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Quasi-artinian groups

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Introduction

Aldosray [1] introduced the concept of quasi-artinian Lie algebras generalizing those of soluble Lie algebras and artinian Lie algebras, that is, Lie algebras satisfying the minimal condition for ideals, and left an open question asking whether a semisimple quasi-artinian Lie algebra is always artinian. On the other hand, he introduced the concept of quasi-artinian groups in an analogous way and noted that the corresponding results mentioned in [1] hold for groups. Subsequently Kubo and Honda [2] provided a negative answer to the question above, and moreover gave a condition under which quasi-artinian Lie algebras are soluble (resp. artinian).

In this paper, following the paper [2] we construct a semisimple quasi-artinian group which is neither soluble nor artinian and give a condition under which quasi-artinian groups are soluble (resp. artinian).

We shall prove in Section 2 that the class of quasi-artinian groups is countably recognizable (Proposition 2.2) and that a subgroup with finite index in a quasi-artinian group is quasi-artinian under some conditions (Proposition 2.3). In Section 3 we shall prove that every residually (ω)-central quasi-artinian group is soluble (Theorem 3.3) and that every residually commutable quasi-artinian group is hyperabelian (Theorem 3.7). The main result of Section 4 is that a quasi-artinian group G is artinian if and only if for each normal subgroup N of G G/N satisfies the minimal condition on abelian normal subgroups (Theorem 4.2). In Section 5 we shall give examples showing that the class of quasi-artinian groups is not E-closed (i.e. $\boldsymbol{P}$-closed) and is not s_n-closed.

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1.

Let G be a group. As usual, $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$, $[x, y, z] = [[x, y], z]$ for $x, y, z \in G$. We write inductively

$$D^1(x_1, x_2) = [x_1, x_2],$$

$$D^{n+1}(x_1, \dots, x_{2^{n+1}}) = [D^n(x_1, \dots, x_{2^n}), D^n(x_{2^n+1}, \dots, x_{2^{n+1}})] \quad (n \geq 1),$$

where each $x_i \in G$. For non-empty subsets X, Y, Z of G, we write $X^Y = \langle x^y: x \in X, y \in Y \rangle$, $[X, Y] = \langle [x, y]: x \in X, y \in Y \rangle$, $[X, Y, Z] = [[X, Y], Z]$. We also write $H \leq G$, $H \lhd G$ and $H \lhd^n G$ if H is a subgroup, a normal subgroup and an n-step subnormal subgroup of G respectively. We denote by $\mathrm{Core}_G H$ the core of H in G, that is, the largest normal subgroup of G contained in H. For an ordinal α we denote by $G^{(\alpha)}$, $\zeta_\alpha(G)$ the α-th terms of the (transfinite) derived and upper central series of G respectively. These are inductively defined by $G^{(0)} = G$, $G^{(\alpha+1)} = [G^{(\alpha)}, G^{(\alpha)}]$ and $G^{(\lambda)} = \bigcap_{\alpha<\lambda} G^{(\alpha)}$ for any limit ordinal λ; $\zeta_0(G) = 1$, $\zeta_1(G)$ = the centre of G, $\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta_1(G/\zeta_\alpha(G))$ and $\zeta_\lambda(G) = \bigcup_{\alpha<\lambda} \zeta_\alpha(G)$ for any limit ordinal λ. The upper central series of G terminates and the terminal subgroup is called the hypercentre of G.

Let $\mathfrak{X}$ be a class of groups. A subgroup H of a group G is called a $\lhd^n\mathfrak{X}$-subgroup (resp. $\lhd^n$-subgroup) of G if $H \lhd^n G$ and $H \in \mathfrak{X}$ (resp. $H \lhd^n G$). Min, Min-$\lhd$ and Min-$\lhd^n\mathfrak{X}$ are the classes of groups satisfying the minimal condition for subgroups, normal subgroups and $\lhd^n\mathfrak{X}$-subgroups respectively. The groups in the class Min-$\lhd$ are called artinian groups. $\mathfrak{F}$, $\mathfrak{A}$ and $\mathfrak{P}$ are the classes of finite, abelian and periodic groups respectively. $\acute{\mathrm{E}}(\lhd^n)\mathfrak{A}$ is the class of groups G which have an ascending abelian series of $\lhd^n$-subgroups of G, i.e., an ascending series $1 = G_0 \lhd G_1 \lhd \cdots G_\alpha = G$ in which each factor $G_{\beta+1}/G_\beta$ is abelian and each G_β is a $\lhd^n$-subgroup of G. In particular $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ is called the class of hyperabelian groups.

For a class $\mathfrak{X}$ of groups the classes

$$\mathrm{S}\mathfrak{X}, \quad \mathrm{S}_n\mathfrak{X}, \quad \mathrm{E}\mathfrak{X}, \quad \mathrm{R}\mathfrak{X}, \quad \mathrm{Q}\mathfrak{X}, \quad \mathrm{L}\mathfrak{X}$$

are defined as follows: $G \in \mathrm{S}\mathfrak{X}$ (resp. $\mathrm{S}_n\mathfrak{X}$) if G is a subgroup (resp. normal subgroup) of an $\mathfrak{X}$-group. $G \in \mathrm{E}\mathfrak{X}$ if G has a finite series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ in which each factor G_{i+1}/G_i belongs to $\mathfrak{X}$. $G \in \mathrm{R}\mathfrak{X}$ if to each non-trivial element x of G there corresponds a normal subgroup $N(x)$ not containing x such that $G/N(x) \in \mathfrak{X}$. $G \in \mathrm{Q}\mathfrak{X}$ if G is a homomorphic image of an $\mathfrak{X}$-group. $G \in \mathrm{L}\mathfrak{X}$ if every finite subset of G is contained in an $\mathfrak{X}$-subgroup. In particular $\mathrm{E}\mathfrak{A}$ is the class of soluble groups and is also denoted by $\mathfrak{S}$. The groups in the class $\mathrm{R}\mathfrak{X}$ (resp. $\mathrm{L}\mathfrak{X}$) are called residually (resp. locally) $\mathfrak{X}$-groups. When A is one of S, S_n, E, R, Q, L, we say that a class $\mathfrak{X}$ of groups is A-closed if $\mathrm{A}\mathfrak{X} = \mathfrak{X}$. $\boldsymbol{P}$ and $\boldsymbol{H}$ are also used instead of E and Q respectively.

