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Semilinear boundary value problems on a self-adjoint harmonic space with non-local boundary conditions

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

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Introduction. In the previous paper [2], the author studied semilinear boundary value problems with respect to an ideal boundary on a self-adjoint harmonic space. When applied to the harmonic structure defined by a self-adjoint elliptic operator L on a bounded domain Ω in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, our problem in [2] may be written as

$$(0.1)\qquad \begin{cases} Lu(x) = F(x, u(x)) & \text{on} \quad \Omega, \\ u(\xi) = \tau(\xi) & \text{on} \quad \partial\Omega \setminus \Lambda, \\ \dfrac{\partial u}{\partial n}(\xi) = \beta(\xi, u(\xi)) & \text{on} \quad \Lambda, \end{cases}$$

where F is a function on $\Omega \times \boldsymbol{R}$, Λ is a part of $\partial\Omega$, τ is a given function on $\partial\Omega$ and β is a function on $\Lambda \times \boldsymbol{R}$. The main existence theorem was proved by the so-called monotone-iteration method.

Recently, S. Zheng [3] applied the same method to the following boundary value problem with non-local boundary condition:

$$(0.2)\qquad \begin{cases} Lu(x) = F(x, u(x)) & \text{on} \quad \Omega, \\ u(\xi) = \text{const. (unknown)} & \text{on} \quad \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n} d\sigma = 0. \end{cases}$$

The purpose of the present paper is to formulate a boundary value problem with respect to an ideal boundary on a self-adjoint harmonic space in such a way that both problems of type (0.1) and of type (0.2) are included as special cases and that the monotone-iteration method can be applied to obtain an existence theorem. In order to describe boundary conditions, we introduce the notion of "boundary behavior spaces". A choice of boundary behavior space gives a problem of the following type, which is a generalization of (0.2):

$$
(0.3)\qquad \begin{cases} Lu(x) = F(x, u(x)) & \text{on} \quad \Omega, \\ u(\xi) = \tau(\xi) & \text{on} \quad \Lambda_0, \\ u(\xi) = a_j \text{ (unknown constant)} & \text{on} \quad \Lambda_j, \quad j \in J, \\ \displaystyle\int_{\Lambda_j} \frac{\partial u}{\partial n} d\sigma = \eta_j(a_j), \quad j \in J, & \end{cases}
$$

where $\{\Lambda_j\}_{j\in J}$ is a countable (finite or infinite) family of mutually disjoint components of $\partial\Omega$, $\Lambda_0 = \partial\Omega \setminus \bigcup_{j\in J} \Lambda_j$ and η_j, $j \in J$, are real functions on $\boldsymbol{R}$.

§1. Preliminaries

As in [2], let $(X, \mathscr{H})$ be a self-adjoint P-harmonic space such that X is connected, has a countable base and $1 \in \mathscr{H}(X)$. Let $G(x, y)$ be a symmetric Green function on X and $\sigma: \mathscr{R} \to \mathscr{M}$ be the canonical measure representation associated with G (see [1]). The image sheaf of $\mathscr{R}$ by σ is denoted by $\mathscr{M}_C$.

We denote by Gv the G-potential of $v \in \mathscr{M}_C(X)$ when it exists. Let

$$
\mathscr{M}_{BF} = \{v \in \mathscr{M}_C(X) \mid G|v| \text{ is bounded, } |v|(X) < \infty\},
$$

$$
\mathscr{M}_{EF} = \{v \in \mathscr{M}_C(X) \mid \int_X G|v|d|v| < \infty, \ |v|(X) < \infty\},
$$

$$
\mathscr{Q}_{ZF} = \{Gv \mid v \in \mathscr{M}_{ZF}\}, \quad Z = B \text{ or } E.
$$

($\mathscr{M}_{ZF}$ and $\mathscr{Q}_{ZF}$ are denoted by $\mathscr{M}_{ZFC}$ and $\mathscr{Q}_{ZFC}$ in [2].) Note that $\mathscr{M}_{BF} \subset \mathscr{M}_{EF}$ and $\mathscr{Q}_{BF} \subset \mathscr{Q}_{EF}$. If $f = h + Gv$ with $h \in \mathscr{H}(X)$ and $v \in \mathscr{M}_{EF}$, then $\sigma(f) = v$.

For $f, g \in \mathscr{R}(X)$, the gradient measures $\delta_{[f,g]}$ and δ_f are defined in [1]. We write $D[f]$ for $\delta_f(X)$ and $D[f, g]$ for $\delta_{[f,g]}(X)$ when $\delta_f(X) < \infty$ and $\delta_g(X) < \infty$. Note that $D[f] < \infty$ for any $f \in \mathscr{Q}_{EF}$ (see [1]).

We consider a resolutive compactification X^* of X. Let $\omega = \omega_x$ be the harmonic measure on $\partial^* X = X^* \setminus X$ at $x \in X$. For $\varphi \in L^1(\omega)$, let $H_\varphi(x) = \int_{\partial^* X} \varphi d\omega_x$ $(x \in X)$. Then $H_\varphi \in \mathscr{H}(X)$. As in [2], we consider the linear spaces

$$
\Phi_D = \{\varphi \in L^1(\omega) \mid D[H_\varphi] < \infty\}, \quad \Phi_{BD} = \Phi_D \cap L^\infty(\omega),
$$

which are closed under max. and min. operations. Obviously, constant functions belong to these spaces. We denote by $\mathscr{N}$ the space of all signed measures on $\partial^* X$ which are absolutely continuous with respect to ω.

Given a space of functions or measures, the subset consisting of non-negative elements in the space will be indicated by the upper index $+$; e.g., Φ_{BD}^+, $\mathscr{M}_{BF}^+$, etc.

