HIROSHIMA MATH. J.
14 (1984), 1–13

Asymptotic behavior of solutions to a Stefan problem with obstacles on the fixed boundary

Dedicated to Professor M. Ohtsuka on his 60th birthday

Nobuyuki KENMOCHI

(Received February 22, 1983)

Introduction

In this paper we consider the following problem: Find a curve $x=l(t)>0$ on $[0, \infty)$ and a function $u=u(x, t)$ on $\bar{\Omega}_l$, Ω_l being the set $\{(x, t); 0<x<l(t), 0<t<\infty\}$, satisfying

$$(0.1) \qquad u_t - u_{xx} = 0 \quad \text{in} \quad \Omega_l,$$

$$(0.2) \qquad u(x, 0) = u_0(x) \qquad \text{for} \quad 0 < x < l_o,$$

$$(0.3) \qquad \begin{cases} u(0, t) \geqq g(t) & \text{for} \quad 0 < t < \infty, \\ u_x(0+, t) = 0 & \text{for} \quad u(0, t) > g(t), \\ u_x(0+, t) \leqq 0 & \text{for} \quad u(0, t) = g(t), \end{cases}$$

$$(0.4) \qquad u(l(t), t) = 0 \qquad \text{for} \quad 0 < t < \infty, \text{ and}$$

$$(0.5) \qquad \begin{cases} l'(t)(= (d/dt)l(t)) = -u_x(l(t)-, t) & \text{for} \quad 0 < t < \infty, \\ l(0) = l_o, \end{cases}$$

where l_o is a given positive number, u_o a given initial function and g is an obstacle function given on the fixed boundary $x=0$. This is regarded as a Stefan problem of type different from those treated so far. Recently the author (cf. [8]) employed a method which has evolved in the theory of nonlinear evolution equations involving time-dependent subdifferential operators in Hilbert spaces in order to show that our system admits global solutions to this problem. The purpose of this paper is to study the asymptotic behavior of the global solutions.

As to the usual Stefan problem which is described as a system with (0.3) replaced by the boundary conditions such as $u(0, t)=f(t)$ or $u_x(0+, t)=f(t)$, the existence and uniqueness as well as the asymptotic behavior of the solutions have been studied by many authors. See for instance [2–5, 9]. On the other hand, in case g is a non-negative constant function on $[0, \infty)$, Yotsutani [10, 11] discussed the system (0.1)–(0.5) and gave detailed results concerning the asymptotic behavior of the solutions. However these methods do not directly apply to the above-mentioned system with non-constant obstacle g. In order to investigate the system (0.1)–(0.5) there arise some difficulties because of the nonlinearity in the boundary condition (0.3), which never occur in the case of constant obstacle. In this paper we establish a new method which makes it possible to treat nonlinear boundary condition of the form (0.3), and this is the principal virtue of our approach.

In this paper we restrict ourselves to the obstacle g satisfying either of the following:

(0.6) g is non-increasing on $[0, \infty)$; or

(0.7) g has a non-increasing majorant $\hat{g}$ on $[0, \infty)$,

and discuss three subjects as listed below:

(a) The monotone dependence of solutions on Stefan data $\{l_o, u_o, g\}$ and the uniqueness of solutions.

(b) The asymptotic behavior of the free boundary $x=l(t)$; and sufficient conditions on g in order that $\lim_{t\to\infty} l(t)<\infty$.

(c) The asymptotic behavior of u; and evaluations of $\liminf_{t\to\infty} u(x, t)$ and $\limsup_{t\to\infty} u(x, t)$ in terms of g.

In [8; Theorem 1.3], the uniqueness of the solution was verified for a specific class of initial values. In section 3 of this paper we show that the uniqueness theorem is valid for a more general class of initial values, and that the uniqueness theorem is a direct consequence of the monotone dependence of solutions on Stefan data $\{l_o, u_o, g\}$. The crucial step for the investigation of (b) is to obtain the inequality

$$0 \geqq u_x(0+, t) \geqq Kg(t) \qquad \text{for} \quad \text{a.e. } t \geqq 0 \text{ and some constant } K \leqq 0$$

and to prove that $\lim_{t\to\infty} l(t)<\infty$ under the assumption that (0.6) (resp. (0.7)) holds and $g \in L^1(0, \infty)$ (resp. $\hat{g} \in L^1(0, \infty)$).

The main results of this paper will be given in section 4.

1. Quasi-variational formulation of the problem

Throughout this paper we use the Hilbert space

$$H = L^2(0, \infty)$$

with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)_H$, and the Sobolev space

$$X = W^{1,2}(0, \infty).$$

Given a curve $x=l(t)>0$ on $[0, \infty)$, a function g on $[0, \infty)$ and a point

$t \geqq 0$, we define a function $\phi^t_{l,g}$ on H by

$$\phi^t_{l,g}(z) = \begin{cases} (1/2)|z_x|^2_H & \text{if} \quad z \in K_{l,g}(t), \\ \infty & \text{otherwise}, \end{cases}$$

where

$$K_{l,g}(t) = \{z \in X;\ z(0) \geqq g(t),\ z = 0 \quad \text{on} \quad [l(t), \infty)\}.$$