A group G is said to be semisimple if G has no non-trivial subnormal abelian subgroups.

Let H be a subgroup of a group G. We introduce a new notation: $H \in$ qmin-G if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots$ of normal subgroups of G contained in H there exist $r, s \in \boldsymbol{N}$ such that $[G^{(r)}, N_s] \subseteq N_n$ for any $n \geq 1$, or equivalently there exists $m \in \boldsymbol{N}$ such that $[G^{(m)}, N_m] \subseteq N_n$ for any $n \geq 1$. Then we have a useful result.

LEMMA 1.1. *Let H be a subgroup of a group G. If there exists $m \in \boldsymbol{N}$ such that $G^{(m)} = G^{(\omega)}$, then the following are equivalent*:

(1) $H \in$ qmin-G.

(2) *The set* $\{[G^{(m)}, N]: N \leq H$ *and* $N \lhd G\}$ *satisfies the minimal condition.*

(3) *For every descending chain* $N_1 \supseteq N_2 \supseteq \cdots$ *of normal subgroups of G contained in H, the descending chain* $[G^{(m)}, N_1] \supseteq [G^{(m)}, N_2] \supseteq \cdots$ *terminates.*

PROOF. Put $M = G^{(m)}$.

(1)⇒(2): Let N_i be a normal subgroup of G contained in H for any $i \geq 1$ and suppose that $[M, N_1] \supseteq [M, N_2] \supseteq \cdots$. Since $H \in$ qmin-G, there exists an integer $n \geq 1$ such that $[G^{(n)}, [M, N_n]] \subseteq [M, N_i]$ for any i. By using the three subgroup lemma we have

$$[M, N_n] \subseteq [M, N_n, M] \subseteq [G^{(n)}, [M, N_n]] \subseteq [M, N_i] \quad \text{for any } i \geq n.$$

(2)⇒(3) is trivial.

(3)⇒(1): Let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of normal subgroups of G contained in H. Then there exists $n \in \boldsymbol{N}$ such that $[M, N_n] = [M, N_{n+1}] = \cdots$. Therefore we have $[M, N_n] = \bigcap_{i=1}^{\infty} [M, N_i] \subseteq \bigcap_{i=1}^{\infty} N_i$.

A group G is said to be quasi-artinian if $G \in$ qmin-G. We denote by qmin-$\lhd$ the class of quasi-artinian groups. We note that if G is quasi-artinian then there exists $m \in \boldsymbol{N}$ such that $G^{(m)} = G^{(\omega)}$. Hence as a special case of Lemma 1.1 we obtain

COROLLARY 1.2. *The following are equivalent*:

(1) *G is quasi-artinian.*

(2) *There exists $m \in \boldsymbol{N}$ such that $G^{(m)} = G^{(\omega)}$, and the set* $\{[G^{(m)}, N]: N \lhd G\}$ *satisfies the minimal condition.*

(3) *There exists $m \in \boldsymbol{N}$ such that $G^{(m)} = G^{(\omega)}$, and for every descending chain* $N_1 \supseteq N_2 \supseteq \cdots$ *of normal subgroups of G, the descending chain* $[G^{(m)}, N_1] \supseteq [G^{(m)}, N_2] \supseteq \cdots$ *terminates.*

(4) *There exists $m \in \boldsymbol{N}$ such that for every descending chain* $N_1 \supseteq N_2 \supseteq \cdots$ *of normal subgroups of G, the descending chain* $[G^{(m)}, N_1] \supseteq [G^{(m)}, N_2] \supseteq \cdots$ *terminates.*

The equivalence of (1), (3) and (4) in the statement of Corollary 1.2 was shown by Aldosray [1, Theorem 3.1] and the equivalence of (1) and (2) is a group analogue of [2, Proposition 1.1].

2.

In this section we shall state several results on quasi-artinian groups.

qmin-$\lhd$ is Q-closed ([1, Theorem 3.2(i)]) but is not E-closed (Example 5.1). However we know the following fact.

LEMMA 2.1 ([1, Theorem 3.2(ii)]). *Let* $N \lhd G$. *Then* $G \in$ qmin-$\lhd$ *if one of the following holds*:

(a) $N \in$ qmin-$\lhd$ *and* $G/N \in \mathrm{E}\mathfrak{A}$.

(b) $N \in$ qmin-G *and* $G/N \in$ qmin-$\lhd$.

(c) $N \in$ Min-$\lhd$ *and* $G/N \in$ qmin-$\lhd$.

