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On bounded entire solutions of semilinear elliptic equations

Nichiro KAWANO

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1. Introduction

This paper is concerned with bounded solutions of the second order semilinear elliptic equations

$$\Delta u + \phi(x)u^{\gamma} = 0 \tag{1.1}$$

and

$$\Delta u + \phi(x)e^{u} = 0 \tag{1.2}$$

in the entire Euclidean space R^n, $n \geq 3$, where Δ denotes the n-dimensional Laplace operator, $\phi(x)$ is a locally Hölder continuous function in R^n and γ is a nonzero constant.

The problems of existence and nonexistence of entire solutions of semilinear elliptic equations of the form $\Delta u + f(x, u) = 0$ have been investigated by many authors; see, for example, [3], [4], [6], [8], [9] and [12]. We refer in particular to the recent papers by Ni [8, 9] in which explicit conditions are given which guarantee the existence of bounded entire solutions of (1.1) and (1.2).

Our main objective is to give conditions for the existence of bounded positive entire solutions of (1.1) and bounded entire solutions of (1.2) by means of the method of supersolutions and subsolutions. The principal device in this paper is the construction of spherically symmetric supersolutions and subsolutions for (1.1) and (1.2), and this enables us to prove the following theorems which extend considerably the basic existence results of Ni [8, 9].

THEOREM 1.1. *Suppose there exists a locally Hölder continuous function $\phi^*(t)$ on $[0, \infty)$ such that $|\phi(x)| \leq \phi^*(|x|)$ for all $x \in R^n$ and*

$$\int_0^{\infty} t\phi^*(t)dt < \infty. \tag{1.3}$$

Then (1.1) *with $\gamma \neq 1$ has infinitely many positive solutions which are bounded and bounded away from zero in R^n. Moreover, if either $\phi(x) \geq 0$ or $\phi(x) \leq 0$ for all $x \in R^n$, then equation* (1.1) *possesses infinitely many bounded positive solutions with the property that each of these solutions tends to a positive constant as $|x| \to \infty$.*

THEOREM 1.2. *Suppose $\phi(x)$ satisfies the conditions of Theorem* 1.1. *Then equation* (1.2) *possesses infinitely many bounded solutions in R^n. Moreover, if either $\phi(x)\geq 0$ or $\phi(x)\leq 0$ for all $x\in R^n$, then equation* (1.2) *possesses infinitely many bounded solutions that tend to constants as $|x|\to\infty$.*

We also obtain necessary conditions for equation (1.1) [resp. (1.2)] with one-signed $\phi(x)$ to possess bounded positive entire solutions [resp. bounded entire solutions].

Furthermore, motivated by the observation that there seems to be no previous result concerning entire solutions of systems of semilinear elliptic equations, we make an attempt to extend the results for (1.1) and (1.2) to systems of the types

$$\tag{1.4} \begin{cases} \Delta u + \phi(x)u^{\gamma}v^{\delta} = 0 \\ \Delta v + \psi(x)u^{\mu}v^{\nu} = 0, \end{cases}$$

$$\tag{1.5} \begin{cases} \Delta u + \phi(x)e^{\gamma u+\delta v} = 0 \\ \Delta v + \psi(x)e^{\mu u+\nu v} = 0, \end{cases}$$

where $\phi(x)$ and $\psi(x)$ are locally Hölder continuous functions in R^n, and γ, δ, μ and ν are constants. The desired extension depends on a suitable modification of the supersolution-subsolution method employed to develop existence theory for the single equations (1.1) and (1.2).

2. The equation $\Delta u+\phi(x)u^{\gamma}=0$

2.1. We begin by stating an existence theorem which is basic to our subsequent considerations. Consider the semilinear elliptic equation

$$\tag{2.1} \Delta u + f(x, u) = 0,$$

where $f(x, u)$ is defined on $R^n\times R^1$, is locally Hölder continuous in x with exponent $\lambda\in(0, 1)$ and is continuously differentiable in u.

By a supersolution [resp. subsolution] of equation (2.1) in R^n is meant a function v [resp. w] $\in C^{2+\lambda}_{loc}(R^n)$ satisfying

$$\tag{2.2} \Delta v + f(x, v) \leq 0 \text{ [resp. } \Delta w+f(x, w)\geq 0], \qquad x\in R^n.$$

THEOREM A. (Akô and Kusano [2]) *If there exist a bounded supersolution $v(x)$ and a bounded subsolution $w(x)$ of* (2.1) *in R^n such that*

$$\tag{2.3} w(x) \leq v(x), \qquad x\in R^n,$$

then equation (2.1) *possesses an entire solution $u(x)$ satisfying*

$$w(x) \le u(x) \le v(x), \qquad x \in R^n \tag{2.4}$$

REMARK 2.1. The boundedness of $v(x)$ and $w(x)$ in Theorem A is not necessary (see W. -M. Ni [9, Theorem 2.10]). However, Theorem A in this form suffices for our purposes.

In this section we consider equation (1.1) under the following assumption.

Assumption (A): $\phi(x)$ is locally Hölder continuous in R^n with exponent $\lambda \in (0, 1)$ and there exists a nonnegative function $\phi^*(t)$ on $[0, \infty)$ such that $\phi^* \in C^{\lambda}_{loc}([0, \infty))$ and

$$|\phi(x)| \le \phi^*(|x|), \qquad x \in R^n. \tag{2.5}$$

In order to discuss the existence of bounded positive entire solutions of (1.1) it is convenient to distinguish the following three cases: (i) $\phi(x) \le 0$ in R^n; (ii) $\phi(x) \ge 0$ in R^n; and (iii) $\phi(x)$ has indefinite sign in R^n.

2.2. The case where $\phi(x) \ge 0$ in R^n. In this case our discussion is based on the following theorem.

THEOREM 2.1. *Let assumption* (A) *be satisfied. If the equation*

$$\Delta v + \phi^*(|x|)v^\gamma = 0 \tag{2.6}$$

has a positive solution which is bounded and bounded away from zero in R^n, then equation (1.1) *has a positive entire solution which is bounded and bounded away from zero in R^n.*

The conclusion of this theorem immediately follows from Theorem A since any positive solution of (2.6) is a supersolution of (1.1) and any positive constant is a subsolution of (1.1).

We therefore wish to construct a positive solution of (2.6) which is bounded and bounded away from zero in R^n. It is natural to seek such a solution of (2.6) with spherical symmetry: $v(x) = y(|x|)$. If we put $t = |x|$, then the problem is reduced to the following one-dimensional problem:

$$\begin{cases} y'' + \dfrac{n-1}{t} y' + \phi^*(t) y^\gamma = 0, & t > 0, \\ y(0) = \alpha, \quad y'(0) = 0 \quad (\alpha > 0). \end{cases} \tag{2.7}$$

As easily verified, solving (2.7) is equivalent to solving the integral equation

$$y(t) = \alpha - \frac{1}{n-2}\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s) y^\gamma(s)\,ds, \quad t \geqq 0. \tag{2.8}$$

Our idea is to regard this as an operator equation $y=\mathscr{F}y$ with $\mathscr{F}$ defined by

$$\mathscr{F}y(t) = \alpha - \frac{1}{n-2}\int_0^t \left[1-\left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)y^\gamma(s)ds, \tag{2.9}$$

and to solve it by means of the fixed point theorem of Schauder-Tychonoff.

THEOREM 2.2. *Let $\gamma \neq 1$. In addition to assumption* (A) *suppose that* (1.3) *holds. Then, for some positive constant α,* (2.7) *has a positive solution which is bounded and bounded away from zero for $t \geq 0$.*

PROOF. We distinguish the three cases: $\gamma>1$; $0<\gamma<1$; and $\gamma<0$.

(i) Superlinear case: $\gamma>1$. Let $\alpha>0$ be *small* enough so that

$$1 - \frac{\alpha^{\gamma-1}}{n-2}\int_0^\infty s\phi^*(s)ds > 0, \tag{2.10}$$

and put

$$k(\alpha) = \alpha - \frac{\alpha^\gamma}{n-2}\int_0^\infty s\phi^*(s)ds > 0. \tag{2.11}$$

Let $C[0, \infty)$ denote the locally convex space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on every compact subinterval of $[0, \infty)$. Consider the set

$$Y = \{y \in C[0, \infty): k(\alpha) \leq y(t) \leq \alpha \quad for \quad t \geq 0\}, \tag{2.12}$$

which is a closed convex subset of $C[0, \infty)$. Now, we show that the operator $\mathscr{F}$ defined by (2.9) maps Y into itself. If $y \in Y$, then obviously $\mathscr{F}y(t) \leq \alpha$ and

$$\begin{aligned}\mathscr{F}y(t) &\geq \alpha - \frac{1}{n-2}\int_0^t s\phi^*(s)y^\gamma(s)ds \\ &\geq \alpha - \frac{\alpha^\gamma}{n-2}\int_0^t s\phi^*(s)ds \geq k(\alpha), \quad t \geq 0.\end{aligned}$$

(ii) Sublinear case: $0<\gamma<1$. Let $\alpha>0$ be *large* enough so that (2.10) holds and define $k(\alpha)$ by (2.11). If we let Y be the same subset of $C[0, \infty)$ as defined in (i), then it readily follows that $\mathscr{F}$ maps Y into itself.

(iii) Singular case: $\gamma<0$. Let $\alpha>0$ be so *large* that

$$\alpha^{1-\gamma} - \frac{1-\gamma}{n-2}\int_0^\infty s\phi^*(s)ds > 0, \tag{2.13}$$

and define

$$A(t) = \alpha^{1-\gamma} - \frac{1-\gamma}{n-2}\int_0^t s\phi^*(s)ds. \tag{2.14}$$

Then, $0<[A(t)]^{1/(1-\gamma)}\leq\alpha$, $t\geq 0$, and we can consider the set

$$Y=\{y\in C[0,\infty): [A(t)]^{1/(1-\gamma)}\leq y(t)\leq\alpha \quad for \quad t\geq 0\}. \tag{2.15}$$

It can be shown that $\mathscr{F}$ maps Y into itself. Indeed, clearly $\mathscr{F}y(t)\leq\alpha$ and

$$\begin{aligned}\mathscr{F}y(t) &\geq \alpha-\frac{1}{n-2}\int_0^t s\phi^*(s)y^\gamma(s)ds\\ &\geq \alpha-\frac{1}{n-2}\int_0^t s\phi^*(s)[A(s)]^{\gamma/(1-\gamma)}ds\\ &= [A(t)]^{1/(1-\gamma)}, \quad t\geq 0.\end{aligned}$$

To see the last equality it suffices to consider the function

$$P(t)=\alpha-\frac{1}{n-2}\int_0^t s\phi^*(s)[A(s)]^{\gamma/(1-\gamma)}ds-[A(t)]^{1/(1-\gamma)}$$

and show that $P(0)=0$ and $P'(t)=0$ for $t>0$. In this case, we define $k(\alpha)$ by

$$k(\alpha)=\lim_{t\to\infty}[A(t)]^{1/(1-\gamma)}=\left(\alpha^{1-\gamma}-\frac{1-\gamma}{n-2}\int_0^\infty s\phi^*(s)\,ds\right)^{1/(1-\gamma)}. \tag{2.16}$$

We have thus seen that $\mathscr{F}$ is a self-map of Y in each of the cases (i), (ii) and (iii).

Next, we show the continuity of the operator $\mathscr{F}$. If $y_m\in Y$ $(m=1, 2,\dots)$ and $y_m(t)\to y(t)$ as $m\to\infty$ uniformly on every compact subinterval of $[0,\infty)$, then $y\in Y$ and we have

$$|\mathscr{F}y_m(t)-\mathscr{F}y(t)|\leq\frac{1}{n-2}\int_0^t s\phi^*(s)|y_m^\gamma(s)-y^\gamma(s)|ds \tag{2.17}$$

for $t\geq 0$. With the use of (1.3) and Lebesgue's dominated convergence theorem, it follows from (2.17) that, in each of the cases (i), (ii) and (iii), $\mathscr{F}y_m(t)\to\mathscr{F}y(t)$ as $m\to\infty$ uniformly on every compact subinterval of $[0,\infty)$. Finally, we check the relative compactness of $\mathscr{F}Y$. In fact, $\mathscr{F}Y$ is clearly uniformly bounded at every point of $[0,\infty)$, and from the relation

$$|(\mathscr{F}y)'(t)|=\left|\int_0^t\left(\frac{s}{t}\right)^{n-1}\phi^*(s)y^\gamma(s)ds\right|\leq C\int_0^t\phi^*(s)ds,$$

where $C=\alpha^\gamma$ if $\gamma>0$ and $C=[k(\alpha)]^\gamma$ if $\gamma<0$, it follows that $\mathscr{F}y$ is equicontinuous at every point in $[0,\infty)$.