Clearly, the function $\phi^t_{l,g}$ is proper, lower semi-continuous (l.s.c.) and convex on H and the effective domain $D(\phi^t_{l,g})$ is exactly the set $K_{l,g}(t)$. The subdifferential $\partial\phi^t_{l,g}$ is a multivalued operator in H. For the definition and general properties of subdifferential operators, we refer to [1]. We then consider the Cauchy problem $CP(\phi^t_{l,g};\ u_o)$ on $[0, T]$:

$$CP(\phi^t_{l,g};\ u_o) \begin{cases} -u'(t) \in \partial\phi^t_{l,g}(u(t)),\ 0 < t < T, \\ u(0) = u_o \end{cases}$$

where $0 < T < \infty$, the initial-value u_o is given in H and $u'(t)$ denotes the strong derivative $(d/dt)u(t)$ in H of $u(t)$. By a solution u of $CP(\phi^t_{l,g};\ u_o)$ on $[0, T]$ we mean the H-valued function satisfying

$$(1.1) \quad u \in C([0, T]; H) \cap L^2(0, T; X) \cap W^{1,2}(\delta, T; H) \cap L^\infty(\delta, T; X) \quad \text{for every} \quad 0 < \delta < T,$$

$$u(0) = u_o,$$

$$-u'(t) \in \partial\phi^t_{l,g}(u(t)) \qquad \text{for} \quad \text{a.e. } t \in [0, T].$$

Also, we say that u is a solution to $CP(\phi^t_{l,g};\ u_o)$ on $[0, \infty)$, if it is a solution to $CP(\phi^t_{l,g};\ u_o)$ on $[0, T]$ for every finite $T > 0$.

We identify a function $u = u(x, t)$ in $L^2((0, \infty) \times (0, T))$ with an H-valued function $u = u(t)$ on $(0, T)$ in such a way that

$$[u(t)](x) = u(x, t) \qquad \text{for} \quad 0 < x < \infty \quad \text{and} \quad t \in (0, T).$$

Using the functions $\phi^t_{l,g}$ we give a quasi-variational formulation of the problem (0.1)–(0.5).

DEFINITION 1.1. Let $0 < l_o < \infty$, $u_o \in H$ and g a function on $[0, \infty)$. Then we say that a pair $\{l, u\}$ of a positive function l in $C([0, \infty))$ and a function u in $C([0, \infty); H)$ is a solution to $QV(l_o, u_o, g)$, if it fulfills the following conditions:

(QV1) u is a solution to $CP(\phi^t_{l,g};\ u_o)$ on $[0, \infty)$.

(QV2) $l \in W^{1,2}(\delta, T)$ for every $0 < \delta < T < \infty$, $l(0) = l_o$ and

$$(1.2) \qquad l'(t) = -u_x(l(t)-, t) \quad \text{for a.e. } t \geqq 0.$$

Existence of a solution to $QV(l_o, u_o, g)$ can then be discussed under the following assumptions (A) (or (A)') and (B):

(A) $g \geqq 0$ on $[0, \infty)$ and $g \in W^{1,2}(0, T)$ for every finite $T>0$.

(A)' $g \geqq 0$ on $[0, \infty)$ and g is non-increasing on $[0, \infty)$.

(B) $0<l_o<\infty$, $u_o \in H$, $u_o \geqq 0$ a.e. on $[0, \infty)$, and $u_o=0$ a.e. on $[l_o, \infty)$.

In fact, we have:

THEOREM 1.1 (cf. [8; *Theorems* 1.1, 1.2]). *Suppose* (A) (*or* (A)') *and* (B) *hold. Then we have*:

(i) *$QV(l_o, u_o, g)$ admits at least one solution $\{l, u\}$ such that $t^{1/2}l' \in L^2(0, T)$, $t^{1/2}u' \in L^2(0, T; H)$ for every finite $T>0$, the mapping $t \to t|u_x(\cdot, t)|_H^2$ is bounded on $(0, T]$ for every finite $T>0$, and $u(0, t) \geqq g(t)$ for all $t>0$.*

(ii) *If in addition $u_o \in X$ and $u_o(0) \geqq g(0)$, then $QV(l_o, u_o, g)$ has at least one solution $\{l, u\}$ such that $l \in W^{1,2}(0, T)$, $u \in W^{1,2}(0, T; H)$ for every finite $T>0$, the mapping $t \to |u_x(\cdot, t)|_H$ is locally bounded on $[0, \infty)$, and $u(0, t) \geqq g(t)$ for all $t \geqq 0$.*

(iii) *If $\{l, u\}$ is a solution to $QV(l_o, u_o, g)$, then l is non-decreasing on $[0, \infty)$ and u is non-negative on $[0, \infty) \times (0, \infty)$.*

REMARK 1.1. According to the results given in [7; Chapter 1], $CP(\phi_{l,g}^t; u_o)$ admits one and only one solution on $[0, \infty)$ under conditions (A) (or (A)') and (B). Also the solution u is continuous in $(x, t) \in [0, \infty) \times (0, \infty)$, since

$$u \in W^{1,2}(\delta, T; H) \cap L^\infty(\delta, T; X) \ (\subset C([0, \infty) \times [\delta, T]))$$

for every $0<\delta<T<\infty$ by (1.1). Moreover we note (cf. [8; §1]) that a function $u: [0, \infty) \to H$ is a solution of $CP(\phi_{l,g}^t; u_o)$ if and only if (1.1) holds and u satisfies the following system:

$$\begin{aligned}
&u_t(\cdot, t) - u_{xx}(\cdot, t) = 0 \quad \text{in } L^2(0, l(t)) \quad \text{for a.e. } t \geqq 0,\\
&u(\cdot, 0) = u_o \quad \text{in } H,\\
&u(\cdot, t) = 0 \quad \text{on } [l(t), \infty) \quad \text{for all } t \geqq 0,\\
&u(0, t) \geqq g(t) \quad \text{for all } t > 0,\\
&u_x(0+, t) = 0 \quad \text{for a.e. } t \in \{t \geqq 0; u(0, t) > g(t)\},\\
&u_x(0+, t) \leqq 0 \quad \text{for a.e. } t \in \{t \geqq 0; u(0, t) = g(t)\}.
\end{aligned}$$