§ 2. Boundary behavior spaces

A subset Ψ of Φ_D will be called a *boundary behavior space* if it satisfies the following four conditions:

(Ψ.1) Ψ is a linear subspace of Φ_D;

(Ψ.2) Ψ is closed under max. and min. operations;

(Ψ.3) for each $\psi \in \Psi^+$, there is a sequence $\{\psi_n\}$ in $\Psi_B^+ \equiv \Psi^+ \cap L^\infty(\omega)$ such that $\psi_n \leqq \psi$ for each n, $\psi_n \to \psi$ ω-a.e. on $\partial^* X$ and $D[H_{\psi_n} - H_\psi] \to 0$ $(n \to \infty)$;

(Ψ.4) if $\psi_n \in \Psi$, $n = 1, 2, \ldots$, $\psi_n \to \psi \in \Phi_D$ ω-a.e. on $\partial^* X$ and $D[H_{\psi_n} - H_\psi] \to 0$ $(n \to \infty)$, then $\psi \in \Psi$.

Note that (Ψ.3) follows from (Ψ.2) if $1 \in \Psi$; we may take $\psi_n = \min(\psi, n)$ in this case.

EXAMPLE 2.1. Let Λ be an ω-measurable subset of $\partial^* X$ and write

$$\Phi_D(\Lambda) = \{\varphi \in \Phi_D \mid \varphi = 0 \ \omega\text{-a.e. on } \partial^* X \setminus \Lambda\}.$$

Then $\Phi_D(\Lambda)$ is a boundary behavior space. In particular, Φ_D (the case $\omega(\partial^* X \setminus \Lambda) = 0$) and $\{0\}$ (the case $\omega(\Lambda) = 0$) are boundary behavior spaces. More generally, let $\psi_0 \in \Phi_{BD}$ and $\psi_0 \geqq 0$ ω-a.e. on $\partial^* X \setminus \Lambda$. Then

$$\Psi = \Phi_D(\Lambda) + \boldsymbol{R}\psi_0 \equiv \{\varphi + c\psi_0 \mid \varphi \in \Phi_D(\Lambda),\ c \in \boldsymbol{R}\} \tag{2.1}$$

is a boundary behavior space. In fact, (Ψ.1) is obvious and (Ψ.4) is easily verified. To show (Ψ.2) let $\psi = \varphi + c\psi_0$ with $\varphi \in \Phi_D(\Lambda)$ and $c \in \boldsymbol{R}$. If $c \leqq 0$ then $\psi^+ \in \Phi_D(\Lambda) \subset \Psi$ and if $c > 0$ then $\psi^+ = \max(\varphi, -c\psi_0) + c\psi_0$ and $\max(\varphi, -c\psi_0) \in \Phi_D(\Lambda)$. Thus ($\Psi$.2) holds. If $\psi = \varphi + c\psi_0 \geqq 0$, then for $n > |c| \, \|\psi_0^-\|_\infty$, $\psi_n = \min(\varphi, n) + c\psi_0 \in \Psi_B^+$ and $\{\psi_n\}$ has the properties stated in (Ψ.3).

EXAMPLE 2.2. Let $\{\Lambda_j\}_{j \in J}$ be a finite or countably infinite family of mutually disjoint ω-measurable subsets of $\partial^* X$ such that $\omega(\Lambda_j) > 0$ and the characteristic function χ_j of Λ_j belongs to Φ_D for every $j \in J$. If J is a finite set, then let

$$\Phi_D^c(\{\Lambda_j\}_{j \in J}) = \{\textstyle\sum_{j \in J} a_j \chi_j \mid a_j \in \boldsymbol{R},\ j \in J\}.$$

If J is an infinite set, then we define

$$\Phi_D^c(\{\Lambda_j\}_{j \in J}) = \mathrm{Cl}\,\{\textstyle\sum_{j \in J'} a_j \chi_j \mid a_j \in \boldsymbol{R},\ j \in J',\ J'\text{: finite} \subset J\},$$

where Cl means the closure with respect to the convergence given in (Ψ.4). Any element of $\Phi_D^c(\{\Lambda_j\}_{j \in J})$ is of the form $\sum_{j \in J} a_j \chi_j$, $a_j \in \boldsymbol{R}$. It is easy to see that $\Phi_D^c(\{\Lambda_j\}_{j \in J})$ is a boundary behavior space. (To verify (Ψ.3), we may use [2; Lemma 2.3] and [1; Lemma 7.5] and show that $\psi \in \Phi_D^c(\{\Lambda_j\}_{j \in J})^+$ implies $\min(\psi, n) \in \Phi_D^c(\{\Lambda_j\}_{j \in J})^+$.)

Let $\Lambda' = \bigcup_{j \in J} \Lambda_j$ and $\Lambda_0 = \partial^* X \setminus \Lambda'$. For $\psi_0 = \varphi_0 + a_0$ with $\varphi_0 \in \Phi_{BD}(\Lambda_0)$ and $a_0 \in \boldsymbol{R}$ such that $\psi_0 \geqq 0$ on $\partial^* X$,

$$\Psi = \Phi_D^c(\{\Lambda_j\}_{j \in J}) + \boldsymbol{R}\psi_0 \tag{2.2}$$

is a boundary behavior space. Note that in case $\chi_{\Lambda'}$ (the characteristic function of Λ') belongs to Φ_D (in particular, in case J is finite), we may write $\psi_0 = \varphi_0 + a_0 \chi_{\Lambda'}$.

Also, for an ω-measurable set $\Lambda \subset \Lambda_0$, $\Phi_D^c(\{\Lambda_j\}_{j \in J}) + \Phi_D(\Lambda)$ is a boundary behavior space.

