \int_F \Phi(x-y')dy'.$$

Thus assertion (ii) follows from (2.4) and (2.5).

Finally, we prove (iii). Let $x, \tilde{x} \in E(0, \varepsilon) \cap B^{(n)}(0, r_0/4)\backslash\{0\}$ with $|x'| \leqq |\tilde{x}'|$. Then

$$\begin{aligned}\tilde{V}(x) - \tilde{V}(\tilde{x}) &= \int_{|y'-x'|\leqq d} \Phi(x-y')dy' - \int_{|y'-\tilde{x}'|\leqq d} \Phi(\tilde{x}-y')dy' \\ &\quad + \int_{F_1} \{\Phi(x-y') - \Phi(\tilde{x}-y')\}dy' \\ &\quad + \int_{F_2} \Phi(\tilde{x}-y')dy' - \int_{F_3} \Phi(\tilde{x}-y')dy' \\ &= I_1(x) - I_1(\tilde{x}) + I_2(x, \tilde{x}) + I_3(x, \tilde{x}) - I_4(x, \tilde{x}),\end{aligned}$$

where $d=r_0-|x'|$, $F_1=F$, $F_2=\{y';\ |y'-x'|>d,\ |y'-\tilde{x}'|\leqq d\}$ and $F_3=\{y';\ |y'-x'|\leqq d, |y'-\tilde{x}'|>d\}$. Since $d\geqq 3r_0/4$ and so $F_1\subset B^{(k)}(0, r_0)\backslash B^{(k)}(0, r_0/2)$, by ($\Phi$–3) with $\lambda=k$, the absolute value of the integrand of I_2 is dominated by $C|x-\tilde{x}|$ for every $y'\in F_1$. Therefore $|I_2(x, \tilde{x})|\leqq C|x-\tilde{x}|$. Since $\Phi(x-y')$ is bounded for $(x, y')\in B^{(n)}(0, r_0/4)\times\{B^{(k)}(0, r_0)\backslash B^{(k)}(0, r_0/2)\}$ and $m_k(F_2)=m_k(F_3)\leqq C|x-\tilde{x}|$, we see that $|I_3(x, \tilde{x})|\leqq C|x-\tilde{x}|$ and $|I_4(x, \tilde{x})|\leqq C|x-\tilde{x}|$. As above, we have

$$I_1(x) = r_0^k\int_0^{d/|x^*|} \rho^{k-1}(1+\rho)^{-k}p(\rho;\ x^*/|x^*|,\ \Phi)d\rho,$$

so that

$$\begin{aligned}|I_1(x)-I_1(\tilde{x})| &\leqq r_0^k\left|\int_{d/|\tilde{x}^*|}^{d/|x^*|} \rho^{k-1}(1+\rho)^{-k}p(\rho;\ x^*/|x^*|,\ \Phi)d\rho\right| \\ &\quad + r_0^k\int_0^{d/|\tilde{x}^*|} \rho^{k-1}(1+\rho)^{-k}|p(\rho;\ x^*/|x^*|,\ \Phi) - p(\rho;\ \tilde{x}^*/|\tilde{x}^*|,\ \Phi)|d\rho \\ &= I_1^{(1)}(x, \tilde{x}) + I_2^{(1)}(x, \tilde{x}).\end{aligned}$$

Now (2.5) implies

$$I_1^{(1)}(x, \tilde{x}) \leqq C\left|\int_{d/|\tilde{x}^*|}^{d/|x^*|} \rho^{k-1}(1+\rho)^{-k-1}d\rho\right| \leqq C|x-\tilde{x}|,$$

since $d\geqq 3r_0/4$. By (Φ–3) with $\lambda=k$, we have

$$\begin{aligned}&|\Phi(r_0(1+\rho)^{-1}(-\rho y'+x^*/|x^*|)) - \Phi(r_0(1+\rho)^{-1}(-\rho y'+\tilde{x}^*/|\tilde{x}^*|))| \\ &\quad\leqq Cr_0^k(1+\rho)^{-1}|(x^*/|x^*|) - (\tilde{x}^*/|\tilde{x}^*|)|\end{aligned}$$

for every y', $|y'|=1$, so that

$$I_2^{(1)}(x, \tilde{x}) \leqq C|(x^*/|x^*|) - (\tilde{x}^*/|\tilde{x}^*|)|.$$

Therefore

$$|\tilde{V}(x)-\tilde{V}(\tilde{x})| \leqq C\{|x-\tilde{x}| + |(x^*/|x^*|) - (\tilde{x}^*/|\tilde{x}^*|)|\}.$$

Note that the constant C in this last expression depends only on M_2, M_3, r_0 and ε. Thus assertion (iii) follows, since $\operatorname{dist}(E(0, \varepsilon)\backslash B^{(n)}(0, r_0/8), R^k)>0$ and thus by (Φ–3) $\tilde{V}$ is 1-Hölder continuous on $E(0, \varepsilon)\backslash B^{(n)}(0, r_0/8)$ with Hölder constant depending only on M_3, r_0 and ε.

COROLLARY 2.2. *Let a be a unit vector with $a^*\neq 0$. If Φ satisfies (Φ–2), (Φ–3) and (Φ–4) with $\lambda=k$, then $\tilde{V}$ is a 1-Hölder continuous function on $L(0, a)\cap B^{(n)}(0, r_0)$ with Hölder constant depending only on M_2, M_3, a and r_0.*

Similarly we obtain

LEMMA 2.2′. *Assume that Φ satisfies (Φ–1) and (Φ–4). Let a be a unit vector with $a^*\neq 0$.*

(i) *If $\lambda>k$, then*

$$\lim_{x\to 0, x\in L(0,a)} |x|^{\lambda-k}\tilde{V}(x; \Phi) = r_0^{\lambda}|a^*|^{k-\lambda}\int_0^{\infty}\rho^{k-1}(1+\rho)^{-\lambda}p(\rho; a^*/|a^*|, \Phi)d\rho.$$

(ii) *If $\lambda=k$, then*

$$\lim_{x\to 0, x\in L(0,a)} (\log|x|)^{-1}\tilde{V}(x; \Phi) = -r_0^k\int_{|y'|=1}\Phi(-r_0 y')dm_{k-1}(y').$$

2.3. Limits and Hölder continuity in special case

PROPOSITION 2.2. *Let a be a unit vector with $a^*\neq 0$. Assume that S satisfies α_0-condition at 0 and Φ satisfies (Φ–2), (Φ–3) and (Φ–4) with $\lambda=k$.*

(i) *If a signed measure σ on S satisfies $(\sigma-1)$ with $A\in R$, then*

$$\lim_{x\to 0, x\in L(0,a)} V_{\Phi}^{\sigma}(x) = V_{\Phi}^{\sigma}(0) + Ar_0^k\int_0^{\infty}\rho^{k-1}(1+\rho)^{-k}p(\rho; a^*/|a^*|, \Phi)d\rho.$$

(ii) *Suppose a signed measure σ on S satisfies $(\sigma-2)$ with $A\in R$ and $\alpha_1>0$. Let $0<\varepsilon\leqq 1$ and let $\beta=\min\{\alpha_0, \alpha_1\}$ in case $\min\{\alpha_0, \alpha_1\}<1$; $0<\beta<1$ in case $\min\{\alpha_0, \alpha_1\}=1$; $\beta=1$ in case $\min\{\alpha_0, \alpha_1\}>1$. If, in addition, $\Phi\in C^1(B^{(n)}(0, 4r_0)\backslash\{0\})$ and it satisfies (Φ–6) with $\lambda=k$, then there exists a positive number C depending only on A, K_2, K_3, M_0, M_2, M_5, r_0, α_0, α_1, β and ε such that*

$$|V_{\Phi}^{\sigma}(x)-V_{\Phi}^{\sigma}(\tilde{x})| \leqq C\{|x-\tilde{x}|^{\beta} + |(x^*/|x^*|) - (\tilde{x}^*/|\tilde{x}^*|)|\}$$

for all $x, \tilde{x}\in E(0, \varepsilon)\cap B^{(n)}(0, r')\backslash\{0\}$ with $r'>0$ depending only on K_2, α_0 and ε; in particular, V_{Φ}^{σ} is a β-Hölder continuous function on $L(0, a)\cap B^{(n)}(0, r')$ with $r'>0$ depending only on K_2, α_0 and a and with Hölder constant depending only on A, K_2, K_3, M_0, M_2, M_5, a, r_0, α_0, α_1 and β.

PROOF. For $x \in L(0, a) \cap B^{(n)}(0, r')$, where $r' = r(L(0, a), S)$, we write

$$V_{\Phi}^{\sigma}(x) = \int_S \{\Phi(x-y) - \Phi(x-y')\} d(\sigma - A\mu_S)(y) \tag{2.6}$$
$$+ \int_S \Phi(x-y') d(\sigma - A\mu_S)(y)$$
$$+ A\{V(\Phi, \nu_S)(x) + Y(x; \Phi) + \tilde{V}(x; \Phi)\}.$$

Since $|\Phi(x-y) - \Phi(x-y')| \leqq C|y|^{\alpha_0 - k}$ for $x \in L(0, a) \cap B^{(n)}(0, r')$ and $y \in S$ by (S–1) and (Φ–3) with $\lambda = k$, and hence by (σ–1)

$$\int_S |y|^{\alpha_0 - k} d|\sigma - A\mu_S|(y) < \infty.$$

Lebesgue's dominated convergence theorem implies

$$\lim_{x \to 0, x \in L(0,a)} \int_S \{\Phi(x-y) - \Phi(x-y')\} d(\sigma - A\mu_S)(y)$$
$$= \int_S \Phi(-y) d(\sigma - A\mu_S)(y) = V(\Phi, \sigma - A\mu_S)(0),$$

because $\Phi(-y') = 0$ by (Φ–2). In order to estimate the second integral on the right of (2.6), let $g(r) = |\sigma - A\mu_S|(S(0, r))$ and $\varepsilon(r) = \sup_{0 < \rho \leqq r} \rho^{-k} g(\rho)$. Then for $0 < r < r'$, we have

$$\int_{S(0,r)} |\Phi(x-y')| d|\sigma - A\mu_S|(y)$$
$$\leqq C|x^*| \left\{ (|x^*| + r)^{-k-1} g(r) + \int_0^r (|x^*| + \rho)^{-k-2} g(\rho) d\rho \right\}$$
$$\leqq C\left\{ 1 + \int_0^\infty \rho^k (1 + \rho)^{-k-2} d\rho \right\} \varepsilon(r),$$

since $|\Phi(x-y')| \leqq C|x^*|(|x^*| + |y'|)^{-k-1}$ for $x \in L(0, a) \cap B^{(n)}(0, r')$ and $y \in S$ by (Φ–2) and Lemma 1.2. Therefore

$$\left| \int_S \Phi(x-y') d(\sigma - A\mu_S)(y) \right| \leqq C\varepsilon(r) + \int_{S \setminus S(0,r)} |\Phi(x-y')| d|\sigma - A\mu_S|(y).$$

Hence, the second integral on the right of (2.6) tends to zero as $x \to 0$ along $L(0, a)$, because $\Phi(-y') = 0$ and $\lim_{r \downarrow 0} \varepsilon(r) = 0$ by (σ–1). Since

$$V_{\Phi}^{\sigma}(0) = V(\Phi, \sigma - A\mu_S)(0) + A\{V(\Phi, \nu_S)(0) + Y(0; \Phi)\},$$

assertion (i) follows from Corollary 2.1, (i) of Lemma 2.1 and (ii) of Lemma 2.2.

Next, for $x \in B^{(n)}(0, r_0) \setminus S$, we write

$$V_{\Phi}^{\sigma}(x) = V(\Phi, \sigma - A\mu_S)(x) + A\{V(\Phi, \nu_S)(x) + Y(x; \Phi) + \tilde{V}(x; \Phi)\}. \tag{2.7}$$

Applying (ii) of Proposition 2.1 with $\gamma=k+\alpha_1$ and σ replaced by $\sigma-A\mu_S$, we see that the first term on the right of (2.7) is β-Hölder continuous on $E(0, \varepsilon)\cap B^{(n)}(0, r')$ with $r'=r(E(0, \varepsilon), S)$. Thus assertion (ii) follows from Corollary 2.1, (ii) of Lemma 2.1 and (iii) of Lemma 2.2.

COROLLARY 2.3. *Assume that* $k=n-1$. *Let* $0<\varepsilon\leqq 1$, $E_+(0, \varepsilon)=\{x\in E(0, \varepsilon); x_n>0\}$ *and* S *be as in Proposition* 2.2. *Assume that* (Φ–2), (Φ–3) *and* (Φ–4) *with* $\lambda=n-1$ *are valid for* Φ.