Therefore $QV(l_o, u_o, g)$ is understood to be a weak formulation of the problem (0.1)–(0.5).

REMARK 1.2. Let $\{l, u\}$ be a solution to $QV(l_o, u_o, g)$. If we write $f(t)$ for $u(0, t)$, then $\{l, u\}$ is regarded as a solution to the usual Stefan problem with (0.3) replaced by the boundary condition $u(0, t)=f(t)$ for $0<t<\infty$. Thus it follows from [2; Theorem 1] that the solution $\{l, u\}$ has the following properties:

(i) u_t and u_{xx} are continuous on Ω_l.

(ii) $u_x(l(t)-, t)$ exists and (1.2) holds for all $t>0$.

(iii) l is continuously differentiable on $(0, \infty)$.

We here recall the expressions of the free boundary $x=l(t)$ which are useful in the later argument.

LEMARK 1.1 (cf. [2–4]). *Let $\{l, u\}$ be a solution to $QV(l_o, u_o, g)$. Then*:

$$l(t) = l(s) + \int_0^\infty u(x, s)dx - \int_0^\infty u(x, t)dx - \int_s^t u_x(0+, \tau)d\tau \tag{1.3}$$

$$\textit{for every } 0 < s \leqq t < \infty.$$

$$l(t)^2 = l(s)^2 + 2\int_0^\infty xu(x, s)dx - 2\int_0^\infty xu(x, t)dx + 2\int_s^t u(0, \tau)d\tau \tag{1.4}$$

$$\textit{for every } 0 \leqq s \leqq t < \infty.$$

2. Some lemmas

We first prepare the following lemma.

LEMMA 2.1. *Let $0<T<\infty$, k a constant, l a function in $C([0, T])$ with $l>0$ on $[0, T]$, and let v, w be functions in $C([0, T]; H)\cap W^{1,2}(\delta, T; H)\cap L^\infty(\delta, T; X)$ such that $v_{xx}, w_{xx}\in L^2(D_\delta)$ for every $\delta\in(0, T)$, where $D_\delta=\{(x, t); 0<x<l(t), \delta<t<T\}$. Assume further that*

$$w_t - w_{xx} \leqq v_t - v_{xx} \ \textit{a.e. on } \{(x, t); 0 < x < l(t), 0 < t < T\},$$

$$w(x, 0) \leqq v(x, 0) + k \quad \textit{for a.e. } x \geqq 0,$$

$$w \leqq v + k \quad \textit{on } \{(x, t); l(t) \leqq x < \infty, 0 < t \leqq T\}, \ \textit{and}$$

$$(w_x(0+, t) - v_x(0+, t))(w(0, t) - v(0, t) - k)^+ \geqq 0 \quad \textit{for a.e. } t\in[0, T].$$

Then

$$w \leqq v + k \quad \textit{on } [0, \infty)\times(0, T].$$

PROOF. Note that v and w are continuous on $[0, \infty)\times(0, T]$. Since the support of $(w-v-k)^+$ is contained in $\{(x, t); 0\leqq x\leqq l(t), 0\leqq t\leqq T\}$, it follows from the assumptions that

$$\begin{aligned}
&(1/2)(d/dt)|(w(t)-v(t)-k)^+|_H^2 \\
&= \int_0^\infty (w_t(x,t)-v_t(x,t))(w(x,t)-v(x,t)-k)^+ dx \\
&\leqq \int_0^{l(t)} (w_{xx}(x,t)-v_{xx}(x,t))(w(x,t)-v(x,t)-k)^+ dx \\
&= -\int_0^{l(t)} (w_x(x,t)-v_x(x,t))\{(w(x,t)-v(x,t)-k)^+\}_x dx \\
&\quad + (w_x(l(t)-,t)-v_x(l(t)-,t))(w(l(t),t)-v(l(t),t)-k)^+ \\
&\quad - (w_x(0+,t)-v_x(0+,t))(w(0,t)-v(0,t)-k)^+ \\
&\leqq 0 \qquad \text{for a.e. } t\in[0,T].
\end{aligned}$$

Integrating this over the subinterval $[\delta, s]$ with $0<\delta\leqq s\leqq T$, we get

$$|(w(s)-v(s)-k)^+|_H \leqq |(w(\delta)-v(\delta)-k)^+|_H.$$

Since $|(w(\delta)-v(\delta)-k)^+|_H \to |(w(0)-v(0)-k)^+|_H=0$ as $\delta\downarrow 0$, we obtain $|(w(s)-v(s)-k)^+|_H=0$ for all $s\in[0,T]$. Thus we have the conclusion.

