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Boundedness of singular integral operators of Calderón type (IV)

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1. Introduction

We denote by L^p $(1 \leqq p \leqq \infty)$ the L^p-space on the real line $\boldsymbol{R}$ with norm $\|\cdot\|_p$ with respect to the 1-dimensional Lebesgue measure $|\cdot|$. We denote by S^∞ the totality of rapidly decreasing functions on $\boldsymbol{R}$. We say that a locally integrable function $f(x)$ is of bounded mean oscillation if $\|f\|_{BMO} = \sup (1/|I|) \int_I |f(x) - m_I f|\, dx < \infty$, where $m_I f = (1/|I|) \int_I f(x)\, dx$ and the supremum is taken over all finite intervals I. The space BMO of functions of bounded mean oscillation, modulo constants, is a Banach space with norm $\|\cdot\|_{BMO}$. For $0 < \delta \leqq 1$ and a complex-valued kernel $K(x, y)$ $(x, y \in \boldsymbol{R})$, we define $\omega_\delta(K)$ by the infimum over all A's with the following three inequalities:

$$|K(x, y)| \leqq A/|x-y| \quad (x \neq y)$$

$$|K(x, y) - K(x', y)| \leqq A|x-x'|^\delta/|x-y|^{1+\delta} \quad (|x-x'| \leqq |x-y|/2,\ x \neq y)$$

$$|K(x, y) - K(x, y')| \leqq A|y-y'|^\delta/|x-y|^{1+\delta} \quad (|y-y'| \leqq |x-y|/2,\ x \neq y).$$

(If such an A does not exist, we put $\omega_\delta(K) = \infty$.) We say that $K(x, y)$ is a Calderón-Zygmund kernel (CZ-kernel), if $\omega_\delta(K) < \infty$ for some $0 < \delta \leqq 1$,

$$Kf(x) = \int_{-\infty}^{\infty} K(x, y) f(y)\, dy = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} K(x, y) f(y)\, dy$$

exists almost everywhere (a.e.) for any $f \in L^2$ and $\|K\| = \sup \{\|Kf\|_2/\|f\|_2 ; f \in L^2\} < \infty$. For a CZ-kernel $K(x, y)$, a complex-valued function $h(x)$ and a real-valued function $\phi(x)$, we put

$$K[h, \phi](x, y) = K(x, y) h\left\{\frac{\phi(x) - \phi(y)}{x - y}\right\}.$$

Calderón [1] showed that $K[h, \phi]$ is a CZ-kernel if $K(x, y) = 1/(x-y)$, $\phi' \in L^\infty$ and $h(x)$ is extended as an entire function, where "$\phi' \in L^\infty$" implies that $\phi(x)$ is differentiable a.e. and its derivative is essentially bounded. Coifman-David-Meyer [4] showed that Calderón's theorem is valid with the above condition on $h(x)$ replaced by "$h \in S^\infty$". The author [7] showed that their theorem is valid with "$\phi' \in L^\infty$" replaced by "$\phi' \in BMO$". The purpose of this paper is to show an analogous property for CZ-kernels $K(x, y)$ defined by pseudo-differential operators of classic order 0.

Given a non-negative integer n, we say that an infinitely differentiable function $\tau(x, \xi)$ in $\boldsymbol{R} \times \boldsymbol{R}$ is a symbol of (classic) order n if, to any pair (p, q) of non-negative integers, there corresponds a constant $C(p, q)$ such that

$$(1) \qquad |\partial_x^p \partial_\xi^q \tau(x, \xi)| \leqq C(p, q)(1+|\xi|)^{n-q} \quad (x, \xi \in \boldsymbol{R}).$$

We denote by $C(p, q; \tau)$ the infimum of $C(p, q)$'s satisfying (1) and put $\mathfrak{C}(\tau) = \{C(p, q; \tau)\}_{(p,q)}$. We write $\mathfrak{C}(\tau) \leqq \mathfrak{C}_0 = \{C_0(p, q)\}_{(p,q)}$ if $C(p, q; \tau) \leqq C_0(p, q)$ for any pair (p, q). The pseudo-differential operator $\tau(x, D)$ from S^∞ to C^∞ associated with $\tau(x, \xi)$ is defined by

$$\tau(x, D)f(x) = \int_{-\infty}^{\infty} e^{ix\xi}\, \tau(x, \xi) \hat{f}(\xi)\, d\xi \quad (f \in S^\infty),$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$ and C^∞ the totality of infinitely differentiable functions on $\boldsymbol{R}$. We say that $K(x, y)$ is defined by $\tau(x, D)$ if

$$(2) \qquad Kf(x) = \tau(x, D)f(x) \quad \text{a.e.} \quad (f \in S^\infty).$$

Let us note that, for $K(x, y)$ defined by a pseudo-differential operator of order 0, there exists a sequence $(K_m)_{m=1}^\infty$ of CZ-kernels such that $\lim_{m\to\infty} K_m(x, y) = K(x, y)$ a.e. in $\boldsymbol{R} \times \boldsymbol{R}$ and $\sup_m \|K_m\| < \infty$ ([3, p. 83]). We show

THEOREM 1. *For any $0 < \delta \leqq 1$, there exists a positive integer n_δ depending only on δ with the following property: If $K(x, y)$ is a CZ-kernel with $\omega_\delta(K) < \infty$ and $\rho_K(n_\delta) < \infty$, then $K[h, \phi]$ is also a CZ-kernel as long as $\phi' \in BMO$ and $h \in S^\infty$, where*

$$\rho_K(n_\delta) = \sup \{\|K[t^n, \psi]\|;\ n = 0, 1, \dots, n_\delta,\ \|\psi'\|_\infty \leqq 1\ (\psi' \in L^\infty)\}.$$

As an application of this theorem, we show

THEOREM 2. *Let $K(x, y)$ be a CZ-kernel defined by a pseudo-differential operator of order 0. Then $K[h, \phi]$ is also a CZ-kernel as long as $\phi' \in BMO$ and $h \in S^\infty$.*

2. Known facts

We use C for absolute constants. Throughout the paper, we fix $0 < \delta \leqq 1$ and use C_δ for constants depending only on δ. The values of C, C_δ differ in general from one occasion to another. We write by $L_R'^\infty$ the totality of real-valued functions $f(x)$ with $f' \in L^\infty$. For a kernel $K(x, y)$ with $\omega_\delta(K) < \infty$, we define an operator K^* by

$$K^*f(x) = \sup\left\{\left|\int_{\varepsilon<|x-y|<\eta} K(x, y)f(y)dy\right| ; 0 < \varepsilon < \eta\right\} \quad (f \in L^2).$$

The norm $\|K^*\|$ is analogously defined to $\|K\|$. We say that $K(x, y)$ is a δ-CZ-kernel if it is a CZ-kernel with $\omega_\delta(K) < \infty$. For $\phi \in S^\infty$ and a pseudo-differential operator $\tau(x, D)$, we inductively define operators $[\phi, \tau(\cdot, D)]_n$ $(n \geqq 1)$ from S^∞ to C^∞ by:

$$[\phi, \tau(\cdot, D)]_1 f(x) = \phi(x)\tau(x, D)f(x) - \tau(x, D)(\phi f)(x) \quad (f \in S^\infty),$$

$$[\phi, \tau(\cdot, D)]_n f(x) = \phi(x)[\phi, \tau(\cdot, D)]_{n-1} f(x) - [\phi, \tau(\cdot, D)]_{n-1}(\phi f)(x) \quad (n \geqq 2, f \in S^\infty).$$

Here are some known facts necessary for the proof of our theorems.