All the conditions of the Schauder-Tychonoff fixed point theorem are satisfied, and hence the mapping $\mathscr{F}$ has a fixed point y in Y. This fixed point $y=y(t)$ is a solution of the integral equation (2.8), and so it is a solution of the initial value problem (2.7) with the required boundedness property. This completes the proof of Theorem 2.2.

Combining Theorem 2.1 with Theorem 2.2 yields the following existence theorem for (1.1)

THEOREM 2.3. *Let* $\gamma \neq 1$ *and in addition to assumption* (A) *suppose that* (1.3) *holds. Then, equation* (1.1) *has infinitely many bounded positive entire solutions which converge to positive constants as* $|x| \to \infty$.

PROOF. By Theorem 2.2 there are positive constants α and $k(\alpha)$ such that the initial value problem (2.7) has a solution $y(t)$ satisfying $k(\alpha) \leq y(t) \leq \alpha$ for $t \geq 0$. Since $y'(t) < 0$ for $t > 0$, the limit $\eta = \lim_{t \to \infty} y(t) \geq k(\alpha)$ exists. The function $v(x) = y(|x|)$ is a solution of (2.6), and hence a supersolution of (1.1), whereas the constant η is a subsolution of (1.1). From Theorem A it follows that (1.1) has an entire solution $u(x)$ such that $\eta \leq u(x) \leq v(x)$ in R^n. It is clear that $\lim_{|x| \to \infty} u(x) = \eta$. It is not hard to see that there exist infinitely many values of $\alpha > 0$ which yield different positive entire solutions of (1.1). This completes the proof of Theorem 2.3.

REMARK 2.2. In the case where $\gamma = 1$, that is, equation (1.1) is linear, we have a weaker conclusion: *If*

$$\int_0^\infty t\phi^*(t)dt < n - 2, \tag{2.18}$$

then equation (1.1) *with* $\gamma = 1$ *has infinitely many bounded positive entire solutions which tend to positive constants as* $|x| \to \infty$.

Whether (2.18) can be replaced by (1.3) or not is unknown to us.

2.3. In this subsection we assume that $\phi(x) \leq 0$ for $x \in R^n$. The following theorem is parallel to Theorem 2.1.

THEOREM 2.4. *Let assumption* (A) *be satisfied. If the equation*

$$\Delta w - \phi^*(|x|)w^\gamma = 0 \tag{2.19}$$

has a positive solution which is bounded and bounded away from zero in R^n, *then equation* (1.1) *has a positive entire solution which is bounded and bounded away from zero* R^n.

This is an immediate consequence of Theorem A. In fact, since $\phi(x) \leq 0$ in R^n, every positive constant is a supersolution of (1.1) in R^n, and in view of (2.5) any positive solution of (2.19) is a positive subsolution of (1.1) in R^n.

According to Theorem 2.4, the problem under study for (1.1) is reduced to the problem of finding a bounded positive solution of (2.19). As in subsection 2.2, we seek a spherically symmetric solution $w(x) = z(|x|)$ of (2.19). Putting $t = |x|$, we get the following ODE problem:

$$\text{(2.20)} \qquad \begin{cases} z'' + \dfrac{n-1}{t} z' - \phi^*(t) z^\gamma = 0, & t > 0, \\ z(0) = \beta, \quad z'(0) = 0 \quad (\beta > 0), \end{cases}$$

which in turn is equivalent to the integral equation

$$\text{(2.21)} \qquad z(t) = \beta + \frac{1}{n-2} \int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right] \phi^*(s) z^\gamma(s)\, ds, \quad t \geq 0.$$

We will solve (2.21) by reducing it to an operator equation $z = \mathscr{G}z$ with $\mathscr{G}$ defined by

$$\mathscr{G}z(t) = \beta + \frac{1}{n-2} \int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right] \phi^*(s) z^\gamma(s)\, ds,$$

and applying the fixed point theorem of Schauder-Tychonoff.

THEOREM 2.5. *In addition to assumption* (A) *suppose that* (1.3) *is satisfied. Then, for some positive constant* β, *equation* (2.20) *has a bounded positive solution which is bounded away from zero for* $t \geq 0$.

PROOF. The proof of this theorem is analogous to that of Theorem 2.2.

(i) Superlinear case: $\gamma > 1$. We take a positive constant β so *small* that

$$\text{(2.22)} \qquad \beta^{1-\gamma} - \frac{\gamma - 1}{n-2} \int_0^\infty t\phi^*(t)\, dt > 0.$$

Define the function $B(t)$ by

$$\text{(2.23)} \qquad B(t) = \beta^{1-\gamma} - \frac{\gamma - 1}{n-2} \int_0^t s\phi^*(s)\, ds$$

and put

$$\text{(2.24)} \qquad l(\beta) = \lim_{t \to \infty} [B(t)]^{1/(1-\gamma)}.$$

Noting that $\beta \leq [B(t)]^{1/(1-\gamma)}$, $t \geq 0$, we consider

$$\text{(2.25)} \qquad Z = \{z \in C[0, \infty) \colon \beta \leq z(t) \leq [B(t)]^{1/(1-\gamma)} \quad \textit{for} \quad t \geq 0\},$$

which is a closed convex subset of locally convex space $C[0, \infty)$. If $z \in Z$, then $\beta \leq \mathscr{G}z(t)$ and as in case (iii) in the proof of Theorem 2.2, we have

$$\begin{aligned} \mathscr{G}z(t) &\leq \beta + \frac{1}{n-2} \int_0^t s\phi^*(s) [B(s)]^{\gamma/(1-\gamma)} ds \\ &= [B(t)]^{1/(1-\gamma)}, \quad t \geq 0. \end{aligned}$$

It follows that $\mathscr{G}$ maps Z into itself.

(ii) Linear case: $\gamma = 1$. Let β be any fixed positive constant and put

$$B(t) = \beta \exp\left(\frac{1}{n-2}\int_0^t s\phi^*(s)ds\right), \tag{2.26}$$

$$l(\beta) = \lim_{t\to\infty} B(t). \tag{2.27}$$

Define

$$Z = \{z \in C[0, \infty): \beta \le z(t) \le B(t) \quad \text{for} \quad t \ge 0\}.$$

Then $\mathscr{G}$ maps Z into itself. In fact, if $z \in Z$, then $\beta \le \mathscr{G}z(t)$, and

$$\begin{aligned}\mathscr{G}z(t) &\le \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)z(s)ds \\ &\le \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)B(s)ds \\ &= B(t), \quad t \ge 0.\end{aligned}$$

(iii) Sublinear case: $0<\gamma<1$. Put

$$B(t) = \beta^{1-\gamma} + \frac{1-\gamma}{n-2}\int_0^t s\phi^*(s)ds, \tag{2.28}$$

$$l(\beta) = \lim_{t\to\infty} [B(t)]^{1/(1-\gamma)}, \tag{2.29}$$

where β is any positive constant. Then $\beta \le [B(t)]^{1/(1-\gamma)}$, $t \ge 0$. Define Z by (2.25). If $z \in Z$, then $\beta \le \mathscr{G}z(t)$, $t \ge 0$, and proceeding as in case (i), we have

$$\begin{aligned}\mathscr{G}z(t) &\le \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)[B(s)]^{\gamma/(1-\gamma)}ds \\ &= [B(t)]^{1/(1-\gamma)}, \quad t \ge 0.\end{aligned}$$

This implies that $\mathscr{G}$ maps Z into itself.

(iv) Singular case: $\gamma<0$. Put

$$l(\beta) = \beta + \frac{\beta^\gamma}{n-2}\int_0^\infty s\phi^*(s)ds, \tag{2.30}$$

where β is any positive constant, and define

$$Z = \{z \in C[0, \infty): \beta \le z(t) \le l(\beta) \quad \textit{for} \quad t \ge 0\}. \tag{2.31}$$

Then it follows that $\mathscr{G}$ is a self-map of Z. In fact, if $z \in Z$, then obviously $\beta \le \mathscr{G}z(t)$, and we have

$$\begin{aligned}\mathscr{G}z(t) &\le \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)z^\gamma(s)ds \\ &\le \beta + \frac{\beta^\gamma}{n-2}\int_0^t s\phi^*(s)ds \le l(\beta), \quad t \ge 0.\end{aligned}$$

We have thus shown that in each of the cases (i)–(iv), $\mathscr{G}$ maps Z into itself. Furthermore, it can easily be shown that $\mathscr{G}$ is continuous and that $\mathscr{G}Z$ is relatively compact in $C[0, \infty)$. Therefore, applying the Schauder-Tychonoff fixed point theorem, we conclude that the integral equation (2.21) has at least one solution in Z. This implies that the initial value problem (2.20) has a bounded positive solution which is bounded away from zero. This completes the proof of Theorem 2.5.

Theorems 2.4 and 2.5 yield the following theorem which is the main result of this subsection.

THEOREM 2.6. *In addition to assumption* (A) *suppose that* (1.3) *is satisfied. Then,* (1.1) *has infinitely many positive entire solutions which tend to positive constants as* $|x| \to \infty$.

PROOF. By Theorem 2.5, in each of the cases (i)–(iv), we can take positive numbers β and $l(\beta)$ in such a way that the initial value problem (2.20) has a positive solution $z(t)$ satisfying $\beta \leq z(t) \leq l(\beta)$ for $t \geq 0$. Since $z'(t) > 0$, $t > 0$, the positive limit $\lim_{t\to\infty} z(t) = \zeta \leq l(\beta)$ exists. The functions $v(x) = \zeta$ and $w(x) = z(|x|)$ are, respectively, a supersolution and a subsolution of (1.1) in R^n, and so, from Theorem 2.4 and Theorem A it follows that (1.1) has a solution $u(x)$ such that $w(x) \leq u(x) \leq v(x)$ in R^n. Obviously, $\lim_{|x|\to\infty} u(x) = \zeta$. As easily verified, there exist infinitely many values of $\beta > 0$ which yield different entire solutions of (1.1). This finishes the proof.

2.4. We now turn to the case where $\phi(x)$ is not of constant sign.

THEOREM 2.7. *Let assumption* (A) *be satisfied. If the equations* (2.6) *and* (2.19) *possess bounded positive solutions* $v(x)$ *and* $w(x)$, *respectively, such that* $w(x) \leq v(x)$ *in* R^n, *then there exists at least one solution* $u(x)$ *of* (1.1) *satisfying*

$$w(x) \leq u(x) \leq v(x), \quad x \in R^n. \tag{2.32}$$

This theorem follows from Theorem A, since $v(x)$ and $w(x)$ are respectively, a supersolution and a subsolution of (1.1) in R^n.