Let $\mathfrak{X}$ be any class of groups. We recall that $\mathrm{L}_{\aleph_0}\mathfrak{X}$ is the class of groups G such that every countable subset of G is contained in an $\mathfrak{X}$-subgroup of G. $\mathfrak{X}$ is called countably recognizable if $\mathfrak{X}$ is $\mathrm{L}_{\aleph_0}$-closed, that is, $\mathfrak{X} = \mathrm{L}_{\aleph_0}\mathfrak{X}$. It is well known that many interesting classes of groups which are not L-closed are $\mathrm{L}_{\aleph_0}$-closed (cf. [6, pp. 104–110]). For example, $\mathrm{E}\mathfrak{A}$, $\mathfrak{N}$, Max, Max-$\lhd$, Min, Min-$\lhd$, etc. are countably recognizable. Though qmin-$\lhd$ is not L-closed (Remark 3.9) we have

PROPOSITION 2.2. qmin-$\lhd$ *is countably recognizable.*

PROOF. Let $G \notin$ qmin-$\lhd$. It follows from Corollary 1.2 that for any $m \in \boldsymbol{N}$ there exists a descending chain $N_1 > N_2 > \cdots$ of normal subgroups of G such that $[G^{(m)}, N_i] > [G^{(m)}, N_{i+1}]$ for any $i \geq 1$. Choose x_i to be any element of $[G^{(m)}, N_i] \backslash [G^{(m)}, N_{i+1}]$. Now we can write $x_i = \Pi_j [y_{ij}, n_{ij}]^{\varepsilon_{ij}}$ where $y_{ij} \in G^{(m)}$, $n_{ij} \in N_i$ and $\varepsilon_{ij} = \pm 1$. Since $G^{(m)} = \langle \mathrm{D}^m(g_1, \dots, g_{2^m}) \colon g_k \in G \rangle$ we can also write $y_{ij} = \Pi_k \mathrm{D}^m(g_{ijk1}, \dots, g_{ijk2^m})$. Let X be a subgroup of G which contains the countable set $\{n_{ij}, g_{ijkl}\}_{i,j,k,l}$. Since $x_i \in [X^{(m)}, X \cap N_i] \backslash [X^{(m)}, X \cap N_{i+1}]$, we have $[X^{(m)}, X \cap N_i] > [X^{(m)}, X \cap N_{i+1}]$ for any $i \geq 1$. Hence $X \notin$ qmin-$\lhd$ by Corollary 1.2 and so $G \notin \mathrm{L}_{\aleph_0}$(qmin-$\lhd$). It follows that qmin-$\lhd$ is countably recognizable.

The class of artinian groups Min-$\lhd$ is not s-closed and is not even s_n-closed (cf. [5, p. 153]). However Wilson showed that a subgroup with finite index in an artinian group is artinian (cf. [5, Theorem 5.21] or [7, 3.1.8]). Though qmin-$\lhd$ is not s_n-closed (Example 5.2), we shall show that a subgroup with finite index in a quasi-artinian group is quasi-artinian under some conditions.

DEFINITION. We say that a group G has the property (P) if $[A, B] \cap [A, C] = [A, B \cap C]$ holds for any three normal subgroups A, B, C of G.

PROPOSITION 2.3. *Let G be a quasi-artinian group and let H be a subgroup with finite index in G. If $G/\mathrm{Core}_G H$ is soluble and $\mathrm{Core}_G H$ has the property (P), then H is quasi-artinian.*

PROOF. Suppose that $H \notin$ qmin-$\lhd$. Set $C = \mathrm{Core}_G H$. Then C is of finite index in G. Since H/C is soluble Lemma 2.1 implies that $C \notin$ qmin-$\lhd$. By hypothesis there exists $m \in N$ such that $G^{(m)} = G^{(\omega)} \subseteq C$. Hence we have $G^{(m)} = C^{(m)}$, say N. Since $C \notin$ qmin-C it follows from Corollary 1.2 that there exists a minimal element $[N, K]$ of the non-empty set $\{[N, L]: L \lhd G,\ L \leq C$ and $L \notin$ qmin-$C\}$.

Let $\mathscr{S}$ be the set of all non-empty finite subsets X of G with the following property: if

$$K_1 > K_2 > \cdots \tag{1}$$

is a strictly descending chain of C-admissible subgroups of K such that $[N, K_1] > [N, K_2] > \cdots$, then

$$[N, K] = [N, K_i^X] \tag{2}$$

for all i. Let T be a transversal to C in G. Then $G = CT$. For any chain (1) the relation $K_i \lhd C$ implies that $K_i^T \lhd G$. Also $K_i^T \leq K$ since $K \lhd G$ and so $[N, K_i^T] \leq [N, K]$. If $[N, K_i^T] < [N, K]$, then $K_i^T \in$ qmin-C by minimality of $[N, K]$ and therefore $[N, K_j] = [N, K_{j+1}] = \cdots$ for some $j \geq i$, in view of Lemma 1.1. By this contradiction $[N, K_i^T] = [N, K]$ for all i. Thus $T \in \mathscr{S}$ and $\mathscr{S}$ is not empty.

We now select a minimal element X of $\mathscr{S}$. If $x \in X$, then $Xx^{-1} \in \mathscr{S}$ because $N, K \lhd G$. Therefore Xx^{-1} is a minimal element of $\mathscr{S}$ containing 1. Hence we may assume that $1 \in X$. Now if $X = \{1\}$, the equation (2) shows that $K \in$ qmin-C. It follows that X has at least two elements. Consequently the set

$$Y = X \backslash \{1\}$$

is non-empty. Therefore Y does not belong to $\mathscr{S}$ by minimality of X.