LEMMA 3 (The Calderón-Zygmund decomposition: Journé [6, p. 12]). *Let* $f \in L^1$ *and* $\lambda > 0$. *Then there exists a sequence* $\{J_k\}_{k=1}^\infty$ *of mutually disjoint finite intervals such that, with* $J = \cup_{k=1}^\infty J_k$,

$$|J| \leqq \|f\|_1/\lambda, \ m_{J_k}|f| \leqq 2\lambda \ (k \geqq 1), \ |f(x)| \leqq \lambda \quad a.e. \quad in \quad J^c.$$

LEMMA 4 (cf. Journé [6, Chap. 4]). *For a kernel* $K(x, y)$, $\|K^*\| \leqq C_\delta\{\|K\| + \omega_\delta(K)\}$.

The following lemma is a version of David's theorem [6, p. 110]. Since the proof is analogous, we omit the proof.

LEMMA 5. *Let* $B \geqq 0$ *and let* $L(x, y)$ *be a kernel with the following property: To every finite open interval* I, *there corresponds a pair* (E_I, L_I) *of a Borel set* E_I *in* I *with* $|E_I| \leqq 2|I|/3$ *and a kernel* $L_I = L_I(x, y)$ *such that*

$$\|L_I^*\| \leqq B, \ \omega_\delta(L_I) \leqq B$$

and

$$L_I(x, y) = L(x, y) \quad (x, y \in I - E_I).$$

Then $\|L^*\| \leqq C_\delta\{B + \omega_\delta(L)\}$.

LEMMA 6 (Coifman-Meyer [2]). *Let* $\phi \in S^\infty$ *and let* $\tau(x, D)$ *be a pseudo-differential operator of order* $n \geqq 1$. *Then* $[\phi, \tau(\cdot, D)]_n$ *is uniquely extended as a bounded operator from* L^2 *to itself and the norm is dominated by* $D_n(\tau)\|\phi'\|_\infty^n$, *where* $D_n(\tau)$ *is a constant depending only on* n *and* $\mathfrak{C}(\tau)$.

LEMMA 7 (Coifman-Meyer [2]). *Let* $H(x, y) = 1/(x-y)$. *Then*

$$\|H[t^n, \phi]\| \leqq D_n\|\phi'\|_\infty^n \quad (n \geqq 0, \phi \in L_R'^\infty),$$

where D_n is a constant depending only on n.

3. Proof of Theorem 1

In this section, we prove Theorem 1. We begin by showing some lemmas.

LEMMA 8. *Let $K(x, y)$ be an η-CZ-kernel* $(0<\eta\leqq 1)$, *$h(t)$ a function in L^∞ with $h'\in L^\infty$ and let $\phi\in L_R'^\infty$. Then $\omega_\eta(K[h, \phi])\leqq C\omega_\eta(K)\{\|h\|_\infty+\|h'\|_\infty\|\phi'\|_\infty\}$. If $0<\eta<1$ and $\phi(x)$ is a real-valued function with $\phi'\in BMO$, then the above inequality is valid with $\|\phi'\|_\infty$ and C replaced by $\|\phi'\|_{BMO}$ and a constant depending only on η, respectively.*

PROOF. Since the first assertion is easily shown, we give only the proof of the second assertion. We have $|K[h, \phi](x, y)|\leqq\omega_\eta(K)\|h\|_\infty/|x-y|$ $(x\neq y)$. Let (x, x', y) be a triple of real numbers with $0<|x-x'|\leqq|x-y|/2$. Then

$$\begin{aligned}
Q &= |K[h, \phi](x, y) - K[h, \phi](x', y)| \\
&\leqq |K(x, y) - K(x', y)|\left|h\left\{\frac{\phi(x)-\phi(y)}{x-y}\right\}\right| \\
&\qquad + |K(x', y)|\left|h\left\{\frac{\phi(x)-\phi(y)}{x-y}\right\} - h\left\{\frac{\phi(x')-\phi(y)}{x'-y}\right\}\right| \\
&\leqq \omega_\eta(K)\|h\|_\infty|x-x'|^\eta/|x-y|^{1+\eta} \\
&\qquad + \{\omega_\eta(K)\|h'\|_\infty/|x'-y|\}\left|\frac{\phi(x)-\phi(y)}{x-y} - \frac{\phi(x')-\phi(y)}{x'-y}\right|.
\end{aligned}$$

To estimate $Q'=|(\phi(x)-\phi(y))/(x-y)-(\phi(x')-\phi(y))/(x'-y)|$, we consider the interval Y with endpoints x, x' and put $\tilde{\phi}(s)=\phi(s)-(m_Y\phi')s$. Let ν be the smallest integer such that $2^m|Y|\geqq 2|x-y|$ $(m\geqq 1)$ and let $\tilde{Y}$ be the interval with midpoint x and of length $2^\nu|Y|$. Then we have $\nu\leqq C\log(|x-y|/|x-x'|)$ and $|m_Y\phi'-m_{\tilde{Y}}\phi'|\leqq C\nu\|\phi'\|_{BMO}$ (cf. [5, p. 142]). Thus

$$\begin{aligned}
Q' &= \left|\frac{\tilde{\phi}(x)-\tilde{\phi}(y)}{x-y} - \frac{\tilde{\phi}(x')-\tilde{\phi}(y)}{x'-y}\right| \\
&= \left|\frac{(x'-x)}{(x-y)(x'-y)}(\tilde{\phi}(x)-\tilde{\phi}(y)) + \frac{\tilde{\phi}(x)-\tilde{\phi}(x')}{x'-y}\right| \\
&\leqq C|x-x'|/(x-y)^2\cdot\int_{\tilde{Y}}|\phi'(s)-m_Y\phi'|ds + C/|x-y|\cdot\int_Y|\phi'(s)-m_Y\phi'|ds \\
&\leqq C\nu\|\phi'\|_{BMO}|x-x'|/|x-y|.
\end{aligned}$$

Consequently we have, with a constant C_η' depending only on η,

$$\begin{aligned}
Q &\leqq \omega_\eta(K)\|h\|_\infty|x-x'|^\eta/|x-y|^{1+\eta} + C\omega_\eta(K)\|h'\|_\infty\|\phi'\|_{BMO}\nu|x-x'|/(x-y)^2 \\
&\leqq C_\eta'\omega_\eta(K)\{\|h\|_\infty + \|h'\|_\infty\|\phi'\|_{BMO}\}|x-x'|^\eta/|x-y|^{1+\eta}.
\end{aligned}$$

In the same manner, we have, for any triple (x, y, y') with $0<|y-y'|\leqq|x-y|/2$,

$$\begin{aligned}&|K[h, \phi](x, y) - K[h, \phi](x, y')|\\ &\leqq C'_\eta\omega_\eta(K)\{\|h\|_\infty + \|h'\|_\infty\|\phi'\|_{BMO}\}|y-y'|^\eta/|x-y|^{1+\eta}.\end{aligned}$$

Hence the required inequality holds. Q.E.D.