Now suppose that (1.3) holds. We put

$$k(\alpha) = \alpha - \frac{\alpha^\gamma}{n-2}\int_0^\infty t\phi^*(t)dt, \quad l(\beta) = \left(\beta^{1-\gamma} + \frac{1-\gamma}{n-2}\int_0^\infty t\phi^*(t)dt\right)^{1/(1-\gamma)}$$

if either $\gamma > 1$ or $0 < \gamma < 1$, and

$$k(\alpha) = \left(\alpha^{1-\gamma} - \frac{1-\gamma}{n-2}\int_0^\infty t\phi^*(t)dt\right)^{1/(1-\gamma)}, \quad l(\beta) = \beta + \frac{\beta^\gamma}{n-2}\int_0^\infty t\phi^*(t)dt$$

if $\gamma < 0$. In either case positive numbers α and β can be taken so that the above

$k(\alpha)$ and $l(\beta)$ are positive and

$$\beta < l(\beta) \leq k(\alpha) < \alpha. \tag{2.33}$$

From the proof of Theorems 2.2 and 2.5 we see that to such α and β there correspond a solution $y(t)$ of (2.7) and a solution $z(t)$ of (2.20). In view of (2.33) we have

$$z(t) \leq y(t) \quad \text{for} \quad t \geq 0. \tag{2.34}$$

The functions $v(x)=y(|x|)$ and $w(x)=z(|x|)$ are, respectively, a solution of (2.6) and a solution of (2.19) satisfying $w(x) \leq v(x)$ in R^n, and Theorem 2.7 guarantees the existence of an entire solution $u(x)$ of (1.1) satisfying (2.32). It is easy to check that there exist infinitely many pairs of positive numbers α and β which produce different entire solutions of (1.1). We have thus proved the following

THEOREM 2.8. *Let $\gamma \neq 1$. In addition to assumption* (A) *suppose that* (1.3) *is satisfied. Then, equation* (1.1) *possesses infinitely many positive entire solutions which are bounded and bounded away from zero in R^n.*

Concerning the linear equation (1.1) ($\gamma=1$) we have the following existence theorem.

THEOREM 2.9. *Suppose that assumption* (A) *and* (2.18) *are satisfied. Then, equation* (1.1) *with $\gamma=1$ has infinitely many positive entire solutions which are bounded and bounded away from zero in R^n.*

REMARK 2.3. Theorem 1.1 stated in the introduction follows from Theorems 2.3, 2.6 and 2.8.

We now present a variant of Theorems 2.8 and 2.9 which is suggested by Ni [9]. We write $x=(x_1, x_2) \in R^m \times R^{n-m}$, $m \geq 3$.

THEOREM 2.10. *Suppose there exists a positive and locally Hölder continuous function $\phi^*(t)$ on $[0, \infty)$ satisfying*

$$|\phi(x)| \leq \phi^*(|x_1|), \quad x \in R^n \tag{2.35}$$

Suppose that

$$\int_0^\infty t\phi^*(t)dt < \infty \quad \text{if } \gamma \neq 1$$

and

$$\int_0^\infty t\phi^*(t)dt < m-2 \quad \text{if } \gamma = 1.$$

Then (1.1) *has infinitely many positive entire solutions which are bounded and bounded away from zero in* R^n. *Moreover, if either* $\phi(x)\geq 0$ *or* $\phi(x)\leq 0$ *for all* $x\in R^n$, *then* (1.1) *has infinitely many bounded positive entire solutions which tend to positive constants uniformly in* x_2 *as* $|x_1|\to\infty$.

PROOF. Consider the equations

$$\Delta_m\tilde{v} + \phi^*(|x_1|)\tilde{v}^\gamma = 0,$$
$$\Delta_m\tilde{w} - \phi^*(|x_1|)\tilde{w}^\gamma = 0,$$

where Δ_m denote the Laplace operator in R^m. By the proof of Theorems 2.2 and 2.5 these equations have spherically symmetric positive solutions $\tilde{v}(x_1)$, $\tilde{w}(x_1)$ which are bounded and bounded away from zero in R^m and satisfy $\tilde{w}(x_1)\leq\tilde{v}(x_1)$, $x_1\in R^m$. Define the functions $v(x)$ and $w(x)$ in R^n as follows:

$$v(x) = v(x_1, x_2) = \tilde{v}(x_1), \quad w(x) = w(x_1, x_2) = \tilde{w}(x_1).$$

Then, $\Delta v=\Delta_m\tilde{v}$, $\Delta w=\Delta_m\tilde{w}$ and in view of (2.35) we have

$$\Delta v + \phi(x)v^\gamma \leq \Delta_m\tilde{v} + \phi^*(|x_1|)\tilde{v}^\gamma = 0,$$
$$\Delta w + \phi(x)w^\gamma \geq \Delta_m\tilde{w} - \phi^*(|x_1|)\tilde{w}^\gamma = 0$$

in R^n. This implies that $v(x)$ and $w(x)$ are, respectively, a supersolution and a subsolution of (1.1) in R^n. Since $w(x)\leq v(x)$, $x\in R^n$, the conclusion follows from Theorem A.

3. The equation $\Delta u+\phi(x)e^u=0$

The purpose of this section is to establish the existence of bounded entire solutions of equation (1.2). As in the preceding section we distinguish the three cases: (i) $\phi(x)\geq 0$ in R^n; (ii) $\phi(x)\leq 0$ in R^n; and (iii) $\phi(x)$ is not one-signed in R^n.

3.1. The case where $\phi(x)\geq 0$ in R^n.

THEOREM 3.1. *Let assumption* (A) *be satisfied. If the equation*

$$\Delta v + \phi^*(|x|)e^v = 0 \tag{3.1}$$

has a bounded solution in R^n, *then* (1.2) *has a bounded solution in* R^n.

Since a bounded solution of (3.1) is a supersolution and any constant is a subsolution of (1.2) in R^n, the conclusion of this theorem follows from Theorem A.

Consider the initial value problem:

$$(3.2)\qquad \begin{cases} y'' + \dfrac{n-1}{t} y' + \phi^*(t)e^y = 0, & t > 0, \\ y(0) = \alpha, \quad y'(0) = 0, \end{cases}$$

where α is a constant. If $y(t)$ is a bounded solution of (3.2), then the function $v(x)=y(|x|)$ is a bounded solution of (3.1). In order to solve (3.2) we transform it into the integral equation

$$(3.3)\qquad y(t) = \alpha - \frac{1}{n-2}\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)e^{y(s)}ds, \quad t \geq 0.$$

Let α be fixed. Putting

$$(3.4)\qquad k(\alpha) = \alpha - \frac{e^\alpha}{n-2}\int_0^\infty s\phi^*(s)ds,$$

we consider the set

$$Y = \{y \in C[0, \infty): k(\alpha) \leq y(t) \leq \alpha \quad for \quad t \geq 0\}$$

and define the operator $\mathscr{F}$: $Y \to C[0, \infty)$ by

$$\mathscr{F}y(t) = \alpha - \frac{1}{n-2}\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)e^{y(s)}ds.$$

Then, $\mathscr{F}$ maps Y into itself. For, if $y \in Y$, then

$$\begin{aligned} \alpha \geq \mathscr{F}y(t) &\geq \alpha - \frac{1}{n-2}\int_0^t s\phi^*(s)e^{y(s)}ds \\ &\geq \alpha - \frac{e^\alpha}{n-2}\int_0^t s\phi^*(s)ds \geq k(\alpha), \quad t \geq 0. \end{aligned}$$

The continuity of $\mathscr{F}$ follows from the inequality

$$|\mathscr{F}y_m(t) - \mathscr{F}y(t)| \leq \frac{1}{n-2}\int_0^t s\phi^*(s)|e^{y_m(s)} - e^{y(s)}|ds,$$

and the relative compactness of $\mathscr{F}Y$ follows from the relation

$$|(\mathscr{F}y)'(t)| \leq \int_0^t \phi^*(s)e^{y(s)}ds \leq e^\alpha \int_0^t \phi^*(s)ds.$$

Consequently, Schauder-Tychonoff's fixed point theorem is applicable, and (3.3) (and hence (3.2)) has a bounded solution. Proceeding as in the proof of Theorem 2.3, we can prove the following

THEOREM 3.2. *Suppose assumption* (A) *is satisfied. If* (1.3) *holds, then equation* (1.2) *has infinitely many bounded entire solutions which tend to constants as* $|x| \to \infty$.

3.2. The case where $\phi(x) \leq 0$ in R^n.

THEOREM 3.3. *Let assumption* (A) *be satisfied. If the equation*

$$\Delta w - \phi^*(|x|)e^w = 0 \tag{3.5}$$

has a bounded solution in R^n, *then* (1.2) *has a bounded solution in* R^n.

Because of $\phi(x) \leq 0$ and (2.5), a bounded solution of (3.5) is a subsolution and any constant is a supersolution of (1.2) in R^n. Therefore, the conclusion of Theorem 3.3 follows from Theorem A.

We seek a spherically symmetric solution $w(x) = z(|x|)$ of (3.5). This reduces to the initial value problem:

$$\begin{cases} z'' + \dfrac{n-1}{t} z' - \phi^*(t)e^z = 0, \quad t > 0, \\ z(0) = \beta, \quad z'(0) = 0, \end{cases} \tag{3.6}$$

where β is constant. The equivalent integral equation is

$$z(t) = \beta + \frac{1}{n-2}\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)e^{z(s)}ds, \quad t \geq 0. \tag{3.7}$$

Define the operator $\mathscr{G}$ by

$$\mathscr{G}z(t) = \beta + \frac{1}{n-2}\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)e^{z(s)}ds, \quad t \geq 0. \tag{3.8}$$

Now, if (1.3) holds, then we choose β such that

$$e^{-\beta} - \frac{1}{n-2}\int_0^\infty s\phi^*(s)ds > 0. \tag{3.9}$$

Put

$$B(t) = \log\left(e^{-\beta} - \frac{1}{n-2}\int_0^t s\phi^*(s)ds\right)^{-1} \tag{3.10}$$

and consider the set

$$Z = \{z \in C[0, \infty) : \beta \leq z(t) \leq B(t) \quad for \quad t \geq 0\}.$$

If $z \in Z$, then we have for $t \geq 0$

$$\begin{aligned} \beta \leq \mathscr{G}z(t) &\leq \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)e^{z(s)}ds \\ &\leq \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)e^{B(s)}ds \end{aligned}$$

$$= \beta + \frac{1}{n-2}\int_0^t s\phi^*(s)\left(e^{-\beta} - \frac{1}{n-2}\int_0^s r\phi^*(r)\,dr\right)^{-1} ds$$

$$= \log\left(e^{-\beta} - \frac{1}{n-2}\int_0^t s\phi^*(s)ds\right)^{-1} = B(t)\,.$$

This shows that $\mathscr{G}$ maps Z into itself. The continuity of $\mathscr{G}$ and the relative compactness of $\mathscr{G}Z$ are obvious. Hence, there exists a fixed point z of $\mathscr{G}$ in Z, which is a solution of (3.6).

THEOREM 3.4. *Suppose assumption* (A) *is satisfied. If* (1.3) *holds, then* (1.2) *has infinitely many bounded entire solutions which tend to constants as* $|x|\to\infty$.

The proof of this theorem is similar to that of Theorem 2.6.

3.3. The case where $\phi(x)$ is not of constant sign in R^n.

THEOREM 3.5. *Let assumption* (A) *be satisfied. If* (3.1) *and* (3.5) *have bounded solutions* $v(x)$ *and* $w(x)$, *respectively, such that* $w(x)\le v(x)$, $x\in R^n$, *then* (1.2) *has a bounded entire solution* $u(x)$ *such that*

$$w(x) \le u(x) \le v(x), \quad x\in R^n. \tag{3.11}$$

Let α be a fixed constant. Choose a constant β so that (3.9) holds and put $l(\beta)=\lim_{t\to\infty} B(t)$, where $B(t)$ is defined by (3.10), that is

$$l(\beta) = \log\left(e^{-\beta} - \frac{1}{n-2}\int_0^\infty s\phi^*(s)ds\right)^{-1}. \tag{3.12}$$

Since $l(\beta)\to-\infty$ as $\beta\to-\infty$, we can take β so that

$$\beta < l(\beta) \le k(\alpha) < \alpha, \tag{3.13}$$

where $k(\alpha)$ is given by (3.4). Let $y(t)$ and $z(t)$ denote the solutions of (3.2) and (3.6), respectively. Then, $\beta\le z(t)\le l(\beta)\le k(\alpha)\le y(t)\le\alpha$ for $t\le 0$. If we define the functions $v(x)$ and $w(x)$ by

$$v(x) = y(|x|) \quad \text{and} \quad w(x) = z(|x|),$$

then, $v(x)$ and $w(x)$ are, respectively, solutions of (3.1) and (3.5) satisfying $w(x)\le v(x)$ in R^n. Theorem 3.5 then implies that (1.2) has an entire solution $u(x)$ satisfying (3.11). Thus we obtain the following theorem.