(i) *If a signed measure* σ *on* S *satisfies* (σ–1) *with* $A\in R$, *then*

$$\lim_{x\to 0, x\in E_+(0,\varepsilon)} V_\Phi^\sigma(x)$$
$$= V_\Phi^\sigma(0) + Ar_0^{n-1}\int_0^\infty \rho^{n-2}(1+\rho)^{-n+1}p(\rho; e_n, \Phi)d\rho.$$

(ii) *Suppose a signed measure* σ *on* S *satisfies* (σ–2) *with* $k=n-1$, $A\in R$ *and* $\alpha_1>0$. *If, in addition,* $\Phi\in C^1(B^{(n)}(0, 4r_0)\backslash\{0\})$ *and it satisfies* (Φ–6) *with* $\lambda=n-1$, *then* V_Φ^σ *is* β-*Hölder continuous on* $E_+(0, \varepsilon)\cap B^{(n)}(0, r')$ *with* $r'>0$ *depending only on* K_2, α_0 *and* ε, *where* β *is the same as in Proposition* 2.2. *The Hölder constant depends only on* A, K_2, K_3, M_0, M_2, M_5, α_0, α_1, β *and* ε.

By a slight modification of the proof of Proposition 2.2, we obtain

PROPOSITION 2.2′. *Let* $\lambda\geqq k$ *and let* a *and* S *be as in Proposition* 2.2. *Assume that* (Φ–3) *and* (Φ–4) *hold for* Φ *and that a signed measure* σ *on* S *satisfies* (σ–1). *If* $\lambda>k$, *then*

$$\lim_{x\to 0, x\in L(0,a)} |x|^{\lambda-k}V_\Phi^\sigma(x)$$
$$= Ar_0^\lambda|a^*|^{k-\lambda}\int_0^\infty \rho^{k-1}(1+\rho)^{-\lambda}p(\rho; a^*/|a^*|, \Phi)d\rho$$

and if $\lambda=k$, *then*

$$\lim_{x\to 0, x\in L(0,a)}(\log|x|)^{-1}V_\Phi^\sigma(x) = -Ar_0^k\int_{|y'|=1}\Phi(-r_0y')dm_{k-1}(y').$$

PROOF. We prove only the case $\lambda>k$. For $x\in L(0, a)\cap B^{(n)}(0, r')$, where $r'=r(L(0, a), S)$, we write

$$(2.8)\qquad |x|^{\lambda-k}V_\Phi^\sigma(x) = |x|^{\lambda-k}\int_S \Phi(x-y)d(\sigma-A\mu_S)(y)$$
$$+ A|x|^{\lambda-k}\{V(\Phi, \nu_S)(x) + Y(x; \Phi)\} + A|x|^{\lambda-k}\tilde{V}(x; \Phi).$$

As above, let $g(r)=|\sigma-A\mu_S|(S(0, r))$ and $\varepsilon(r)=\sup_{0<\rho\leqq r}\rho^{-k}g(\rho)$. Then for $0<r<r'$, we have

$$
\begin{aligned}
&|x|^{\lambda-k}\int_{S(0,r)}|\Phi(x-y)|d|\sigma-A\mu_S|(y)\\
&\leqq C|x|^{\lambda-k}\left\{(|x|+r)^{-\lambda}g(r)+\int_0^r(|x|+\rho)^{-\lambda-1}g(\rho)d\rho\right\}\\
&\leqq C\left\{|x|^{\lambda-k}r^k(|x|+r)^{-\lambda}+|x|^{\lambda-k}\int_0^r\rho^k(|x|+\rho)^{-\lambda-1}d\rho\right\}\varepsilon(r)\\
&\leqq C\left\{1+\int_0^\infty\rho^k(1+\rho)^{-\lambda-1}d\rho\right\}\varepsilon(r),
\end{aligned}
$$

since $|\Phi(x-y)|\leqq C(|x|+|y|)^{-\lambda}$ for $x\in L(0,a)\cap B^{(n)}(0,r')$ and $y\in S$. Therefore the first term on the right of (2.8) tends to zero as $x\to 0$ along $L(0,a)$, because $\lim_{r\downarrow 0}\varepsilon(r)=0$ by (σ–1). Similarly, as $x\to 0$ along $L(0,a)$, the second term on the right of (2.8) also tends to zero. Hence, by (i) of Lemma 2.2′ the assertion of the first part is proved.

§3. Hölder continuity and limits of directional derivatives on non-tangential sets

In this section we prove the Hölder continuity of directional derivatives of Φ-potentials on a non-tangential line terminating at the origin (cf. [9; Theorem 18]).

Throughout this section we assume that S satisfies α_0-condition at 0. Let $T(0)$ (resp. $N(0)$) be the set of all tangent (resp. normal) vectors to S at 0, i.e., $T(0)=\{t\in R^n;\ |t|=1,\ t^*=0\}$, $N(0)=\{n\in R^n;\ |n|=1,\ n'=0\}$.

3.1. Tangential derivatives

THEOREM 3.1. *Let $0<\varepsilon\leqq 1$. Suppose $\Phi\in C^1(B^{(n)}(0,4r_0)\setminus\{0\})$ and it satisfies (Φ–6). If a signed measure σ on S satisfies (σ–2) with $A\in R$ and $\alpha_1>0$ and if $k+\min\{\alpha_0,\alpha_1\}>\lambda+1$, then for each $i=1,\dots,k$, $D_iV_\Phi^\sigma$ is β-Hölder continuous on $E(0,\varepsilon)\cap B^{(n)}(0,r')\setminus\{0\}$ with $r'>0$ depending only on K_2, α_0 and ε, where $\beta=k+\min\{\alpha_0,\alpha_1\}-\lambda-1$, if $k+\min\{\alpha_0,\alpha_1\}-\lambda-1<1$; $0<\beta<1$, if $k+\min\{\alpha_0,\alpha_1\}-\lambda-1=1$; $\beta=1$, if $k+\min\{\alpha_0,\alpha_1\}-\lambda-1>1$. The Hölder constant depends only on A, K_2, K_3, M_0, M_5, r_0, α_0, α_1, β, ε and λ. Furthermore,*

$$
\begin{aligned}
\lim_{x\to 0,x\in E(0,\varepsilon)}D_iV_\Phi^\sigma(x) &= V(D_i\Phi,\sigma-A\mu_S)(0)\\
&+A\{V(D_i\Phi,\nu_S)(0)+Y(0;D_i\Phi)\}\\
&-A\int_{|y'|=r_0}\Phi(-y')\langle\nu(y'),e_i\rangle dm_{k-1}(y'),
\end{aligned}
$$

where $\nu(y')$ is the unit outer normal at y' to the boundary $\partial B^{(k)}(0,r_0)$ of $B^{(k)}(0,r_0)$ in R^k. The same assertions hold for $(d/dt)V_\Phi^\sigma$ for any $t\in T(0)$ with D_i and e_i replaced by d/dt and t.

PROOF. For simplicity, let $E=E(0,\varepsilon)\cap B^{(n)}(0,r')\setminus\{0\}$, where $r'=r(E(0,\varepsilon),S)$. Since $D_iV_\Phi^\sigma(x)=\int_S D_i\Phi(x-y)d\sigma(y)=V(D_i\Phi,\sigma)(x)$ for $x\in B^{(n)}(0,r_0)\setminus S$, it is enough to show that $V(D_i\Phi,\sigma)$ is β-Hölder continuous on E. Consider the measures $\sigma_0=\sigma-A\mu_S$ and $\mu_1=(J_k\Psi\circ\Psi^{-1})^{-1}\mu_S$ on S. Then

$$V(D_i\Phi,\sigma)(x)=V(D_i\Phi,\sigma_0)(x)+A\{V(D_i\Phi,\nu_S)(x)+V(D_i\Phi,\mu_1)(x)\}.$$

By (Φ–6) and (σ–2), Proposition 2.1 implies that $V(D_i\Phi,\sigma_0)$ is β-Hölder continuous on E. By Corollary 2.1, $V(D_i\Phi,\nu_S)$ is also β-Hölder continuous there. We rewrite $V(D_i\Phi,\mu_1)$ as follows:

$$\begin{aligned}
V(D_i\Phi,\mu_1)(x) &= \int_{|y'|\leqq r_0} D_i\Phi(x-\Psi(y'))dy' \\
&= -\int_{|y'|\leqq r_0}\frac{\partial}{\partial y_i}\Phi(x-\Psi(y'))dy' \\
&\quad -\sum_{j=k+1}^n\int_{|y'|\leqq r_0} D_j\Phi(x-\Psi(y'))\frac{\partial\psi_j}{\partial y_i}(y')dy' \\
&= -\int_{|y'|=r_0}\Phi(x-\Psi(y'))\langle\nu(y'),e_i\rangle dm_{k-1}(y') \\
&\quad -\sum_{j=k+1}^n\int_S D_j\Phi(x-y)\Big(\frac{\partial\psi_j}{\partial y_i}\circ\Psi^{-1}(y)\Big)d\mu_1(y).
\end{aligned}$$

Since $\Phi\in C^1(B^{(n)}(0,4r_0)\setminus\{0\})$ and $r'<r_0$,

$$x\longrightarrow\int_{|y'|=r_0}\Phi(x-\Psi(y'))\langle\nu(y'),e_i\rangle dm_{k-1}(y')$$

is a C^1-function on $B^{(n)}(0,r')$ and hence it is β-Hölder continuous there. Finally, for $\sigma_j=((\partial\psi_j/\partial y_i)\circ\Psi^{-1})\mu_1$, by ($S$–1) we have

$$|\sigma_j|(S(0,r))\leqq K_2\int_{|y'|\leqq r}|y'|^{\alpha_0}dy'=Cr^{k+\alpha_0},\quad 0\leqq r\leqq r_0.$$

Hence again by Proposition 2.1, each $V(D_j\Phi,\sigma_j)$ is β-Hölder continuous on E. Thus $D_iV_\Phi^\sigma$ is β-Hölder continuous on E. Note that the Hölder constant depends only on the values stated in the theorem.

If $t\in T(0)$, then

$$(d/dt)V_\Phi^\sigma=V(d\Phi/dt,\sigma)=\sum_{i=1}^k t_iV(D_i\Phi,\sigma)$$

on $B^{(n)}(0,r_0)\setminus S$, so that this is β-Hölder continuous on E.

As to the limit, we write $D_iV_\Phi^\sigma(x)$ as follows: For $x\in B^{(n)}(0,r_0)\setminus S$,

$$D_iV_\Phi^\sigma(x)=V(D_i\Phi,\sigma_0)(x)+A\{V(D_i\Phi,\nu_S)(x)+Y(x;D_i\Phi)+\tilde V(x;D_i\Phi)\}. \tag{3.1}$$

As $x\to0$, $x\in E(0,\varepsilon)\setminus\{0\}$, by Proposition 2.1 and its corollary, $V(D_i\Phi,\sigma_0)(x)$ and

$V(D_i\Phi, \nu_S)(x)$ converge to $V(D_i\Phi, \sigma_0)(0)$ and $V(D_i\Phi, \nu_S)(0)$, respectively. By Lemma 2.1 $Y(x; D_i\Phi)$ converges to $Y(0; D_i\Phi)$. Finally, since

$$\tilde{V}(x; D_i\Phi) = -\int_{|y'|=r_0} \Phi(x-y')\langle \nu(y'), e_i\rangle dm_{k-1}(y')$$

and this is continuous on $B^{(n)}(0, r')$, it tends to $-\int_{|y'|=r_0} \Phi(-y')\langle \nu(y'), e_i\rangle dm_{k-1}(y')$. Thus the theorem is proved.

REMARK 3.1. In a way similar to the proof of (i) of Proposition 2.1, we can show that if $\lambda < k-1$, $|D_i\Phi(x)| \leqq C|x|^{-\lambda-1}$ $(1 \leqq i \leqq n)$ and $V_{\lambda+1}^{|\sigma|}(0) < \infty$, then

$$\lim_{x\to 0, x\in E(0,\varepsilon)} D_i V_\Phi^\sigma(x) = V(D_i\Phi, \sigma)(0) \qquad (1 \leqq i \leqq n).$$

In particular, this equality holds in case $\lambda < k-1$ in the above theorem.

REMARK 3.2. In a way similar to the proof of Proposition 2.2′, we can see that if $\Phi \in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$, it satisfies ($\Phi$–6), a signed measure σ on S satisfies (σ–1) and $\lambda > k-1$, then

$$\lim_{x\to 0, x\in E(0,\varepsilon)} |x|^{\lambda-k+1} D_i V_\Phi^\sigma(x) = 0$$

for $0 < \varepsilon \leqq 1$ and $i = 1, \dots, k$.