LEMMA 9. *There exist two constants n_δ and M_δ depending only on δ such that, for any δ-CZ-kernel $K(x, y)$,*

$$\|K[t^n, \phi]^*\| \leqq \{\rho_K(n_\delta)+\omega_\delta(K)\}M_\delta^n\|\phi'\|_\infty^n \quad (n\geqq n_\delta,\ \phi\in L_R'^\infty). \tag{3}$$

PROOF. We choose $n_\delta\geqq 1$ and M_δ later. Put

$$\rho_K^*(n) = \sup\{\|K[t^j, \psi]^*\|;\ j = 0, 1,\dots, n,\ \|\psi'\|_\infty \leqq 1\ (\psi\in L_R'^\infty)\} \quad (n\geqq 0).$$

Then we have

$$\rho_K^*(n) = \sup\{\|K[t^j, \psi]^*\|;\ j = 0, 1,\dots, n,\ \|\psi'\|_\infty \leqq 1 \quad (\psi\in S^\infty)\}. \tag{4}$$

To see this, for $\psi\in L_R'^\infty$, we choose a sequence $(\psi_l)_{l=1}^\infty$ in S^∞ so that $\lim_{l\to\infty}\psi_l(x)=\psi(x)$ $(x\in\boldsymbol{R})$ and $\|\psi_l'\|_\infty\leqq\|\psi'\|_\infty$ $(l\geqq 1)$. Then, for any $f\in L^2$, $0\leqq j\leqq n$ and $x\in\boldsymbol{R}$, $K[t^j, \psi]^*f(x)\leqq\liminf_{l\to\infty}K[t^j, \psi_l]^*f(x)$. Hence Fatou's lemma shows that $\|K[t^j, \psi]^*\|\leqq\sup\{\|K[t^j, \lambda]^*\|;\ \|\lambda'\|_\infty\leqq\|\psi'\|_\infty\ (\lambda\in S^\infty)\}$ $(0\leqq j\leqq n)$, which gives that $\rho_K^*(n)$ is dominated by the quantity in the right-hand side of (4). Since the inverse inequality evidently holds, we have (4).

Now let $n\geqq n_\delta$. For a while we assume that $\rho_K^*(m)<\infty$ for all $m\geqq 0$ and estimate $\rho_K^*(n)$. To do this, we choose $\psi\in S^\infty$ so that $\|\psi'\|_\infty\leqq 1$. With $L=K[t^n, \psi]$, we shall associate pairs $\{(E_I, L_I)\}_I$ as in Lemma 5. Given a finite open interval $I=(a, b)$, we may assume that $\psi(a)\leqq\psi(b)$; otherwise we deal with $-\psi(x)$. We define $\theta(x)$ by

$$\theta(x) = \begin{cases}\psi(a) & (x\leqq a)\\ \inf\{\lambda(x);\ \lambda\geqq\psi \text{ on } I,\ \lambda'\geqq -v/2,\ \lambda\in S^\infty\} & (a<x\leqq b)\\ \theta(b) & (x>b),\end{cases} \tag{5}$$

where $v=\|\psi'\|_\infty$. Let

$$E_I = \{x\in I;\ \theta(x)\neq\psi(x)\}. \tag{6}$$

Since $-v/2\leqq\theta'(x)\leqq v$ everywhere and $E_I\subset\{x\in I;\ \theta'(x)=-v/2\}$, we have

$$\begin{aligned}0 \leqq \theta(b)-\theta(a) &= \int_I\theta'(x)dx = \int_{E_I}+\int_{I-E_I}\\ &\leqq -v|E_I|/2 + v|I-E_I| = v(|I|-3|E_I|/2),\end{aligned}$$

and hence $|E_I| \leqq 2|I|/3$. We put $L_I = K[t^n, \theta]$. Using Lemma 8 with $h(t) = \{(\operatorname{sign} t) \min(|t|, 1)\}^n$, we have $\omega_\delta(L_I) = \omega_\delta(K[h, \theta]) \leqq Cn\omega_\delta(K)$. To estimate $\|L_I^*\|$, we put

$$(7) \qquad \tilde{\theta}(x) = \theta(x) - \sigma v(x-a) \quad (\sigma = 1/4).$$

Then $\|\tilde{\theta}'\|_\infty \leqq 1-\sigma$ and

$$L_I = \sum_{j=0}^n \binom{n}{j} (\sigma v)^{n-j} (1-\sigma)^j K[t^j, \tilde{\theta}/(1-\sigma)].$$

Hence we have

$$\begin{aligned} \|L_I^*\| &\leqq \sum_{j=0}^n \binom{n}{j} (\sigma v)^{n-j} (1-\sigma)^j \|K[t^j, \tilde{\theta}/(1-\sigma)]^*\| \\ &\leqq \sum_{j=0}^n \binom{n}{j} \sigma^{n-j} (1-\sigma)^j \rho_K^*(j) \leqq (1-\sigma)^n \rho_K^*(n) + \rho_K^*(n-1). \end{aligned}$$

Thus the pair (E_I, L_I) satisfies the conditions in Lemma 5 with $B = (1-\sigma)^n \rho_K^*(n) + \rho_K^*(n-1) + Cn\omega_\delta(K)$. By Lemma 5, we have, with a constant M_δ^*,

$$(8) \quad \|K[t^n, \psi]^*\| \leqq C_\delta\{B + \omega_\delta(L)\} \leqq M_\delta^*\{(1-\sigma)^n \rho_K^*(n) + \rho_K^*(n-1) + n\omega_\delta(K)\}.$$

Since $\psi \in S^\infty$ is arbitrary as long as $\|\psi'\|_\infty \leqq 1$, (4) shows that $\rho_K^*(n)$ is dominated by the last quantity in (8). Now we choose $n_\delta \geqq 1$ so that $M_\delta^*(1-\sigma)^{n_\delta} \leqq 1/2$. Then we have

$$\begin{aligned} (9) \quad \rho_K^*(n) &\leqq (2M_\delta^*)\{\rho_K^*(n-1) + n\omega_\delta(K)\} \leqq \cdots \\ &\leqq (2M_\delta^*)^{n-n_\delta} \rho_K^*(n_\delta) + \{(2M_\delta^*)n + (2M_\delta^*)^2(n-1) + \cdots + (2M_\delta^*)^{n-n_\delta} n_\delta\} \omega_\delta(K) \\ &\leqq \{\rho_K^*(n_\delta) + \omega_\delta(K)\} C_\delta^n. \end{aligned}$$

To remove the assumption that $\rho_K^*(m) < \infty$ for all $m \geqq 0$, we consider $K_\varepsilon(x, y) = K(x, y)\mu_\varepsilon(x-y)$ $(0 < \varepsilon \leqq 1/2)$, where $\mu_\varepsilon(s)$ is the even function on $\boldsymbol{R}$ defined by

$$\mu_\varepsilon(s) = \begin{cases} 0 & (0 \leqq s \leqq \varepsilon) \\ (1/\varepsilon)(s-\varepsilon) & (\varepsilon < s \leqq 2\varepsilon) \\ 1 & (2\varepsilon < s \leqq 1/\varepsilon) \\ \varepsilon(2/\varepsilon - s) & (1/\varepsilon < s \leqq 2/\varepsilon) \\ 0 & (s > 2/\varepsilon). \end{cases}$$

Then elementary calculus yields that $\omega_\delta(K_\varepsilon) \leqq C\omega_\delta(K)$, $\rho_{K_\varepsilon}^*(m) < \infty$ $(0 < \varepsilon \leqq 1/2, m \geqq 0)$. We put $\tilde{\rho}(l) = \sup_{0 < \varepsilon \leqq 1/2} \rho_{K_\varepsilon}^*(l)$ $(l \geqq 0)$ and show that

$$(10) \qquad \rho_K^*(l) \leqq \tilde{\rho}(l) \leqq \rho_K^*(l) + C\omega_\delta(K).$$