THEOREM 3.6. *In addition to assumption* (A) *suppose that* (1.3) *is satisfied. Then equation* (1.2) *has infinitely many bounded entire solutions.*

Combining Theorems 3.2, 3.4 and 3.6 yields Theorem 1.2 stated in the introduction. We conclude this section with a result which corresponds to Theorem 2.10.

THEOREM 3.7. *Let* $x=(x_1, x_2)\in R^m\times R^{n-m}$, $m\geq 3$. *Suppose there exists a positive locally Hölder continuous function* $\phi^*(t)$ *on* $[0, \infty)$ *satisfying* (2.35). *If* (1.3) *holds, then* (1.2) *has infinitely many bounded entire solutions. Moreover, if either* $\phi(x)\geq 0$ *or* $\phi(x)\leq 0$ *for all* $x\in R^n$, *then* (1.2) *has infinitely many bounded entire solutions, each of which tends to a constant uniformly in* x_2 *as* $|x_1|\to\infty$.

REMARK 3.1. The facts mentioned above are also true of the equation

$$\Delta u + \phi(x)e^{\alpha u} = 0,$$

where α is a nonzero constant, since by putting $v=\alpha u$, it is reduced to

$$\Delta v + \alpha\phi(x)e^{v} = 0.$$

4. Necessary conditions

In this section we are interested in necessary conditions in order that: (i) equation (1.1) possesses a bounded positive entire solution; and (ii) equation (1.2) possesses a bounded entire solution.

THEOREM 4.1. *Suppose that either* $\phi(x)\geq 0$ *or* $\phi(x)\leq 0$, $x\in R^n$, *and there exists a continuous function* $\phi^*(t)$ *on* $[0, \infty)$ *satisfying*

$$|\phi(x)| \geq \phi_*(|x|) \geq 0, \quad x\in R^n. \tag{4.1}$$

If (1.1) *has a positive entire solution which is bounded and bounded away from zero in* R^n, *then*

$$\int_0^\infty t\phi_*(t)dt < \infty. \tag{4.2}$$

PROOF. (i) The case where $\phi(x)\geq 0$ in R^n. We assume that (1.1) has a positive entire solution $u(x)$ which is bounded and bounded away from zero in R^n.

Let $\bar{u}(t)$ denote the spherical mean of $u(x)$ over the sphere $S_t=\{x\in R^n: |x|=t\}$, i.e.,

$$\bar{u}(t) = \frac{1}{\omega_n t^{n-1}}\int_{S_t} u(x)ds,$$

where ω_n denotes the surface area of the unit sphere S_1. The spherical mean of $u(x)$ satisfies the following relation (Lemma 2 of [10, p. 69]):

$$\overline{\Delta u} = \Delta\bar{u} = t^{1-n}\frac{d}{dt}\left(t^{n-1}\frac{d}{dt}\bar{u}(t)\right). \tag{4.3}$$

Taking the spherical mean of (1.1) over S_t and using (4.3) and (4.1), we obtain

$$t^{1-n}(t^{n-1}\bar{u}'(t))' + \phi_*(t)\overline{u^\gamma}(t) \le 0, \quad t > 0.$$

where $'=d/dt$. By the boundedness assumption on $u(x)$ there exist positive constants k and K such that $k \le \overline{u^\gamma}(t) \le K$. Therefore, we have

$$t^{1-n}(t^{n-1}\bar{u}'(t))' + k\phi_*(t) \le 0, \quad t > 0.$$

It is easy to see that the above inequality is equivalent to

$$(4.4) \qquad (t^{3-n}(t^{n-2}\bar{u}(t))')' + kt\phi_*(t) \le 0, \quad t > 0.$$

Since

$$(t^{3-n}(t^{n-2}\bar{u}(t))')' \le -kt\phi_*(t) \le 0, \quad t > 0,$$

$t^{3-n}(t^{n-2}\bar{u}(t))'$ is nonincreasing for $t>0$. On the other hand, it can be shown that $t^{3-2}(t^{n-2}\bar{u}(t))' \ge 0$, $t>0$. In fact, if there exists $t_0>0$ such that $-c = t_0^{3-n}(t_0^{n-2}\bar{u}(t_0))' < 0$, then we have

$$t^{3-n}(t^{n-2}\bar{u}(t))' \le -c < 0, \quad t \ge t_0,$$

or

$$(4.5) \qquad (t^{n-2}\bar{u}(t))' \le -ct^{n-3}, \quad t \ge t_0.$$

Integrating (4.5) over $[t_0, t]$, we obtain

$$\begin{aligned} t^{n-2}\bar{u}(t) - t_0^{n-2}\bar{u}(t_0) &\le -c\int_{t_0}^{t} s^{n-3}ds \\ &= \frac{c}{n-2}(t_0^{n-2} - t^{n-2}), \end{aligned}$$

which implies that $\bar{u}(t)<0$ eventually, since the last term tends to $-\infty$ as $t\to\infty$. But this contradicts the positivity of $\bar{u}(t)$, and so we have $(t^{n-2}\bar{u}(t))' \ge 0$, $t>0$, as claimed. From (4.4) we have

$$kt\phi_*(t) \le -(t^{3-n}(t^{n-2}\bar{u}(t))')'.$$

Integrating this over $[t_1, t]$, $t_1>0$ and noting that $t^{3-n}(t^{n-2}\bar{u}(t))' \ge 0$, we obtain

$$\begin{aligned} k\int_{t_1}^{t} s\phi^*(s)ds &\le t_1^{3-n}(t_1^{n-2}\bar{u}(t_1))' - t^{3-2}(t^{n-2}\bar{u}(t))' \\ &\le t_1^{3-n}(t_1^{n-2}\bar{u}(t_1))', \quad t \ge t_1. \end{aligned}$$

This shows that (4.2) holds.

(ii) The case where $\phi(x) \le 0$ in R^n. The spherical mean $\bar{u}(t)$ of $u(x)$ satisfies the inequality

$$t^{1-n}(t^{n-1}\bar{u}'(t))' - \phi_*(t)\overline{u^\gamma}(t) \geq 0, \quad t > 0.$$

There exist positive constants l and L such that $l \leq \overline{u^\gamma}(t) \leq L$. Hence, we have

$$t^{1-n}(t^{n-1}\bar{u}'(t))' - l\phi_*(t) \geq 0, \quad t > 0, \tag{4.6}$$

which is equivalent to

$$(t^{3-n}(t^{n-2}\bar{u}(t))')' - lt\phi_*(t) \geq 0, \quad t > 0. \tag{4.7}$$

This implies that $(t^{3-n}(t^{n-2}\bar{u}(t))')' \geq 0$, so that $t^{3-n}(t^{n-2}\bar{u}(t))'$ is nondecreasing for $t>0$. Moreover, $t^{3-n}(t^{n-2}\bar{u}(t))'$ is bounded above. To see this we assume to the contrary that for any $G>0$, there exists $t_G>0$ such that

$$t^{3-n}(t^{n-2}\bar{u}(t))' \geq G, \quad t \geq t_G. \tag{4.8}$$

Dividing (4.8) by t^{3-n} and integrating over $[t_G, t]$, we obtain

$$t^{n-2}\bar{u}(t) - t_G^{n-2}\bar{u}(t_G) \geq \frac{G}{n-2}(t^{n-2} - t_G^{n-2}), \quad t \geq t_G,$$

which implies

$$\bar{u}(t) \geq \frac{G}{n-2} + \left(\bar{u}(t_G) - \frac{G}{n-2}\right)t_G^{n-2}t^{2-n}, \quad t \geq t_G.$$

Since G is an arbitrary positive constant, it follows that

$$\limsup_{t\to\infty} \bar{u}(t) = \infty,$$

which contradicts the boundedness of $\bar{u}(t)$.

Therefore, there exists a positive constant M such that

$$t^{3-n}(t^{n-2}\bar{u}(t))' \leq M, \quad t > 0. \tag{4.9}$$

Let t_0 be fixed. Integrating (4.7) over $[t_0, t]$ and using (4.9), we have

$$\begin{aligned} l\int_{t_0}^{t} s\phi^*(s)ds &\leq t^{3-n}(t^{n-2}\bar{u}(t))' - t_0^{3-n}(t_0^{n-2}\bar{u}(t_0))' \\ &\leq M - t_0^{3-n}(t_0^{n-2}\bar{u}(t_0))', \quad t \geq t_0. \end{aligned}$$

This implies (4.2) and the proof is complete.

If $\phi(x) \geq 0$ in R^n and $\gamma < 1$ in (1.1), then a stronger result is obtained.

THEOREM 4.2. *Let $\phi(x) \geq 0$ in R^n and $\gamma < 1$. Suppose there exists a continuous function $\phi_*(t)$ on $[0, \infty)$ satisfying* (4.1). *If* (1.1) *has a positive entire solution, then* (4.2) *holds.*

PROOF. (i) The case where $0<\gamma<1$. Let $u(x)$ be a positive entire solution of (1.1). As in Kitamura and Kusano [5] we put

$$U(t)=\frac{1}{\omega_n t^{n-1}}\int_{S_t}[u(x)]^{1-\gamma}dS.$$

From Lemma 2 of [10] it follows that

$$t^{1-n}(t^{n-1}U'(t))'=\frac{1}{\omega_n t^{n-1}}\int_{S_t}\Delta[u(x)]^{-\gamma}dS.$$

By an easy calculation, we obtain

$$\begin{aligned}\Delta u^{1-\gamma}&=-\gamma(1-\gamma)u^{-\gamma-1}|\nabla u|^2+(1-\gamma)u^{-\gamma}\Delta u\\&\leq(1-\gamma)u^{-\gamma}\Delta u=-(1-\gamma)\phi(x)\\&\leq-(1-\gamma)\phi_*(|x|).\end{aligned}$$

Hence, we have

$$t^{1-n}(t^{n-1}U'(t))'\leq-(1-\gamma)\phi_*(t),\quad t>0,$$

or

$$(t^{3-n}(t^{n-2}U(t))')'\leq-(1-\gamma)t\phi_*(t),\quad t>0.$$

Proceeding exactly as in the part (i) of the proof of Theorem 4.1, we get the desired conclusion.

(ii) The case where $\gamma<0$. The following proof is motivated by Kusano and Swanson [7]. Let α be a fixed positive constant and put $v(x)=u(x)+\alpha$. We have

$$\Delta v+\phi_*(|x|)v^\gamma\leq 0,\quad x\in R^n, \tag{4.10}$$

Since

$$\Delta u+\phi(x)u^\gamma\geq\Delta v+\phi(x)v^\gamma\geq\Delta v+\phi_*(|x|)v^\gamma.$$

We take the spherical mean of (4.10) and make use of Jensen's inequality: $\overline{v^\gamma}\geq\bar{v}^\gamma$. (Note that the function v^γ with negative γ is convex.) Then we have

$$t^{1-n}(t^{n-1}\bar{v}'(t))'+\phi_*(t)\bar{v}^\gamma(t)\leq 0,\quad t>0,$$

or equivalently

$$(t^{3-n}(t^{n-2}\bar{v}(t))')'+t\phi_*(t)\bar{v}^\gamma(t)\leq 0,\quad t>0. \tag{4.11}$$

Then, $(t^{3-n}(t^{n-2}\bar{v}(t))')'\leq 0$, $t>0$, and an integration of this inequality shows that there exist constants c_1 and t_1 such that

$$(t^{n-2}\bar{v}(t))'\leq c_1 t^{n-3},\quad t\geq t_1. \tag{4.12}$$

Integrating (4.12) again over $[t_1, t]$, we obtain

$$t^{n-2}\bar{v}(t) - t_1^{n-2}\bar{v}(t_1) \leq \frac{c_1}{n-2}(t^{n-2}-t_1^{n-2}), \quad t \geq t_1,$$

which gives

$$\bar{v}(t) \leq \frac{c_1}{n-2} + \left(\bar{v}(t_1) - \frac{c_1}{n-2}\right)t_1^{n-2}t^{2-n}, \quad t \geq t_1.$$

This implies that there exists a positive constant C such that

$$\bar{v}(t) \leq C, \quad t \geq t_1.$$

Since $\gamma<0$, we have $\bar{v}^{\gamma}(t) \geq C^{\gamma}$, $t \geq t_1$, and combining this with (4.11), we get

$$(t^{3-n}(t^{n-2}\bar{v}(t))')' + C^{\gamma} t\phi_*(t) \leq 0, \quad t \leq t_1.$$

Integrating the above and noting that $t^{3-n}(t^{n-2}\bar{v}(t))'$ is positive, we see that

$$C^{\gamma}\int_{t_1}^{\infty} t\phi_*(t)dt < \infty,$$

thereby completing the proof of Theorem 4.2.