REMARK 3.3. If $\lambda \geqq k-1$ and if we replace (σ–2) by (σ–1) in the theorem, then $\lim_{x\to 0, x\in E(0,\varepsilon)} D_i V_\Phi^\sigma(x)$ does not exist in general as the following example shows:

EXAMPLE 3.1. Let $S = B^{(k)}(0, 1)$, $\lambda \geqq k-1$ and $E = \{x; x_1 = \cdots = x_{n-1} = 0, x_n > 0\}$. Let a non-negative function f be defined by

$$f(x') = \begin{cases} (-\log|x'|)^{-1}, & \text{if } x' \in F, \\ 0, & \text{if } x' \in S\setminus F, \end{cases}$$

where $F = \{x' \in S; 0 < x_1 \leqq 1/2, x_2^2 + \cdots + x_k^2 \leqq x_1^2\}$. Then $d\sigma = f d\mu_S = f dy'$ on S satisfies (σ–1) with $A = 0$ and

$$\lim_{x\to 0, x\in E} D_1 V_\lambda^f(x) = \infty.$$

In fact, for $h > 0$, we have

$$\begin{aligned} D_1 V_\lambda^f(he_n) &= \lambda \int_S y_1 |he_n - y'|^{-\lambda-2} f(y') dy' \\ &= \lambda \int_F y_1 (-\log|y'|)^{-1} (h^2 + |y'|^2)^{-(\lambda+2)/2} dy', \end{aligned}$$

so that Fatou's lemma implies

$$\liminf_{h\downarrow 0} D_1 V_\lambda^f(he_n) \geqq C\int_0^{1/2} \rho^{k-\lambda-2}(-\log\rho)^{-1}d\rho = \infty,$$

since $\lambda \geqq k-1$.

3.2. Limits of normal derivatives

The following two theorems are immediate consequences of Propositions 2.2 and 2.2′ applied to $d\Phi/dn$ in place of Φ.

THEOREM 3.2. *Let $\lambda=k-1$ and $n\in N(0)$. Assume that $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$, that it satisfies ($\Phi$–4) and ($\Phi$–6) with $\lambda=k-1$ and that (Φ–5) with $\lambda=k-1$ holds for $d\Phi/dn$ in place of $D_i\Phi$. If a signed measure σ on S satisfies $(\sigma-1)$ with $A\in R$, then*

$$\begin{aligned}\frac{d}{dn}V_\Phi^\sigma(0) &= \lim_{x\to 0, x\in L(0,n)} \frac{d}{dn}V_\Phi^\sigma(x)\\ &= V\Big(\frac{d\Phi}{dn}, \sigma\Big)(0) + Ar_0^k\int_0^\infty \rho^{k-1}(1+\rho)^{-k}p\Big(\rho\,;\, n, \frac{d\Phi}{dn}\Big)d\rho.\end{aligned}$$

REMARK 3.4. In case $\lambda<k-1$, it is easy to see that if $|(d\Phi/dn)(x)|\leqq C|x|^{-\lambda-1}$ and $V_{\lambda+1}^{|\sigma|}(0)<\infty$, then

$$\frac{d}{dn}V_\Phi^\sigma(0) = \lim_{x\to 0, x\in L(0,n)} \frac{d}{dn}V_\Phi^\sigma(x) = V\Big(\frac{d\Phi}{dn}, \sigma\Big)(0).$$

THEOREM 3.2′. *Let $n\in N(0)$ and a be a unit vector with $a^*\neq 0$. Assume that $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$, that it satisfies ($\Phi$–4) and ($\Phi$–6) and that a signed measure σ on S satisfies (σ–1) with $A\in R$.*

(i) *If $\lambda>k-1$, then*

$$\begin{aligned}&\lim_{x\to 0, x\in L(0,a)} |x|^{\lambda-k+1}\frac{d}{dn}V_\Phi^\sigma(x)\\ &= Ar_0^{\lambda+1}|a^*|^{k-\lambda-1}\int_0^\infty \rho^{k-1}(1+\rho)^{-\lambda-1}p\Big(\rho\,;\, a^*/|a^*|, \frac{d\Phi}{dn}\Big)d\rho.\end{aligned}$$

(ii) *If $\lambda=k-1$, then*

$$\begin{aligned}&\lim_{x\to 0, x\in L(0,a)} (\log|x|)^{-1}\frac{d}{dn}V_\Phi^\sigma(x)\\ &= -\,Ar_0^k\int_{|y'|=1}\frac{d\Phi}{dn}(-r_0y')dm_{k-1}(y').\end{aligned}$$

(iii) *If $\lambda=k-1$ and (Φ–5) holds for $d\Phi/dn$ in place of $D_i\Phi$, then*

$$\lim_{x\to 0, x\in L(0,a)} \frac{d}{dn}V_\Phi^\sigma(x)$$

$$= V\left(\frac{d\Phi}{dn}, \sigma\right)(0) + Ar_0^k \int_0^\infty \rho^{k-1}(1+\rho)^{-k} p\left(\rho; a^*/|a^*|, \frac{d\Phi}{dn}\right) d\rho.$$

REMARK 3.5. In case Φ is defined on $R^n\backslash\{0\}$ and is homogeneous of order $-\lambda$; i.e., $\Phi(hx)=h^{-\lambda}\Phi(x)$ for all $h>0$ and all $x\neq 0$, then

$$p(\rho;\, w,\, \Phi) = r_0^{-\lambda}(1+\rho)^\lambda \int_{|y'|=1} \Phi(-\rho y'+w)dm_{k-1}(y')$$

for a unit vector w in R^n. Write

$$q(\rho;\, w,\, \Phi) = \int_{|y'|=1} \Phi(-\rho y'+w)dm_{k-1}(y').$$

Let a, n and σ be as in Theorem 3.2′. Assume that $\Phi\in C^1(R^n\backslash\{0\})$ and it satisfies (Φ–6) for $0<|x|\leqq|\tilde{x}|$. Then (i) and (iii) of Theorem 3.2′ are written as follows.

(i) If $\lambda>k-1$, then

$$\lim_{x\to 0,\, x\in L(0,a)} |x|^{\lambda-k+1} \frac{d}{dn} V_\Phi^\sigma(x) = A|a^*|^{k-\lambda-1} \int_0^\infty \rho^{k-1} q\left(\rho;\, a^*/|a^*|,\, \frac{d\Phi}{dn}\right) d\rho;$$

(iii) If $\lambda=k-1$ and $(\Phi-5)$ holds for $d\Phi/dn$ in place of $D_i\Phi$, then

$$\lim_{x\to 0,\, x\in L(0,a)} \frac{d}{dn} V_\Phi^\sigma(x) = V\left(\frac{d\Phi}{dn}, \sigma\right)(0) + A\int_0^\infty \rho^{k-1} q\left(\rho;\, a^*/|a^*|,\, \frac{d\Phi}{dn}\right) d\rho.$$

In case $\Phi(x)=|x|^{-\lambda}$, similar results were obtained in [3; Satz 3]. In this special case note that

$$\int_0^\infty \rho^{k-1} q(\rho;\, w,\, \Phi)d\rho = \pi^{k/2}\,\Gamma((\lambda-k)/2)/\Gamma(\lambda/2)$$

for any $w\in N(0)$, if $\lambda>k$.

3.3. Hölder continuity of directional derivatives

THEOREM 3.3 *Let s be a unit vector and let $n_s=|s^*|^{-1}s^*$ in case $s^*\neq 0$. Assume that a signed measure σ on S satisfies (σ–2) with $A\in R$ and $\alpha_1>0$ and that $\Phi\in C^2(B^{(n)}(0, 4r_0)\backslash\{0\})$ and it satisfies (Φ–4) and (Φ–7) with $\lambda=k-1$, and furthermore assume that (Φ–5) with $\lambda=k-1$ holds for $d\Phi/dn_s$ in place of $D_i\Phi$ in case $s^*\neq 0$. Let $0<\varepsilon\leqq 1$ and β be as in* (ii) *of Proposition* 2.2. *Then there is a positive number C depending only on A, K_2, K_3, M_0, M_0', M_4, M_6, r_0, α_0, α_1, β and ε such that*

$$\left|\frac{d}{ds} V_\Phi^\sigma(x) - \frac{d}{ds} V_\Phi^\sigma(\tilde{x})\right| \leqq C\{|x-\tilde{x}|^\beta + |(x^*/|x^*|)-(\tilde{x}^*/|\tilde{x}^*|)|\}$$

for all $x, \tilde{x} \in E(0, \varepsilon) \cap B^{(n)}(0, r')\setminus\{0\}$ *with* $r'>0$ *depending only on* K_2, α_0 *and* ε. *In particular, for any unit vector a with* $a^* \neq 0$, $(d/ds)V_\Phi^\sigma$ *is* β*-Hölder continuous on* $L(0, a) \cap B^{(n)}(0, r')$ *with* $r'>0$ *depending only on* K_2, α_0 *and a. The Hölder constant depends only on* A, K_2, K_3, M_0, M_0', M_4, M_6, a, r_0, α_0, α_1 *and* β.

PROOF. By (Φ–7) and Lemma 1.3, (Φ–1), (Φ–3) and (Φ–6) with $\lambda=k-1$ are valid for Φ, where M_1, M_3 and M_5 appearing in these inequalities depend only on M_0, M_0', M_6 and r_0. For $x \in E(0, \varepsilon) \cap B^{(n)}(0, r')\setminus\{0\}$ with $r'=r(E(0, \varepsilon), S)$, we write

$$\frac{d}{ds}V_\Phi^\sigma(x) = |s'|\frac{d}{dt_s}V_\Phi^\sigma(x) + |s^*|\frac{d}{dn_s}V_\Phi^\sigma(x),$$

where $t_s=|s'|^{-1}s'$, provided $s'\neq 0$; if $s'=0$ (resp. $s^*=0$), then we set $|s'|(d/dt_s)V_\Phi^\sigma=0$ (resp. $|s^*|(d/dn_s)V_\Phi^\sigma=0$). If $s'\neq 0$, then it follows from Theorem 3.1 that $(d/dt_s)V_\Phi^\sigma$ is β-Hölder continuous on $E(0, \varepsilon) \cap B^{(n)}(0, r')\setminus\{0\}$. If $s^*\neq 0$, then applying (ii) of Proposition 2.2 with Φ and $\lambda=k$ replaced by $d\Phi/dn_s$ and $\lambda=k-1$, respectively, we obtain the desired estimate for $|s^*|(d/dn_s)V_\Phi^\sigma$. Therefore the assertions of the theorem are valid.

COROLLARY 3.1. *Let* $k=n-1$ *and s be a unit vector. Assume that* $\Phi \in C^2(B^{(n)}(0, 4r_0)\setminus\{0\})$, *that it satisfies* ($\Phi$–4) *and* ($\Phi$–7) *with* $\lambda=n-2$ *and that* (Φ–5) *with* $\lambda=n-2$ *holds for* $D_n\Phi$ *in case* $s_n\neq 0$. *Let* β, ε *and* σ *be as in Theorem* 3.3. *Then* $(d/ds)V_\Phi^\sigma$ *is a* β*-Hölder continuous function on* $E_+(0, \varepsilon) \cap B^{(n)}(0, r')$ *for some* $r'>0$.

3.4. Applications to double layer potentials

For r_1 with $0<r_1<r_0$, suppose $S(0, r_1)$ is a C^1-surface and $\Phi \in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$. For every $y \in S(0, r_1)$, take a unit normal vector n_y to S at y such that each component of n_y is a Borel measurable function of y on $S(0, r_1)$ and

$$|n_y-n_0| \leqq C|y|^{\alpha_0} \tag{3.2}$$

for some $C>0$. For a signed measure σ on $S(0, r_1)$, we define $W_\Phi^\sigma(x)=\int_S (d/dn_y)\Phi(x-y)d\sigma(y)$ and call W_Φ^σ a double layer Φ-potential of σ. If $x \in B^{(n)}(0, r_0)\setminus S$, then

$$W_\Phi^\sigma(x) = \textstyle\sum_{i=1}^n V(D_i\Phi, \sigma_i)(x), \tag{3.3}$$

where $d\sigma_i(y)=-\langle n_y, e_i\rangle d\sigma(y)$ for $i=1,\dots, n$. Furthermore, by (3.2), it is easily seen that if σ satisfies (σ–1) with $A \in R$, then (σ–2) is valid for σ_i $(1\leqq i\leqq k)$ with $A=A_i=0$ and α_1 replaced by α_0, and (σ–1) is valid for σ_i $(k+1\leqq i\leqq n)$ with $A_i=-A\langle n_0, e_i\rangle$ in place of A; if σ satisfies (σ–2) with $A \in R$ and $\alpha_1>0$, then (σ–2) holds for σ_i $(1\leqq i\leqq n)$ with $A=A_i$ and α_1 replaced by $\min\{\alpha_0, \alpha_1\}$. Thus the following two propositions are consequences of Theorems 3.1, 3.2 and 3.3.

PROPOSITION 3.1. *Let $\lambda=k-1$ and a be a unit vector with $a^*\neq 0$. Assume that $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$ and it satisfies (Φ–4), (Φ–5) and (Φ–6) with $\lambda=k-1$. If a signed measure σ on $S(0, r_1)$ satisfies (σ–1), then*

$$\lim_{x\to 0, x\in L(0,a)} W_\Phi^\sigma(x)$$

exists.

PROPOSITION 3.2. *Let a and λ be as in Proposition 3.1. Assume that $\Phi\in C^2(B^{(n)}(0, 4r_0)\setminus\{0\})$ and it satisfies (Φ–4), (Φ–5) and (Φ–7) with $\lambda=k-1$. If (σ–2) is valid for a signed measure σ on $S(0, r_1)$, then W_Φ^σ is β-Hölder continuous on $L(0, a)\cap B^{(n)}(0, r')$ for some $r'>0$, where β is as in* (ii) *of Proposition* 2.2.