We have, for any $f\in L^2$, $\psi\in L_R'^\infty$, $0\leqq j\leqq l$ and $x\in \boldsymbol{R}$, $K[t^j, \psi]^*f(x)\leqq \liminf_{\varepsilon\to 0} K_\varepsilon[t^j, \psi]^*f(x)$. Hence Fatou's lemma shows that $\|K[t^j, \psi]^*f\|\leqq \sup_{0<\varepsilon\leqq 1/2}\|K_\varepsilon[t^j, \psi]^*f\|$, which gives the first inequality in (10). For any $0<\varepsilon\leqq 1/2$, $0<\eta'<\eta''$, we have

$$\begin{aligned}
&\left|\int_{\eta'<|x-y|<\eta''} K_\varepsilon[t^j, \psi](x, y)f(y)dy\right| \\
&\quad\leqq \left|\int_{\eta'<|x-y|<\eta'', \varepsilon<|x-y|<2\varepsilon}\right| + \left|\int_{\eta'<|x-y|<\eta'', 2\varepsilon<|x-y|<1/\varepsilon}\right| \\
&\qquad + \left|\int_{\eta'<|x-y|<\eta'', 1/\varepsilon<|x-y|<2/\varepsilon}\right| \quad (=R_1+R_2+R_3, \text{ say}).
\end{aligned}$$

We have

$$\begin{aligned}
R_1 &\leqq \int_{\varepsilon<|x-y|<2\varepsilon} |K_\varepsilon[t^j, \psi](x, y)f(y)|dy \\
&\leqq \omega_\delta(K_\varepsilon)\|\psi'\|_\infty^j \int_{\varepsilon<|x-y|<2\varepsilon} |f(y)|/|x-y|dy \leqq C\omega_\delta(K)\|\psi'\|_\infty^j \mathfrak{M}f(x),
\end{aligned}$$

where $\mathfrak{M}f(x)$ denotes the maximal function of $f(x)$ [6, p. 7]. We have analogously $R_3\leqq C\omega_\delta(K)\|\psi'\|_\infty^j\mathfrak{M}f(x)$. We can write $R_2=\left|\int_{\tilde\eta'<|x-y|<\tilde\eta''} K[t^j, \psi]\cdot (x, y)f(y)dy\right|$ with some pair $(\tilde\eta', \tilde\eta'')$, and hence $R_2\leqq K[t^j, \psi]^*f(x)$. Thus $K_\varepsilon[t^j, \psi]^*f(x)\leqq K[t^j, \psi]^*f(x)+C\omega_\delta(K)\|\psi'\|_\infty^j\mathfrak{M}f(x)$, which shows $\|K_\varepsilon[t^j, \psi]^*\| \leqq \|K[t^j, \psi]^*\|+C\omega_\delta(K)\|\psi'\|_\infty^j$ (cf. [6, p. 7]). This inequality yields the second inequality in (10). Consequently (10) holds.

Since $\rho_{K_\varepsilon}^*(m)<\infty$ for all $m\geqq 0$, (9) is valid with $K(x, y)$ replaced by $K_\varepsilon(x, y)$. Since $0<\varepsilon\leqq 1/2$ is arbitrary, we have, by (10) and $\omega_\delta(K_\varepsilon)\leqq C\omega_\delta(K)$,

$$\rho_K^*(n)\leqq \tilde\rho(n)\leqq \{\tilde\rho(n_\delta)+C\omega_\delta(K)\}C_\delta^n\leqq \{\rho_K^*(n_\delta)+C\omega_\delta(K)\}C_\delta^n.$$

By Lemmas 4 and 8, we have $\rho_K^*(n_\delta)\leqq C_\delta\{\rho_K(n_\delta)+n_\delta\omega_\delta(K)\}$. Hence we have, with a constant M_δ depending only on δ, $\rho_K^*(n)\leqq\{\rho_K(n_\delta)+\omega_\delta(K)\}M_\delta^n$ $(n\geqq n_\delta)$.
Since $\|K[t^n, \phi]^*\|=\|K[t^n, \phi/\|\phi'\|_\infty]^*\|\,\|\phi'\|_\infty^n$, we have (3). Q. E. D.

LEMMA 10. *There exists a constant $N_\delta\geqq 1$ depending only on δ such that, for any CZ-kernel $K(x, y)$,*

$$\|K[e^{it}, \phi]^*\|\leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}(1+\|\phi'\|_\infty^{N_\delta}\}\quad (\phi\in L_R'^\infty),$$

where n_δ is the constant in Lemma 9.

PROOF. We put

$$\kappa(\alpha)=\sup\{\|K[e^{it}, \psi]^*\|;\ \|\psi'\|_\infty\leqq\alpha\ (\psi\in L_R'^\infty)\}\quad(\alpha\geqq 1).$$

Then, in the same manner as in the proof of (4), we have $\kappa(\alpha)=\sup\{\|K[e^{it}, \psi]^*\|;$

$\|\psi'\|_\infty \leqq \alpha$ $(\psi \in S^\infty)\}$. Lemma 9 shows that $\kappa(\alpha)<\infty$ for all $\alpha \geqq 1$ and $\kappa(1) \leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}$. To estimate $\kappa(\alpha)$ $(\alpha>1)$, we choose $\psi \in S^\infty$ so that $\|\psi'\|_\infty \leqq \alpha$. With $L=K[e^{it}, \psi]$, we shall associate pairs $\{(E_I, L_I)\}_I$ as in Lemma 5. Given $I=(a, b)$, we may assume that $\psi(a) \leqq \psi(b)$. We define $\theta(x)$, E_I and $\tilde{\theta}(x)$ by (5), (6) and (7), respectively and put $L_I=K[e^{it}, \theta]$. Then $\|\tilde{\theta}'\|_\infty \leqq (1-\sigma)\alpha$ and $\|L_I^*\|=\|K[e^{it}, \tilde{\theta}]^*\| \leqq \kappa((1-\sigma)\alpha)$. Lemma 8 gives $\omega_\delta(L_I) \leqq C\alpha\omega_\delta(K)$. Thus the pair (E_I, L_I) satisfies the conditions in Lemma 5 with $B=\kappa((1-\sigma)\alpha)+C\alpha\omega_\delta(K)$. By Lemma 5, we have

$$(11) \qquad \|K[e^{it}, \psi]^*\| \leqq C_\delta\{B+\omega_\delta(K[e^{it}, \psi])\} \leqq C_\delta\{\kappa((1-\sigma)\alpha) + \alpha\omega_\delta(K)\}.$$

Since $\psi \in S^\infty$ is arbitrary as long as $\|\psi'\|_\infty \leqq \alpha$, $\kappa(\alpha)$ is dominated by the last quantity in (11). Consequently, we have, with a constant $N_\delta \geqq 1$,

$$\kappa(\alpha) \leqq \alpha^{N_\delta}\{\kappa(1)+\omega_\delta(K)\} \leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}\{1+\alpha^{N_\delta}\} \quad (\alpha \geqq 1). \qquad \text{Q.E.D.}$$

LEMMA 11. *Let $K(x, y)$ be a δ-CZ-kernel such that $\rho_K(n_\delta)<\infty$. Then $K[e^{it}, \phi]$ is also a δ-CZ-kernel as long as $\phi \in L_R'^\infty$.*

PROOF. By Lemma 8, we have $\omega_\delta(K[e^{it}, \phi]) \leqq C\omega_\delta(K)\{1+\|\phi'\|_\infty\}$. Lemma 10 shows that $\|K[e^{it}, \phi]^*\|<\infty$. Hence it is sufficient to show that $K[e^{it}, \phi]f(x)$ exists a.e. for any $f \in L^2$.