Combining Theorems 2.3, 2.6 and 4.1, we have the following theorem giving a necessary and sufficient condition for (1.1) to have a bounded positive entire solution which is bounded away from zero in R^n.

THEOREM 4.3. *Let either $\phi(x) \geq 0$ or $\phi(x) \leq 0$, $x \in R^n$. Suppose there exist a locally Hölder continuous function $\phi^*(t)$ on $[0, \infty)$ and a constant c $(0<c<1)$ satisfying*

$$c\phi^*(|x|) \leq |\phi(x)| \leq \phi^*(|x|), \quad x \in R^n. \tag{4.13}$$

Then, (1.3) *is a necessary and sufficient condition for* (1.1) *with $\gamma \neq 1$ to have a positive entire solution which is bounded and bounded away from zero in R^n.*

With regard to the sublinear equation (1.1) with nonnegative $\phi(x)$ a stronger result follows from Theorems 2.3 and 4.2.

THEOREM 4.4. *Let $\gamma<1$ and $\phi(x) \geq 0$, $x \in R^n$. Suppose there exist a function $\phi^*(t)$ and a constant c $(0<c<1)$ satisfying* (4.13). *Then* (1.3) *is a necessary and sufficient condition for* (1.1) *to have a positive entire solution.*

Applying the technique used in the proof of Theorem 4.1 to (1.2), we have the following theorem.

THEOREM 4.5. *Suppose that either $\phi(x) \geq 0$ or $\phi(x) \leq 0$, $x \in R^n$, and there* *exists a continuous function* $\phi_*(t)$ *satisfying* (4.1). *If* (1.2) *has a bounded entire solution, then* (4.2) *holds.*

The following theorem follows from Theorems 3.2, 3.4 and 4.5.

THEOREM 4.6. *Let either* $\phi(x)\geq 0$ *or* $\phi(x)\leq 0$, $x\in R^n$. *Suppose there exist a locally Hölder continuous function* $\phi^*(t)$ *on* $[0, \infty)$ *and a constant* c $(0<c<1)$ *such that* (4.13) *holds. Then,* (1.3) *is a necessary and sufficient condition for* (1.2) *to have a bounded entire solution.*

Now, we consider the equation

$$\Delta u + \phi(x)e^{-u} = 0. \tag{4.14}$$

THEOREM 4.7. *Let* $\phi(x)\geq 0$, $x\in R^n$. *Suppose there exists a function* $\phi_*(t)$ *satisfying* (4.1). *If* (4.14) *has an entire solution which is bounded below, then* (4.2) *holds.*

PROOF. Let $u(x)$ be a solution which is bounded below in R^n. Then, there exists a constant α satisfying $v(x)=u(x)+\alpha>0$ in R^n, and $v(x)$ satisfies the equation

$$\Delta v + \phi(x)e^{\alpha}e^{-v} = 0. \tag{4.15}$$

Taking the spherical mean of (4.15) over the sphere S_t and noting that e^{-v} is a convex function of v, we have

$$t^{1-n}(t^{n-1}\bar{v}'(t))' + e^{\alpha}\phi_*(t)e^{-\bar{v}(t)} \leq 0, \quad t > 0.$$

Hence,

$$(t^{3-n}(t^{n-2}\bar{v}(t))')' + e^{\alpha}t\phi_*(t)e^{-\bar{v}(t)} \leq 0, \quad t > 0. \tag{4.16}$$

Integrating the inequality $(t^{3-n}(t^{n-2}\bar{v}(t))')'\leq 0$, which is a consequence of (4.16), from $t_1>0$ to t, we see that $\bar{v}(t)$ is bounded, that is $\bar{v}(t)\leq k$, $t\geq t_1$, for some constant k. Combining this with (4.16), we obtain

$$(t^{3-n}(t^{n-2}\bar{v}(t))')' + e^{\alpha-k}t\phi_*(t) \leq 0, \quad t \geq t_1. \tag{4.17}$$

Since $(t^{n-2}\bar{v}(t))'\geq 0$ for $t>t_1$ (see the proof of Theorem 4.1), an integration of (4.17) yields

$$e^{\alpha-k}\int_{t_1}^{\infty} t\phi_*(t)dt < \infty.$$

This completes the proof.

THEOREM 4.8. *Let* $\phi(x)\geq 0$, $x\in R^n$. *Suppose there exist a locally Hölder continuous function* $\phi^*(t)$ *on* $[0, \infty)$ *and a positive constant* c *satisfying* (4.13).

Then (1.3) *is a necessary and sufficient condition in order that*: (i) *the equation*

$$\Delta u + \phi(x)e^{\alpha u} = 0, \quad \alpha < 0,$$

possesses an entire solution which is bounded below: *and* (ii) *the equation*

$$\Delta u - \phi(x)e^{\alpha u} = 0, \quad \alpha > 0,$$

possesses an entire solution which is bounded above.

This follows from Theorem 4.7 and Remark 3.1.

5. Systems of elliptic equations

In this section we consider elliptic systems of the form

$$\begin{cases} \Delta u + F(x, u, v) = 0 \\ \Delta v + G(x, u, v) = 0 \end{cases} \tag{5.1}$$

in R^n, where $F(x, u, v)$ and $G(x, u, v)$ are defined on $R^n \times R^1 \times R^1$, are locally Hölder continuous in x with exponent λ and are continuously differentiable in u and v.

Our objective here is to extend the existence theory developed in Sections 2 and 3 to the elliptic systems (1.4) and (1.5), which are specializations of (5.1). This can be done, since the previous supersolution-subsolution method (Theorem A) can be so extended as to apply directly to systems of the form (5.1).

By a *super-supersolution* of (5.1) in R^n is meant a vector function $(\check{u}, \check{v}) \in C^{2+\lambda}_{loc}(R^n) \times C^{2+\lambda}_{loc}(R^n)$ satisfying the differential inequalities

$$\begin{cases} \Delta \check{v} + F(x, \check{u}, \check{v}) \le 0 \\ \Delta \check{v} + G(x, \check{u}, \check{v}) \le 0 \end{cases} \tag{5.2}$$

in R^n. A vector function $(\hat{u}, \hat{v}) \in C^{2+\lambda}_{loc}(R^n) \times C^{2+\lambda}_{loc}(R^n)$ satisfying

$$\begin{cases} \Delta \hat{u} + F(x, \hat{u}, \hat{v}) \ge 0 \\ \Delta \hat{v} + G(x, \hat{u}, \hat{v}) \ge 0 \end{cases} \tag{5.3}$$

in R^n is called a *sub-subsolution* of (5.1) in R^n. A *super-subsolution* of (5.1) in R^n is a vector function $(\check{u}, \hat{v}) \in C^{2+\lambda}_{loc}(R^n) \times C^{2+\lambda}_{loc}(R^n)$ which satisfies the following inequalities in R^n:

$$\begin{cases} \Delta \check{u} + F(x, \check{u}, \hat{v}) \le 0 \\ \Delta \hat{v} + G(x, \check{u}, \hat{v}) \ge 0. \end{cases} \tag{5.4}$$

A *sub-supersolution* of (5.1) can be defined analogously.

Our basic existence theorems for (5.1) follow.

THEOREM 5.1. *Suppose* $F_v(x, u, v) \geq 0$ *and* $G_u(x, u, v) \geq 0$ *in* $R^n \times R^1 \times R^1$. *If there exist a bounded super-supersolution* $(\check{u}(x), v(x))$ *and a bounded sub-subsolution* $(\hat{u}(x), \hat{v}(x))$ *of* (5.1) *in* R^n *such that*

$$\hat{u}(x) \leq \check{u}(x), \quad \hat{v}(x) \leq \check{v}(x), \quad x \in R^n, \tag{5.5}$$

then system (5.1) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying*

$$\hat{u}(x) \leq u(x) \leq \check{u}(x), \quad \hat{v}(x) \leq v(x) \leq \check{v}(x), \quad x \in R^n. \tag{5.6}$$

THEOREM 5.2. *Suppose* $F_v(x, u, v) \leq 0$ *and* $G_u(x, u, v) \leq 0$ *in* $R^n \times R^1 \times R^1$. *If there exist a bounded super-subsolution* $(\check{u}(x), \hat{v}(x))$ *and a bounded sub-supersolution* $(\hat{u}(x), \check{v}(x))$ *of* (5.1) *in* R^n *such that* (5.5) *holds, then system* (5.1) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying* (5.6).

We give a detailed proof of Theorem 5.1, which is based on the following lemma.

LEMMA 5.1. *Let* B_R *be a ball with radius* $R>0$ *in* R^n. *If the hypotheses of Theorem* 5.1 *are satisfied, then there exist vector functions* $(\check{u}_R(x), \check{v}_R(x))$ *and* $(\hat{u}_R(x), \hat{v}_R(x))$ *with the following properties*:

(i) $(\check{u}_R(x), \check{v}_R(x))$ *and* $(\hat{u}_R(x), \hat{v}_R(x))$ *are both of class* $C^{2+\lambda}(\bar{B}_R) \times C^{2+\lambda}(\bar{B}_R)$;

(ii) $(\check{u}_R(x), \check{v}_R(x))$ *and* $(\hat{u}_R(x), \hat{v}_R(x))$ *satisfy* (5.1) *in* B_R; *and*

(iii) $$\hat{u}(x) \leq \hat{u}_R(x) \leq \check{u}_R(x) \leq \check{u}(x),$$
$$\hat{v}(x) \leq \hat{v}_R(x) \leq \check{v}_R(x) \leq \check{v}(x), \quad x \in \bar{B}_R.$$

PROOF OF LEMMA 5.1. We proceed as in Sattinger [11, Theorem 2.1]. Take a vector function $(f, g) \in C^{2+\lambda}_{loc}(R^n) \times C^{2+\lambda}_{loc}(R^n)$ such that

$$\hat{u}(x) \leq f(x) \leq \check{u}(x), \quad \hat{v}(x) \leq g(x) \leq \check{v}(x), \quad x \in R^n,$$

and consider the boundary value problem

$$\begin{cases} \Delta u + F(x, u, v) = 0 \\ \Delta v + G(x, u, v) = 0 \quad \text{in} \quad B_R, \end{cases} \tag{5.7}$$
$$u(x) = f(x), \quad v(x) = g(x) \quad \text{on} \quad \partial B_R.$$

Since F and G are continuously differentiable in u and v there exist positive constants K_1, K_2 satisfying

$$F_u(x, u, v) + K_1 \geq 0, \quad G_v(x, u, v) + K_2 \geq 0, \quad \text{in} \quad B_R \times I_1 \times I_2, \tag{5.8}$$

where $I_1=[\inf_{B_R}\hat{u}(x), \sup_{B_R}\check{u}(x)]$, $I_2=[\inf_{B_R}\hat{v}(x), \sup_{B_R}\check{v}(x)]$.