§ 3. Problem setting

Given a subspace Σ of Φ_D, let

$$\mathscr{R}_Z(\Sigma) = \{H_\varphi + g \mid \varphi \in \Sigma,\ g \in \mathscr{Q}_{ZF}\}, \quad Z = E \text{ or } B.$$

We consider a mapping $F: \mathscr{R}_Z(\Sigma) \to \mathscr{M}_{ZF}$ and the equation

$$\sigma(u) + F(u) = 0 \qquad \text{on} \quad X. \tag{3.1}$$

Given a mapping $\beta: \Sigma \to \mathscr{N}$, a function $\tau \in \Phi_D$ and a boundary behavior space Ψ, we consider the following boundary condition for $u = H_\varphi + g \in \mathscr{R}_Z(\Sigma)$:

$$\text{(B)} \quad \begin{cases} \text{(B-1)} \quad \varphi - \tau \in \Psi \\ \text{(B-2)} \quad D[u, H_\psi] - \int_X H_\psi \, d\sigma(u) + \int_{\partial^* X} \psi \, d\beta(\varphi) = 0 \quad \text{for all } \psi \in \Psi_B, \end{cases}$$

where $\Psi_B = \Psi \cap \Phi_{BD}$.

The problem to find $u = H_\varphi + g \in \mathscr{R}_Z(\Sigma)$ satisfying (3.1) and (B) will be denoted by $P_Z(\Sigma, F, \beta, \tau, \Psi)$.

EXAMPLE 3.1. The problem discussed in [2] is $P_B(\Phi_{BD}, F, \beta, \tau, \Phi_D(\Lambda))$ with $\tau \in \Phi_{BD}$. More generally, if Ψ is given by (2.1), then condition (B) may be written as

$$\begin{cases} \varphi = \tau + c\psi_0 \quad \omega\text{-a.e. on } \partial^* X \setminus \Lambda \quad \text{for some } c \in \boldsymbol{R}, \\ \text{a } \textit{normal derivative of } u = \beta(\varphi) \quad \text{on } \Lambda \text{ (cf. [2])}, \\ D[u, H_{\psi_0}] - \int_X H_{\psi_0} d\sigma(u) + \int_{\partial^* X} \psi_0 d\beta(\varphi) = 0. \end{cases} \tag{3.2}$$

If $\psi_0 = \chi_{\Lambda'}$ for some ω-measurable subset Λ' of $\partial^* X$ such that $\omega(\Lambda' \setminus \Lambda) > 0$, then the last condition in (3.2) may be written as

$$\mathrm{Flux}_{\Lambda'} u = \int_{\Lambda'} d\beta(\varphi),$$

where

$$\mathrm{Flux}_{\Lambda'}\, u = -D[u, H_{\chi_{\Lambda'}}] + \int_X H_{\chi_{\Lambda'}} d\sigma(u).$$

In particular, if $\omega(\Lambda)=0$ and $\psi_0=1$, then (3.2) is reduced to

$$\begin{cases} \varphi = \tau + \text{const. (unknown)} \quad \omega\text{-a.e. on } \partial^* X, \\ \mathrm{Flux}_{\partial^* X}\, u = \int_{\partial^* X} d\beta(\varphi). \end{cases}$$

EXAMPLE 3.2. Let $\{\Lambda_j\}_{j\in J}$ be as in Example 2.2. Let $\{\eta_j\}_{j\in J}$ be a family of real functions on $\boldsymbol{R}$ such that

$$(3.3) \qquad \sum_{j\in J} \sup_{|t|\leqq M} |\eta_j(t)| < +\infty \qquad \text{for any} \quad M > 0.$$

Let $\Sigma=\Phi_{BD}(\Lambda_0)+\Phi_D^c(\{\Lambda_j\}_{j\in J})\cap\Phi_{BD}+\boldsymbol{R}$, and for $\varphi=\varphi^{(0)}+\sum_{j\in J} a_j\chi_j+a$ with $\varphi^{(0)}\in\Phi_{BD}(\Lambda_0)$, $\sum_{j\in J} a_j\chi_j\in\Phi_D^c(\{\Lambda_j\}_{j\in J})\cap\Phi_{BD}$ and $a\in\boldsymbol{R}$ let

$$(3.4) \qquad \beta(\varphi) = \sum_{j\in J} \frac{\eta_j(a_j+a)}{\omega_{x_0}(\Lambda_j)} \chi_j \omega_{x_0} \quad (x_0\in X\colon \text{fixed}).$$

By (3.3), β maps Σ into $\mathscr{N}$. If Ψ is given by (2.2) with $\psi_0=\varphi_0+a_0\geqq 0$ ($\varphi_0\in\Phi_{BD}(\Lambda_0)$, $a_0\in\boldsymbol{R}$), and if $\tau=\tau_0+b$ with $\tau_0\in\Phi_{BD}(\Lambda_0)$ and $b\in\boldsymbol{R}$, then condition (B) is written as

$$\begin{cases} \varphi = \tau + c(\varphi_0+a_0) \quad \omega\text{-a.e. on } \Lambda_0 \quad \text{for some } c\in\boldsymbol{R}, \\ \varphi = a'_j \text{ (const.)} \quad \omega\text{-a.e. on } \Lambda_j \quad \text{for each } j\in J, \\ \sum_{j\in J} (a'_j-b-ca_0)\chi_j \in \Phi_D^c(\{\Lambda_j\}_{j\in J}), \\ \mathrm{Flux}_{\Lambda_j} u = \eta_j(a'_j) \quad \text{for each } j\in J, \\ D[u, H_{\varphi_0}] - \int_X (H_{\varphi_0}+a_0)d\sigma(u) + a_0 \sum_{j\in J} \eta_j(a'_j) = 0. \end{cases}$$

In particular, in case $\psi_0=0$, the above boundary condition is reduced to

$$\begin{cases} \varphi = \tau \quad \omega\text{-a.e. on } \Lambda_0, \\ \varphi = a'_j \text{ (const.)} \quad \omega\text{-a.e. on } \Lambda_j, \quad j\in J, \\ \sum_{j\in J} (a'_j-b)\chi_j \in \Phi_D^c(\{\Lambda_j\}_{j\in J}), \\ \mathrm{Flux}_{\Lambda_j}\, u = \eta_j(a'_j), \quad j\in J. \end{cases}$$

Thus the problem $P_Z(\Sigma, F, \beta, \tau, \Phi_D^c(\{\Lambda_j\}_{j\in J}))$ with above Σ, β and τ is a problem of type (0.3).