Also, we have

PROPOSITION 3.3. *Let $\lambda>k-1$ and a be as in Proposition 3.1. Assume that $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$ and it satisfies (Φ–4) and (Φ–6). If (σ–1) is valid for a signed measure σ on $S(0, r_1)$, then $|x|^{\lambda-k+1}W_\Phi^\sigma(x)$ converges to a finite value, as $x\to 0$, $x\in L(0, a)$.*

In fact, by virtue of (3.3), it suffices to prove that $|x|^{\lambda-k+1}V(D_i\Phi, \sigma_i)(x)$ converges to a finite value for $i=1,\dots, n$. If $k+1\leqq i\leqq n$, then the existence of the limit follows from (i) of Theorem 3.2′, since σ_i satisfies (σ–1) with $A=A_i$, as shown above. If $1\leqq i\leqq k$, then by Remark 3.2 we obtain

$$\lim_{x\to 0, x\in L(0,a)} |x|^{\lambda-k+1}V(D_i\Phi, \sigma_i)(x) = 0,$$

since, as is seen above, (σ–2) holds for σ_i with $A=0$ and α_1 replaced by $\min\{\alpha_0, \alpha_1\}$ and thus (σ–1) holds for σ_i with $A=0$. Hence the assertion is proved.

§4. Existence of derivatives on the surface

In this section we are concerned with differentiability of V_Φ^σ at 0 (cf. [9; Theorem 18] and [11; Theorem 2]). Note that the existence of normal derivatives of V_Φ^σ at 0 was already given in Theorem 3.2. As is easily seen (cf. [2; Satz 4]), if $\lambda<k-1$ and $|\sigma|(\{y; |y-x|\leqq r\})\leqq Cr^k$ for $|x-x^0|<r_0$ and $0\leqq r\leqq r_0$, then $D_iV_\Phi^\sigma(x)=\int D_i\Phi(x-y)d\sigma(y)$ for $|x-x^0|<r_0$. Thus we consider only the case $\lambda\geqq k-1$.

Throughout this section, we assume that S satisfies α_0-condition at 0, $\Phi\in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$ and it satisfies (Φ–6).

4.1. Lemmas

LEMMA 4.1. *Let $k-1 \leqq \lambda < k$, let $t \in T(0)$ and g_i $(i=k+1,\dots,n)$ be real valued Borel measurable functions defined on an open interval of R^1 containing zero such that*

$$|g_i(h)| \leqq M|h|^{1+\alpha} \quad \text{for some} \quad M > 0 \quad \text{and} \quad \alpha > 0. \tag{4.1}$$

Suppose that a signed measure σ on S satisfies (σ–2) with $A \in R$ and $\alpha_1 > 0$, and that there are numbers $L > 0$ and α_2, $0 < \alpha_2 \leqq 1$, such that

$$|\sigma|(B^{(n)}(x, \rho)) \leqq L\rho^{k-1+\alpha_2} \tag{σ–3$)'$}$$

for every $x \in S$ and $0 \leqq \rho \leqq r_0$. If $(k+\alpha_1-1)(k+\alpha_2-\lambda-1) > \lambda$, then $V(d\Phi/dt, \sigma - A\mu_S)(0)$ exists and

$$\lim_{h\to 0} h^{-1}\{V(\Phi, \sigma - A\mu_S)(x(h)) - V(\Phi, \sigma - A\mu_S)(0)\} = V\left(\frac{d\Phi}{dt}, \sigma - A\mu_S\right)(0),$$

where $x(h) = ht + g_{k+1}(h)e_{k+1} + \cdots + g_n(h)e_n$.

PROOF. For simplicity, let $\sigma_0 = \sigma - A\mu_S$. First we note

$$k + \alpha_1 > \lambda + 1, \tag{4.2}$$

since $k+\alpha_2-\lambda-1 \leqq \alpha_2 \leqq 1$ and $(k+\alpha_1-1)(k+\alpha_2-\lambda-1) > \lambda$. By ($\Phi$–6), $|(d/dt)\Phi(x)| \leqq C|x|^{-\lambda-1}$. Hence, by ($\sigma$–2) and (4.2) we have

$$\int_S \left|\frac{d\Phi}{dt}(-y)\right| d|\sigma_0|(y) < \infty.$$

Thus $V(d\Phi/dt, \sigma_0)(0)$ exists. Moreover, by using (Φ–3) and (4.1), we easily see that

$$\lim_{h\to 0} h^{-1}\int_{S\setminus S(0,r)} \{\Phi(x(h)-y) - \Phi(-y)\}d\sigma_0(y) = \int_{S\setminus S(0,r)} \frac{d\Phi}{dt}(-y)d\sigma_0(y)$$

for $0 < r < r_0$. Thus to obtain the assertion of the lemma, it is sufficient to show that

$$\lim_{r\downarrow 0} \limsup_{h\to 0} \left| h^{-1}\int_{S(0,r)} \{\Phi(x(h)-y) - \Phi(-y)\}d\sigma_0(y)\right| = 0.$$

To see this, take r, $0 < r \leqq r_0$, such that $4nMr^\alpha \leqq 1$, (4.1) is valid and $|x(h)| \leqq 2|h|$ for h with $|h| \leqq r/4$. Let $F_1 = \{y \in S(0, r); |ht - y| \leqq |h|/2\}$, $F_2 = S(0, r)\setminus F_1$ and put

$$I_i(h) = h^{-1}\int_{F_i} \{\Phi(x(h)-y) - \Phi(-y)\}d\sigma_0(y), \quad i = 1, 2,$$

for $|h| \leqq r/4$. Observe that for $y \in F_2$, $|y| \leqq 3|ht-y|$, so that by (4.1)

$$\begin{aligned}|x(h)-y| &\geqq |ht-y| - |x(h)-ht| \geqq |ht-y| - nM|h|^{1+\alpha} \\ &\geqq |ht-y| - 4^{-1}|h| \geqq 2^{-1}|ht-y| \geqq 6^{-1}|y|.\end{aligned}$$

Hence by (Φ–3) we have

$$|\{\Phi(x(h)-y) - \Phi(-y)\}/h| \leqq C|y|^{-\lambda-1}$$

for $y \in F_2$, since $|x(h)| \leqq 2|h|$. Thus by (σ–2)

$$\begin{aligned}|I_2(h)| &\leqq C\int_{F_2} |y|^{-\lambda-1}d|\sigma_0|(y) \leqq C\int_{S(0,r)} |y|^{-\lambda-1}d|\sigma_0|(y) \\ &\leqq Cr^{k+\alpha_1-\lambda-1},\end{aligned}$$

so that by (4.2)

$$\lim_{r\downarrow 0} \limsup_{h\to 0} |I_2(h)| = 0.$$

Next, we consider $I_1(h)$. Since S is represented by Lipschitz functions,

(4.3) $$|y-z| \leqq C|y'-z'| \qquad \text{for} \quad y,\ z \in S.$$

Let $x_h = \Psi(ht)$. If $y \in F_1$, then $|ht-y'| \leqq |h|/2 \leqq |y|$ and by (4.3) $|x_h-y| \leqq C|ht-y'| \leqq C|x(h)-y|$, so that

$$|\Phi(x(h)-y) - \Phi(-y)| \leqq C|x_h-y|^{-\lambda}.$$

Thus it is enough to show that

(4.4) $$\lim_{h\to 0} |h|^{-1}\int_{F_1} |x_h-y|^{-\lambda}d|\sigma_0|(y) = 0.$$

For this purpose, take $\beta > 0$ such that $(1+\beta)(k+\alpha_2-\lambda-1) > 1$ and $k+\alpha_1-\lambda-1 > \beta\lambda$. Then $(1+\beta)(k-\lambda) > 1$, since $\alpha_2 \leqq 1$ and thus $k-\lambda \geqq k+\alpha_2-\lambda-1$. Let $F_3 = \{y \in S(0, r);\ |x_h-y| \leqq |h|^{1+\beta}\}$. Then it follows from ($\sigma$–3)$'$ that

$$|h|^{-1}\int_{F_3} |x_h-y|^{-\lambda}d|\sigma|(y) \leqq C|h|^{(1+\beta)(k+\alpha_2-\lambda-1)-1} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.$$

Next, since $|ht-y'| \leqq |x_h-\Psi(y')|$, we have

$$\begin{aligned}|h|^{-1}\int_{F_3} |x_h-y|^{-\lambda}d\mu_S(y) &\leqq C|h|^{-1}\int_{|ht-y'|\leqq|h|^{1+\beta}} |ht-y'|^{-\lambda}dy' \\ &= C|h|^{(1+\beta)(k-\lambda)-1} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.\end{aligned}$$

Finally, by (σ–2) we have

$$|h|^{-1}\int_{F_1\setminus F_3}|x_h-y|^{-\lambda}d|\sigma_0|(y)\leqq C|h|^{k+\alpha_1-(1+\beta)\lambda-1}\longrightarrow 0 \quad \text{as} \quad h\longrightarrow 0.$$

Thus (4.4) holds, and the proof is complete.

Applying the lemma with $\sigma=\nu_S$ which satisfies (σ–2) with $A=0$ and $\alpha_1=\alpha_0$ and (σ–3)$'$ with $\alpha_2=1$, we obtain the following corollary.

COROLLARY 4.1. *Let $k-1\leqq\lambda<k$ and t, g_i ($i=k+1,\dots,n$) and $x(h)$ be as in Lemma 4.1. If $(k+\alpha_0-1)(k-\lambda)>\lambda$, then $V(d\Phi/dt,\nu_S)(0)$ exists and*

$$\lim_{h\to 0}h^{-1}\{V(\Phi,\nu_S)(x(h))-V(\Phi,\nu_S)(0)\}=V\left(\frac{d\Phi}{dt},\nu_S\right)(0).$$

LEMMA 4.2. *Let t, λ, g_i ($i=k+1,\dots,n$) and $x(h)$ be as in Lemma 4.1. If $\min\{\alpha,\alpha_0\}>(\lambda-k+1)/(k-\lambda)$, then*

$$\lim_{h\to 0}h^{-1}\int_{|y'|\leqq r_0}\{\Phi(x(h)-\Psi(y'))-\Phi(-\Psi(y'))-\Phi(ht-y')+\Phi(-y')\}dy'$$
$$=Y\left(0;\frac{d\Phi}{dt}\right).$$

PROOF. For simplicity, let $H(x,y)=\Phi(x-y)-\Phi(-y)$. As in the proof of Lemma 4.1, we see that

$$\lim_{h\to 0}h^{-1}\int_{r<|y'|\leqq r_0}\{H(x(h),\Psi(y'))-H(ht,y')\}dy'$$
$$=\int_{r<|y'|\leqq r_0}\left\{\frac{d\Phi}{dt}(-\Psi(y'))-\frac{d\Phi}{dt}(-y')\right\}dy'.$$

Thus it suffices to prove that

$$\lim_{r\downarrow 0}\limsup_{h\to 0}\left|h^{-1}\int_{|y'|\leqq r}\{H(x(h),\Psi(y'))-H(ht,y')\}dy'\right|=0.$$

Choose r, $0<r<r_0$, such that $|\Psi(y')-y'|\leqq 4^{-1}|y'|$ on $B^{(k)}(0,r)$, (4.1) is valid and $|x(h)|\leqq 2|h|$ for h with $|h|\leqq r/4$, and choose M large enough so that (S–1) with $K_2=M$ is valid. Let $\beta=\min\{\alpha,\alpha_0\}$ and $0<|h|\leqq r/4$. Set

$$I(h)=|h|^{-1}\int_{|y'|\leqq r}|H(x(h),\Psi(y'))-H(ht,y')|dy',$$

$$I_1(h)=|h|^{-1}\int_{|y'|\leqq 3|h|}|\Phi(-\Psi(y'))-\Phi(-y')|dy',$$

$$I_2(h)=|h|^{-1}\int_{|ht-y'|\leqq 3|h|}|\Phi(x(h)-\Psi(y'))-\Phi(ht-y')|dy',$$

$$I_3(h) = |h|^{-1} \int_F |\Phi(x(h) - \Psi(y')) - \Phi(ht - \Psi(y'))| dy'$$

and

$$I_4(h) = |h|^{-1} \int_F |H(ht, \Psi(y')) - H(ht, y')| dy',$$

where $F = \{y';\ 2|h| \leqq |y'| \leqq r,\ 2|h| \leqq |ht - y'|\}$. Then $I(h) \leqq 2I_1(h) + 2I_2(h) + I_3(h) + I_4(h)$. By ($S$–1) and ($\Phi$–3), the integrand of I_1 is dominated by $C|y'|^{\alpha_0-\lambda}$, so that

$$I_1(h) \leqq C|h|^{-1} \int_0^{3|h|} \rho^{k+\alpha_0-\lambda-1} d\rho = C|h|^{k+\alpha_0-\lambda-1}.$$