Now we consider the following iteration scheme:

$$\begin{cases} (\Delta-K_1)u_m = -[F(x, u_{m-1}, v_{m-1})+K_1u_{m-1}] \\ (\Delta-K_2)v_m = -[G(x, u_{m-1}, v_{m-1})+K_2v_{m-1}] \quad \text{in} \quad B_R, \end{cases}$$
$$u_m(x)=f(x), \quad v_m(x)=g(x) \quad \text{on} \quad \partial B_R, \quad m=1, 2,\dots.$$

If we put $(u_0(x), v_0(x))=(\check{u}(x), \check{v}(x))$, then $(u_1(x), v_1(x))$ is well-defined and (5.2) implies

$$\begin{cases} (\Delta-K_1)(u_1-u_0) = -[\Delta\check{u}+F(x, \check{u}, \check{v})] \geq 0 \\ (\Delta-K_2)(v_1-v_0) = -[\Delta\check{v}+G(x, \check{u}, \check{v})] \geq 0 \quad \text{in} \quad B_R, \end{cases}$$
$$u_1(x)=f(x), \quad v_1(x)=g(x) \quad \text{on} \quad \partial B_R.$$

Hence, by the maximum principle

$$u_0(x)\geq u_1(x), \quad v_0(x)\geq v_1(x), \quad x\in B_R. \tag{5.10}$$

Put $\tilde{F}(x, u, v)=F(x, u, v)+K_1u$, $\tilde{G}(x, u, v)=G(x, u, v)+K_2v$. Then, by (5.8), $\tilde{F}(x, u, v)$ and $\tilde{G}(x, u, v)$ are nondecreasing in u and v. Therefore, if we assume that $u_{m-1}(x)\geq u_m(x)$, $v_{m-1}(x)\geq v_m(x)$ in B_R, then from (5.9) we see that

$$\begin{cases} (\Delta-K_1)(u_{m+1}-u_m) = -[\tilde{F}(x, u_m, v_m)-\tilde{F}(x, u_{m-1}, v_{m-1})]\geq 0 \\ (\Delta-K_2)(v_{m+1}-v_m) = -[\tilde{G}(x, u_m, v_m)-\tilde{G}(x, u_{m-1}, v_{m-1})]\geq 0 \text{ in } B_R, \end{cases} \tag{5.11}$$
$$u_{m+1}(x)=u_m(x)=f(x), \quad v_{m+1}(x)=v_m(x)=g(x) \quad \text{on} \quad \partial B_R,$$

and again by the maximum principle we have

$$u_m(x)\geq u_{m+1}(x), \quad v_m(x)\geq v_{m+1}(x), \quad x\in B_R. \tag{5.12}$$

Since (5.10) holds, by incuction we get a sequence $\{(u_m(x), v_m(x))\}_{m=1}^{\infty}$ satisfying (5.12) for each m. We denote this sequence by $\{(\check{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$. If we put $(u_0(x), v_0(x))=(\hat{u}(x), \hat{v}(x))$, then (5.9) yields a sequence $\{(\hat{u}_m(x), \hat{v}_m(x))\}_{m=1}^{\infty}$ with the property that

$$\hat{u}_m(x)\leq\hat{u}_{m+1}(x) \quad \text{and} \quad \hat{v}_m(x)\leq\hat{v}_{m+1}(x) \quad \text{in} \quad B_R, \quad m=1, 2,\dots.$$

On the other hand, in view of (5.5), we have

$$\begin{cases} (\Delta-K_1)(\check{u}_1-\hat{u}_1) = -[\tilde{F}(x,\check{u}, \check{v}) - \tilde{F}(x, \hat{u}, \hat{v})]\leq 0 \\ (\Delta-K_2)(\check{v}_1-\hat{v}_1) = -[\tilde{G}(x, \check{u}, \check{v}) - \tilde{G}(x, \hat{u}, \hat{v})]\leq 0 \quad \text{in} \quad B_R, \end{cases}$$

$$\check{u}_1(x) = \hat{u}_1(x) = f(x), \quad \check{v}_1(x) = \hat{v}_1(x) = g(x) \quad \text{on} \quad \partial B_R.$$

Hence, the maximum principle implies that $\check{u}_1(x) \geq \hat{u}_1(x)$ and $\check{v}_1(x) \geq \hat{v}_1(x)$ in B_R. By induction it is easily seen that the sequences $\{(\check{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$ and $\{(\hat{u}_m(x), \hat{v}_m(x))\}_{m=1}^{\infty}$ satisfy

$$\begin{aligned} &\hat{u} \leq \hat{u}_1 \cdots \leq \hat{u}_m \leq \cdots \leq \check{u} \cdots \leq \check{u}_1 \leq \check{u} \\ &\hat{v} \leq \hat{v}_1 \leq \cdots \leq \hat{v}_m \leq \cdots \leq \check{v}_m \leq \cdots \leq \check{v}_1 \leq \check{v} \text{ in } B_R. \end{aligned} \tag{5.13}$$

Therefore, $\{(\check{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$ and $\{(\hat{u}_m(x), \hat{v}_m(x))\}_{m=1}^{\infty}$ converge pointwise to some vector functions $(\check{u}_R(x), \check{v}_R(x))$ and $(\hat{u}_R(x), \hat{v}_R(x))$ respectively in $\bar{B}_R$.

Now we prove that both $(\check{u}_R(x), \check{v}_R(x))$ and $(\hat{u}_R(x), \hat{v}_R(x))$ are solutions of the boundary value problem (5.7). Since $(\check{u}(x), \check{v}(x))$ and $(\hat{u}(x), \hat{v}(x))$ are bounded on B_R, there exist positive constants L_1 and M_1 such that

$$\begin{aligned} &\|F(x, \check{u}_m, \check{v}_m)\|_{L^p(B_R)} \leq L_1, \ \|\check{u}_m\|_{L^p(B_R)} \leq L_1, \\ &\|G(x, \check{u}_m, \check{v}_m)\|_{L^p(B_R)} \leq M_1, \ \|\check{v}_m\|_{L^p(B_R)} \leq M_1, \end{aligned}$$

for all m. Moreover, $\|f\|_{W_p^2(B_R)}$ and $\|g\|_{W_p^2(B_R)}$ are bounded for any $p>1$. Hence, by the L^p-estimates of Agmon-Douglis-Nirenberg [1, Theorem 15.2 and its Corollary] with choice $p=n/(1-\lambda)$, there exist positive constants L_2 and M_2 independent of m such that

$$\|\check{u}_m\|_{C^{1+\lambda}(B)} \leq L_2, \ \|\check{v}_m\|_{C^{1+\lambda}(B_R)} \leq M_2. \tag{5.14}$$

This implies that $F(x, \check{u}_m(x), \check{v}_m(x))$ and $G(x, \check{u}_m(x), \check{v}_m(x))$ are Hölder continuous with exponent λ in $\bar{B}_R$ and their Hölder norms are independent of m. From the Schauder estimates

$$\begin{aligned} &\|\check{u}_m\|_{C^{2+\lambda}(\bar{B}_R)} \leq L(\|F\|_{C^\lambda(\bar{B}_R)} + \|f\|_{C^{2+\lambda}(\partial B_R)} + \|\check{u}_m\|_{C^0(\bar{B}_R)}), \\ &\|\check{v}_m\|_{C^{2+\lambda}(\bar{B}_R)} \leq M_3(G\|_{C^\lambda(\bar{B}_R)} + \|g\|_{C^{2+\lambda}(\partial B_R)} + \|\check{v}_m\|_{C^0(\bar{B}_R)}) \end{aligned}$$

with constants L_3 and M_3 independent of m, it follows that the sequence $\{(\check{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$ is bounded in $C^{2+\lambda}(\bar{B}_R) \times C^{2+\lambda}(\bar{B}_R)$. Since the injection $C^{2+\lambda}(\bar{B}_R) \times C^{2+\lambda}(\bar{B}_R) \to C^2(\bar{B}_R) \times C^2(\bar{B}_R)$ is compact and $\{(\check{u}_m(x), \check{v}m(x))\}_{m=1}^{\infty}$ is a monotone sequence, $\{(\check{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$ converges in $C^2(\bar{B}_R) \times C^2(\bar{B}_R)$ to $(\check{u}_R(x), \check{v}_R(x))$. From (5.9) it clearly follows that $(\check{u}_R(x), \check{v}_R(x))$ is a solution of the boundary value problem (5.7) and of course belongs to $C^{2+\lambda}(\bar{B}_R) \times C^{2+\lambda}(\bar{B}_R)$.

In an analogous way it can be shown that $(\hat{u}_R(x), \hat{v}_R(x))$ is also a solution of the boundary value problem (5.7) of class $C^{2+\lambda}(\bar{B}_R) \times C^{2+\lambda}(\bar{B}_R)$. Furthermore, from (5.13) the following relations are valid

$$\hat{u}(x) \leq \hat{u}_R(x) \leq \check{u}_R(x) \leq \check{u}(x), \quad \hat{v}(x) \leq \hat{v}_R(x) \leq \check{v}_R(x) \leq \check{v}(x), \quad x \in \bar{B}_R.$$

This completes the proof of Lemma 5.1.

PROOF OF THEOREM 5.1. By Lemma 5.1, for each ball $B_R=\{x\in R^n: |x|<R\}$, $R=1, 2\dots$, there exists a solution $(u_R(x), v_R(x))$ of (5.7) satisfying

$$\hat{u}(x)\le u_R(x)\le \check{u}(x), \quad \hat{v}(x)\le v_R(x)\le \check{v}(x), \quad x\in \bar{B}_R. \tag{5.15}$$

Consider the sequence $\{(u_R(x), v_R(x))\}_{R=1}^{\infty}$. We wish to show that this sequence contains a subsequence converging to a desired entire solution of (5.1).

Let $S>0$ be any fixed integer and let $R\ge S+1$. We claim that there exist constants L_4 and M_4 independent of R such that

$$\|u_R\|_{C^{2+\lambda}(B_S)}\le L_4 \quad\text{and}\quad \|v_R\|_{C^{2+\lambda}(B_S)}\le M_4. \tag{5.16}$$

According to (5.15), $\|F(x, u_R, v_R)\|_{L^p(B_{S+1})}$, $\|G(x, u_R, v_R)\|_{L^p(B_{S+1})}$, $\|u_R\|_{L^p(B_{S+1})}$ and $\|v_R\|_{L^p(B_{S+1})}$ are uniformly bounded. Therefore, by the interior L^p-estimates of Agmon-Douglis-Nirenberg with choice $p=n/(1-\lambda)$, there exist positive constants L_5 and M_5 independent of R such that

$$\|u_R\|_{C^{1+\lambda}(B_{S+\rho})}\le L_5 \text{ and } \|v_R\|_{C^{1+\lambda}(B_{S+\rho})}\le M_5, \tag{5.17}$$

where ρ is any constant with $0<\rho<1$. This implies that the functions $F_R(x)=F(x, u_R(x), v_R(x))$ and $G_R(x)=G(x, u_R(x), v_R(x))$ are uniformly Hölder continuous with exponent λ in $\bar{B}_{S+\rho}$. Applying the Schauder interior estimates, we have

$$\begin{aligned}\|u_R\|_{C^{2+\lambda}(B_S)}&\le C_1(\|u_R\|_{C^0(B_{S+\rho})}+\|F_R\|_{C^\lambda(B_{S+\rho})}),\\ \|v_R\|_{C^{2+\lambda}(B_S)}&\le C_2(\|v_R\|_{C^0(B_{S+\rho})}+\|G_R\|_{C^\lambda(B_{S+\rho})}),\end{aligned} \tag{5.18}$$

where C_1 and C_2 are independent of R, giving the required estimtes (5.16).

Because of the compactness of the injection $C^{2+\lambda}(B_1)\times C^{2+\lambda}(B_1)\to C^2(B_1)\times C^2(B_1)$, $\{(u_R(x), v_R(x))\}_{R=1}^{\infty}$ has a subsequence $\{(u_{R_{j1}}(x), v_{R_{j1}}(x))\}_{j=1}^{\infty}$ which converges in $C^2(B_1)\times C^2(B_1)$ to a vector function $(u^1(x), v^1(x))$. Obviously $(u^1(x), v^1(x))$ satisfies (5.1) in B_1 and

$$\hat{u}(x)\le u^1(x)\le \check{u}(x), \quad \hat{v}(x)\le v^1(x)\le \check{v}(x), \quad x\in \bar{B}_1.$$

In a similar way $\{(u_{R_{j1}}(x), v_{R_{j1}}(x))\}_{j=1}^{\infty}$ has a subsequence $\{(u_{R_{j2}}(x), v_{R_{j2}}(x))\}_{j=1}^{\infty}$ which converges to a vector function $(u^2(x), v^2(x))$ in $C^2(B_2)\times C^2(B_2)$. Repeating this procedure, we obtain for each $k=1, 2,\dots$ a sequence $\{(u_{R_{jk}}(x), v_{R_{jk}}(x))\}_{j=1}^{\infty}$ which converges in $C^2(B_k)\times C^2(B_k)$ and is a subsequence of $\{(u_{R_{jk-1}}(x), v_{R_{jk-1}}(x))\}_{j=1}^{\infty}$. Let $(u^k(x), v^k(x))=\lim_{j\to\infty}(u_{R_{jk}}(x), v_{R_{jk}}(x))$. Then, $(u^k(x), v^k(x))$ satisfies (5.1) in B_k and

$$\hat{u}(x)\le u^k(x)\le \check{u}(x), \quad \hat{v}(x)\le v^k(x)\le \check{v}(x), \quad x\in \bar{B}_k.$$

Moreover, $(u^k(x), v^k(x))|_{B_{k-1}}=(u^{k-1}(x), v^{k-1}(x))$. Accordingly, we define $(U(x), V(x))$ in R^n such that

$$(U(x),\ V(x)) = (u^k(x),\ v^k(x)) \quad \text{if} \quad x \in B_k.$$

Then $(U(x), V(x))$ is obviously a solution of system (5.1) in R^n satisfying

$$\hat{u}(x) \le U(x) \le \check{u}(x), \quad \hat{v}(x) \le V(x) \le \check{v}(x), \quad x \in R^n.$$

This completes the proof of theorem 5.1.