§4. Comparison principle

By slightly modifying the proof of [2; Theorem 2.1], we obtain the following comparison principle.

THEOREM 1. *Let Σ be a subspace of Φ_D, Ψ be a boundary behavior space and suppose F: $\mathscr{R}_Z(\Sigma)\to\mathscr{M}_C(X)$ and β: $\Sigma\to\mathscr{N}$ satisfy the following monotonicity conditions*:

(F.M) *For any open set U in X, if $f_1, f_2\in\mathscr{R}_Z(\Sigma)$ and $f_1\leqq f_2$ on U, then $F(f_1)\leqq F(f_2)$ on U;*

(β.M; Ψ) *For any $\psi\in\Psi_B^+$, if $\varphi_1, \varphi_2\in\Sigma$ and $\varphi_1\leqq\varphi_2$ ω-a.e. on $\{\xi\in\partial^*X|\psi(\xi)>0\}$, then $\int\psi d\beta(\varphi_1)\leqq\int\psi d\beta(\varphi_2)$.*

Suppose $u=H_\varphi+g\in\mathscr{R}_Z(\Sigma)$ and $v=H_\tau+q\in\mathscr{R}_Z(\Sigma)$ satisfy

(a) $\sigma(u)+F(u)\geqq\sigma(v)+F(v)$ *on* X,

(b) $(\varphi-\tau)^-\in\Psi$,

(c) $$D[u, H_\psi]-\int_X H_\psi\, d\sigma(u)+\int\psi d\beta(\varphi)$$
$$\geqq D[v, H_\psi]-\int_X H_\psi\, d\sigma(v)+\int\psi d\beta(\tau)\quad \textit{for all } \psi\in\Psi_B^+.$$

Then,

(i) *in case $1\notin\Psi$ or $1\notin\Sigma$, we have $u\geqq v$ on X;*

(ii) *in case $1\in\Psi$ and $1\in\Sigma$, we have either $u\geqq v$ on X or $v=u+c$ with a constant $c>0$; the latter occurs only when $F(u+c)=F(u)$ and $\int\psi d\beta(\varphi)=\int\psi d\beta(\varphi+c)$ for any $\psi\in\Psi_B$.*

Outline of the proof: Put $f=(u-v)^-$ and $\varphi_0=(\varphi-\tau)^-$. By [2; Lemma 2.3], $f=H_{\varphi_0}+g$ with $g\in\mathscr{D}_E$, where

$$\mathscr{D}_E=\{Gv\,|\,v\in\mathscr{M}_C(X),\ \int_X G|v|d|v|<\infty\}.$$

By assumption $\varphi_0\in\Psi^+$. Hence by (Ψ.3) there is a sequence $\{\varphi_n\}$ in Ψ_B^+ such that $\varphi_n\leqq\varphi_0$, $\varphi_n\to\varphi_0$ ω-a.e. on ∂^*X and $D[H_{\varphi_n}-H_{\varphi_0}]\to 0$ $(n\to\infty)$. Let $f_n=H_{\varphi_n}+\max(g, -H_{\varphi_n})$. We can easily see that $g_n=\max(g, -H_{\varphi_n})$ belongs to $\mathscr{D}_E$ for each n. Obviously, $0\leqq f_n\leqq f$. Then, by the same arguments as in the proof of [2; Theorem 2.1], we obtain our theorem.

REMARK 4.1. Condition (β.1) in [2] implies condition (β.M; $\Phi_D(\Lambda)$) for $\Sigma=\Phi_{BD}$.

REMARK 4.2. In case $\Psi=\Phi_D^c(\{A_j\}_{j\in J})$ or $\Psi=\Phi_D^c(\{A_j\}_{j\in J})+\boldsymbol{R}$, if β is given by η_j: $\boldsymbol{R}\to\boldsymbol{R}$, $j\in J$, as in Example 3.2, then (β. M; Ψ) is equivalent to the condition that every η_j is monotone non-decreasing.

COROLLARY. *Let $\lambda\in\mathscr{M}_C^+(X)$, $\alpha\in\mathscr{N}^+$ and Ψ be a boundary behavior space. If $u=H_\varphi+g\in\mathscr{R}_E(\Phi_D\cap L^1(\alpha))$ satisfies*

(a) $\sigma(u)+u\lambda\geqq 0$ *on* X,

(b) $\varphi^-\in\Psi$,

(c) $D[u, H_\psi]-\int_X H_\psi\, d\sigma(u)+\int\psi\varphi\, d\alpha\geqq 0$ *for all* $\psi\in\Psi_B^+$,

then

(i) *in case* $1\notin\Psi$ *or* $\lambda\neq 0$ *or* $\alpha\neq 0$, *we have* $u\geqq 0$ *on* X;
(ii) *in case* $1\in\Psi$, $\lambda=0$ *and* $\alpha=0$, *we have either* $u\geqq 0$ *on* X *or* $u\equiv$ *const.* <0.

This corollary is obtained by applying the theorem with $\Sigma=\Phi_D\cap L^1(\alpha)$, $F(f)=f\lambda$, $\beta(\varphi)=\varphi\alpha$ and $Z=E$.

REMARK 4.3. For Theorem 1 and its corollary, condition (Ψ.4) for Ψ is not necessary.

§5. Linear problems

As in [2], we first give an existence and uniqueness theorem for linear problems. Let $\lambda\in\mathscr{M}_{BF}^+$ and $\alpha\in\mathscr{N}^+$ be given. For each $\varphi\in\Phi_D$ with $H_\varphi\in L^2(\lambda)$, there exists a unique $u\in\mathscr{R}(X)$ such that $\sigma(u)+u\lambda=0$ on X and $u-H_\varphi\in\mathscr{Q}_{EF}$ ([2; pp. 43–44]). This u is denoted by H_φ^λ. As in [2], we consider the space

$$\Phi_D^{\lambda,\alpha}=\{\varphi\in\Phi_D\,|\,H_\varphi\in L^2(\lambda),\ \varphi\in L^2(\alpha)\}$$

and a semi-norm (a norm if either $\lambda\neq 0$ or $\alpha\neq 0$)

$$\|\varphi\|_{D,\lambda,\alpha}=\{D[H_\varphi^\lambda]+\int(H_\varphi^\lambda)^2 d\lambda+\int\varphi^2 d\alpha\}^{1/2}$$

on $\Phi_D^{\lambda,\alpha}$. Note that $\Phi_{BD}\subset\Phi_D^{\lambda,\alpha}$.