Let $f \in L^2$ and $\psi \in S^\infty$. Then

$$\int_{|x-y|>\varepsilon} K[t, \psi](x, y)f(y)dy = \int_{\varepsilon<|x-y|<1} K(x, y)\{\psi'(y)+O(x-y)\}f(y)dy$$
$$+ \int_{|x-y|>1} K[t, \psi](x, y)f(y)dy \quad (0<\varepsilon<1).$$

Since $K(x, y)$ is a CZ-kernel, this shows that $K[t, \psi]f(x)$ exists a.e.. Note that

$$(12) \qquad |x;\, K[t, \psi]^*g(x)>\lambda| \leqq C_\delta(\rho_K(1)+\omega_\delta(K))\{\|\psi'\|_1\|g\|_1/\lambda\}^{1/2} \quad (\lambda>0;\ \ \psi', g \in L^1).$$

(See for example [7, Lemma 11].) Using this inequality, we show that $K[t, \phi]f(x)$ exists a.e. in a finite open interval I. Let I^* be an interval with the same midpoint as I and of length $3|I|$. We denote by $\chi(x)$, $\chi^*(x)$ the characteristic functions of I, I^*, respectively. Since $K[t, \phi]\{(1-\chi)f\}(x)$ exists everywhere in I, we show that $K[t, \phi](\chi f)(x)$ exists a.e. in I. Note that, for $x \in I$, $K[t, \phi]\cdot(\chi f)(x)$ exists if and only if $K[t, \chi^*\phi](\chi f)(x)$ exists. Choose a sequence $(\psi_n)_{n=1}^\infty$ in S^∞ so that $\lim_{n\to\infty}\|\chi^*\phi-\psi_n\|_1=0$. Then we have $\{x \in I;\ K[t, \chi^*\phi](\chi f)(x)$ does not exist$\} \subset \{x \in I;\ \liminf_{n\to\infty} K[t, \chi^*\phi-\psi_n]^*f(x)>0\}$. Inequality (12) shows that the measure of the second set equals zero, and hence $K[t, \phi](\chi f)(x)$ exists a.e. in I. Thus $K[t, \phi]f(x)$ exists a.e. in I. Since I is arbitrary, $K[t, \phi]f(x)$ exists a.e..

Lemma 9 shows that, for any finite interval I,

$$\int_I \{\textstyle\sum_{n=0}^{\infty}(1/n!)K[t^n,\phi]^*f(x)\}dx \leqq \sum_{n=0}^{\infty}(1/n!)\sqrt{|I|}\|K[t^n,\phi]^*f\|_2$$
$$\leqq \sqrt{|I|}\,\|f\|_2 \textstyle\sum_{n=0}^{\infty}(1/n!)\|K[t^n,\phi]^*\| < \infty,$$

and hence $\sum_{n=0}^{\infty}(1/n!)K[t^n,\phi]^*f(x)<\infty$ a.e. in I. Thus the Lebesgue dominated convergence theorem shows that $K[e^{it},\phi]f(x)=\sum_{n=0}^{\infty}(i^n/n!)K[t^n,\phi]f(x)$ exists a.e. in I. Since I is arbitrary, $K[e^{it},\phi]f(x)$ exists a.e.. Q.E.D.

Now we give the proof of Theorem 1. Since $\omega_\eta(K)$ is increasing with respect to η, we may assume that $\delta<1$. By Lemma 8, we have

$$\omega_\delta(K[h,\phi]) \leqq C_\delta\omega_\delta(K)\{\|h\|_\infty+\|h'\|_\infty\|\phi'\|_{BMO}\}.$$

To estimate $\|K[h,\phi]^*\|$, we discuss $\|K[e^{it},\phi]^*\|$. With $L=K[e^{it},\phi]$, we associate pairs $\{(E_I,L_I)\}_I$ as in Lemma 5. Given a finite open interval I, we use the preceding notation I^*, $\chi^*(x)$. Since $K[e^{it},\phi]=e^{iu}K[e^{it},\underline{\phi}]$ $(u=m_{I^*}\phi',\ \underline{\phi}(x)=\phi(x)-ux)$, we may assume that $m_{I^*}\phi'=0$. We put $\phi_*(x)=\phi'(x)\chi^*(x)$. Then $\|\phi_*\|_1\leqq C\|\phi'\|_{BMO}|I|$. By Lemma 3 $(\lambda=C\|\phi'\|_{BMO})$, there exists a sequence $\{J_k\}_{k=1}^{\infty}$ of mutually disjoint finite intervals such that, with $J=\bigcup_{k=1}^{\infty}J_k$,

$$\begin{cases} |J|\leqq |I|/10, \quad m_{J_k}|\phi_*|\leqq C\|\phi'\|_{BMO} \quad (k\geqq 1) \\ |\phi_*(x)|\leqq C\|\phi'\|_{BMO} \quad \text{a.e. in } J^c. \end{cases}$$

We define $\theta_*(x)$ and $\theta(x)$ by

$$\text{(13)}\qquad \begin{cases} \theta_*(x)=m_{J_k}\phi_* \ (x\in J_k,\ k\geqq 1), \quad \theta_*(x)=\phi_*(x) \quad (x\in J^c) \\ \theta(x)=\phi(d)+\displaystyle\int_d^x\theta_*(s)ds \quad (d\text{: a point in } I\backslash J). \end{cases}$$

Then $\|\theta'\|_\infty=\|\theta_*\|_\infty\leqq C\|\phi'\|_{BMO}$. We put $E_I=I\cap J$ and $L_I=K[e^{it},\theta]$. Then $\omega_\delta(L_I)\leqq C\omega_\delta(K)\{1+\|\phi'\|_{BMO}\}$. Lemma 10 shows that $\|L_I^*\|\leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}\cdot\{1+\|\phi'\|_{BMO}^{N_\delta}\}$. Thus the pair (E_I,L_I) satisfies the conditions in Lemma 5 with $B=C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}\{1+\|\phi'\|_{BMO}^{N_\delta}\}$. We have

$$\|K[e^{it},\phi]^*\|\leqq C_\delta\{B+\omega_\delta(K[e^{it},\phi])\}\leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}\{1+\|\phi'\|_{BMO}^{N_\delta}\}.$$

Since $K[h,\phi]=C\int_{-\infty}^{\infty}\hat{h}(\xi)K[e^{it},\xi\phi]d\xi$, we have

$$\text{(14)}\qquad \|K[h,\phi]^*\|\leqq C\int_{-\infty}^{\infty}|\hat{h}(\xi)|\,\|K[e^{it},\xi\phi]^*\|d\xi$$
$$\leqq C_\delta\{\rho_K(n_\delta)+\omega_\delta(K)\}\int_{-\infty}^{\infty}|\hat{h}(\xi)|\{1+(|\xi|\|\phi'\|_{BMO})^{N_\delta}\}d\xi<\infty,$$

It remains to show that $K[h, \phi]f(x)$ exists a.e. for any $f\in L^2$. Given $f\in S^\infty$, we begin by discussing $K[e^{it}, \phi]f$. For a finite open interval I, we use the preceding notation I^*, $\chi(x)$ and $\chi^*(x)$. We show that $K[e^{it}, \phi]f(x)$ exists a.e. in I. To do this, it is sufficient to show that $K[e^{it}, \phi](\chi f)(x)$ exists a.e. in I. We may assume that $m_{I^*}\phi'=0$. Let $0<\eta<1/10$. For any ε $(0<\varepsilon<\eta)$, there exists a sequence $\{J_k\}_{k=1}^\infty$ of mutually disjoint finite intervals such that, with $\phi_*=\phi'\chi^*$ and $J^\varepsilon=\bigcup_{k=1}^\infty J_k$,

$$\begin{cases} |J^\varepsilon|\leqq \varepsilon|I|,\ m_{J_k}|\phi_*|\leqq (C/\varepsilon)\|\phi'\|_{BMO} \quad (k\geqq 1) \\ |\phi_*(x)|\leqq (C/\varepsilon)\|\phi'\|_{BMO} \quad \text{a.e.}\quad \text{in}\quad J^{\varepsilon c}. \end{cases}$$