Next, using (Φ–3), we have

$$\begin{aligned} I_2(h) &\leqq C|h|^{-1} \int_{|ht-y'| \leqq 3|h|} |ht - y'|^{-\lambda} |x(h) - \Psi(y')|^{-1} \\ &\qquad \times \{\textstyle\sum_{i=k+1}^n (g_i(h) - \psi_i(y'))^2\}^{1/2} dy' \\ &\leqq C|h|^{\beta} \int_0^{3|h|} \rho^{k-\lambda-1} \{\rho^2 + (2nM|h|^{1+\beta})^2\}^{-1/2} d\rho \\ &\leqq C|h|^{\beta(k-\lambda)-(\lambda-k+1)} \int_0^{3(2nM|h|^{\beta})^{-1}} \rho^{k-\lambda-1} (1+\rho)^{-1} d\rho \\ &\leqq C|h|^{\beta(k-\lambda)-(\lambda-k+1)} \log (r/|h|), \end{aligned}$$

because $\{\sum_{i=k+1}^n (g_i(h) - \psi_i(y'))^2\}^{1/2} \leqq \sum_{i=k+1}^n (|g_i(h)| + |\psi_i(y')|) \leqq 2nM|h|^{1+\beta}$ on $B^{(k)}(0, 4|h|)$ by (S–1) and (4.1), and thus by the monotonicity of $t \to t(A^2 + t^2)^{-1/2}$,

$$\begin{aligned} &|x(h) - \Psi(y')|^{-1} \{\textstyle\sum_{i=k+1}^n (g_i(h) - \psi_i(y'))^2\}^{1/2} \\ &\quad \leqq 2nM|h|^{1+\beta} \{|ht - y'|^2 + (2nM|h|^{1+\beta})^2\}^{-1/2}. \end{aligned}$$

By (Φ–3) and (4.1), the integrand of I_3 is dominated by $C|h|^{1+\alpha}|ht - y'|^{-\lambda-1}$, so that

$$\begin{aligned} I_3(h) &\leqq C|h|^{\alpha} \int_F |ht - y'|^{-\lambda-1} dy' \leqq C|h|^{\alpha} \int_{2|h|}^{r} \rho^{k-\lambda-2} d\rho \\ &\leqq C|h|^{k+\alpha-\lambda-1} \log (r/|h|). \end{aligned}$$

Finally, applying the mean value theorem, by (S–1) and (Φ–6), we see that the integrand of I_4 is dominated by $C|h|\,|y'|^{\alpha_0-\lambda-1}$ on F, since $2^{-1}|y'| \leqq |ht - y'| \leqq 2|y'|$ on F. Thus

$$I_4(h) \leqq C \int_{|y'| \leqq r} |y'|^{\alpha_0-\lambda-1} dy' = Cr^{k+\alpha_0-\lambda-1}.$$

Since $\min\{\alpha, \alpha_0\} > \lambda - k + 1$ by assumption, the lemma is proved.

4.2. Existence of derivatives at the origin

In [9; Theorem 15] Ohtsuka proved that if f satisfies a Hölder condition at 0, then the tangential derivative $(d/dt)V_1^f$ of a single layer Newtonian potential V_1^f exists. In this connection the following problem is raised by him ([9; p. 56]): In R^3, let S be a 2-dimensional C^1-surface which satisfies α_0-condition at 0 ($\in S$) and let $t \in T(0)$. If the origin is a Lebesgue point of order $\alpha_1 > 0$ of $\sigma = f\mu_S$, i.e.,

$$\int_{S(0,r)} |f(y) - A| d\mu_S(y) = O(r^{2+\alpha_1}) \quad \text{as} \quad r \downarrow 0$$

with some $A \in R$, then does the tangential derivative $(d/dt)V_1^f$ exist at 0?

First we give a negative answer to the problem and next a condition under which the assertion holds.

EXAMPLE 4.1. Let $S = B^{(k)}(0, 1)$ and $\alpha_1 > 0$. Put $r_i = 2^{-i}$ and $\delta_i = 2^{-(k+\alpha_1)i/(k-1)}$ $(i = k, k+1, \dots)$. Let f be a function on S defined as follows:

$$f(x') = \begin{cases} |x' - r_i e_1|^{-1}, & \text{if} \quad |x' - r_i e_1| \leqq \delta_i \ (i = k,\ k+1, \dots), \\ 0, & \text{otherwise.} \end{cases}$$

Then for $i = k, k+1, \dots$, $V_{k-1}^f(r_i e_1) = \infty$ and

$$\int_{S(0,r)} f(y')dy' \leqq Cr^{k+\alpha_1} \qquad \text{for all} \quad r,\ 0 < r < 2^{-k}.$$

In fact, if $2^{-i} \leqq r < 2^{-i+1}$, then

$$\int_{S(0,r)} f(y')dy' \leqq \int_{S(0,2^{-i+1})} f(y')dy' \leqq C \textstyle\sum_{j=i-1}^{\infty} \delta_j^{k-1} \leqq Cr^{k+\alpha_1}.$$

THEOREM 4.1. *Let t, λ, σ, g_i $(i = k+1, \dots, n)$ and $x(h)$ be as in Lemma* 4.1. *If* $\min\{\alpha, \alpha_0\} > (\lambda - k + 1)/(k - \lambda)$ *and* $(k + \alpha_1 - 1)(k + \alpha_2 - \lambda - 1) > \lambda$, *then*

$$\begin{aligned} &\lim_{h\to 0} \{V_\Phi^\sigma(x(h)) - V_\Phi^\sigma(0)\}/h \\ &= V\Big(\frac{d\Phi}{dt}, \sigma - A\mu_S\Big)(0) + A\Big\{V\Big(\frac{d\Phi}{dt}, \nu_S\Big)(0) + Y\Big(0; \frac{d\Phi}{dt}\Big)\Big\} \\ &\quad - A\int_{|y'|=r_0} \Phi(-y')\langle \nu(y'), t\rangle dm_{k-1}(y'). \end{aligned}$$

PROOF. Writing

$$\begin{aligned} &\{V_\Phi^\sigma(x(h)) - V_\Phi^\sigma(0)\}/h \\ &= h^{-1}\{V(\Phi, \sigma - A\mu_S)(x(h)) - V(\Phi, \sigma - A\mu_S)(0)\} \\ &\quad + Ah^{-1}\{V(\Phi, \nu_S)(x(h)) - V(\Phi, \nu_S)(0)\} \end{aligned}$$

$$+ Ah^{-1}\int_{|y'|\leqq r_0}\{\Phi(x(h)-\Psi(y'))-\Phi(-\Psi(y'))-\Phi(ht-y')+\Phi(-y')\}dy'$$
$$+ Ah^{-1}\{\tilde{V}(ht;\Phi)-\tilde{V}(0;\Phi)\},$$

we see that the assertion of the theorem follows from Lemmas 4.1, 4.2 and 1.4 and Corollary 4.1.

In this theorem let all $g_i \equiv 0$. Then, by the aid of Theorem 3.1, we obtain

COROLLARY 4.2. *Let* $0<\varepsilon\leqq 1$, λ *and* σ *be as in Lemma* 4.1. *If* $\alpha_0>(\lambda-k+1)/(k-\lambda)$ *and* $(k+\alpha_1-1)(k+\alpha_2-\lambda-1)>\lambda$, *then the partial derivatives* $D_iV_\Phi^\sigma$ $(i=1,\dots,k)$ *exist at* 0 *and*

$$D_iV_\Phi^\sigma(0) = \lim_{x\to 0, x\in E(0,\varepsilon)} D_iV_\Phi^\sigma(x).$$

Moreover,

$$\frac{d}{dt}V_\Phi^\sigma(0)=\lim_{x\to 0, x\in E(0,\varepsilon)}\frac{d}{dt}V_\Phi^\sigma(x)$$

and

$$\frac{d}{dt}V_\Phi^\sigma(0)=\sum_{i=1}^k t_iD_iV_\Phi^\sigma(0)$$

for any $t\in T(0)$.

REMARK 4.1. Assume that a Borel measurable function f on S satisfies

$$|f(x)-A|\leqq C|x|^{\alpha_1}, \quad \text{whenever} \quad x\in S$$

for some $A\in R$, $C>0$ and $\alpha_1>0$. Then $\sigma=f\mu_S$ satisfies $(\sigma$–$3)'$ with $\alpha_2=1$. Thus, in case $\lambda=k-1$, [9; Theorem 15] is a special case of this Corollary 4.2, since the assumptions on α_0 and α_1 are nothing but $\alpha_0>0$ and $\alpha_1>0$.

For a Lebesgue measurable function f on R^n such that $\int|x-y|^{-\lambda}|f(y)|dy\not\equiv\infty$, we define a domain Φ-potential of f by $\int\Phi(x-y)f(y)dy$ and denote it by $U_\Phi^f(x)$. Since domain Φ-potentials can be considered as the restrictions of single layer Φ-potentials in R^{n+1}, we obtain

COROLLARY 4.3. *Let f be a Lebesgue measurable function on* R^n *such that* $f=0$ *outside* $B^{(n)}(0,r_0)$. *Assume that for a point* x^0 *with* $|x^0|<r_0$, *there are numbers* A, $C>0$ *and* $\alpha_1>0$ *such that*

$$|f(x)-A|\leqq C|x-x^0|^{\alpha_1}$$

for every x. *If* $n+\alpha_1>\lambda+1$, *then the partial derivatives* $D_iU_\Phi^f$, $i=1,\dots,n$, *exist at* x^0 *and*

$$D_i U_\Phi^f(x^0) = \int_{B^{(n)}(0,r_0)\setminus\Omega} D_i\Phi(x^0-y)f(y)dy$$
$$+ \int_\Omega D_i\Phi(x^0-y)\{f(y)-A\}dy$$
$$- A\int_{\partial\Omega} \Phi(x^0-y)\langle\nu(y),\ e_i\rangle dm_{n-1}(y)$$

for every domain Ω with C^1-boundary $\partial\Omega$ such that $x^0\in\Omega\subset B^{(n)}(0, r_0)$, where $\nu(y)$ denotes the unit outer normal to $\partial\Omega$ at y.

Next, we consider the existence of directional derivatives at 0. Since Theorem 3.2′ shows that normal derivatives of single layer Φ-potentials do no exist in general in case $\lambda > k-1$, we consider only the case $\lambda = k-1$.

THEOREM 4.2 (*cf.* [11; *Theorem* 2]). *Let $\lambda = k-1$ and s be a unit vector. Assume that Φ satisfies (Φ–4) and (Φ–5), and a signed measure σ on S satisfies (σ–2) with $\alpha_1>0$ and (σ–3)′ with $\alpha_2>0$. If $\alpha_2(k+\alpha_1-1)>k-1$, then the derivative $(d/ds)V_\Phi^\sigma$ in the direction s exists at 0 and*

$$\frac{d}{ds}V_\Phi^\sigma(0) = |s'|\frac{d}{dt_s}V_\Phi^\sigma(0) + |s^*|\frac{d}{dn_s}V_\Phi^\sigma(0)$$
$$= \textstyle\sum_{i=1}^k s_i D_i V_\Phi^\sigma(0) + |s^*|\frac{d}{dn_s}V_\Phi^\sigma(0),$$

where t_s and n_s are as in 3.3.

PROOF. If $s^*=0$, then the assertion is obtained in Corollary 4.2. If $s'=0$, then the existence of the normal derivative is proved in Theorem 3.2. Thus in the sequel we assume that $s'\neq 0$ and $s^*\neq 0$. Then for $h>0$, we write

$$I(h) = \{V_\Phi^\sigma(hs) - V_\Phi^\sigma(hs^*)\}/h + \{V_\Phi^\sigma(hs^*) - V_\Phi^\sigma(0)\}/h.$$

By the mean value theorem we find a point $x_1(h)$ on the segment between hs and hs^* and a point $x_2(h)$ on the segment between hs^* and the origin such that

$$I(h) = |s'|\frac{d}{dt_s}V_\Phi^\sigma(x_1(h)) + |s^*|\frac{d}{dn_s}V_\Phi^\sigma(x_2(h)).$$

Since $x_1(h)\in E(0, |s^*|)$, we have by Corollary 4.2

$$\lim_{h\downarrow 0}\frac{d}{dt_s}V_\Phi^\sigma(x_1(h)) = \frac{d}{dt_s}V_\Phi^\sigma(0).$$

Since $x_2(h)\in L(0, n_s)$, by Theorem 3.2 we obtain

$$\lim_{h\downarrow 0}\frac{d}{dn_s}V_\Phi^\sigma(x_2(h)) = \frac{d}{dn_s}V_\Phi^\sigma(0).$$

Thus the proof is complete.

4.3. Counter examples

We here show that in case $k-1<\lambda<k$ and $0<\alpha_0<(\lambda-k+1)/(k-\lambda)$ or in case $k-1<\lambda$ and $0<\alpha_1 \leqq \lambda-k+1$ the partial derivative $D_1 V_\lambda^f$ does not exist in general even if $\sigma=f\mu_S$ satisfies (σ–2) and (σ–3)$'$.