For any chain (1) we define

$$L_i = K_i \cap K_i^Y.$$

Now $K_i^g \lhd C^g = C$ for all g in G and so $L_i \lhd C$. Also $L_i \geq L_{i+1}$ and $[N, L_i] \geq [N, L_{i+1}]$. Suppose that $[N, L_i] = [N, L_{i+1}]$. Since $X \in \mathscr{S}$, we must have $[N, K] = [N, K_{i+1}^X]$ and

$$\begin{aligned}[N, K_i] &= [N, K_i] \cap [N, K_{i+1}^X] = [N, K_i] \cap [N, K_{i+1} K_{i+1}^Y] \\ &= [N, K_i] \cap ([N, K_{i+1}][N, K_{i+1}^Y]) = [N, K_{i+1}]([N, K_i] \cap [N, K_{i+1}^Y]) \\ &= [N, K_{i+1}][N, K_i \cap K_{i+1}^Y] \subseteq [N, K_{i+1}][N, L_i] = [N, K_{i+1}],\end{aligned}$$

using that C has the property (P). Thus $[N, K_i] = [N, K_{i+1}]$, which is not the case. Hence $[N, L_i] > [N, L_{i+1}]$ for all i. Therefore $[N, K] = [N, L_i^X]$ for all i, which shows that

$$[N, K_i] = [N, K_i] \cap [N, L_i^X] = [N, K_i] \cap ([N, L_i][N, L_i^Y])$$
$$= [N, L_i]([N, K_i] \cap [N, L_i^Y]) \subseteq [N, L_i].$$

Hence $[N, K_i]=[N, L_i]$. By definition of L_i it follows that

$$[N, K_i^Y] = [N, K_i^X] = [N, K]$$

for all i, and so $Y \in \mathscr{S}$, which is a contradiction.

3.

In this section we shall first give classes $\mathfrak{X}$ of groups such that qmin-$\lhd \cap \mathfrak{X} =$ E$\mathfrak{A}$, and secondly give classes $\mathfrak{Y}$ of groups such that qmin-$\lhd \cap \mathfrak{Y} \leq$ É$(\lhd)\mathfrak{A}$.

A group G is said to be residually central if

$$x \notin [G, x]$$

for each non-trivial element x of G. We denote by $\mathfrak{R}$ the class of residually central groups. $\mathfrak{R}$ is S, L and R-closed and every Z-group is residually central. So L$\mathfrak{R} \leq \mathfrak{R}$. Following [2], we generalize the notion of residually central groups.

DEFINITION. We say that a group G is residually (ω)-central if

$$x \notin [G^{(\omega)}, x]^G$$

for each non-trivial element x of G, and denote by $\mathfrak{R}_{(\infty)}$ the class of residually (ω)-central groups. It is clear that $\mathfrak{R}_{(\infty)}$ is S and R-closed and that RE$\mathfrak{A} \cup \mathfrak{R} \leq \mathfrak{R}_{(\infty)}$ But $\mathfrak{R}_{(\infty)}$ is not L-closed (see the statement after Remark 3.5).

We first prove a simple result.

LEMMA 3.1. *Let H be a subgroup of a group G and let Z be a subgroup of the centralizer of H in G. If x is an element in G such that $x \in [H, x]^G Z \setminus Z$, then there is a non-trivial element c of $[H, x]^G$ such that $c \in [H, c]^G$.*

PROOF. By hypothesis we can write $x=cz$ where $c \in [H, x]^G$ and $z \in Z$. Since $x \notin Z$, we see that $c \neq 1$. Let h be any element of H. Then

$$[h, x] = [h, z][h, c]^z = [h, c]^z.$$

Hence $c \in [H, x]^G = [H, c]^G$.

Since all free groups are residually nilpotent (cf. [7, 6.1.9]), $\mathfrak{R}_{(\infty)}$ is not Q-closed. However there is the following weak form of Q-closedness.

PROPOSITION 3.2. *Let G be a residually (ω)-central group and let N be a* *normal subgroup of G contained in the hypercentre of G. If there exists* $n \in \boldsymbol{N}$ *such that* $G^{(n)} = G^{(\omega)}$, *then* G/N *is residually* (ω)*-central.*

PROOF. Let $Z_\alpha = \zeta_\alpha(G)$. Since N is contained in the hypercentre of G, it is sufficient to prove that $G/N \cap Z_\alpha$ is residually (ω)-central for every ordinal α. Suppose that α is the first ordinal for which this is false. Then $\alpha > 0$ and there exists an element x such that $x \in [G^{(n)}, x]^G(N \cap Z_\alpha)$ but $x \notin N \cap Z_\alpha$. Assume that α is not a limit ordinal. Then $x(N \cap Z_{\alpha-1})$ does not belong to $N \cap Z_\alpha/N \cap Z_{\alpha-1}$ which is a subgroup of the centre of $G/N \cap Z_{\alpha-1}$, but it does belong to

$$[(G/N \cap Z_{\alpha-1})^{(\omega)}, x(N \cap Z_{\alpha-1})]^{G/N \cap Z_{\alpha-1}}(N \cap Z_\alpha/N \cap Z_{\alpha-1}).$$

Lemma 3.1 may therefore be applied to the group $G/N \cap Z_{\alpha-1}$ and we conclude that this group is not residually (ω)-central. By this contradiction α is a limit ordinal and $x \in [G^{(n)}, x]^G(N \cap Z_\beta)$ for some $\beta < \alpha$. But $G/N \cap Z_\beta$ is residually (ω)-central, and so $x \in N \cap Z_\beta \leq N \cap Z_\alpha$, our final contradiction.

Now we shall give the first of main results in this section, which is a group analogue of [2, Theorem 2.3].