COROLLARY 1. *Let $0<T<\infty$, l a function in $C([0,T])$ with $l>0$ on $[0,T]$, g a non-negative function on $[0,T]$ and let u_o be a non-negative function in H. Let u be the solution to $CP(\phi^t_{l,g}; u_o)$ on $[0,T]$. Then $u\geqq 0$ on $[0,\infty)\times(0,T]$. In addition, if $u_o\in L^\infty(0,\infty)$ and $g\in L^\infty(0,T)$, then*

$$u \leqq \max\{|u_o|_{L^\infty(0,\infty)}, |g|_{L^\infty(0,T)}\} \qquad \text{on} \quad [0,\infty)\times(0,T].$$

PROOF. Recalling Remark 1.1 and applying Lemma 2.1 with $w=0$, $v=u$ and $k=0$, we have $u\geqq 0$ on $[0,\infty)\times(0,T]$. Next, the application of Lemma 2.1 with $w=u$, $v=0$ and $k=\max\{|u_o|_{L^\infty(0,\infty)}, |g|_{L^\infty(0,T)}\}$ implies $u\leqq k$ on $[0,\infty)\times(0,T]$.

COROLLARY 2. *Let $0<T<\infty$, l_1, l_2 a pair of functions in $C([0,T])$ with $0<l_1\leqq l_2$ on $[0,T]$, g_1, g_2 a pair of non-negative functions on $[0,T]$ and let $u_{1,o}$, $u_{2,o}$ be non-negative functions in H. Further let u_1 and u_2 be the solutions of $CP(\phi^t_{l_1,g_1}; u_{1,o})$ and $CP(\phi^t_{l_2,g_2}; u_{2,o})$ on $[0,T]$, respectively. If $g_1\leqq g_2$ on $[0,T]$ and $u_{1,o}\leqq u_{2,o}$ a.e. on $[0,\infty)$, then*

$$u_1 \leqq u_2 \qquad \text{on } [0,\infty)\times(0,T].$$

PROOF. We infer from Remark 1.1 and Corollary 1 that

$$u_{1,t}-u_{1,xx}=0=u_{2,t}-u_{2,xx} \quad \text{a.e. on } \{(x,t);\, 0<x<l_1(t),\, 0<t<T\},$$

$$u_1(x,0)=u_{1,o}(x)\leqq u_{2,o}(x)=u_2(x,0) \qquad \text{for a.e. } x\geqq 0,$$

$$u_1=0\leqq u_2 \qquad \text{on} \quad \{(x,t);\, l_1(t)\leqq x<\infty,\, 0<t\leqq T\}, \quad \text{and}$$

$$u_{1,x}(0+, t) = 0,\ u_{2,x}(0+, t) \leqq 0 \qquad \text{for a.e. } t \in \{t \geqq 0;\ u_1(0, t) > u_2(0, t)\}.$$

Hence, the application of Lemma 2.1 with $w=u_1$, $v=u_2$ and $k=0$ yields the desired conclusion.

Next we recall a notion of the convergence for proper l.s.c. convex functions. Given a family $\{\psi^t;\ 0 \leqq t < \infty\}$ of proper l.s.c. convex functions on H and a proper l.s.c. convex function ψ^∞ on H, we say that ψ^t converges on H to ψ^∞ in the sense of Mosco as $t \to \infty$, if the following two conditions (a) and (b) hold:

(a) If $z_n \to z$ weakly in H and $t(n) \to \infty$ (as $n \to \infty$), then

$$\liminf_{n\to\infty} \psi^{t(n)}(z_n) \geqq \psi^\infty(z).$$

(b) For each $z \in D(\psi^\infty)$ and each sequence $\{t(n)\}$ with $t(n) \to \infty$ there is a sequence $\{z_n\}$ such that $z_n \to z$ in H and $\psi^{t(n)}(z_n) \to \psi^\infty(z)$.

The following lemma is elementary.

LEMMA 2.2. *Let l be a positive non-decreasing function in $C([0, \infty))$ and g a non-increasing function on $[0, \infty)$ with $c=\lim_{t\to\infty} g(t) > -\infty$. Then $D(\phi^s_{l,g}) \subset D(\phi^t_{l,g})$, $\phi^s_{l,g}(z) = \phi^t_{l,g}(z)$ for $0 \leqq s \leqq t < \infty$ and $z \in D(\phi^s_{l,g})$* and *$\phi^t_{l,g}$ converges on H to ϕ^∞ in the sense of Mosco as $t \to \infty$, where ϕ^∞ is the function on H defined by*

$$(2.1) \qquad \phi^\infty(z) = \begin{cases} (1/2)|z_x|_H^2 & \text{if } z \in X,\ z(0) \geqq c \text{ and } z = 0 \text{ on } [l_\infty, \infty), \\ \infty & \text{otherwise}, \end{cases}$$

and $l_\infty = \lim_{t\to\infty} l(t)$; in (2.1), the restriction that $z=0$ on $[l_\infty, \infty)$ is not necessary in the case of $l_\infty = \infty$.