The proof of Theorem 5.2 is based on the following lemma.

LEMMA 5.2. *Let B_R be a ball with radius $R>0$ in R^n. If the hypotheses of Theorem 5.2 are satisfied, then there exist vector functions $(\check{u}_R(x), \hat{v}_R(x))$ and $(\hat{u}_R(x), \check{v}_R(x))$ with the following properties*:

(i) *$(\check{u}_R(x), \hat{v}_R(x))$ and $(\hat{u}_R(x), \check{v}_R(x))$ are both of class $C^{2+\lambda}(\bar{B}_R)\times C^{2+\lambda}(\bar{B}_R)$*;

(ii) *$(\check{u}_R(x), \hat{v}_R(x))$ and $(\hat{u}_R(x), \check{v}_R(x))$ satisfy (5.1) in B_R; and*

(iii)
$$\hat{u}(x) \le \hat{u}_R(x) \le \check{u}_R(x) \le \check{u}(x),$$
$$\hat{v}(x) \le \hat{v}_R(x) \le \check{v}_R(x) \le \check{v}(x), \quad x \in \bar{B}_R.$$

To prove Lemma 5.2 we employ the same iteration scheme (5.9) with constants K_1 and K_2 satisfying (5.8). Let $\tilde{F}(x, u, v)$ and $\tilde{G}(x, u, v)$ be as in the proof of Lemma 5.1. Then, $\tilde{F}(x, u, v)$ is nondecreasing in u and nonincreasing in v, and $\tilde{G}(x, u, v)$ is nonincreasing in u and nondecreasing in v. Using this fact and the maximum principle, we can show that the iteration scheme (5.9) with $(u_0(x), v_0(x))=(\check{u}(x), \hat{v}(x))$ and $(u_0(x), v_0(x))=(\hat{u}(x), \check{v}(x))$ produces sequences of vector functions $\{(\check{u}_m(x), \hat{v}_m(x))\}_{m=1}^{\infty}$ and $\{(\hat{u}_m(x), \check{v}_m(x))\}_{m=1}^{\infty}$, respectively, satisfying (5.13). The desired functions are obtained as the limits of these sequences:

$$(\check{u}_R(x), \hat{v}_R(x)) = \lim_{m\to\infty} (\check{u}_m(x), \hat{v}_m(x)), \quad (\hat{u}_R(x), \check{v}_R(x)) = \lim_{m\to\infty} (\hat{u}_m(x), \check{v}_m(x)).$$

In what follows we consider the systems

$$(5.19) \qquad \begin{cases} \Delta u + \phi(x)f(u, v) = 0 \\ \Delta v + \psi(x)g(u, v) = 0, \end{cases}$$

$$(5.20) \qquad \begin{cases} \Delta u - \phi(x)f(u, v) = 0 \\ \Delta v - \psi(x)g(u, v) = 0, \end{cases}$$

which are specializations of (5.1). We assume that $\phi(x)$ and $\psi(x)$ are nonnegative and locally Hölder continuous (with exponent λ) in R^n, and that $f(u, v)$ and $g(u, v)$ are positive and continuously differentiable in u and v.

Suppose that there exist nonnegative locally Hölder continuous functions $\phi_*(x)$, $\phi^*(x)$, $\psi_*(x)$ and $\psi^*(x)$ in R^n such that

$$\phi_*(x) \leq \phi(x) \leq \phi^*(x), \quad \psi_*(x) \leq \psi(x) \leq \psi^*(x), \quad x \in R^n. \tag{5.21}$$

It is easy to see that under condition (5.21) a vector function $(\check{u}(x), \check{v}(x))$ satisfying

$$\begin{cases} \Delta \check{u} + \phi^*(x) f(\check{u}, \check{v}) = 0 \\ \Delta \check{v} + \psi^*(x) g(\check{u}, \check{v}) = 0 \end{cases} \tag{5.22}$$

in R^n is a super-supersolution of (5.19) in R^n, and a function $(\hat{u}(x), \hat{v}(x))$ satisfying

$$\begin{cases} \Delta \hat{u} + \phi_*(x) f(\hat{u}, \hat{v}) = 0 \\ \Delta \hat{v} + \psi_*(x) g(\hat{u}, \hat{v}) = 0 \end{cases} \tag{5.23}$$

in R^n is a sub-subsolution of (5.19) in R^n. Similarly, functions $(\check{u}(x), \hat{v}(x))$ and $(\hat{u}(x), \check{v}(x))$ satisfying

$$\begin{cases} \Delta \check{u} + \phi^*(x) f(\check{u}, \hat{v}) = 0 \\ \Delta \hat{v} + \psi_*(x) g(\check{u}, \hat{v}) = 0 \end{cases} \tag{5.24}$$

and

$$\begin{cases} \Delta \hat{u} + \phi_*(x) f(\hat{u}, \check{v}) = 0 \\ \Delta \check{v} + \psi^*(x) g(\hat{u}, \check{v}) = 0 \end{cases} \tag{5.25}$$

in R^n are, respectively, a super-subsolution and a sub-supersolution of (5.19) in R^n. The following theorem immediately follows from the above observation and Theorems 5.1 and 5.2.

THEOREM 5.3. (i) *Suppose* $f_v(u, v) \geq 0$ *and* $g_u(u, v) \geq 0$. *If* (5.22) *and* (5.23) *possess bounded solutions* $(\check{u}(x), \check{v}(x))$ *and* $(\hat{u}(x), \hat{v}(x))$ *satisfying* (5.5), *then* (5.19) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying* (5.6).

(ii) *Suppose* $f_v(u, v) \leq 0$ *and* $g_u(u, v) \leq 0$. *If* (5.24) *and* (5.25) *possess bounded solutions* $(\check{u}(x), \hat{v}(x))$ *and* $(\hat{u}(x), \check{v}(x))$ *satisfying* (5.5), *then* (5.19) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying* (5.6).

Likewise, under condition (5.21), vector functions $(\check{u}(x), \check{v}(x))$ and $(\hat{u}(x), \hat{v}(x))$ satisfying

$$\begin{cases} \Delta \check{u} - \phi_*(x) f(\check{u}, \check{v}) = 0 \\ \Delta \check{v} - \psi_*(x) g(\check{u}, \check{v}) = 0 \end{cases} \tag{5.26}$$

and

$$(5.27)\qquad \begin{cases} \Delta\hat{u} - \phi^*(x)f(\hat{u}, \hat{v}) = 0 \\ \Delta\hat{v} - \psi^*(x)g(\hat{u}, \hat{v}) = 0 \end{cases}$$

in R^n are, respectively, a super-supersolution and a sub-subsolution of (5.20) in R^n, and $(\check{u}(x), \hat{v}(x))$ and $(\hat{u}(x), \check{v}(x))$ satisfying

$$(5.28)\qquad \begin{cases} \Delta\check{u} - \phi_*(x)f(\check{u}, \hat{v}) = 0 \\ \Delta\hat{v} - \psi^*(x)g(\check{u}, \hat{v}) = 0 \end{cases}$$

and

$$(5.29)\qquad \begin{cases} \Delta\hat{u} - \phi^*(x)f(\hat{u}, \check{v}) = 0 \\ \Delta\check{v} - \psi_*(x)g(\hat{u}, \check{v}) = 0 \end{cases}$$

in R^n are, respectively, a super-subsolution and a sub-supersolution of (5.20) in R^n. This observation combined with Theorems 5.1 and 5.2 yield the following result.

THEOREM 5.4. (i) *Suppose* $f_v(u, v) \geq 0$ *and* $g_u(u, v) \geq 0$. *If* (5.28) *and* (5.29) *possess bounded solutions* $(\check{u}(x), \hat{v}(x))$ *and* $(\hat{u}(x), \check{v}(x))$ *satisfying* (5.5), *then* (5.20) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying* (5.6).

(ii) *Suppose* $f_v(u, v) \leq 0$ *and* $g_u(u, v) \leq 0$. *If* (5.26) *and* (5.27) *possess bounded solutions* $(\check{u}(x), \check{v}(x))$ *and* $(\hat{u}(x), \hat{v}(x))$ *satisfying* (5.5), *then* (5.20) *possesses a bounded entire solution* $(u(x), v(x))$ *satisfying* (5.6).

6. Systems of elliptic equations (continued)

Let us now apply the above existence theorems to the specific systems (1.4) and (1.5) with one-signed coefficients $\phi(x)$ and $\psi(x)$.

THEOREM 6.1. *Suppose that* $\delta \geq 0$, $\mu \geq 0$, $\gamma + \delta > 1$ *and* $\mu + \nu > 1$. *Suppose moreover that there exist locally Hölder continuous functions* $\phi^*(t)$ *and* $\psi^*(t)$ *on* $[0, \infty)$ *such that*

$$(6.1)\qquad 0 \leq \phi(x) \leq \phi^*(|x|), \quad 0 \leq \psi(x) \leq \psi^*(|x|), \quad x \in R^n.$$

If

$$(6.2)\qquad \int_0^\infty t\phi^*(t)dt < \infty \quad \textit{and} \quad \int_0^\infty t\psi^*(t)dt < \infty,$$

then system (1.4) *has infinitely many positive entire solutions* $(u(x), v(x))$ *such that* $u(x)$ *and* $v(x)$ *are bounded and tend to positive constants as* $|x| \to \infty$.