We define $\theta_*^\varepsilon(x)$ and $\theta^\varepsilon(x)$ in the same manner as in (13). Then $\theta^\varepsilon\in L_R'^\infty$. Let $J^{*\varepsilon}=\bigcup_{k=1}^\infty J_k^*$, where J_k^* is an interval with the same midpoint as J_k and of length $2|J_k|$. Then, for any $x\in I\backslash J^{*\varepsilon}$,

$$\begin{aligned} M^\varepsilon(x) &= \int_{-\infty}^{\infty} |K[e^{it}, \phi](x, y)-K[e^{it}, \theta^\varepsilon](x, y)|\,|(\chi f)(y)|dy \\ &\leqq \omega_\delta(K)\int_{-\infty}^{\infty} |\phi(y)-\theta^\varepsilon(y)|/(x-y)^2\cdot|(\chi f)(y)|dy \\ &= \omega_\delta(K)\textstyle\sum_{k=1}^\infty \displaystyle\int_{J_k} |\phi(y)-\theta^\varepsilon(y)|/(x-y)^2\cdot|(\chi f)(y)|dy \\ &\leqq \omega_\delta(K)\|f\|_\infty \textstyle\sum_{k=1}^\infty \displaystyle\int_{J_k\cap I}\left\{\int_{J_k}|\phi'(s)-m_{J_k}\phi'|ds\right\}/(x-y)^2dy \\ &\leqq \omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}\textstyle\sum_{k=1}^\infty |J_k|\displaystyle\int_{J_k} 1/(x-y)^2dy, \end{aligned}$$

and hence

$$\begin{aligned} &\int_{I\backslash J^{*\varepsilon}} M^\varepsilon(x)dx \\ &\quad\leqq \omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}\textstyle\sum_{k=1}^\infty |J_k|\displaystyle\int_{J_k^{*c}}dx\int_{J_k} 1/(x-y)^2dy \\ &\quad\leqq 2\omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}\textstyle\sum_{k=1}^\infty\displaystyle\int_{J_k}dy \leqq 2\varepsilon|I|\omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}. \end{aligned}$$

We have, with $I^\varepsilon=\{x\in I\backslash J^{*\varepsilon};\ M^\varepsilon(x)\leqq\sqrt{\varepsilon}\}$,

$$\begin{aligned} |I^\varepsilon| &\geqq |I|-2\sqrt{\varepsilon}|I|\omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}-|J^{*\varepsilon}| \\ &\geqq |I|\{1-2\sqrt{\varepsilon}\omega_\delta(K)\|f\|_\infty\|\phi'\|_{BMO}-2\varepsilon\}. \end{aligned}$$

Since $K[e^{it}, \theta^\varepsilon](\chi f)(x)$ exists a.e. in I for any $0<\varepsilon<\eta$, $K[e^{it}, \phi](\chi f)(x)$ exists a.e. in $I_\eta=\bigcap_{j=1}^\infty I^{\varepsilon_j}$, where $\varepsilon_j=2^{-j}\eta$. Since $\lim_{\eta\to 0}|I_\eta|=|I|$, $K[e^{it}, \phi](\chi f)(x)$ exists a.e. in I. Consequently, $K[e^{it}, \phi]f(x)$ exists a.e.. Since $f\in S^\infty$ is arbitrary, $K[e^{it}, \phi]f(x)$ exists a.e. for any $f \in L^2$.

Let $f \in L^2$. By (14), we have, for any finite interval I,

$$\int_I \left\{\int_{-\infty}^{\infty} |\hat{h}(\xi)| K[e^{it}, \xi\phi]^* f(x) d\xi\right\} dx \leqq \sqrt{|I|}\, \|f\|_2 \int_{-\infty}^{\infty} |\hat{h}(\xi)|\, \|K[e^{it}, \xi\phi]^*\| d\xi < \infty,$$

and hence $\int_{-\infty}^{\infty} |\hat{h}(\xi)| K[e^{it}, \xi\phi]^* f(x) d\xi < \infty$ a.e.. This yields that $K[h, \phi]f(x)$ exists a.e..

4. Proof of Theorem 2

Let $K(x, y)$ be a CZ-kernel defined by a pseudo-differential operator $\sigma(x, D)$ of order 0. Then $K(x, y)$ is a 1-CZ-kernel [3, p. 87]. By Theorem 1, it is sufficient to show that $\rho_K(n_1) < \infty$. We shall deduce this fact from Lemmas 6 and 7. We write simply $\mathfrak{C}(\sigma) = \mathfrak{C}_0 = \{C_0(p, q)\}_{(p,q)}$.

Let $\beta(s)$ be a non-negative even function in S^∞ such that

$$\|\beta\|_\infty \leqq 4, \quad \|\beta\|_1 = 2, \quad \operatorname{supp}(\beta) \subset \{1/2 \leqq |s| \leqq 1\},$$

where supp (β) denotes the support of $\beta(s)$. We put $\beta_m(s) = (1/m)\beta(s/m)$, $\beta_{m,n}(s) = |s|^{n+1}\beta_m(s)$ $(m \geqq 1,\ n \geqq 0)$. We easily see that

$$\|\beta_{m,n}^{(n+1)}\|_1 \leqq \Lambda_n, \quad |\beta_{m,n}^{(q)}(s)| \leqq \Gamma_{n,q}(1+|s|)^{n-q} \quad (n,\ q \geqq 0),$$

where $\Lambda_n = 2^{n+1}(n+1)! \max\{\|\beta^{(j)}\|_1;\ 0 \leqq j \leqq n+1\}$ and $\Gamma_{n,q} = 2^q(n+1)! \times \max\{\|\beta^{(j)}\|_\infty;\ 0 \leqq j \leqq q\}$.

LEMMA 12. *Suppose that $\sigma(x, \xi)$ satisfies $\sigma(x, \xi) = 0$ $(|\xi| \geqq m)$ for some positive integer m. We inductively define two sequences $(\sigma_n^\iota(x, \xi))_{n=1}^\infty$ $(\iota = \pm)$* of symbols by

$$\sigma_n^\iota(x, \xi) = \int_0^\xi \sigma_{n-1}^\iota(x, s) ds - b_{n-1}^\iota(x) \int_0^\xi \beta_{m,n-1}(s) ds \quad (\iota = \pm,\ n \geqq 1,\ \sigma_0^\iota = \sigma),$$

where

$$b_{n-1}^\iota(x) = \left\{\int_0^{\iota\infty} \sigma_{n-1}^\iota(x, s) ds\right\} \Big/ \left\{\int_0^{\iota\infty} \beta_{m,n-1}(s) ds\right\}.$$

Then $\sigma_n^\iota(x, \xi)$ is a symbol of order n with $\mathfrak{C}(\sigma_n^\iota) \leqq \mathfrak{C}_n = \{C_n(p, q)\}_{(p,q)}$ and $\sigma_n^\iota(x, \xi) = 0$ $(\iota\xi \geqq m)$ for any $\iota = \pm$, $n \geqq 1$, where $C_n(p, q)$ depends only on n, $\Gamma_{n-1,q}$ and and $C_0(j, k)$ $(0 \leqq j \leqq p,\ 0 \leqq k \leqq q)$.