LEMMA 2.3. *Let l be a positive non-decreasing function in $C([0, \infty))$ with $l_\infty = \lim_{t\to\infty} l(t) < \infty$, g a non-negative non-increasing function on $[0, \infty)$ with $c = \lim_{t\to\infty} g(t)$, and let u_o be a function in H satisfying $u_o = 0$ a.e. on $[l(0), \infty)$. Then $CP(\phi^t_{l,g}; u_o)$ has one and only one solution u on $[0, \infty)$ such that $u \in L^\infty(0, \infty; H)$ and*

$$u(t) \longrightarrow u_\infty \quad \text{in} \quad X \quad \text{as} \quad t \longrightarrow \infty,$$

where

$$(2.2) \qquad u_\infty(x) = \begin{cases} c(1 - x/l_\infty) & \text{for } 0 \leqq x \leqq l_\infty, \\ 0 & \text{for } l_\infty < x < \infty. \end{cases}$$

PROOF. The existence and uniqueness of the solution follow from [7; Theorems 1.1.1, 1.2.1] and Corollary 1 to Lemma 2.1 implies that $u \in L^\infty(0, \infty; H)$. Moreover, using Lemma 2.2 for the case of $l_\infty < \infty$, we can apply the result of [6; Theorem 1] to obtain that $u(t)$ converges to some u_∞ in H as $t \to \infty$ and $\phi^t_{l,g}(u(t)) \to \phi^\infty(u_\infty) = \min \phi^\infty$ as $t \to \infty$, where ϕ^∞ is as mentioned in

(2.1). Hence $u(t)\to u_\infty$ in X as $t\to\infty$ and we see that u_∞ is expressed as in (2.2).

LEMMA 2.4. *Let l be a positive non-decreasing function in $C([0, \infty))$, $g=0$ on $[0, \infty)$ and let u_o be a non-negative function in H such that $u_o=0$ a.e. on $[l(0), \infty)$. Then $CP(\phi^t_{l,g}; u_o)$ has one and only one solution u on $[0, \infty)$ such that*

$$u(x, t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \text{ uniformly in } x \in [0, \infty).$$

PROOF. The existence and uniqueness of the solution follow from [7; Theorems 1.1.1, 1.2.1]. By the definition of subdifferential $\partial\phi^t_{l,g}$ we have

$$(u'(\tau), u(\tau)-z)_H \leqq \phi^\tau_{l,g}(z) - \phi^\tau_{l,g}(u(\tau)) \qquad \text{for} \quad z \in D(\phi^\tau_{l,g}),$$

for a.e. $\tau\geqq 0$. Letting $z=0$, then integrating the resultant inequality over $[\delta, t]$, $0<\delta<t<\infty$, with respect to τ and finally letting $\delta\to 0$, we get

$$|u(t)|^2_H + \int_0^t |u_x(\cdot, \tau)|^2_H d\tau \leqq |u_o|^2_H \qquad \text{for} \quad t \geqq 0.$$

Therefore $u \in L^\infty(0, \infty; H)$ and in view of Lemma 2.2 we can apply the result [6; Theorem 1] to show that $u(t)\to u_\infty$ weakly in H and $\phi^t_{l,g}(u(t))\to\phi^\infty(u_\infty)=\min\phi^\infty$ $(=0)$, where ϕ^∞ is as defined by (2.1) with $c=0$. From this it follows that $u_\infty=0$ and $u(x, t)\to 0$ as $t\to\infty$ uniformly in $x\in[0, \infty)$.

3. Monotone dependence on Stefan data

In this section we establish the following result.

THEOREM 3.1. *Let $\{l_o, u_o, g\}$ and $\{\bar{l}_o, \bar{u}_o, \bar{g}\}$ be two Stefan data satisfying (A) (or (A)') and (B). Let $\{l, u\}$ and $\{\bar{l}, \bar{u}\}$ be solutions to $QV(l_o, u_o, g)$ and $QV(\bar{l}_o, \bar{u}_o, \bar{g})$, respectively. Suppose that $l_o\leqq\bar{l}_o$, $u_o\leqq\bar{u}_o$ a.e. on $[0, \infty)$ and $g\leqq\bar{g}$ on $[0, \infty)$. Then*

(3.1) $$l \leqq \bar{l} \text{ on } [0, \infty) \quad \text{and} \quad u \leqq \bar{u} \text{ on } [0, \infty)\times(0, \infty).$$

PROOF. (Case 1) Assuming $l_o<\bar{l}_o$, we first show that $l<\bar{l}$ on $[0, \infty)$ and $u\leqq\bar{u}$ on $[0, \infty)\times(0, \infty)$. For the contrary suppose that there is a positive number t_o such that $l(t_o)=\bar{l}(t_o)$ and $l(t)<\bar{l}(t)$ for $0\leqq t<t_o$. Then Corollary 2 to Lemma 2.1 implies that $u\leqq\bar{u}$ on $[0, \infty)\times(0, t_o]$. Now denoting $u(0, t)$ (resp. $\bar{u}(0, t)$) by $f(t)$ (resp. $\bar{f}(t)$), we see that $f\leqq\bar{f}$ on $(0, t_o]$, and that $\{l, u\}$ (resp. $\{\bar{l}, \bar{u}\}$) is the solution to the usual Stefan problem with the boundary condition $u(0, t)=f(t)$ (resp. $\bar{u}(0, t)=\bar{f}(t)$) for $0<t\leqq t_o$. Hence it follows from a well-known result concerning the monotone dependence of solutions on Stefan data (see [2]) that $l<\bar{l}$ on $[0, t_o]$. This is a contradiction. Thus $l<\bar{l}$ on $[0, \infty)$ and $u\leqq\bar{u}$ on $[0, \infty)\times(0, \infty)$.