Proof. We first construct a positive super-supersolution $(\check{u}(x), \check{v}(x))$ of (1.4) as a spherically symmetric solution of the system

$$(6.3)\qquad \begin{cases} \Delta\check{u} + \phi^*(|x|)\check{u}^{\gamma}\check{v}^{\delta} = 0 \\ \Delta\check{v} + \psi^*(|x|)\check{u}^{\mu}\check{v}^{\nu} = 0 \end{cases}$$

in R^n. If $(y(t), z(t))$ is a solution of the system of ordinary differential equations

$$(6.4)\qquad \begin{cases} y'' + \dfrac{n-1}{t}y' + \phi^*(t)y^{\gamma}z^{\delta} = 0 \\ z'' + \dfrac{n-1}{t}z' + \psi^*(t)y^{\mu}z^{\nu} = 0, \quad t > 0, \end{cases}$$

satisfying the initial conditions

$$(6.5)\qquad y(0) = z(0) = \alpha > 0, \quad y'(0) = z'(0) = 0,$$

α being a constnat, then the function $(\check{u}(x), \check{v}(x))$ with $\check{u}(x)=y(|x|)$, $\check{v}(x)=z(|x|)$ is a solution of (6.4). To solve (6.4)–(6.5) the Schauder-Tychonoff fixed point theorem is used. Let $\alpha>0$ be small enough so that

$$(6.6)\qquad 1 - \frac{\alpha^{\gamma+\delta-1}}{n-2}\int_0^{\infty} t\phi^*(t)\,dt > 0, \quad 1 - \frac{\alpha^{\mu+\nu-1}}{n-2}\int_0^{\infty} t\psi^*(t)\,dt > 0$$

and put

$$(6.7)\qquad l(\alpha) = \alpha - \frac{\alpha^{\gamma+\delta}}{n-2}\int_0^{\infty} t\phi^*(t)\,dt, \quad m(\alpha) = \alpha - \frac{\alpha^{\mu+\nu}}{n-2}\int_0^{\infty} t\psi^*(t)\,dt.$$

Consider the set defined by

$$X = \{(y, z) \in C[0, \infty) \times C[0, \infty): l(\alpha) \leq y(t) \leq \alpha,\ m(\alpha) \leq z(t) \leq \alpha,\ t \geq 0\}.$$

X is a closed convex subset of the locally convex space $C[0, \infty) \times C[0, \infty)$ of all continuous vector functions on $[0, \infty)$ with the topology of uniform convergence on every compact subinterval of $[0, \infty)$. Define the operator $\mathscr{F}: X \to C[0, \infty) \times C[0, \infty)$ by $\mathscr{F}(y, z) = (y^*, z^*)$, where

$$(6.8)\qquad \begin{cases} y^*(t) = \alpha - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)y^{\gamma}(s)z^{\delta}(s)\,ds \\ z^*(t) = \alpha - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\psi^*(s)y^{\mu}(s)z^{\nu}(s)\,ds, \quad t \geq 0. \end{cases}$$

Applying an argument similar to that used in the proof of Theorem 2.2, we can show that $\mathscr{F}$ maps X into itself, that $\mathscr{F}$ is continuous and that the image set $\mathscr{F}X$ is relatively commpact in $C[0, \infty) \times C[0, \infty)$. Therefore, the Schauder-Tychonoff theorem implies that $\mathscr{F}$ has a fixed point $(y, z) \in X$. This fixed point is a solution of the system of integral equations

$$(6.9)\quad \begin{cases} y(t) = \alpha - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)y^{\gamma}(s)z^{\delta}(s)ds \\ z(t) = \alpha - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\psi^*(s)y^{\mu}(s)z^{\nu}(s)ds \end{cases}$$

for $t \geq 0$. By differentiation of (6.9) we see that $(y(t), z(t))$ is a solution of (6.4)–(6.5). Since

$$y'(t) = -\int_0^t \left(\frac{s}{t}\right)^{n-1} \phi^*(s)y^{\gamma}(s)z^{\delta}(s)ds < 0,$$

$$z'(t) = -\int_0^t \left(\frac{s}{t}\right)^{n-2} \psi^*(s)y^{\mu}(s)z^{\nu}(s)ds < 0,$$

there exist positive constants η and ζ such that $\lim_{t\to\infty} y(t)=\eta \geq l(\alpha)$ and $\lim_{t\to\infty} z(t)=\zeta \geq m(\alpha)$. Define $\hat{u}(x)=\eta$ and $\hat{v}(x)=\zeta$. Then, the constant vector function $(\hat{u}(x), \hat{v}(x))=(\eta, \zeta)$ is a sub-subsolution of (1.4) in R^n and satisfies (5.5). Applying (i) of Theorem 5.3, we conclude that (1.4) has an entire solution $(u(x), v(x))$ such that

$$\eta \leq u(x) \leq y(|x|), \quad \zeta \leq v(x) \leq z(|x|), \quad x \in R^n.$$

Obviously, $\lim_{|x|\to\infty} u(x)=\eta$ and $\lim_{|x|\to\infty} v(x)=\zeta$. The conclusion of the theorem follows, since there exist infinitely many constants $\alpha>0$ satisfying (6.6).

Next we consider the system

$$(6.10)\quad \begin{cases} \Delta u - \phi(x)v^{\delta} = 0 \\ \Delta v - \psi(x)u^{\gamma} = 0 \end{cases}$$

in R^n, where δ and γ are constants such that $\delta>1$ and $\gamma>1$.

THEOREM 6.2. *Suppose there exist locally Hölder continuous functions* $\phi^*(t)$, $\phi_*(t)$, $\psi^*(t)$ *and* $\psi_*(t)$ *on* $[0, \infty)$ *such that*

$$(6.11)\quad 0 \leq \phi_*(|x|) \leq \phi(x) \leq \phi^*(|x|), \quad 0 \leq \psi_*(|x|) \leq \psi(x) \leq \psi^*(|x|), \quad x \in R^n.$$

If (6.2) *holds, then* (6.10) *has infinitely many bounded positive entire solutions* $(u(x), v(x))$ *such that* $u(x)$ *and* $v(x)$ *are bounded and bounded away from zero.*

PROOF. A bounded positive solution $(\check{u}(x), \hat{v}(x))$ of

$$(6.12)\quad \begin{cases} \Delta \check{u} - \phi_*(|x|)\hat{v}^{\delta} = 0 \\ \Delta \hat{v} - \psi^*(|x|)\check{u}^{\gamma} = 0 \end{cases}$$

is a super-subsolution of (6.10) and a bounded positive solution $(\hat{u}(x), \check{v}(x))$ of

$$\left\{\begin{array}{l} \Delta\hat{u} - \phi^*(|x|)\check{v}^\delta = 0 \\ \Delta\check{v} - \psi_*(|x|)\hat{u}^\gamma = 0 \end{array}\right. \tag{6.13}$$

is a sub-subsolution of (6.10). We wish to find such $(\check{u}(x), \hat{v}(x))$ and $(\hat{u}(x), \check{v}(x))$ so that they are spherically symmetric and satisfy (5.5).

Consider the following initial value problem:

$$\left\{\begin{array}{l} y'' + \dfrac{n-1}{t}y' - \phi_*(t)z^\delta = 0 \\ z'' + \dfrac{n-1}{t}z' - \psi^*(t)y^\gamma = 0, \quad t > 0, \end{array}\right. \tag{6.14}$$

$$y(0) = \alpha_1 > 0, \quad z(0) = \beta_1 > 0, \quad y'(0) = z'(0) = 0.$$

We take α_1 and β_1 so *small* that

$$\left\{\begin{array}{l} \dfrac{(2\beta_1)^\delta}{n-2}\displaystyle\int_0^\infty t\phi_*(t)\,dt \le \alpha_1 \\ \dfrac{(2\alpha_1)^\gamma}{n-2}\displaystyle\int_0^\infty t\psi^*(t)\,dt \le \beta_1. \end{array}\right. \tag{6.15}$$

and consider the set X_1 defined by

$$X_1 = \{(y, z) \in C[0, \infty) \times C[0, \infty) \colon \alpha_1 \le y(t) \le 2\alpha_1, \beta_1 \le z(t) \le 2\beta_1, t \ge 0\}.$$

It is easy to verify that if we define the operator $\mathscr{F}_1$ by $\mathscr{F}_1(y, z) = (y^*, z^*)$, where

$$\left\{\begin{array}{l} y^*(t) = \alpha_1 + \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\phi_*(s)z^\delta(s)\,ds \\ z^*(t) = \beta_1 + \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1 - \left(\frac{s}{t}\right)^{n-2}\right]\psi^*(s)y^\gamma(s)\,ds, \quad t \ge 0, \end{array}\right. \tag{6.16}$$

then $\mathscr{F}_1$ is continuous and maps X_1 into a compact subset of X_1. It follows that $\mathscr{F}_1$ has a fixed point $(y, z) \in X_1$, which is a solution of (6.14). Putting $\check{u}(x) = y(|x|)$ and $\hat{v}(x) = z(|x|)$, we obtain a radial super-subsolution $(\check{u}(x), \hat{v}(x))$ of (6.10) in R^n satisfying

$$\alpha_1 \le u(x) \le 2\alpha_1, \quad \beta_1 \le v(x) \le 2\beta_1, \quad x \in R^n. \tag{6.17}$$

Likewise, by choosing positive constants α_2 and β_2 so *small* that

$$\left\{\begin{array}{l} \dfrac{(2\beta_2)^\delta}{n-2}\displaystyle\int_0^\infty t\phi^*(t)\,dt \le \alpha_2 \\ \dfrac{(2\alpha_2)^\gamma}{n-2}\displaystyle\int_0^\infty t\psi_*(t)\,dt \le \beta_2 \end{array}\right. \tag{6.18}$$

and arguing as above, we see that there exists a radial sub-supersolution $(\hat{u}(x), \check{v}(x))$ of (6.10) in R^n such that

$$\alpha_2 \leq u(x) \leq 2\alpha_2, \quad \beta_2 \leq v(x) \leq 2\beta_2, \quad x \in R^n. \tag{6.19}$$

If we choose α_1, α_2 β_1 and β_2 so that $2\alpha_2 \leq \alpha_1$ and $2\beta_1 \leq \beta_2$, then the functions $(\check{u}(x), \hat{v}(x))$ and $(\hat{u}(x), \check{v}(x))$ satisfy (5.5), and by (i) of Theorem 5.4, equation (6.10) possesses an entire solution $(u(x), v(x))$ satisfying (5.6). This finishes the proof.

REMARK 6.1. Theorem 6.2 is true even if either $\delta=1$ or $\gamma=1$. Suppose $\delta=1$ and $\phi(x)>0$ in R^n. Then the first component of a solution $(u(x), v(x))$ of (6.10) satisfies the fourth order elliptic equation

$$\Delta\left(\frac{1}{\phi(x)}\Delta u\right) - \psi(x)u^\gamma = 0 \tag{6.20}$$

in R^n. Therefore, the condition (6.2) is sufficient for the existence of a positive entire solution of (6.20) which is bounded and bounded away from zero.

Finally, we consider the elliptic system (1.5).

THEOREM 6.3. *Suppose that γ, δ, μ and ν are nonnegative and that $\phi(x)$ and $\psi(x)$ satisfy condition* (6.1). *If* (6.2) *holds, then* (1.5) *possesses infinitely many bounded entire solutions.*

PROOF. We seek a super-supersolution of (1.5) as a solution of the system

$$\begin{cases} \Delta \check{u} + \phi^*(|x|)e^{\gamma\check{u}+\delta\check{v}} = 0 \\ \Delta \check{v} + \psi^*(|x|)e^{\mu\check{u}+\nu\check{v}} = 0. \end{cases} \tag{6.21}$$

The initial value problem associated with (6.21) in R^1 is the following:

$$\begin{cases} y'' + \dfrac{n-1}{t}y' + \phi^*(t)e^{\gamma y+\delta z} = 0 \\[2ex] z'' + \dfrac{n-1}{t}z' + \psi^*(t)e^{\mu y+\nu z} = 0, \quad t > 0, \end{cases} \tag{6.22}$$
$$y(0) = \alpha, \quad z(0) = \beta, \quad y'(0) = z'(0) = 0,$$

where α and β are constants. Put

$$A = \alpha - \frac{e^{\gamma\alpha+\delta\beta}}{n-2}\int_0^\infty t\phi^*(t)dt, \qquad B = \beta - \frac{e^{\mu\alpha+\nu\beta}}{n-2}\int_0^\infty t\psi^*(t)dt,$$

and consider the subset of $C[0, \infty) \times C[0, \infty)$

$$X = \{(y, z) \in C[0, \infty) \times C[0, \infty) \colon A \leq y(t) \leq \alpha,\ B \leq z(t) \leq \beta,\ t \geq 0\}.$$

Define the operator $\mathscr{G}$ by $\mathscr{G}(y, z) = (y^*, z^*)$, where

$$\begin{cases} y^*(t) = \alpha - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\phi^*(s)e^{\gamma y(s)+\delta z(s)}ds \\ z^*(t) = \beta - \dfrac{1}{n-2}\displaystyle\int_0^t s\left[1-\left(\frac{s}{t}\right)^{n-2}\right]\psi^*(s)e^{\mu y(s)+\nu z(s)}ds, \quad t \geq 0. \end{cases}$$

Then, $\mathscr{G}$ is a continuous operator mapping X into a compact subset of X, and so $\mathscr{G}$ has a fixed point (y, z) in X. The function $(\check{u}(x), \check{v}(x)) = (y(|x|), z(|x|))$ is a super-supersolution of (1.5) in R^n. On the other hand, the constant function $(\hat{u}(x), \hat{v}(x)) = (A, B)$ is a sub-subsolution of (1.5) in R^n. The conclusion of the theorem now follows from (i) of Theorem 5.3.

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Department of Mathematics,
Faculty of Education,
Miyazaki University