EXAMPLE 4.2. Let $k-1<\lambda<k$ and $\lambda-k+1<\alpha_0<(\lambda-k+1)/(k-\lambda)$. Let $S=\{x;\, x_{k+1}=|x_1|^{1+\alpha_0}, x_{k+2}=\cdots=x_n=0, |x'|\leqq 1\}$. Then $V_\lambda(x)=\int_S |x-y|^{-\lambda} d\mu_S(y)$ is not differentiable with respect to x_1 at 0.

To see this, we write

$$\{V_\lambda(he_1) - V_\lambda(0)\}/h = h^{-1}\int_{S\setminus S(0,r)} (|he_1-y|^{-\lambda} - |y|^{-\lambda}) d\mu_S(y)$$

$$+ h^{-1}\int_{|y'|\leqq r} (|he_1-\Psi(y')|^{-\lambda} - |\Psi(y')|^{-\lambda})\{J_k\Psi(y') - 1\} dy'$$

$$+ h^{-1}\int_{|y'|\leqq r} (|he_1-\Psi(y')|^{-\lambda} - |\Psi(y')|^{-\lambda}) dy'$$

for $0<4h<r$, where $\Psi(y')=(y_1,\ldots, y_k, |y_1|^{1+\alpha_0}, 0,\ldots, 0)$. Denote the terms on the right by $I_1(h)$, $I_2(h)$ and $I_3(h)$, respectively. Clearly,

$$\lim_{h\downarrow 0} I_1(h) = -\int_{S\setminus S(0,r)} \frac{\partial}{\partial y_1}(|y|^{-\lambda}) d\mu_S(y).$$

As in the proof of Lemma 4.1, we can show

$$\limsup_{h\downarrow 0} |I_2(h)| \leqq Cr^{k+\alpha_0-\lambda-1}.$$

To evaluate I_3, we write it as follows:

$$I_3(h) = h^{-1}\int_{|y'|\leqq r} (|he_1-y'|^{-\lambda} - |y'|^{-\lambda}) dy'$$

$$+ h^{-1}\int_{F_1} (|y'|^{-\lambda} - |\Psi(y')|^{-\lambda}) dy'$$

$$+ h^{-1}\int_{F_1} (|he_1-\Psi(y')|^{-\lambda} - |he_1-y'|^{-\lambda}) dy'$$

$$+ h^{-1}\int_{F_2} \{|he_1-\Psi(y')|^{-\lambda} - |\Psi(y')|^{-\lambda} - |he_1-y'|^{-\lambda} + |y'|^{-\lambda}\} dy'$$

$$= J(h) + J_1(h) + J_2(h) + J_3(h),$$

where $F_1=\{y';\, |y'-he_1|\leqq 2h \text{ or } |y'|\leqq 2h\}$ and $F_2=S(0,r)\setminus F_1$. Applying Lemma 1.4 with $\Phi(x)=|x|^{-\lambda}$, $n=k$ and $i=1$, we obtain

$$\lim_{h\downarrow 0} J(h) = -\int_{|y'|=r} |y'|^{-\lambda}\langle\nu(y'), e_1\rangle dm_{k-1}(y') = 0.$$

Since the integrand of J_1 is non-negative and dominated by $C|y'|^{\alpha_0-\lambda}$, we have

$$0 \leqq J_1(h) \leqq Ch^{-1}\int_{|y'|\leqq 3h} |y'|^{\alpha_0-\lambda}dy' = Ch^{k+\alpha_0-\lambda-1},$$

so that $\lim_{h\downarrow 0} J_4(h)=0$, since $k+\alpha_0>\lambda+1$. As in the proof of Lemma 4.2, we see that $\limsup_{h\downarrow 0} |J_3(h)| \leqq Cr^{k+\alpha_0-\lambda-1}$.

We now show that $\lim_{h\downarrow 0} J_2(h)=-\infty$. Since the integrand is non-positive, by changing variables, we obtain

$$\begin{aligned}
-J_2(h) &\geqq h^{-1}\int_{|he_1-y'|\leqq h/2} (|he_1-y'|^{-\lambda} - |he_1-\Psi(y')|^{-\lambda})dy' \\
&\geqq h^{-1}\int_{|u|\leqq h/2} [|u|^{-\lambda} - \{|u|^2 + (h^2/4)^{1+\alpha_0}\}^{-\lambda/2}]du \\
&= Ch^{-1}\int_0^{h/2} \rho^{k-1}[\rho^{-\lambda} - \{\rho^2 + (h^2/4)^{1+\alpha_0}\}^{-\lambda/2}]d\rho \\
&\geqq Ch^{\alpha_0(k-\lambda)-(\lambda-k+1)}.
\end{aligned}$$

Thus $\lim_{h\downarrow 0} J_2(h)=-\infty$, because $\alpha_0(k-\lambda)<(\lambda-k+1)$, and hence

$$\lim_{h\downarrow 0} \{V_\lambda(he_1) - V_\lambda(0)\}/h = -\infty,$$

which implies that V_λ is not differentiable with respect to x_1 at 0.

In case $k-1<\lambda<k$ and $0<\alpha_0\leqq\lambda-k+1$, let S and f be as in Example 3.1. As in the proof of that example it can be easily seen that

$$\lim_{h\downarrow 0} D_1V_\lambda^f(-he_1) = \infty.$$

Hence V_λ^f is not differentiable with respect to x_1 at 0.

EXAMPLE 4.3. Let $k-1<\lambda$ and $0<\alpha_1\leqq\lambda-k+1$. Let $S=B^{(k)}(0, 1)$. Consider a non-negative continuous function f on S such that it is equal to $|x'|^{\alpha_1}$ in $F=\{x';\ 0\leqq x_1\leqq 1/2,\ x_2^2+\cdots+x_k^2\leqq x_1^2\}$, equal to zero if $x_1<0$ and $f(x')\leqq|x'|^{\alpha_1}$ everywhere. Then $D_1V_\lambda^f$ does not exist at 0.

In fact, we show that

$$\lim_{h\downarrow 0} D_1V_\lambda^f(-he_1) = \infty.$$

Since f is non-negative, we have

$$\begin{aligned}
D_1V_\lambda^f(-he_1) &= \lambda\int_S (h+y_1)|he_1+y'|^{-\lambda-2}f(y')dy' \\
&\geqq \lambda\int_F y_1|y'|^{\alpha_1}|he_1+y'|^{-\lambda-2}dy'
\end{aligned}$$

for $h>0$. Thus

$$\liminf_{h\downarrow 0} D_1 V_\lambda^f(-he_1) \geqq C\int_F |y'|^{\alpha_1-\lambda-1}dy' = \infty,$$

since $k+\alpha_1 \leqq \lambda+1$ and $|y'|\leqq 2y_1$ on F.

§5. Hölder continuity of derivatives on the surface

In this section we study the Hölder continuity of derivatives of Φ-potentials on the surface S when $\lambda=k-1$ under the condition that S is a C^1-surface satisfying uniform α_0-condition (cf. [6; Chap. II, §7] and [11; Theorem 3]).

5.1. Surface with uniform α_0-condition

In what follows we assume that S is a C^1-surface, i.e., $\psi_i \in C^1(B^{(k)}(0, r_0))$ $(i=k+1,\dots, n)$. For any $x\in S$, let $T(x)$ (resp. $N(x)$) be the set of unit tangent (resp. normal) vectors to S at x. For each $x\in S$, applying Gram-Schmidt orthogonalization process to the vectors

$$\left(1, 0,\dots, 0, \frac{\partial\psi_{k+1}}{\partial x_1}(x'),\dots, \frac{\partial\psi_n}{\partial x_1}(x')\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\left(0,\dots, 0, 1, \frac{\partial\psi_{k+1}}{\partial x_k}(x'),\dots, \frac{\partial\psi_n}{\partial x_k}(x')\right),$$
$$\left(-\frac{\partial\psi_{k+1}}{\partial x_1}(x'),\dots, -\frac{\partial\psi_{k+1}}{\partial x_k}(x'), 1, 0,\dots, 0\right),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\left(-\frac{\partial\psi_n}{\partial x_1}(x'),\dots, -\frac{\partial\psi_n}{\partial x_k}(x'), 0,\dots, 0, 1\right),$$

we obtain an orthonormal system $\{s_1(x),\dots, s_n(x)\}$ such that $s_1(x),\dots, s_k(x)\in T(x)$ and $s_{k+1}(x),\dots, s_n(x)\in N(x)$. Note that $s_i(0)=e_i$, $i=1,\dots, n$, and each s_i is continuous on S. We denote by $A(x)$ the orthogonal matrix such that $s_i(x)=e_iA(x)$ for $i=1,\dots, n$. Let $0<r_2<r_0$. For each $x\in S(0, r_2)$, there exists an open neighborhood V_x of x such that $S\cap C\ell(V_x)$ is expressed by C^1-functions $\psi_{k+1}(\xi'; x),\dots, \psi_n(\xi'; x)$ with tangent-normal system of coordinates ξ, that is,

$$S \cap C\ell(V_x) = \{x+\Psi(\xi'; x)A(x);\ |\xi'| \leqq r_3\},$$

where $\Psi(\xi'; x)=(\xi', \psi_{k+1}(\xi'; x),\dots, \psi_n(\xi'; x))$ and r_3 is a positive number independent of $x\in S(0, r_2)$. As in §1,

$$d\mu_S(x+\Psi(\xi'; x)A(x)) = J_k\Psi(\xi'; x)d\xi'$$

for $|\xi'|\leqq r_3$ with a continuous function $J_k\Psi(\xi'; x)$ $(\geqq 1)$.

For $0<\alpha_0\leqq 1$, we say that S satisfies α_0-condition uniformly on $S(0, r_2)$, if there is a positive number K such that

$$\sum_{i,j}\left(\frac{\partial\psi_j}{\partial\xi_i}(\xi';\, x)\right)^2 \leqq K|\xi'|^{2\alpha_0}$$

for every $x\in S(0, r_2)$ and $|\xi'|\leqq r_3$. Then as in §1 we can find $K_4>0$ such that

$$\text{(S–3)} \qquad |\psi_i(\xi';\, x)| \leqq K_4|\xi'|^{1+\alpha_0}$$

and

$$0 \leqq J_k\Psi(\xi';\, x) - 1 \leqq K_4|\xi'|^{\alpha_0}$$

for every $x\in S(0, r_2)$, $|\xi'|\leqq r_3$ and $i=k+1,\dots, n$. We note that in case $k=n-1$, $S(0, r_2)$ is a Liapunov surface with a Liapunov function $\varepsilon(t)=Ct^{\alpha_0}$, if S satisfies α_0-condition uniformly (see [6; Chap. I, §1] and [12; p. 18]).

We denote by $S(x, \rho)$ the set $\{x+\Psi(\xi';\, x)A(x);\, |\xi'|\leqq\rho\}$ for $0\leqq\rho\leqq r_3$ and $x\in S(0, r_2)$.

LEMMA 5.1. *Let $0<\alpha_0\leqq 1$ and $0<r_2<r_0$. If S satisfies α_0-condition uniformly on $S(0, r_2)$, then there is a positive number C depending only on K_4 such that for every x, $\tilde{x}\in S(0, r_2)$,*

$$|\langle s_i(x),\, s_j(\tilde{x})\rangle| \leqq C|x-\tilde{x}|^{\alpha_0},\ i\neq j,$$

$$1 - \langle s_i(x),\, s_i(\tilde{x})\rangle \leqq C|x-\tilde{x}|^{2\alpha_0}$$

and so

$$|s_i(x) - s_i(\tilde{x})| \leqq C|x - \tilde{x}|^{\alpha_0}.$$

This lemma is easily obtained from the construction of $s_i(x)$'s.

For $x\in S(0, r_2)$ and $z\in R^n$, let $z^*(x)=(zA(x)^{-1})^*$, and for $0<\varepsilon\leqq 1$ and $0<r\leqq r_3$, let $E(x, r, \varepsilon)=B^{(n)}(x, r)\cap\{y;\, |(y-x)^*(x)|\geqq\varepsilon|y-x|\}$.

LEMMA 5.2. *Let $0<r_2<r_0$ and $0<\varepsilon\leqq 1$. Assume that S satisfies α_0-condition uniformly on $S(0, r_2)$. Then there are positive numbers C and r depending only on K_4, r_3, α_0 and ε such that*

$$\text{(5.1)} \qquad |z-x| + |x-y| \leqq C|z-y|$$

for every $x\in S(0, r_2)$, $y\in S$ and $z\in E(x, r, \varepsilon)$.

By virtue of Lemma 1.2 and the uniform α_0-condition, the assertion holds.

5.2. A remark on Hölder continuity

LEMMA 5.3. *Let $0<\alpha\leqq 1$, S be a k-dimensional Lipschitz surface as in §1 and f be a Borel measurable function on S. Then the following statements are mutually equivalent:*

(i) *The function f is α-Hölder continuous on $S \cap B^{(n)}(0, r)$ for some r,* $0<r \leqq r_0$.

(ii) *There are positive numbers ρ_1, ρ_2 and C such that*

$$\int_{S \cap B^{(n)}(x,\rho)} |f(y)-f(x)| d\mu_S(y) \leqq C\rho^{k+\alpha}$$

for every $x \in S \cap B^{(n)}(0, \rho_1)$ and $0 \leqq \rho \leqq \rho_2$.