(Case 2) We next consider the case when $l_o = \bar{l}_o$. In this case, a sequence $\{l_{o,n}\}$ can be taken such that $\bar{l}_o < l_{o,n+1} < l_{o,n}$ and $l_{o,n} \to \bar{l}_o$ as $n \to \infty$. Evidently, $\{l_{o,n}, \bar{u}_o, \bar{g}\}$ satisfies (A) (or (A)′) and (B). Therefore, denoting by $\{l_n, u_n\}$ the solution to $QV(l_{o,n}, \bar{u}_o, \bar{g})$ that is given by Theorem 1.1, we infer from the proof of Case 1 that

$$(3.2)\qquad \begin{aligned} &\bar{l} < l_{n+1} < l_n \text{ on } [0, \infty),\ \bar{u} \leqq u_{n+1} \leqq u_n \text{ on } [0, \infty) \times (0, \infty), \\ &l < l_n \text{ on } [0, \infty), \quad \text{and} \quad u \leqq u_n \text{ on } [0, \infty) \times (0, \infty). \end{aligned}$$

Also, taking account of (1.3) of Lemma 1.1, we have

$$(3.3)\qquad \begin{aligned} l_n(t) - \bar{l}(t) = l_n(s) - \bar{l}(s) &+ \int_0^\infty (u_n(x, s) - \bar{u}(x, s))dx - \int_0^\infty (u_n(x, t) \\ &- \bar{u}(x, t))dx - \int_s^t (u_{n,x}(0+, \tau) - \bar{u}_x(0+, \tau))d\tau \end{aligned}$$

for $0 < s \leqq t < \infty$. Moreover we have

$$(3.4)\qquad u_{n,x}(0+, \tau) \geqq \bar{u}_x(0+, \tau) \qquad \text{for} \quad \text{a.e. } \tau \geqq 0.$$

In fact, $u_{n,x}(0+, \tau) = 0$ and $\bar{u}_x(0+, \tau) \leqq 0$ for a.e. $\tau \in \{\tau \geqq 0;\ u_n(0, \tau) > \bar{u}(0, \tau)\ (\geqq \bar{g}(\tau))\}$, and (3.2) yields

$$\begin{aligned} u_{n,x}(0+, \tau) &= \lim_{x \downarrow 0} (u_n(x, \tau) - u_n(0, \tau))/x \\ &\geqq \lim_{x \downarrow 0} (\bar{u}(x, \tau) - \bar{u}(0, \tau))/x = \bar{u}_x(0+, \tau) \end{aligned}$$

for a.e. $\tau \in \{\tau \geqq 0;\ u_n(0, \tau) = \bar{u}(0, \tau)\}$. Hence (3.4) holds. Consequently, (3.3), (3.2) and (3.4) together imply

$$0 < l_n(t) - \bar{l}(t) \leqq l_n(s) - \bar{l}(s) + \int_0^\infty (u_n(x, s) - \bar{u}(x, s))dx$$

for $0 < s \leqq t < \infty$. Letting $s \downarrow 0$ in this inequality, we have

$$0 < l_n(t) - \bar{l}(t) \leqq l_{o,n} - \bar{l}_o \qquad \text{for} \quad t \geqq 0 \quad \text{and} \quad n.$$

This shows that l_n converges to $\bar{l}$ uniformly on $[0, \infty)$ as $n \to \infty$. Thus we can apply the convergence result [7; Theorem 2.7.1] (or [8; Proposition 3.3]) to obtain

$$u_n \longrightarrow \bar{u} \quad \text{in} \quad C([0, T]; H) \quad \text{as} \quad n \longrightarrow \infty$$

for every finite $T > 0$. Therefore (3.1) follows from (3.2).

Corollary. *Suppose* (A) (or (A)′) *and* (B) *hold. Then* $QV(l_o, u_o, g)$ *has at most one solution.*

4. Asymptotic behavior of the solutions

The first main result of the present paper is stated as follows.

THEOREM 4.1. *Suppose* (A)′ *and* (B) *hold. Let* $\{l, u\}$ *be the solution to* $QV(l_o, u_o, g)$. *Then* $l_\infty = \lim_{t\to\infty} l(t) < \infty$ *if and only if* $g \in L^1(0, \infty)$. *Moreover, in this case,* $u(\cdot, t)\to 0$ *in* X *as* $t\to\infty$.

In order to prove this theorem, we need the following

LEMMA 4.1. *Suppose* (A)′ *and* (B) *hold. Let* $\{l, u\}$ *be the solution to* $QV(l_o, u_o, g)$. *Then there are numbers* $T>0$ *and* $0<\delta<l_o$ *such that*

$$u(x, t) \geqq (1-x/\delta)g(t) \qquad \text{for} \quad x\in[0, \delta] \quad \text{and} \quad t \geqq T. \tag{4.1}$$