PROOF. The symbol $\sigma_0^+ = \sigma$ is of order 0 and satisfies $\mathfrak{C}(\sigma_0^+) = \mathfrak{C}_0$ and $\sigma_0^+(x, \xi) = 0$ $(\xi \geqq m)$. Suppose that $\sigma_{n-1}^+(x, \xi)$ satisfies the required conditions. Then we have, for any pair (p, q) with $q \geqq 1$,

$$\begin{aligned}|\partial_x^p\partial_\xi^q\sigma_n^+(x,\xi)| &\leqq |\partial_x^p\partial_\xi^{q-1}\sigma_{n-1}^+(x,\xi)| + |D^p b_{n-1}^+(x)D^{q-1}\beta_{m,n-1}(\xi)| \\ &\leqq C_{n-1}(p,q-1)(1+|\xi|)^{n-1-(q-1)} \\ &\quad + \left\{\int_0^m C_{n-1}(p,0)(1+s)^{n-1}ds\Big/\int_{m/2}^\infty \beta_{m,n-1}(s)ds\right\}|D^{q-1}\beta_{m,n-1}(\xi)| \\ &\leqq C_{n-1}(p,q-1)(1+|\xi|)^{n-q} + 4^n C_{n-1}(p,0)\Gamma_{n-1,q-1}(1+|\xi|)^{n-1-(q-1)},\end{aligned}$$

and hence

$$(15)\qquad |\partial_x^p\partial_\xi^q\sigma_n^+(x,\xi)| \leqq C_n(p,q)(1+|\xi|)^{n-q} \quad (x,\xi\in\boldsymbol{R}),$$

where

$$(16)\qquad C_n(p,q) = C_{n-1}(p,\tilde{q}) + 4^n C_{n-1}(p,0)\Gamma_{n-1,\tilde{q}} \quad (\tilde{q}=\max\{q-1,0\}).$$

Here note that (15) is valid for any pair $(p, 0)$ with $C_n(p, 0)$ defined by (16). Thus σ_n^+ is of order n and satisfies $\mathfrak{C}(\sigma_n^+)\leqq\mathfrak{C}_n=\{C_n(p,q)\}_{(p,q)}$, where each $C_n(p,q)$ is inductively defined by (16). Since $\sigma_{n-1}^+(x,\xi)=\beta_{m,n-1}(\xi)=0$ $(\xi\geqq m)$, we have $\sigma_n^+(x,\xi)=0$ $(\xi\geqq m)$. Thus σ_n^+ satisfies the required conditions. In the same manner, we see that σ_n^- satisfies the required conditions with $\mathfrak{C}_n$ defined by (16).

Q.E.D.

LEMMA 13. *Suppose that $\sigma(x,\xi)$ satisfies $\sigma(x,\xi)=0$ $(|\xi|\geqq m)$ for some $m\geqq 1$. Then*

$$(17)\qquad \|K[t^n,\phi]\| \leqq \hat{D}_n(\mathfrak{C}_0)\|\phi'\|_\infty^n \quad (n\geqq 0,\ \phi\in S^\infty),$$

where $\hat{D}_n(\mathfrak{C}_0)$ is a constant depending only on n and $\mathfrak{C}_0$.

PROOF. In the case $n=0$, (17) evidently holds. Let $n\geqq 1$. By (2), we have

$$K(x,y) = \int_{-\infty}^\infty e^{i(x-y)\xi}\sigma(x,\xi)d\xi.$$

Repeating the integration by parts, we have

$$\begin{aligned}\int_0^{\iota\infty} e^{i(x-y)\xi}\sigma_n(x,\xi)d\xi &= [-i(x-y)]^{-n}\int_0^{\iota\infty} e^{i(x-y)\xi}\sigma(x,\xi)d\xi \\ &\quad - \textstyle\sum_{j=1}^n [-i(x-y)]^{-j} b_{n-j}^\iota(x)\displaystyle\int_0^{\iota\infty} e^{i(x-y)\xi}\beta_{m,n-j}(\xi)d\xi \quad (\iota=\pm),\end{aligned}$$

and hence

$$\begin{aligned}(18)\qquad K[t^n,\phi](x,y) &= (-i)^n\int_0^\infty e^{i(x-y)\xi}\sigma_n^+(x,\xi)d\xi(\phi(x)-\phi(y))^n \\ &\quad + (-i)^n\int_{-\infty}^0 e^{i(x-y)\xi}\sigma_n^-(x,\xi)d\xi(\phi(x)-\phi(y))^n\end{aligned}$$

$$+\ \sum_{j=1}^{n}(-i)^{n-j}\int_0^{\infty} e^{i(x-y)\xi}\beta_{m,n-j}(\xi)d\xi b_{n-j}^{+}(x)(\phi(x)-\phi(y))^n/(x-y)^j$$

$$+\ \sum_{j=1}^{n}(-i)^{n-j}\int_{-\infty}^{0} e^{i(x-y)\xi}\beta_{m,n-j}(\xi)d\xi b_{n-j}^{-}(x)(\phi(x)-\phi(y))^n/(x-y)^j$$

$$=(-i)^n\int_0^{\infty} e^{i(x-y)\xi}\sigma_n^{+}(x,\xi)d\xi(\phi(x)-\phi(y))^n$$

$$+\ (-i)^n\int_{-\infty}^{0} e^{i(x-y)\xi}\sigma_n^{-}(x,\xi)d\xi(\phi(x)-\phi(y))^n$$

$$+\ i\sum_{j=1}^{n}\int_0^{\infty} e^{i(x-y)\xi}\beta_{m,n-j}^{(n-j+1)}(\xi)d\xi b_{n-j}^{+}(x)(\phi(x)-\phi(y))^n/(x-y)^{n+1}$$

$$+\ i\sum_{j=1}^{n}\int_{-\infty}^{0} e^{i(x-y)\xi}\beta_{m,n-j}^{(n-j+1)}(\xi)d\xi b_{n-j}^{-}(x)(\phi(x)-\phi(y))^n/(x-y)^{n+1}$$

$$(=L_n^{+}(x,y)+L_n^{-}(x,y)+i\sum_{j=1}^{n}L_{n-j}^{+}(x,y)+i\sum_{j=1}^{n}L_{n-j}^{-}(x,y),\ \text{say}).$$

Note that $\|b_{n-j}^{\iota}\|_{\infty}\leqq 4^{n-j}C_{n-j}(0,0)$ and recall that $\|\beta_{m,n-j}^{(n-j+1)}\|_1\leqq A_{n-j}$ $(\iota=\pm,\ 1\leqq j\leqq n)$. By Lemmas 7 and 12, we have

$$\begin{aligned}(19)\qquad \|L_{n-j}^{\iota}\| &\leqq \|H[t^n,\phi]\|\,\|b_{n-j}^{\iota}\|_{\infty}\|\beta_{m,n-j}^{(n-j+1)}\|_1\\ &\leqq \{C4^{n-j}C_{n-j}(0,0)A_{n-j}\}\|\phi'\|_{\infty}^{n}\quad(\iota=\pm,\ 1\leqq j\leqq n).\end{aligned}$$

To estimate $\|L_n^{\iota}\|$ $(\iota=\pm)$, we choose a non-negative function $\gamma\in C^{\infty}$ so that $\gamma(s)=1$ $(s\in[0,\infty))$, $\mathrm{supp}(\gamma)\subset[-1/2,\infty)$, and put

$$L_{n,\gamma}^{\iota}(x,y)=(-i)^n\int_{-\infty}^{\infty} e^{i(x-y)\xi}\sigma_n^{\iota}(x,\xi)\gamma(\iota\xi)d\xi(\phi(x)-\phi(y))^n\quad(\iota=\pm).$$