PROOF. Since ψ_i's representing S are Lipschitz functions, there are positive numbers C ($\geqq 1$) and ρ_0 such that

$$C^{-1}\rho^k \leqq \mu_S(B^{(n)}(x, \rho)) \leqq C\rho^k \tag{5.2}$$

for every $x \in S \cap B^{(n)}(0, \rho_0)$ and $0 \leqq \rho \leqq \rho_0$, Thus it can be easily seen that (i) implies (ii). Suppose (ii) is valid. Let x, $\tilde{x} \in S \cap B^{(n)}(0, \rho_1)$ with $|x-\tilde{x}| \leqq \rho_2/2$. Then by (5.2) we obtain

$$\begin{aligned} &C^{-1}|f(x)-f(\tilde{x})|\,|x-\tilde{x}|^k \\ &\leqq |f(x)-f(\tilde{x})|\mu_S(B^{(n)}(x,|\,x-\tilde{x}|)) \\ &\leqq |(f-f(x))\mu_S|(B^{(n)}(x,\,|x-\tilde{x}|)) \\ &\quad + |(f-f(\tilde{x}))\mu_S|(B^{(n)}(\tilde{x},\,2|x-\tilde{x}|)) \leqq C|x-\tilde{x}|^{k+\alpha}. \end{aligned}$$

Thus (ii) implies (i).

REMARK 5.1. Let S be as in Lemma 5.3 and f, g be Borel measurable functions on S. Let $0<r<r_0$ and $0<\alpha \leqq 1$. If

$$\int_{S \cap B^{(n)}(x,\rho)} |f(y)-g(x)| d\mu_S(y) \leqq C\rho^{k+\alpha}$$

for all $x \in S \cap B^{(n)}(0, r)$ and $\rho \geqq 0$, then it follows from [5; Chap. II, Theorem 2.9.7] that $f=g$ μ_S-a.e. on $S \cap B^{(n)}(0, r)$, because, as in the proof of Lemma 5.3, μ_S satisfies the diametric regularity condition (see [5; Chap. II, 2.8.8]). Thus, by the above lemma, g is α-Hölder continuous on $S \cap B^{(n)}(0, r)$, so that we may assume that f is α-Hölder continuous there, when we consider the single layer Φ-potential of f.

5.3. Boundedness of derivatives

In the rest of this section, we assume that S satisfies α_0-condition uniformly on $S(0, r_2)$ for $0<r_2<r_0$.

LEMMA 5.4. *Let $\lambda=k-1>0$ and $0<\varepsilon \leqq 1$. Assume that $\Phi \in C^1(B^{(n)}(0, 4r_0)\setminus\{0\})$ and it satisfies (Φ–4) and (Φ–6) with $\lambda=k-1$ and*

$$(\Phi\text{–}8) \qquad \left|\frac{\partial}{\partial \xi_i}\Phi(\xi A(x))\right| \leqq M_7|\xi^*|\,|\xi|^{-k-1},\ i=k+1,\dots,n,$$

for every $x\in S(0, r_2)$ *and* $\xi\in R^n$ *with* $0<|\xi|\leqq r_3$. *If* f *is* α_1*-Hölder continuous on* S, *then there exists a positive number* C *depending only on* K_4, M_5, M_7, r_3, α_0, α_1, ε, $\max_S|f|$ *and the Hölder constant of* f *such that*

$$\left|\frac{\partial}{\partial x_j}V^f_\Phi(z)\right| \leqq C$$

for all $z\in\bigcup_{x\in S(0,r_2)}E(x, r, \varepsilon)\backslash S(0, r_2)$ *and* $j=1,\dots, n$, *where* r *is the number given by Lemma* 5.2.

PROOF. For simplicity, let $E_x=E(x, r, \varepsilon)\backslash\{x\}$ *for* $x\in S(0, r_2)$ and put $D=\bigcup_{x\in S(0,r_2)}E_x$. Since $D_jV^f_\Phi(z)=\sum_{i=1}^n(d/ds_i)V^f_\Phi(z)\langle e_j, s_i\rangle$ for $z\in E_x$, where $s_i=s_i(x)$, it is sufficient to prove that $\sup_{z\in D}|(d/ds_i)V^f_\Phi(z)|<\infty$. Let $\tilde\Phi(\xi)=\Phi(\xi A(x))$. Then $\tilde\Phi$ also satisfies $(\Phi\text{–}6)$.

If $z\in E_x$, $x\in S(0, r_2)$, then we write

$$\begin{aligned}\frac{d}{ds_i}V^f_\Phi(z) &= \int_{S\backslash S(x,r_3)}\frac{d\Phi}{ds_i}(z-y)f(y)\,d\mu_S(y)\\
&\quad+\int_{S(x,r_3)}\frac{d\Phi}{ds_i}(z-y)\{f(y)-f(x)\}\,d\mu_S(y)\\
&\quad+f(x)\int_{|\eta'|\leqq r_3}\frac{\partial\tilde\Phi}{\partial\xi_i}(\xi-\Psi(\eta';x))\{J_k\Psi(\eta';x)-1\}\,d\eta'\\
&\quad+f(x)\int_{|\eta'|\leqq r_3}\left\{\frac{\partial\tilde\Phi}{\partial\xi_i}(\xi-\Psi(\eta';x))-\frac{\partial\tilde\Phi}{\partial\xi_i}(\xi-\eta')\right\}d\eta'\\
&\quad+f(x)\int_{|\eta'|\leqq r_3}\frac{\partial\tilde\Phi}{\partial\xi_i}(\xi-\eta')\,d\eta',\end{aligned}$$

where $z-x=\xi A(x)$. It is clear that the first term on the right is bounded on $\tilde D=\{(x, z);\ x\in S(0, r_2)$ and $z\in E_x\}$ with a bound depending only on M_5 and ε. By using $(\Phi\text{–}6)$ and (5.1), we see that the absolute value of the second term is majorized by a constant which depends only on K_4, M_5, r_3, α_0, α_1, ε and the Hölder constant of f. Similarly, the third and the fourth terms are bounded on $\tilde D$ with a bound depending only on K_4, M_5, r_3, α_0, ε and $\max_S|f|$. If $1\leqq i\leqq k$, then the last term on the right is equal to

$$-f(x)\int_{|\eta'|=r_3}\tilde\Phi(\xi-\eta')\langle\nu(\eta'), e_i\rangle\,dm_{k-1}(\eta'),$$

whose absolute value is dominated by a constant depending only on M_5, r_3, ε and $\max_S|f|$. If $k+1\leqq i\leqq n$, then by (i) of Lemma 2.2 with Φ replaced by $(\partial/\partial\xi_i)\tilde\Phi$, the absolute value of the last term on the right is majorized by a positive number depending only on M_7, r_3, ε and $\max_S |f|$. Thus the assertion of the lemma is obtained.

From Theorems 3.2 and 4.2, Corollary 4.2 and this lemma we derive the following corollary.

COROLLARY 5.1. *Under the same assumptions as in the lemma, there exists a positive number C depending only on K_4, M_5, M_7, r_3, α_0, α_1, $\max_S |f|$ and the Hölder constant of f such that*

$$\left|\frac{d}{ds}V_\Phi^f(x)\right| \leqq C$$

for all unit vectors s and $x \in S(0, r_2)$.

5.4. Hölder continuity on the surface

THEOREM 5.1 (cf. [9; *Theorem* 20]). *Let $\lambda = k-1$ and s be a unit vector. Assume that $\Phi \in C^2(B^{(n)}(0, 4r_0)\backslash\{0\})$ and it satisfies (Φ–4), (Φ–7) with $\lambda = k-1$ and (Φ–8). If a function f is α_1-Hölder continuous on S and* $\min\{\alpha_0, \alpha_1\} < 1$, *then the derivative $(d/ds)V_\Phi^f$ in the direction s is* $\min\{\alpha_0, \alpha_1\}$*-Hölder continuous on $S(0, r_2)$. The Hölder constant depends only on K_4, M_6, M_7, r_3, α_0, α_1,* $\max_S |f|$ *and the Hölder constant of f.*

REMARK 5.2. In case $n=3$ and $\Phi(x) = |x|^{-1}$, this theorem is reduced to [9; Theorem 20].

PROOF. Let $\beta = \min\{\alpha_0, \alpha_1\}$. By Theorem 4.2,

$$\frac{d}{ds}V_\Phi^f(x) = \textstyle\sum_{i=1}^k \langle s, s_i(x)\rangle \dfrac{d}{ds_i(x)}V_\Phi^f(x) + |s^*(x)|\dfrac{d}{dn_s(x)}V_\Phi^f(x)$$

for every $x \in S(0, r_2)$, where $n_s(x) = |s^*(x)|^{-1}s^*(x)$ in case $s^*(x) \neq 0$.

First, we prove the Hölder continuity of $(d/ds_i(x))V_\Phi^f(x)$ for $i = 1, \dots, k$. Let $r > 0$ be the number given in Lemma 5.2 for $\varepsilon = 1/2$. Then there is $r_4 > 0$ ($r_4 \leqq r_3$), depending only on K_4 and r_3, such that $x + |x-\tilde{x}|n \in E(\tilde{x}, r, 1/2)$ whenever $n \in N(x)$, x, $\tilde{x} \in S(0, r_2)$ and $|x-\tilde{x}| \leqq r_4$. For x, $\tilde{x} \in S(0, r_2)$ with $|x-\tilde{x}| \leqq r_4$, let $w = x + |x-\tilde{x}|s_{k+1}(x)$ and write

$$\begin{aligned}\frac{d}{ds_i(x)}V_\Phi^f(x) - \frac{d}{ds_i(\tilde{x})}V_\Phi^f(\tilde{x}) &= \left\{\frac{d}{ds_i(x)}V_\Phi^f(x) - \frac{d}{ds_i(x)}V_\Phi^f(w)\right\} \\ &\quad + \{\langle s_i(x), s_i(\tilde{x})\rangle - 1\}\frac{d}{ds_i(\tilde{x})}V_\Phi^f(w) \\ &\quad + \left\{\frac{d}{ds_i(\tilde{x})}V_\Phi^f(w) - \frac{d}{ds_i(\tilde{x})}V_\Phi^f(\tilde{x})\right\} \\ &\quad + \textstyle\sum_{j\neq i} \langle s_i(x), s_j(\tilde{x})\rangle \dfrac{d}{ds_j(\tilde{x})}V_\Phi^f(w).\end{aligned}$$

Since $w \in E(\tilde{x}, r, 1/2)$, by Lemma 5.4 $(d/ds_i(\tilde{x}))V_\Phi^f(w)$ is a bounded function of $(x, \tilde{x}) \in S(0, r_2) \times S(0, r_2)$, so that the second and the fourth terms on the right are dominated by $C|x-\tilde{x}|^\beta$ in absolute value by Lemma 5.1. Since $s_i(x)$ (resp. $s_i(\tilde{x})$), $1 \leqq i \leqq k$, are tangent vectors to S at x (resp. $\tilde{x}$), we see by Theorem 3.1 that the first and the third terms on the right are majorized by $C|x-\tilde{x}|^\beta$ in absolute value. The above constants C depend only on K_4, M_6, M_7, r_3, α_0, α_1, $\max_S |f|$ and the Hölder constant of f. Therefore we obtain

$$\left| \frac{d}{ds_i(x)} V_\Phi^f(x) - \frac{d}{ds_i(\tilde{x})} V_\Phi^f(\tilde{x}) \right| \leqq C|x-\tilde{x}|^\beta$$

for $|x-\tilde{x}| \leqq r_4$ with a constant C of the above type. It follows that $\langle s, s_i(x)\rangle (d/ds_i(x))V_\Phi^f(x)$ is β-Hölder continuous on $S(0, r_2)$ for $i=1,\dots, k$, since $(d/ds_i(x))V_\Phi^f(x)$ is bounded by Corollary 5.1 and $\langle s, s_i(x)\rangle$ is α_0-Hölder continuous by Lemma 5.1.