PROOF. We first choose $T>0$ and $0<\delta<l_o$ satisfying

$$u(x, T) \geqq (1-x/\delta)g(T) \qquad \text{for} \quad x\in[0, \delta]\,;$$

we see from the facts as mentioned in Remark 1.1 that such T and δ exist. Next, we take a sequence $\{g_n\}$ of smooth functions on $[0, \infty)$ such that g_n is non-increasing on $[0, \infty)$, $g_n \leqq g$ on $[0, \infty)$ and $g_n(t)\to g(t)$ as $n\to\infty$ for a.e. $t\geqq 0$. Then, comparing u with the function $v(x, t)=(1-x/\delta)g_n(t)$ on $[0, \delta]\times[T, \infty)$, we have

$$u \geqq v \quad \text{on} \quad [0, \delta] \times [T, \infty). \tag{4.2}$$

In fact, we have

$$v_t - v_{xx} = (1-x/\delta)g_n'(t) \leqq 0 = u_t - u_{xx} \quad \text{on} \quad (0, \delta) \times (T, \infty),$$
$$v(x, T) = (1-x/\delta)g_n(T) \leqq (1-x/\delta)g(T) \leqq u(x, T) \qquad \text{for} \quad 0 \leqq x \leqq \delta,$$
$$v(0, t) = g_n(t) \leqq g(t) \leqq u(0, t) \qquad \text{for} \quad T\leqq t < \infty, \quad \text{and}$$
$$v(\delta, t) = 0 \leqq u(\delta, t) \qquad \text{for} \quad T \leqq t < \infty,$$

so that the maximum principle for the linear heat equation implies (4.2), i.e.

$$u(x, t) \geqq (1-x/\delta)g_n(t) \qquad \text{for} \quad x\in[0, \delta] \text{ and } t\in[T, \infty).$$

Letting $n\to\infty$ in this inequality now gives (4.1).

COROLLARY. *Under the same assumptions as in Lemma* 4.1, *we have*

$$u_x(0+, t) \geqq -\delta^{-1}g(t) \qquad \text{for} \quad \text{a.e. } t\in[T, \infty). \tag{4.3}$$

PROOF. Since $u_x(0+, t)=0$ for a.e. $t\in\{t\geqq 0;\, u(0, t)>g(t)\}$, it suffices to show that (4.3) holds for a.e. $t\in\{t\geqq 0;\ u(0,t)=g(t)\}$. At such point t, Lemma 4.1 yields

$$x^{-1}(u(x,t)-g(t))\geqq -\delta^{-1}g(t).$$

Therefore, letting $x\downarrow 0$ gives (4.3).

PROOF OF THEOREM 4.1. First assume $l_\infty<\infty$. In this case, $u\in L^\infty(0,\infty;H)$ by Corollary 1 to Lemma 2.1. Also, taking account of (1.4) of Lemma 1.1, we have

$$\begin{aligned}2\int_0^t g(\tau)d\tau &\leqq 2\int_0^t u(0,\tau)d\tau\\ &= l(t)^2-l_0^2-2\int_0^\infty xu_o(x)dx+2\int_0^\infty xu(x,t)dx\\ &\leqq l_\infty^2-l_0^2-2\int_0^\infty xu_o(x)dx+2l_\infty^{3/2}|u|_{L^\infty(0,\infty;H)},\end{aligned}$$

which implies that $g\in L^1(0,\infty)$. Since $g(t)\to 0$ as $t\to\infty$, it follows from Lemma 2.3 that $u(\cdot,t)\to 0$ in X as $t\to\infty$.

Conversely, assume $g\in L^1(0,\infty)$. Then by the Corollary to Lemma 4.1 and (1.3) of Lemma 1.1 we have

$$\begin{aligned}l(t) &= l(T)+\int_0^\infty u(x,T)dx-\int_0^\infty u(x,t)dx-\int_T^t u_x(0+,\tau)d\tau\\ &\leqq l(T)+\int_0^\infty u(x,T)dx+\delta^{-1}\int_T^\infty g(\tau)d\tau\end{aligned}$$

for all $t\geqq T$. Thus $l_\infty<\infty$.

THEOREM 4.2. *Suppose* (A) *and* (B) *hold. Let* $\{l,u\}$ *be the solution to* $QV(l_o,u_o,g)$. *Then we have*:

(a) *If g is bounded on* $[0,\infty)$ *and* $l_\infty<\infty$, *then* $g\in L^1(0,\infty)$. *Moreover, if* $g(t)\to 0$ *as* $t\to\infty$, *then* $u(\cdot,t)\to 0$ *in* X *as* $t\to\infty$.

(b) *If there exists a non-increasing function* $\hat{g}$ *in* $L^1(0,\infty)$ *such that* $g\leqq\hat{g}$ *on* $[0,\infty)$, *then* $l_\infty<\infty$.

PROOF. Assertion (a) is obtained in the same way as in the proof of the "only if" part of Theorem 4.1. In order to show (b), consider the solution $\{\bar{l},\bar{u}\}$ to $QV(l_o,u_o,\hat{g})$. Then Theorem 3.1 implies that $l\leqq\bar{l}$ on $[0,\infty)$. Since $\lim_{t\to\infty}\bar{l}(t)<\infty$ by Theorem 4.1, it follows that $l_\infty<\infty$.

Finally, we establish the following result:

THEOREM 4.3. *Suppose* (A) (*or* (A)′) *and* (B) *hold. Let* $\{l,u\}$ *be the solution to* $QV(l_o,u_o,g)$. *Furthermore, set*

$$g_\infty = \liminf_{t\to\infty} g(t) \quad \textit{and} \quad g^\infty = \limsup_{t\to\infty} g(t).$$

Then

(4.4) $\liminf_{t\to\infty} u(x, t) \geqq g_\infty$ *uniformly on each bounded interval of* x, *and*

(4.5) $\limsup_{t\to\infty} u(x, t) \leqq g^\infty$ *uniformly in* $x \in [0, \infty)$.