THEOREM 2 (cf. [2; Theorem 3.1]). *Let Ψ be a boundary behavior space and write $\Psi^{\lambda,\alpha}=\Psi\cap\Phi_D^{\lambda,\alpha}$. Suppose $\mu\in\mathscr{M}_{BF}$ and $\gamma\in\mathscr{N}$ satisfy*

$$[\mu]\qquad \left|\int_X H_\psi d\mu\right|\leqq a(\mu)\|\psi\|_{D,\lambda,\alpha}\qquad\text{for all}\quad \psi\in\Psi^{\lambda,\alpha},$$

$$[\gamma]\qquad \left|\int_{\partial^*X}\psi d\gamma\right|\leqq b(\gamma)\|\psi\|_{D,\lambda,\alpha}\qquad\text{for all}\quad \psi\in\Psi^{\lambda,\alpha}.$$

Then, given $\tau \in \Phi_D^{\lambda,\alpha}$, *there exists a solution* $u = H_\varphi + g \in \mathscr{R}_E(\Phi_D^{\lambda,\alpha})$ *of the linear problem*

$$\begin{cases} \sigma(u) + u\lambda = \mu \quad on \quad X, \\ \varphi - \tau \in \Psi, \\ D[u, H_\psi] - \int_X H_\psi \, d\sigma(u) + \int \psi\varphi \, d\alpha = \int \psi d\gamma \quad for\ all \quad \psi \in \Psi_B. \end{cases}$$

The solution is unique if either $1 \notin \Psi$ *or* $\lambda \neq 0$ *or* $\alpha \neq 0$; *unique up to an additive constant if* $1 \in \Psi$, $\lambda = 0$ *and* $\alpha = 0$. *Furthermore,*

$$D[u]^{1/2} \leqq 2\|\tau\|_{D,\lambda,\alpha} + (2 + \|G\lambda\|_\infty) D[G|\mu|]^{1/2} + a(\mu) + b(\gamma).$$

On account of condition (Ψ.4) for Ψ, we see that $\Psi^{\lambda,\alpha}$ is a Hilbert space with respect to the norm $\|\cdot\|_{D,\lambda,\alpha}$, in case $1 \notin \Psi$ or $\lambda \neq 0$ or $\alpha \neq 0$, and $\Psi^{\lambda,\alpha}/\boldsymbol{R} = \Psi/\boldsymbol{R}$ is a Hilbert space with respect to $\|\psi\|_D = D[H_\psi]^{1/2}$ in case $1 \in \Psi$, $\lambda = 0$ and $\alpha = 0$. In the latter case, $[\mu]$ and $[\gamma]$ imply that $\int_X d\mu = \int_{\partial^* X} d\gamma = 0$. Thus the above theorem can be proved in the same way as [2; Theorem 3.1].

§6. Existence theorem for semilinear problems I

Given Z, Σ, F, β, τ and Ψ as in §3, $v = H_\varphi + g \in \mathscr{R}_Z(\Sigma)$ is called a supersolution (resp. subsolution) of $P_Z(\Sigma, F, \beta, \tau, \Psi)$ if

$$\begin{cases} \sigma(v) + F(v) \geqq 0 \text{ (resp. } \leqq 0) \quad \text{on} \quad X, \\ (\varphi - \tau)^- \in \Psi \text{ (resp. } (\tau - \varphi)^- \in \Psi), \\ D[v, H_\psi] - \int_X H_\psi \, d\sigma(v) + \int \psi d\beta(\varphi) \geqq 0 \text{ (resp. } \leqq 0) \quad \text{for all} \quad \psi \in \Psi_B^+. \end{cases}$$

Now, we introduce a notion of Ψ-admissible space for a boundary behavior space Ψ.

A subset Γ of Φ_D will be said to be *Ψ-admissible* if it satisfies the following two conditions:

(A.1) Γ is a linear subspace of Φ_D containing Ψ,

(A.2) $\varphi_1, \varphi_2 \in \Gamma$ and $\varphi_1^-, \varphi_2^- \in \Psi$ imply $(\varphi_1 + \varphi_2)^- \in \Psi$.

Obviously, Ψ itself is Ψ-admissible.

EXAMPLE 6.1. For $\Psi = \Phi_D(\Lambda) + \boldsymbol{R}\psi_0$ with $\psi_0 \in \Phi_{BD}$ such that $\psi_0 \geqq 0$ ω-a.e. on $\partial^* X \setminus \Lambda$,

$$\Gamma = \{\varphi \in \Phi_D \mid \varphi = a\psi_0 \ \omega\text{-a.e. on } \{\xi \in \partial^* X \setminus \Lambda \mid \psi_0(\xi) > 0\} \text{ for some } a \in \boldsymbol{R}\}$$

is Ψ-admissible. In particular, Φ_D is $\Phi_D(\Lambda)$-admissible. (If $\psi_0 \notin \Phi_D(\Lambda)$, then

Φ_D is not Ψ-admissible).

EXAMPLE 6.2. Let $\{\Lambda_j\}_{j\in J}$ be as in Example 2.2. Then $\Phi_D(\Lambda_0)+\Phi_D^c(\{\Lambda_j\}_{j\in J})$ is $\Phi_D^c(\{\Lambda_j\}_{j\in J})$-admissible. If $\chi_{\Lambda'}\in\Phi_D$, then $\Phi_D(\Lambda_0)+\Phi_D^c(\{\Lambda_j\}_{j\in J})+\boldsymbol{R}\chi_{\Lambda'}$ is $(\Phi_D^c(\{\Lambda_j\}_{j\in J})+\boldsymbol{R}\chi_{\Lambda'})$-admissible.