Next, we prove the Hölder continuity of $|s^*(x)|(d/dn_s(x))V_\Phi^f(x)$. For x, $\tilde{x} \in S(0, r_2)$ with $|x-\tilde{x}| \leqq r_4$, we assume that $|s^*(x)| \leqq |s^*(\tilde{x})|$ and $s^*(\tilde{x}) \neq 0$, and put $w = x + |x-\tilde{x}| n_s(x)$ and $z = \tilde{x} + |x-\tilde{x}| n_s(\tilde{x})$. Here we let $n_s(x) = n_s(\tilde{x})$ if $s^*(x) = 0$. Then

$$\begin{aligned}
&|s^*(x)| \frac{d}{dn_s(x)} V_\Phi^f(x) - |s^*(\tilde{x})| \frac{d}{dn_s(\tilde{x})} V_\Phi^f(\tilde{x}) \\
&= |s^*(x)| \left\{ \frac{d}{dn_s(x)} V_\Phi^f(x) - \frac{d}{dn_s(x)} V_\Phi^f(w) \right\} \\
&\quad + \textstyle\sum_{j=1}^n \langle s^*(x) - s^*(\tilde{x}), s_j(\tilde{x})\rangle \displaystyle\frac{d}{ds_j(\tilde{x})} V_\Phi^f(w) \\
&\quad + |s^*(\tilde{x})| \left\{ \frac{d}{dn_s(\tilde{x})} V_\Phi^f(w) - \frac{d}{dn_s(\tilde{x})} V_\Phi^f(z) \right\} \\
&\quad + |s^*(\tilde{x})| \left\{ \frac{d}{dn_s(\tilde{x})} V_\Phi^f(z) - \frac{d}{dn_s(\tilde{x})} V_\Phi^f(\tilde{x}) \right\} = J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Theorems 3.2 and 3.3, we have $|J_1| \leqq C|x-\tilde{x}|^\beta$ and $|J_4| \leqq C|x-\tilde{x}|^\beta$, and by Lemmas 5.1 and 5.4 we have $|J_2| \leqq C|x-\tilde{x}|^\beta$, where the constants C depend only on K_4, M_6, M_7, r_3, α_0, α_1, $\max_S |f|$ and the Hölder constant of f.

To estimate J_3, we observe that

$$|(x-\tilde{x})^*(\tilde{x})| \leqq C|x-\tilde{x}|^{1+\alpha_0}, \tag{5.3}$$

$$\begin{aligned}
1 - |n_s(x)^*(\tilde{x})| &\leqq \textstyle\sum_{i=1}^k |\langle n_s(x), s_i(\tilde{x})\rangle| \\
&\leqq \textstyle\sum_{j=k+1}^n \sum_{i=1}^k |\langle n_s(x), s_j(x)\rangle| \, |\langle s_j(x), s_i(\tilde{x})\rangle| \\
&\leqq C|x-\tilde{x}|^{\alpha_0}
\end{aligned} \tag{5.4}$$

and

(5.5)
$$\begin{aligned}|s^*(\tilde{x})|\,|n_s(x) - n_s(\tilde{x})| &\leqq 2|s^*(x) - s^*(\tilde{x})| \\ &\leqq 2\textstyle\sum_{i=k+1}^{n} |\langle s, s_i(x)\rangle s_i(x) - \langle s, s_i(\tilde{x})\rangle s_i(\tilde{x})| \\ &\leqq C|x-\tilde{x}|^{\alpha_0},\end{aligned}$$

by (S–3) and Lemma 5.1, where the constants C depend only on K_4. Now, put $\tilde{w}=(w-\tilde{x})^*(\tilde{x})$ for simplicity. Since

$$\tilde{w} = (x-\tilde{x})^*(\tilde{x}) + |x-\tilde{x}|n_s(x)^*(\tilde{x}),$$

(5.3) and (5.4) imply that

$$||\tilde{w}| - |x-\tilde{x}|| \leqq C|x-\tilde{x}|^{1+\alpha_0} \tag{5.6}$$

and

(5.7)
$$\begin{aligned}&|(\tilde{w}/|x-\tilde{x}|) - n_s(\tilde{x})| \\ &\leqq |(x-\tilde{x})^*(\tilde{x})|\,|x-\tilde{x}|^{-1} + |n_s(x)^*(\tilde{x}) - n_s(\tilde{x})| \\ &\leqq C|x-\tilde{x}|^{\alpha_0} + |n_s(x) - n_s(\tilde{x})|\,.\end{aligned}$$

From (5.5), (5.6) and (5.7), it follows that

$$\begin{aligned}&|s^*(\tilde{x})|\,|(w-\tilde{x})^*(\tilde{x})/|(w-\tilde{x})^*(\tilde{x})| - (z-\tilde{x})^*(\tilde{x})/|(z-\tilde{x})^*(\tilde{x})||\\ &= |s^*(\tilde{x})|\,|(\tilde{w}/|\tilde{w}|) - n_s(\tilde{x})| \\ &\leqq |s^*(\tilde{x})|\,||\tilde{w}| - |x-\tilde{x}||\,|x-\tilde{x}|^{-1} + |s^*(\tilde{x})|\,|\tilde{w}/|x-\tilde{x}| - n_s(\tilde{x})| \\ &\leqq C|x-\tilde{x}|^{\alpha_0}\end{aligned}$$

with a constant C depending only on K_4. Thus, by Theorem 3.3, $|J_3| \leqq C|x-\tilde{x}|^{\beta}$ with C depending only on the values described in the theorem. The proof of the theorem is now complete.

5.5. A generalization of a theorem of Liapunov

Let $r>0$, $0<\theta<\pi/2$ and a be a unit vector in R^n. For a point x^0 in R^n, we denote by $C(x^0; a, r, \theta)$ the truncated closed cone with vertex at x^0, axis along $L(x^0, a)$, height r and angle θ, that is, the set of all points x satisfying the inequalities

$$|x-x^0|\cos\theta \leqq \langle x - x^0, a\rangle \leqq r.$$

Since S satisfies α_0-condition uniformly on $S(0, r_2)$, by Lemma 5.2 there are positive numbers C and r^* depending only on K_4, r_3, α_0 and θ, such that

$$|z-x| + |x-y| \leqq C|z-y|$$

for every $x\in S(0, r_2)$, $y\in S$ and $z\in C(x; s_{k+1}(x), r^*, \theta)$.

LEMMA 5.5 (*cf.* [9; *Lemma* 9]). *Let* $0<\alpha\leqq 1$ *and* $0<\theta<\pi/2$. *Assume that* g *is a function defined on* $\bigcup_{y\in S(0,r_2)} C(y;\, s_{k+1}(y),\, r^*,\, \theta)\setminus S(0,\, r_2)$ *for which there is a positive number* C_1 *such that*

$$|g(x) - g(\tilde{x})| \leqq C_1|x-\tilde{x}|^\alpha,$$

whenever $x,\ \tilde{x}\in C(y;\, s_{k+1}(y),\, r^*,\, \theta)\setminus\{y\}$ *for some* $y\in S(0,\, r_2)$. *Let* $\bar{g}(x)$ *be equal to* $g(x)$ *on* $\bigcup_{y\in S(0,r_2)} C(y;\, s_{k+1}(y),\, r^*/2,\, \theta/2)\setminus S(0,\, r_2)$ *and defined by*

$$\lim_{z\to x,\, z\in C(x; s_{k+1}(x), r^*, \theta)\setminus\{x\}} g(z)$$

on $S(0, r_2)$. *Then for* r *with* $0<r<r_2$, $\bar{g}$ *is* α-*Hölder continuous on* $\bigcup_{y\in S(0,r)} C(y;\, s_{k+1}(y),\, r^*/2,\, \theta/2)$ *with Hölder constant depending only on* C_1, K_4, r_3, α, α_0 *and* θ.

PROOF. First, we prove that $\bar{g}$ is α-Hölder continuous on $S(0,\, r)$. Since S satisfies α_0-condition uniformly on $S(0,\, r_2)$, it is enough to show that $\bar{g}$ is α-Hölder continuous on S near the origin. If we choose p so large that $\cos\theta/2< p/(p+1)$, then we can find r_5 (>0) depending only on K_4, r_3, α_0 and θ such that

$$x + p|x-\tilde{x}|s_{k+1}(x)\in C(\tilde{x};\, s_{k+1}(\tilde{x}),\, r^*/2,\, \theta/2)$$

for every $x,\ \tilde{x}\in S(0,\, r_5)$. Given $x,\ \tilde{x}\in S(0,\, r_5)$, let $w=x+p|x-\tilde{x}|s_{k+1}(x)$. Then by our assumption

$$|\bar{g}(x)-g(w)| \leqq C_1|x-w|^\alpha \quad\text{and}\quad |g(w)-\bar{g}(\tilde{x})| \leqq C_1|w-\tilde{x}|^\alpha,$$

which imply

$$|\bar{g}(x)-\bar{g}(\tilde{x})| \leqq C_2|x-\tilde{x}|^\alpha,$$

where $C_2=2(1+p)^\alpha C_1$, since $|x-w|\leqq p|x-\tilde{x}|$ and $|w-\tilde{x}|\leqq(1+p)|x-\tilde{x}|$. Thus $\bar{g}$ is α-Hölder continuous on $S(0,\, r)$ with Hölder constant depending only on C_1, K_4, r_3, α, α_0 and θ. Next we prove the assertion of the lemma. For simplicity, we denote the cone $C(y;\, s_{k+1}(y),\, r^*,\, \theta)$ (resp. $C(y;\, s_{k+1}(y),\, r^*/2,\, \theta/2)$) by $C(y)$ (resp. $C^*(y)$) for $y\in S(0,\, r_2)$. For $x,\ \tilde{x}\in\bigcup_{y\in S(0,r)} C^*(y)$, there exist $y,\ \tilde{y}\in S(0,\, r)$ such that $x\in C^*(y)$ and $\tilde{x}\in C^*(\tilde{y})$. If $x\in C(\tilde{y})$, then $|\bar{g}(x)-\bar{g}(\tilde{x})|\leqq C_1|x-\tilde{x}|^\alpha$. Thus suppose $x\notin C(\tilde{y})$ and $\tilde{x}\notin C(y)$. Since $x\in C^*(y)$ and $\tilde{x}\in C^*(\tilde{y})$, we see that

$$|x-y| \leqq C_3|x-\tilde{x}| \quad\text{and}\quad |\tilde{x}-\tilde{y}| \leqq C_3|x-\tilde{x}|,$$

where $C_3=\operatorname{cosec}\theta/2$, so that

$$|\bar{g}(x)-\bar{g}(y)| \leqq C_1|x-y|^\alpha \leqq C_1C_3^\alpha|x-\tilde{x}|^\alpha,$$

$$|\bar{g}(\tilde{x})-\bar{g}(\tilde{y})| \leqq C_1|\tilde{x}-\tilde{y}|^\alpha \leqq C_1C_3^\alpha|x-\tilde{x}|^\alpha$$

and

$$|\bar{g}(y)-\bar{g}(\tilde{y})| \leqq C_4|y-\tilde{y}|^\alpha \leqq C_4(1+2C_3)^\alpha|x-\tilde{x}|^\alpha,$$

where C_4 is the Hölder constant of $\bar{g}$ on $S(0, r)$. Hence,

$$|\bar{g}(x)-\bar{g}(\tilde{x})| \leqq C|x-\tilde{x}|^\alpha.$$

Thus the lemma is proved.

Now we give a generalization of a theorem of Liapunov [6; Chap. II, §7 or Appendix, §1] and [11; Theorem 3]).

THEOREM 5.2. *Let* $k=n-1$, $\lambda=n-2$, $0<\alpha_0<1$ *and* $0<r<r_2$. *Assume that* $\Phi \in C^2(B^{(n)}(0, 4r_0)\backslash\{0\})$, *that it satisfies* ($\Phi$–4) *and* ($\Phi$–7) *with* $\lambda=n-2$ *and that* (Φ–8) *with* $k=n-1$ *holds. Let* K *be a compact set contained in* $\{x=(x', x_n);\ x_n \geqq \psi_n(x'),\ |x'| \leqq r\}$ *or in* $\{x=(x', x_n);\ x_n \leqq \psi_n(x'),\ |x'| \leqq r\}$ *and* $K \subset B^{(n)}(0, 2r_0)$. *If* f *is* α_1*-Hölder continuous on* S, *then the derivative* $(d/ds)V_\Phi^f$ *in any direction* s *can be extended to be* $\min\{\alpha_0, \alpha_1\}$*-Hölder continuous on* K.

PROOF. We prove only the case $K \subset \{x;\ x_n \geqq \psi_n(x'),\ |x'| \leqq r\}$. Let $0<r<r'<r_2$. Then there exists a positive number r_6 such that for $x \in K \cap \{x;\ \mathrm{dist}(x, S) \leqq r_6\}$, the point y_x nearest to S from x belongs to $S(0, r')$, so that $y_x - x$ is a normal to S at y_x and

$$K \cap \{x;\ \mathrm{dist}(x, S) \leqq r_6\} \subset \cup_{y \in S(0,r')} C(y;\ s_n(y), r_6, \theta)$$

for any θ, $0<\theta<\pi/2$. Hence, using Lemma 5.3 and Corollary 3.1, we obtain the assertion.

THEOREM 5.2′. *Under the same assumptions as in Theorem* 5.2, *if* K *is contained in* $\{x;\ x_n \geqq \psi_n(x'),\ |x'| \leqq r\}$, *then* $(d/ds)V_\Phi^f$ *in the direction* s *is* $\min\{\alpha_0, \alpha_1\}$*-Hölder continuous on* K, *provided* $\langle s, s_n(x)\rangle \geqq 0$ *for every* $x \in S \cap K$.

In fact, since $\langle s, s_n(x)\rangle \geqq 0$, by Theorems 3.1, 3.2 and 4.2 we have

$$(d/ds)V_\Phi^f(x) = \lim_{z \to x, z \in C(x; s_n(x), r)} (d/ds)V_\Phi^f(z)$$

for $x \in S \cap K$. Thus by Theorem 5.2 the assertion holds.

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