Then Lemmas 6 and 12 show that

$$\|L_{n,\gamma}^{\iota}\|=\|[\phi,\sigma_n^{\iota}(\cdot,D)\gamma(\iota D)]_n\|\leqq D_n'\|\phi'\|_{\infty}^{n}\quad(\iota=\pm),$$

where D_n' depends only on n, $\mathfrak{C}_0$, $\beta(s)$ and $\gamma(s)$. We have

$$\begin{aligned}&L_{n,\gamma}^{\iota}(x,y)-L_n^{\iota}(x,y)\\ &=\iota(-i)^n\int_{-\iota/2}^{0} e^{i(x-y)\xi}\sigma_n^{\iota}(x,\xi)\gamma(\iota\xi)d\xi(\phi(x)-\phi(y))^n\\ &=\iota i\int_{-\iota/2}^{0} e^{i(x-y)\xi}\partial_{\xi}^{n+1}\{\sigma_n^{\iota}(x,\xi)\gamma(\iota\xi)\}d\xi(\phi(x)-\phi(y))^n/(x-y)^{n+1}\\ &\quad-\iota i\sigma(x,0)(\phi(x)-\phi(y))^n/(x-y)^{n+1},\end{aligned}$$

and hence

$$\begin{aligned}&\|L_{n,\gamma}^{\iota}-L_n^{\iota}\|\\ &\leqq\left\{\left|\int_{-\iota/2}^{0}\|\partial_{\xi}^{n+1}\{\sigma_n^{\iota}(\cdot,\xi)\gamma(\iota\xi)\}\|_{\infty}d\xi\right|+\|\sigma(\cdot,0)\|_{\infty}\right\}\|H[t^n,\phi]\|\end{aligned}$$

$$\leqq D_n''\|\phi'\|_\infty^n \quad (\iota=\pm),$$

where D_n'' depends only on n, $\mathfrak{C}_0$, $\beta(s)$ and $\gamma(s)$. Thus

$$\|L_n^\iota\| \leqq \|L_{n,\gamma}^\iota\| + \|L_{n,\gamma}^\iota - L_n^\iota\| \leqq (D_n' + D_n'')\|\phi'\|_\infty^n \quad (\iota=\pm). \tag{20}$$

Consequently, (18), (19) and (20) show (17) with $D_n(\mathfrak{C}_0)=C\{D_n'+D_n''+D_n 4^n C_n(0, 0)\Lambda_n\}$. Q.E.D.

Now we prove Theorem 2. To do this, we show that

$$\|K[t^n, \phi]\| \leqq D_n^*(\mathfrak{C}_0)\|\phi'\|_\infty^n \quad (n \geqq 0,\ \phi \in L_R'^\infty), \tag{21}$$

where $D_n^*(\mathfrak{C}_0)$ is a constant depending only on n and $\mathfrak{C}_0$.

We define a function $v \in S^\infty$ so that $v(s)=1$ $(s \in [-1/2, 1/2])$ and $\mathrm{supp}(v) \subset [-1, 1]$, and put

$$K_m(x, y) = \int_{-\infty}^{\infty} e^{i(x-y)\xi}\,\sigma(x, \xi)v_m(\xi)d\xi \quad (m \geqq 1),$$

where $v_m(\xi)=v(\xi/m)$. Note that $\sigma(x, \xi)v_m(\xi)$ is of order 0 and satisfies $\mathfrak{C}(\sigma v_m) \leqq \mathfrak{C}_0^*$ for some $\mathfrak{C}_0^*=\{C_0^*(p, q)\}_{(p,q)}$, where each $C_0^*(p, q)$ is independent of m. Also note that $\omega_1(K_m) \leqq C\sum_{j=0}^3 C_0^*(0, j)$ $(m \geqq 1)$ ([3, p. 88]). Let $\psi \in S^\infty$. Then Lemma 13 shows that $\|K_m[t^n, \psi]\| \leqq \hat{D}_n(\mathfrak{C}_0^*)\|\psi'\|_\infty^n$. Lemma 8 yields that $\omega_1(K_m[t^n, \psi]) \leqq C(n+1)\omega_1(K_m)\|\psi'\|_\infty^n$. Hence we have, by Lemma 4,

$$\begin{aligned} \|K_m[t^n, \psi]^*\| &\leqq C\{\hat{D}_n(\mathfrak{C}_0^*)\|\psi'\|_\infty^n + \omega_1(K_m[t^n, \psi])\} \\ &\leqq C\{\hat{D}_n(\mathfrak{C}_0^*) + (n+1)\omega_1(K_m)\}\|\psi'\|_\infty^n \\ &\leqq C\{\hat{D}_n(\mathfrak{C}_0^*) + (n+1)\textstyle\sum_{j=0}^3 C_0^*(0, j)\}\|\psi'\|_\infty^n \quad (=D_n^*(\mathfrak{C}_0)\|\psi'\|_\infty^n,\ \text{say}). \end{aligned} \tag{22}$$

By (2), we have, for any $x \in \boldsymbol{R}$ and $f \in S^\infty$ with $x \notin \mathrm{supp}(f)$, $\lim_{m\to\infty} K_m f(x)=Kf(x)$. Since $\sup_m \omega_1(K_m)<\infty$, the Ascoli-Arzelà theorem yields that $K_m(x, y)$ converges locally uniformly to $K(x, y)$ in $\boldsymbol{R}\times\boldsymbol{R}-\{(x, x);\ x \in \boldsymbol{R}\}$ as $m\to\infty$. By (22) and Fatou's lemma, we have $\|K[t^n, \psi]^*\| \leqq D_n^*(\mathfrak{C}_0)\|\psi'\|_\infty^n$. Given $\phi \in L_R'^\infty$, we can choose a sequence $(\psi_j)_{j=1}^\infty \subset S^\infty$ so that $\lim_{j\to\infty}\psi_j=\phi$ and $\|\psi_j'\|_\infty \leqq \|\phi'\|_\infty$. Hence, again by Fatou's lemma, we have $\|K[t^n, \phi]^*\| \leqq D_n^*(\mathfrak{C}_0)\|\phi'\|_\infty^n$, which shows (21).

By (21), we have immediately $\rho_K(n_1)<\infty$. Thus Theorem 1 yields Theorem 2.

Note. Recently, the author estimated n_δ and obtained that $n_\delta=2$ is sufficient. Perhaps the condition "$\rho_K(n_\delta)<\infty$" is not necessary.

References

[1] A. P. Calderón, Cauchy integrals on Lipschitz curves and related operators, Proc. Nat.

Acad. Sci. USA, **74** (1977), 1324–1327.

[2] R. R. Coifman-Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilinéaires, Ann. Inst. Fourier (Grenoble), **28**-3 (1978), 177–202.

[3] R. R. Coifman-Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque, **57** (1978).

[4] R. R. Coifman-G. David-Y. Meyer, La solution des conjectures de Calderón, Adv. Math., **48** (1983), 144–148.

[5] C. Fefferman, E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1972), 137–193.

[6] J. L. Journé, Calderón-Zygmund operators, Pseudo-differential operators and the Cauchy integral of Calderón, L. N. in Math., 994, Springer, 1983.

[7] T. Murai, Boundedness of singular integral operators of Calderón type (III), Nagoya Math. J., to appear.

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