As a generalization of [2; Theorem 4.1], we have the following

THEOREM 3. *Let Ψ be a boundary behavior space, Γ be a Ψ-admissible subset of Φ_D, F: $\mathscr{R}_B(\Gamma_B)\to\mathscr{M}_{BF}$ and β: $\Gamma_B\to\mathscr{N}$, where $\Gamma_B=\Gamma\cap\Phi_{BD}$. Suppose*

(F.L) *for each $M>0$, there is $\lambda_M\in\mathscr{M}_{BF}^+$ such that*

$$|F(f_1)-F(f_2)|\leqq(f_2-f_1)\lambda_M \quad on \quad X$$

whenever $f_1, f_2\in\mathscr{R}_B(\Gamma_B)$ and $-M\leqq f_1\leqq f_2\leqq M$;
(β.L) *for each $M>0$, there is $\alpha_M\in\mathscr{N}^+$ such that*

$$\left|\int\psi\, d\{\beta(\varphi_1)-\beta(\varphi_2)\}\right|\leqq\int\psi(\varphi_2-\varphi_1)\,d\alpha_M$$

for all $\psi\in\Psi_B^+$, whenever φ_1, $\varphi_2\in\Gamma_B$ and $-M\leqq\varphi_1\leqq\varphi_2\leqq M$ ω-a.e.

Let $\tau\in\Gamma_B$ and suppose there exist a supersolution u_0 and a subsolution v_0 of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that $v_0\leqq u_0$ on X. Then there exist solutions u^* and v^* of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that
(i) *$v_0\leqq v^*\leqq u^*\leqq u_0$;*
(ii) *if u is a solution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that $v_0\leqq u\leqq u_0$, then $v^*\leqq u\leqq u^*$.*

Let us sketch the proof emphasizing the difference from that of [2; Theorem 4.1].

Let $M=\max(\sup_X u_0, -\inf_X v_0)$. If $1\in\Psi$, $F(f)=0$ for all $f\in\mathscr{R}_B(\Gamma_B)$ with $|f|\leqq M$ and $\beta(\varphi)=0$ for all $\varphi\in\Gamma_B$ with $|\varphi|\leqq M$, then choose any $\lambda\in\mathscr{M}_{BF}^+$ with $\lambda\neq 0$. Otherwise, let $\lambda=\lambda_M+|F(0)|$. Put $\alpha=\alpha_M+|\beta(0)|$. Then either $1\notin\Psi$ or $\lambda\neq 0$ or $\alpha\neq 0$.

Starting with the given u_0, we define a sequence $\{u_n\}$ by induction as follows: Suppose $u_0, u_1,\dots, u_{n-1}$ ($n\geqq 1$) are so chosen that each $u_j=H_{\varphi_j}+g_j$, $j=1,\dots, n-1$, is a supersolution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ and $v\leqq u_{n-1}\leqq\cdots\leqq u_1\leqq u_0$ for any subsolution v such that $-M\leqq v\leqq u_0$. As in the proof of [2; Theorem 4.1], we see that $\mu_n=-F(u_{n-1})+u_{n-1}\lambda$ satisfies [μ] and $\gamma_n=\varphi_{n-1}\alpha-\beta(\varphi_{n-1})$ satisfies [γ] in Theorem 2. Also, $\tau\in\Phi_{BD}\subset\Phi_D^{\lambda,\alpha}$. Hence, by Theorem 2, there is $u_n=H_{\varphi_n}+g_n\in\mathscr{R}_E(\Phi_D^{\lambda,\alpha})$ satisfying

$$(6.1)\quad \begin{cases} \sigma(u_n)+u_n\lambda = -F(u_{n-1})+u_{n-1}\lambda \quad \text{on} \quad X, \\ \varphi_n - \tau \in \Psi, \\ D[u_n, H_\psi] - \int_X H_\psi\, d\sigma(u_n) + \int \psi\varphi_n\, d\alpha = \int \psi\varphi_{n-1}\, d\alpha - \int \psi\, d\beta(\varphi_{n-1}) \\ \qquad\qquad \text{for all} \quad \psi \in \Psi_B. \end{cases}$$

By virtue of (A.2) for Γ, we see that $(\varphi_{n-1}-\varphi_n)^- = (\varphi_{n-1}-\tau+\tau-\varphi_n)^- \in \Psi$. Hence, applying the corollary to Theorem 1 to $u_{n-1}-u_n$, we see that $u_n \leqq u_{n-1}$. Similarly, if $v = H_\eta + q \in \mathscr{R}_B(\Gamma_B)$ is a subsolution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that $-M \leqq v \leqq u_0$, then $(\varphi_n-\eta)^- = (\varphi_n-\tau+\tau-\eta)^- \in \Psi$, and hence using (F.L), (β.L) and applying the corollary to Theorem 1 to $u_n - v$, we see that $v \leqq u_n$; in particular $-M \leqq u_n$. It follows that $\varphi_n \in \Phi_{BD}$. Since $\varphi_n - \tau \in \Psi$, $\varphi_n \in \Gamma$ by (A.1) for Γ. Therefore $\varphi_n \in \Gamma_B$. On the other hand, since u_n is bounded, (6.1) implies that $\sigma(u_n) \in \mathscr{M}_{BF}$, so that $u_n \in \mathscr{R}_B(\Gamma_B)$. Then, by (6.1), (F.L) and (β.L), we see that u_n is a supersolution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$.

Thus, we obtain a sequence $\{u_n\}$ of supersolutions of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that $-M \leqq u_n \leqq u_{n-1} \leqq u_0$ for all n and $v \leqq u_n$ for any subsolution v such that $-M \leqq v \leqq u_0$.