PROOF. First we show (4.4). If $g_\infty = 0$, (4.4) is trivial. Assume $g_\infty > 0$. In this case we infer from Theorem 3.1 and (a) of Theorem 4.2 that $l_\infty = \lim_{t\to\infty} l(t) = \infty$. Let ε be any positive number with $\varepsilon < g_\infty$ and L an arbitrary positive number. Then there is a number $t_\varepsilon > 0$ such that

$$l(t) > L \quad \text{and} \quad g(t) > g_\infty - \varepsilon \qquad \text{for} \quad t \geqq t_\varepsilon.$$

We then consider the Cauchy problem $CP(\psi; u(t_\varepsilon))$ on the interval $[t_\varepsilon, \infty)$:

$$-v'(t) \in \partial\psi(v(t)),\ t_\varepsilon < t < \infty,\ v(t_\varepsilon) = u(t_\varepsilon),$$

where ψ is a proper l.s.c. convex function on H defined by

$$\psi(z) = \begin{cases} (1/2)|z_x|_H^2 & \text{if } z \in X,\ z(0) \geqq g_\infty - \varepsilon \text{ and } z = 0 \text{ on } [L, \infty), \\ \infty & \text{otherwise.} \end{cases}$$

By virtue of Lemma 2.3, $CP(\psi; u(t_\varepsilon))$ admits a unique solution v such that

$$v(t) \longrightarrow v_{\varepsilon,L} \quad \text{in} \quad X \quad \text{as} \quad t \longrightarrow \infty,$$

where

$$v_{\varepsilon,L}(x) = \begin{cases} (g_\infty - \varepsilon)(1 - x/L) & \text{for} \quad 0 \leqq x \leqq L, \\ 0 & \text{for} \quad L < x < \infty. \end{cases}$$

Furthermore, Corollary 2 to Lemma 2.1 yields

$$u \geqq v \qquad \text{on} \quad [0, \infty) \times (t_\varepsilon, \infty).$$

Therefore

$$\liminf_{t\to\infty} u(x, t) \geqq v_{\varepsilon,L}(x) \quad \text{uniformly in} \quad x \in [0, \infty),$$

from which we obtain (4.4).

Next we show (4.5). To this end, it suffices to prove (4.5) under the assumption that $g^\infty < \infty$. Let $\varepsilon > 0$ and choose $t_\varepsilon > 0$ such that

$$g(t) < g^\infty + \varepsilon \qquad \text{for} \quad t \geqq t_\varepsilon.$$

Also, let $h = 0$ on $[t_\varepsilon, \infty)$ and let v be the solution to $CP(\phi_{l,h}^t; u(t_\varepsilon))$ on the interval

$[t_\varepsilon, \infty)$. Then it follows from Lemma 2.1 that

$$u \leqq v + g^\infty + \varepsilon \qquad \text{on} \quad [0, \infty) \times (t_\varepsilon, \infty),$$

since

$$u_t - u_{xx} = v_t - v_{xx} = 0 \quad \text{on} \quad \{(x, t);\, 0 < x < l(t),\, t_\varepsilon < t < \infty\},$$

$$u = v = 0 \quad \text{on} \quad \{(x, t);\, l(t) \leqq x < \infty,\, t_\varepsilon < t < \infty\},$$

$$u(t_\varepsilon) = v(t_\varepsilon), \quad \text{and}$$

$$u_x(0+, t) = 0,\, v_x(0+, t) \leqq 0 \qquad \text{for} \quad \text{a.e. } t \in \{t \geqq t_\varepsilon;\, u(0, t) > v(0, t) + g^\infty + \varepsilon\}.$$

Moreover, Lemma 2.4 implies that $v(x, t) \to 0$ as $t \to \infty$ uniformly in $x \in [0, \infty)$, and so

$$\limsup_{t\to\infty} u(x, t) \leqq g^\infty + \varepsilon \qquad \text{uniformly in} \quad x \in [0, \infty).$$

Since ε is arbitrary, we get the desired estimate.

References

[1] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Math. Studies **5**, North-Holland, Amsterdam-London, 1973.

[2] J. R. Cannon and C. D. Hill, Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation, J. Math. Mech. **17** (1967), 1–19.

[3] J. R. Cannon and C. D. Hill, Remarks on a Stefan problem, J. Math. Mech. **17** (1967), 433–441.

[4] J. R. Cannon and M. Primicerio, Remarks on the one-phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary, J. Math. Anal. Appl. **35** (1971), 361–373.

[5] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[6] H. Furuya, K. Miyashiba and N. Kenmochi, Asymptotic behavior of solutions to a class of nonlinear evolution equations, preprint.

[7] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ. **30** (1981), 1–87.

[8] N. Kenmochi, Subdifferential operator approach to a class of free boundary problems, preprint.

[9] L. I. Rubinstein, The Stefan problem, Translations of Mathematical Monographs **27**, Amer. Math. Soc., Providence R. I., 1971.

[10] S. Yotsutani, Stefan problems with the unilateral boundary condition on the fixed boundary I, Osaka J. Math. **19** (1982), 365–403.

[11] S. Yotsutani, Stefan problems with the unilateral boundary condition on the fixed boundary III, preprint.

Department of Mathematics,
Faculty of Education,
Chiba University