THEOREM 3.3. $\text{qmin-}\lhd \cap \mathfrak{X} = \text{E}\mathfrak{A}$ *for any class* $\mathfrak{X}$ *of groups such that* $\text{E}\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{N}_{(\infty)}$.

PROOF. It is sufficient to prove that $\text{qmin-}\lhd \cap \mathfrak{N}_{(\infty)} \leq \text{E}\mathfrak{A}$. Suppose that there exists a group G such that $G \in \text{qmin-}\lhd \cap \mathfrak{N}_{(\infty)} \setminus \text{E}\mathfrak{A}$. Put $N = G^{(\omega)}$. Then $1 \neq N = G^{(n)}$ for some $n \in \boldsymbol{N}$. Since N is perfect we have $\zeta_1(N) = \zeta_2(N) < N$, owing to the Grün's lemma. We note that $x \notin [N, x]^G\zeta_1(N)$ for any $x \in N \setminus \zeta_1(N)$. In fact, if $x \in [N, x]^G\zeta_1(N)$ then since $\zeta_1(N) \leq C_G(N)$ it follows from Lemma 3.1 that there exists a non-trivial $c \in [N, c]^G$, which implies that G is not residually (ω)-central. Now take $x_1 \in N \setminus \zeta_1(N)$. As $\zeta_1(N) = \zeta_2(N)$ we have $\zeta_1(N) < [N, x_1]^G\zeta_1(N)$. Next we take $x_2 \in [N, x_1]^G\zeta_1(N) \setminus \zeta_1(N)$. Then we also have $\zeta_1(N) < [N, x_2]^G\zeta_1(N)$. By repeating this procedure, we can find a sequence $(x_i)_{i=1}^\infty$ of elements of $N \setminus \zeta_1(N)$ such that for any integer $i \geq 1$

$$x_i \notin [N, x_i]^G\zeta_1(N) \quad \text{and} \quad x_{i+1} \in [N, x_i]^G\zeta_1(N).$$

Put $N_i = [N, x_i]^G\zeta_1(N)$. Then $N_i \lhd G$ and $N_i > N_{i+1}$ for any $i \geq 1$. Since G is quasi-artinian, there exists $m \in \boldsymbol{N}$ such that $[N, N_m] \subseteq N_{m+1}$. Using the three subgroup lemma we obtain that

$$[N, x_m]^G \subseteq [N, x_m, N]^G \subseteq [N_m, N] \subseteq N_{m+1}.$$

Therefore $N_m = [N, x_m]^G\zeta_1(N) \subseteq N_{m+1}$, which is a contradiction.

COROLLARY 3.4. (1) $\text{qmin-}\lhd \cap \mathfrak{N} \leq \text{E}\mathfrak{A}$.
(2) $\text{Min-}\lhd \cap \mathfrak{N}_{(\infty)} \leq \text{E}\mathfrak{A}$.

REMARK 3.5. As a finite residually central group is nilpotent (cf. [6, p. 7]) we see that qmin-$\lhd \cap \mathfrak{R} < \mathrm{E}\mathfrak{A}$. By considering an infinite cyclic group we also see that Min-$\lhd \cap \mathfrak{R}_{(\infty)} < \mathrm{E}\mathfrak{A}$.

Corollary 3.4 indicates that every locally nilpotent quasi-artinian group is soluble. But locally soluble quasi-artinian groups need not be soluble. In fact, McLain [3] constructed a locally soluble artinian group which is not soluble. However we shall later obtain that a locally soluble quasi-artinian group is hyperabelian. To do this we need the following

LEMMA 3.6 (Baer). *A group G is hyperabelian if and only if given two sequences $x_0, x_1,\dots$ and $y_0, y_1,\dots$ of elements of G such that*

$$x_{i+1} = [x_i, y_i, x_i],$$

there is an integer $m \geq 0$ such that $x_m = 1$.

PROOF. See [5, Theorem 2.15].

A group G is said to be residually commutable if given a pair of non-trivial elements a and b, there exists a normal subgroup N of G which contains $[a, b]$ but does not contain both a and b. We denote by $\mathfrak{R}_0$ the class of residually commutable groups. $\mathfrak{R}_0$ is S, R and L-closed and every *SI*-group is residually commutable. So $\mathrm{LE}\mathfrak{A} \leq \mathfrak{R}_0$. It is well known that a residually commutable artinian group is hyperabelian (cf. [6, Theorem 8.15]). Now we can strengthen this result. Namely we show the second of main results in this section.

THEOREM 3.7. *A residually commutable quasi-artinian group is hyperabelian.*

PROOF. Let G be residually commutable and quasi-artinian. Suppose that G is not hyperabelian. By making use of Lemma 3.6 we see that there exist two sequences of elements of G $x_0, x_1,\dots$ and $y_0, y_1,\dots$ such that

$$1 \neq x_{i+1} = [x_i, y_i, x_i]$$

for each integer $i \geq 0$. It is easily seen that $x_i \in G^{(i)}$ for any $i \geq 0$. Let $N_0 = G$ and $N_1 = G^{(1)}$. Suppose that for $i \geq 1$ we have constructed a normal subgroup N_i of G containing x_i such that

$$N_i \subseteq [G^{(i-1)}, N_{i-1}].$$

Now, since each $x_j \neq 1$, we see that $[x_i, y_i] \neq 1$. Since G is residually commutable, there is a normal subgroup N of G such that $x_{i+1} = [x_i, y_i, x_i] \in N$, but N does not contain both x_i and $[x_i, y_i]$. On the other hand, we also have $x_{i+1} \in [G^{(i)}, N_i]$. So, set $N_{i+1} = [G^{(i)}, N_i] \cap N$. Then $x_{i+1} \in N_{i+1}$ and $N_i > N_{i+1}$

since either x_i or $[x_i, y_i]$ belongs to $N_i \backslash N$. This construction produces an infinite descending chain of normal subgroups

$$\cdots \supseteq [G^{(i-1)}, N_{i-1}] \supseteq N_i \supseteq [G^{(i)}, N_i] \supseteq N_{i+1} \supseteq [G^{(i+1)}, N_{i+1}] \supseteq \cdots.$$

Consequently $([G^{(i)}, N_i])_{i=1}^{\infty}$ does not terminate. This is impossible by Corollary 1.2.