Let $u^* = \lim_{n\to\infty} u_n$ and $\varphi^* = \lim_{n\to\infty} \varphi_n$. As in the proof of [2; Theorem 4.1], we see that $u^* = H_{\varphi^*} + g^*$ with

$$g^* = -\lim_{n\to\infty} G(F(u_n)) \in \mathscr{Q}_{BF}.$$

Also, with the help of the estimate of $D[u]$ in Theorem 2, we see that $D[H_{\varphi_n} - H_{\varphi_m}] \to 0$ $(n, m \to \infty)$. Since $\varphi_n - \tau \in \Psi$, it follows from ($\Psi$.4) for Ψ that $\varphi^* - \tau \in \Psi$. Hence $\varphi^* \in \Gamma_B$ and $u^* \in \mathscr{R}_B(\Gamma_B)$. Then, again as in the proof of [2; Theorem 4.1], we see that $g^* = -G(F(u^*))$, so that $\sigma(u^*) + F(u^*) = 0$, and

$$D[u^*, H_\psi] - \int_X H_\psi\, d\sigma(u^*) + \int \psi\, d\beta(\varphi^*) = 0$$

holds for any $\psi \in \Psi_B$, which shows that u^* is a solution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$. Obviously, $u^* \leqq u_0$ and $v \leqq u^*$ for any subsolution v with $-M \leqq v \leqq u_0$.

Similarly, starting with v_0, we obtain a solution v^* of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ such that $v_0 \leqq v^*$ and $v^* \leqq u$ for any supersolution u with $v_0 \leqq u \leqq M$. Thus, these u^*, v^* are the required solutions.

REMARK 6.1. In case $\Psi = \Phi_D(\Lambda)$ and $\Gamma = \Phi_D$, only the values of β on Λ are relevent in the boundary condition (B–2), so that condition (β.L) in this case may be replaced by

(β.L; Λ) for any $M > 0$, there is $\alpha_M \in \mathscr{N}^+$ such that

$$|\beta(\varphi_1) - \beta(\varphi_2)| \leqq (\varphi_2 - \varphi_1)\alpha_M \quad \text{on} \quad \Lambda$$

whenever $\varphi_1, \varphi_2 \in \Phi_{BD}$ and $-M \leqq \varphi_1 \leqq \varphi_2 \leqq M$. (Cf. [2; Theorem 4.1].)

REMARK 6.2. In case $\Psi = \Phi_D^c(\{\Lambda_j\}_{j\in J})$ and $\Gamma = \Phi_D(\Lambda_0) + \Phi_D^c(\{\Lambda_j\}_{j\in J})$, or in case $\chi_{\Lambda'} \in \Phi_D$, $\Psi = \Phi_D^c(\{\Lambda_j\}_{j\in J}) + \boldsymbol{R}\chi_{\Lambda'}$ and $\Gamma = \Phi_D(\Lambda_0) + \Psi$, if β is given by (3.4) then condition (β.L) means that each η_j is Lipschitz continuous with Lipschitz constant $A_{M,j} \geqq 0$ on the interval $[-M, M]$ such that $\sum_{j\in J} A_{M,j} < \infty$ for each $M > 0$.

§7. Existence theorem for semilinear problems II

In this section we give some sufficient conditions for the existence of super- and subsolutions.

THEOREM 4. *Let Ψ be a boundary behavior space and Γ be a Ψ-admissible set containing constant functions. For F: $\mathscr{R}_B(\Gamma_B) \to \mathscr{M}_{BF}$ and β: $\Gamma_B \to \mathscr{N}$, suppose there exist $t_0 \in \boldsymbol{R}$, $\mu_0 \in \mathscr{M}_{BF}$, $\alpha_0 \in \mathscr{N}$ and $\psi_0 \in \Psi_B$ satisfying the following conditions*:

(i) $F(f) \geqq \mu_0$ (resp. $\leqq \mu_0$) *for all* $f \in \mathscr{R}_B(\Gamma_B)$ *with* $f \geqq t_0$ (resp. $f \leqq t_0$),

(ii) $\int \psi d\beta(\varphi) \geqq \int \psi d\alpha_0$ (resp. $\int \psi d\beta(\varphi) \leqq \int \psi d\alpha_0$) *for all* $\psi \in \Psi_B^+$ *and for all* $\varphi \in \Gamma_B$ *with* $\varphi \geqq t_0$ (resp. $\varphi \leqq t_0$),

(iii) $D[H_{\psi_0}, H_\psi] + \int_X H_\psi d\mu_0 + \int \psi d\alpha_0 \geqq 0$ (resp. $\leqq 0$) *for all* $\psi \in \Psi_B^+$.

Then, for any $\tau \in \Gamma_B$, *there exists a supersolution* u_0 (resp. *a subsolution* v_0) *of* $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ *such that* $u_0 \geqq t_0$ (resp. $v_0 \leqq t_0$).

PROOF (cf. the proof of [2; Theorem 4.2]). Given $\tau \in \Gamma_B$, let

$$a = \max(t_0, \sup \tau) + \sup \psi_0^- + \sup G\mu_0^+$$

and put

$$u_0 = a + H_{\psi_0} - G\mu_0.$$

By assumption, $a + \psi_0 \in \Gamma_B$, so that $u_0 \in \mathscr{R}_B(\Gamma_B)$. Since

$$u_0 \geqq a - H_{\psi_0^-} - G\mu_0^+ \geqq t_0 \quad \text{and} \quad a + \psi_0 \geqq t_0,$$

$F(u_0) \geqq \mu_0$ and $\beta(a + \psi_0) \geqq \alpha_0$ by (i) and (ii). Hence

$$\sigma(u_0) + F(u_0) \geqq -\mu_0 + \mu_0 = 0 \quad \text{on } X, \quad (a + \psi_0 - \tau)^- = 0 \in \Psi$$

and

$$D[u_0, H_\psi] - \int_X H_\psi d\sigma(u_0) + \int \psi \, d\beta(a + \psi_0)$$

$$\geqq D[H_{\psi_0}, H_\psi] + \int_X H_\psi d\mu_0 + \int \psi d\alpha_0 \geqq 0$$

for all $\psi \in \Psi_B^+$, by (iii). Hence u_0 is a supersolution of $P_B(\Gamma_B, F, \beta, \tau, \Psi)$.
Similar arguments hold for the existence of a subsolution $v_0 \leqq t_0$.