As mentioned in the paragraph after Remark 3.5 we obtain the following

COROLLARY 3.8. *A locally soluble quasi-artinian group is hyperabelian.*

REMARK 3.9. Hyperabelian groups need not be quasi-artinian. In fact, let G_i be a soluble group with derived length i for all $i \geq 1$. Set $G = \mathrm{Dr}_{i=1}^{\infty} G_i$. Then G is hyperabelian and locally soluble (so locally quasi-artinian). But since $G^{(1)} > G^{(2)} > \cdots$ we see that G is not quasi-artinian.

Robinson showed that $\acute{\mathrm{E}}(\lhd^2)\mathfrak{A} \cap \mathfrak{P} \cap \text{Min-}\lhd^2\mathfrak{A} \leq \mathrm{E}\mathfrak{A} \cap \text{Min}$ ([4, Theorem E]). Hence we have the following

COROLLARY 3.10. $\text{qmin-}\lhd \cap \mathfrak{N}_0 \cap \mathfrak{P} \cap \text{Min-}\lhd^2\mathfrak{A} = \mathrm{E}\mathfrak{A} \cap \text{Min}$.

4.

In this section we shall present classes $\mathfrak{X}$ of groups such that $\text{qmin-}\lhd \cap \mathfrak{X} = \text{Min-}\lhd$.

For any class $\mathfrak{X}$ of groups, let $\mathfrak{X}^{\mathrm{Q}}$ denote the largest Q-closed subclass of $\mathfrak{X}$. It is easy to see that for a group G, $G \in \mathfrak{X}^{\mathrm{Q}}$ if and only if $N \lhd G$ implies $G/N \in \mathfrak{X}$.

It is obvious that

$$\text{Min-}\lhd \leq (\text{Min-}\lhd \mathrm{E}\mathfrak{A})^{\mathrm{Q}} \leq (\text{Min-}\lhd \mathfrak{A})^{\mathrm{Q}} \leq (\text{Min-}\lhd(\mathfrak{A} \cap \mathfrak{P}))^{\mathrm{Q}}.$$

For the first and second inclusions we obtain the following

PROPOSITION 4.1. $\text{Min-}\lhd < (\text{Min-}\lhd \mathrm{E}\mathfrak{A})^{\mathrm{Q}} = (\text{Min-}\lhd \mathfrak{A})^{\mathrm{Q}}$.

We state the main result in this section.

THEOREM 4.2. $\text{qmin-}\lhd \cap \mathfrak{X} = \text{Min-}\lhd$ for any class $\mathfrak{X}$ of groups such that $\text{Min-}\lhd \leq \mathfrak{X} \leq (\text{Min-}\lhd \mathfrak{A})^{\mathrm{Q}}$.

Proposition 4.1 and Theorem 4.2 are group analogues of the results on Lie algebras (Propositions 3.1, 3.2 and Theorem 3.3 in [2]) and their proofs can be carried over quite similarly. So we omit the proofs.

Robinson showed that $\mathrm{LE}\mathfrak{A} \cap \text{Min-}\lhd \cap \text{Min-}\lhd^2\mathfrak{A} \leq \mathrm{E}\mathfrak{A} \cap \text{Min}$ ([4, Theorem E*]). Hence we have the following

COROLLARY 4.3. $\mathrm{L}\mathrm{E}\mathfrak{A} \cap \text{qmin-}\lhd \cap (\text{Min-}\lhd^2\mathfrak{A})^{\mathrm{Q}} = \mathrm{E}\mathfrak{A} \cap \text{Min}$.

REMARK 4.4. There exists a group G such that

$$G \in \text{qmin-}\lhd \cap (\text{Min-}\lhd(\mathfrak{A} \cap \mathfrak{P}))^{\mathrm{Q}} \quad \text{but} \quad G \notin \text{Min-}\lhd$$

(Example 5.3). Hence by Theorem 4.2 we see that

$$(\text{Min-}\lhd\mathfrak{A})^{\mathrm{Q}} < (\text{Min-}\lhd(\mathfrak{A} \cap \mathfrak{P}))^{\mathrm{Q}}.$$

5.

In this section we shall present several examples in connection with the results in Sections 2 and 4.