REMARK 7.1. In case $1 \in \Psi$, condition (iii) in Theorem 4 implies

$$(7.1) \qquad \int_X d\mu_0 + \int d\alpha_0 \geqq 0 \qquad (\text{resp.} \leqq 0).$$

REMARK 7.2. If we can find $\mu_0 \in \mathscr{M}_{BF}$ and $\alpha_0 \in \mathscr{N}$ satisfying (i) and (ii) such that $\mu_0 \geqq 0$ and $\alpha_0 \geqq 0$ (resp. $\mu_0 \leqq 0$ and $\alpha_0 \leqq 0$), then (iii) is always satisfied with $\psi_0 = 0$.

REMARK 7.3. In case $\Psi = \Phi_D^c(\{A_j\}_{j=1}^k)$ (i.e., the case where J is a finite set in Example 2.2), if we write $\psi_0 = \sum_{j=1}^k a_j \chi_j$, then condition (iii) is equivalent to

$$(7.2) \qquad \sum_{j=1}^k a_j D[h_j, h_m] + \int_X h_m d\mu_0 + \int_{A_m} d\alpha_0 \geqq 0 \quad (\text{resp.} \leqq 0)$$
$$\text{for all} \quad m = 1, \dots, k,$$

where $h_j = H_{\chi_j}$ $(j = 1, \dots, k)$. If $1 \notin \Psi$, then the matrix $\{D[h_j, h_m]\}_{j,m=1}^k$ is positive definite, and hence we can find $a_1, \dots, a_k$ satisfying (7.2) for given μ_0 and α_0. Therefore, (iii) in Theorem 4 is always satisfied in this case. If $1 \in \Psi$, then there exist $a_1, \dots, a_k$ satisfying (7.2) if and only if (7.1) holds, so that condition (iii) is reduced to (7.1) in this case (cf. [3; Theorem 3]).

Combining Theorem 4 with Theorem 3, we obtain

COROLLARY 1 (cf. [2; Theorem 4.2]). *Let Ψ, Γ be as in Theorem* 4. *Suppose $F: \mathscr{R}_B(\Gamma_B) \to \mathscr{M}_{BF}$ and $\beta: \Gamma_B \to \mathscr{N}$ satisfy* (F.L) *and* (β.L) *in Theorem* 3, *and* (F.M) *and* (β.M; Ψ) *in Theorem* 1 *with $Z = B$ and $\Sigma = \Gamma_B$. If there are $t_0, t_1 \in \boldsymbol{R}$ and $\psi_0, \psi_1 \in \Psi_B$ such that*

$$(7.3) \qquad \begin{cases} D[H_{\psi_0}, H_\psi] + \int_X H_\psi \, dF(t_0) + \int \psi d\beta(t_0) \geqq 0 \\ D[H_{\psi_1}, H_\psi] + \int_X H_\psi \, dF(t_1) + \int \psi d\beta(t_1) \leqq 0 \end{cases} \quad \text{for all} \quad \psi \in \Psi_B^+,$$

then $P_B(\Gamma_B, F, \beta, \tau, \Psi)$ has a solution for any $\tau \in \Gamma_B$; the solution is unique if $1 \notin \Psi$; the solution is unique up to an additive constant if $1 \in \Psi$.

In view of Remark 7.3, in case $\Psi = \Phi_D^c(\{A_j\}_{j=1}^k)$, condition (7.3) is always satisfied if $1 \notin \Psi$, and is reduced to

$$(7.4) \qquad \int_X dF(t_0) + \int d\beta(t_0) \leqq 0 \leqq \int_X dF(t_1) + \int d\beta(t_1)$$

in case $1 \in \Psi$. Thus, in this special case, we can state

COROLLARY 2. *Let* $\Gamma = \Phi_D(\Lambda_0) + \Phi_D^c(\{\Lambda_j\}_{j=1}^k)$ $(\Lambda_0 = \partial^* X \setminus \cup_{j=1}^k \Lambda_j)$ *and suppose* $F: \mathscr{R}_B(\Gamma_B) \to \mathscr{M}_{BF}$ *satisfies* (F.L) *and* (F.M) *with* $Z = B$ *and* $\Sigma = \Gamma_B$. *Suppose* $\eta_j: \boldsymbol{R} \to \boldsymbol{R}$, $j = 1, \ldots, k$, *are monotone non-decreasing and locally Lipschitz continuous, and* $\beta: \Gamma_B \to \mathscr{N}$ *is given by* (3.4). *Let* $\tau \in \Gamma_B$.

(i) *If* $\omega(\Lambda_0) > 0$, *then* $P_B(\Gamma_B, F, \beta, \tau, \Phi_D^c(\{\Lambda_j\}_{j=1}^k))$ *has a unique solution;*

(ii) *If* $\omega(\Lambda_0) = 0$, *then* $P_B(\Gamma_B, F, \beta, \tau, \Phi_D^c(\{\Lambda_j\}_{j=1}^k))$ *has a solution if and only if*

$$(7.4)' \qquad \int_X dF(t_0) + \textstyle\sum_{j=1}^k \eta_j(t_0) \leqq 0 \leqq \int_X dF(t_1) + \sum_{j=1}^k \eta_j(t_1)$$

for some $t_0, t_1 \in \boldsymbol{R}$ $(t_0 \leqq t_1)$; *in this case the solution is unique up to an additive constant.*

REMARK 7.4. By the continuity of the mapping $t \mapsto \int dF(t) + \sum_{j=1}^k \eta_j(t)$, condition (7.4)′ is equivalent to

$$(7.4)'' \qquad \int_X dF(\tilde{t}) + \textstyle\sum_{j=1}^k \eta_j(\tilde{t}) = 0 \qquad \text{for some} \quad \tilde{t} \in \boldsymbol{R}$$

(cf. [3; Corollary to Theorem 3]).

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