EXAMPLE 5.1. Let S be a non-abelian simple group and let $\boldsymbol{Z}$ be the additive group of all integers. Put $G=\boldsymbol{Z}\sim S$, that is, the standard wreath product of $\boldsymbol{Z}$ with S. Let B be the base group of G. Then $B=\oplus_{x\in S}\boldsymbol{Z}_x$ where $\boldsymbol{Z}_x\cong\boldsymbol{Z}$ for each $x\in S$ and $G=B\rtimes S$. For each integer $i\geq 1$ we put $N_i=\oplus_{x\in S}2^i\boldsymbol{Z}_x$. Obviously $N_1>N_2>\cdots$. We note that

$$[ax, b] = [a, b]^x[x, b] = (b^{-1})^x b \quad \text{for} \quad a, b \in B \quad \text{and} \quad x \in S.$$

Let 1_y denote the element of $\boldsymbol{Z}_y$ which is the isomorphic copy of 1. Now take any element b of N_i. Then we can write $b=\Sigma_{y\in S}2^i n_y\cdot 1_y$ where $n_y\in\boldsymbol{Z}$ for each $y\in S$. From the note above we have

$$\begin{aligned}[ax, b] &= (\Sigma 2^i(-n_y)\cdot 1_y)^x + \Sigma 2^i n_y\cdot 1_y \\ &= \Sigma 2^i(n_y - n_{yx^{-1}})\cdot 1_y \in N_i.\end{aligned}$$

Hence $[G, N_i]=[S, N_i]\subseteq N_i$, which shows that $N_i\lhd G$ and $[G^{(m)}, N_i]=[S, N_i]$ for all $m\geq 0$ and $i\geq 1$. Now put $b=2^i\cdot 1_y\in\boldsymbol{Z}_y\cap N_i$. Then for a non-trivial element x in S

$$[x, b] = 2^i(-1_{yx}) + 2^i\cdot 1_y = 2^i(1_y - 1_{yx}) \notin N_{i+1}.$$

Hence we have $[S, N_i]>[S, N_{i+1}]$ for any $i\geq 1$. So for each $m\geq 0$ we have

$$[G^{(m)}, N_i] > [G^{(m)}, N_{i+1}] \qquad \text{for any } i \geq 1,$$

which implies by Corollary 1.2 that G is not quasi-artinian. However it is clear that B and G/B are quasi-artinian. Therefore qmin-$\lhd$ is not E-closed.

EXAMPLE 5.2. Let S be a non-abelian simple group and let S^* be an infinite simple group. Put $G=S\sim S^*$. Then $G=B\rtimes S^*$ where $B=\mathrm{Dr}_{x\in S^*}S_x$ ($S_x\cong S$). Clearly $B^{(1)}=B$ and for every subset T of S^* $[B, \mathrm{Dr}_{x\in T}S_x]=\mathrm{Dr}_{x\in T}S_x$. Let $\{x_1,$

$x_2,\ldots\}$ be a countable subset of S^*. Then for any integer $n\geq 0$

$$[B^{(n)}, \mathrm{Dr}_{x\in S^*\setminus\{x_1\}}S_x] > [B^{(n)}, \mathrm{Dr}_{x\in S^*\setminus\{x_1,x_2\}}S_x] > \cdots.$$

Hence by Corollary 1.2 we see that B is not quasi-artinian.

Next let M be a normal subgroup of G contained in B. Assume that $M\neq 1$. Since M is normal in B we can write $M=\mathrm{Dr}_{x\in T}S_x$ where T is a non-empty subset of S^*. If $T\neq S^*$ then choose an element y of $S^*\setminus T$. For an element x of T we have $S_y=(S_x)^{x^{-1}y}\subseteq M$, a contradiction. Therefore $T=S^*$, that is, $M=B$.

Now we shall show that B is the only non-trivial normal subgroup of G. Let N be a non-trivial normal subgroup of G. Assume that $N\cap B=1$. Any element of N is expressed as $z=ax$ where $a\in B$ and $x\in S^*$. Then $a=\Pi_{x\in T}a_x$ ($a_x\in S_x$) for some finite subset T of S^*. As $T\neq S^*$ there exists an element y of $S^*\setminus T$. Choose $1\neq b_y\in S_y$. Then we have

$$[z, b_y]=[a, b_y]^x[x, b_y]=(b_y^x)^{-1}b_y\in N\cap B.$$

Hence $b_y^x=b_y$, which implies that $x=1$. Therefore $N\subseteq B$ and so $N=1$, a contradiction. Thus $N\cap B\neq 1$. Then since $N\cap B\lhd G$ we have $N\cap B=B$ by the previous paragraph, and $N/B\lneqq G/B$. Since G/B is simple we have $N=B$.

Consequently G is artinian and so quasi-artinian. Therefore qmin-$\lhd$ is not s_n-closed.

EXAMPLE 5.3. There exists a group G satisfying the following conditions:

(1) $G\in$ qmin-$\lhd\cap(\text{Min-}\lhd(\mathfrak{A}\cap\mathfrak{P}))^{\mathrm{Q}}$.

(2) Every subgroup with finite index in G is quasi-artinian.

(3) $G\notin \mathrm{E}\mathfrak{A}\cup\text{Min-}\lhd$.

(4) G has no non-trivial soluble subnormal subgroups.

In fact, let $\boldsymbol{Z}_+$ be the set of all positive integers and let S_∞ be the group of all finitary permutations of $\boldsymbol{Z}_+$, that is, all permutations which move only a finite number of the symbols. Then define $S(n)$ to be the stabilizer in S_∞ of $\{n+1, n+2,\ldots\}$. Clearly $S_n\cong S(n)$. Let $A(n)$ be the image of A_n under the isomorphism. Then $A(5)<A(6)<\cdots$ and $A_\infty=\bigcup_{n\geq 5}A(n)$ is an infinite simple group. Also we have

$$A_\infty\leq S_\infty\leq \mathrm{Sym}(\boldsymbol{Z}_+).$$

For any integer $n\geq 3$ we put $k(n)=2+3+\cdots+n$, and define

$$\alpha=(1, 2)(3, 4, 5)\cdots(k(n)+1, k(n)+2,\ldots, k(n+1))\ldots\in \mathrm{Sym}(\boldsymbol{Z}_+).$$

Since $A_\infty^\alpha=A_\infty$ we define t to be the automorphism of A_∞ induced by α. Set $G=A_\infty\rtimes\langle t\rangle$. As $\langle t\rangle$ is infinite we first see that $G\notin\text{Min-}\lhd$.

We now claim that every subnormal subgroup ($\neq 1$) of G contains A_∞. Let

H be a subnormal subgroup ($\neq 1$) of G. Then there is a finite series $(H_i)_{i\leq n}$ of subgroups of G such that $H=H_n \lhd H_{n-1} \lhd \cdots \lhd H_0=G$. By induction on i we show that $A_\infty \subseteq H_i$. It is trivial for $i=0$. Let $i\geq 0$ and assume that $A_\infty \subseteq H_i$. Suppose that $[A_\infty, H_{i+1}]=1$ and take any element $h=\sigma t^m$ ($\sigma \in A_\infty$) in H_{i+1}. Then $\sigma \in A(k(n))$ for some $n\geq 3$ and put $l=\max\{|m|+2, n\}$. For an element $\tau=(k(l)+1, k(l)+2, k(l)+3)\in A_\infty$ we have

$$1 = [\tau, h] = [\tau, t^m][\tau, \sigma]^{t^m} = \tau^{-1}\tau^{t^m}.$$

Hence $\tau=\tau^{t^m}$ and we may assume that $m\geq 0$. Considering that $k(l)+m+3\leq k(l+1)$ we have

$$\alpha^m\colon k(l) + i \longmapsto k(l) + m + i \quad \text{for} \quad i = 1, 2, 3.$$

Therefore $(k(l)+1,\ k(l)+2,\ k(l)+3)=(k(l)+m+1,\ k(l)+m+2,\ k(l)+m+3)$, which implies $m=0$. So $H_{i+1}\subseteq A_\infty$ and $H_{i+1}\subseteq \zeta_1(A_\infty)=1$, a contradiction. Hence we have $[A_\infty, H_{i+1}]\neq 1$. Since $A_\infty \lhd H_i$ and $H_{i+1}\lhd H_i$, $[A_\infty, H_{i+1}]\lhd H_i \cap A_\infty = A_\infty$. By the simplicity of A_∞ we have $A_\infty=[A_\infty, H_{i+1}]\subseteq [H_i, H_{i+1}]\subseteq H_{i+1}$.

We next prove that every soluble subnormal subgroup of G must be 1. Let H be a soluble subnormal subgroup of G and $H\neq 1$. Then $A_\infty \subseteq H$ by the previous paragraph, which contradicts the simplicity of A_∞.

Let $N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of normal subgroups of G and $N_i\neq 1$. Since $G^{(1)}=A_\infty$ and $A_\infty \subseteq N_n$ for any $n\geq 1$, we have

$$[G^{(1)}, N_1] \subseteq [A_\infty, G] \subseteq A_\infty \subseteq N_n \quad \text{for any} \quad n \geq 1.$$

This says that G is quasi-artinian.

Let H be a subgroup with finite index in G. Since $\mathrm{Core}_G H$ is of finite index in G we have $1\neq \mathrm{Core}_G H \lhd G$, and so $A_\infty \subseteq \mathrm{Core}_G H \subseteq H$. Thus H is normal in G. For any three normal subgroups $M_i\neq 1$ $(1\leq i\leq 3)$ of H we obtain $A_\infty \subseteq M_i$, which plainly implies that

$$[M_1, M_2] = [M_1, M_3] = [M_1, M_2 \cap M_3] = A_\infty.$$

Hence H has the property (P). As G/H is abelian it follows from Proposition 2.3 that H is quasi-artinian.

We finally assert that $G\in(\text{Min-}\lhd(\mathfrak{A}\cap\mathfrak{P}))^{\mathrm{Q}}$. It is trivial that $G\in \text{Min-}\lhd\,\mathrm{E}\mathfrak{A}\leq \text{Min-}\lhd(\mathfrak{A}\cap\mathfrak{P})$. Let $1\neq N\lhd G$. Then $A_\infty\subseteq N$. If $A_\infty\neq N$, then $1\neq N/A_\infty \lhd G/A_\infty \cong \langle t\rangle$. Hence $G/N\in\mathfrak{F}$. If $A_\infty=N$, then $G/N\cong\langle t\rangle\in \text{Min-}\lhd(\mathfrak{A}\cap\mathfrak{P})$. Therefore we obtain our assertion.

From Example 5.3 we deduce that there is a semisimple quasi-artinian group which does not satisfy the minimal condition for normal subgroups.

EXAMPLE 5.4. A group does not necessarily have the property (P). In fact, let Q_8 be the group of Hamilton's quaternions. This is the group consisting of the symbols ± 1, $\pm i$, $\pm j$, $\pm k$ where $-1=i^2=j^2=k^2$ and $ij=k=-ji$, $jk=i=-kj$, $ki=j=-ik$. Now clearly $\langle i\rangle=\{\pm 1, \pm i\}$, $\langle j\rangle=\{\pm 1, \pm j\}$, $\langle k\rangle=\{\pm 1, \pm k\}$ and these are normal in Q_8. Since $[i, j]=-1$ we have $[\langle i\rangle, \langle j\rangle]\supseteq\{\pm 1\}$ and similarly $[\langle i\rangle, \langle k\rangle]\supseteq\{\pm 1\}$. However

$$[\langle i\rangle, \langle j\rangle] \cap [\langle i\rangle, \langle k\rangle] > [\langle i\rangle, \langle j\rangle \cap \langle k\rangle]$$

because $\langle j\rangle \cap \langle k\rangle=\{\pm 1\}=\zeta_1(Q_8)$.

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